Sheet 5: Solutions

Problem 1

$$B(\vec{\beta}) = \begin{pmatrix} \gamma & \gamma \vec{\beta}^T \\ \gamma \vec{\beta} & E_3 + (\gamma - 1)\, \vec{n} \otimes \vec{n}^T \end{pmatrix} \tag{5.1.1}$$

$$R(D) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & D \end{pmatrix} \tag{5.1.2}$$

Note: $[R(D)]^{-1} = R(D^{-1})$ (5.1.3)

1.)

$R(D)\, B(\vec{\beta})\, R(D)^{-1} =$

$$\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \gamma & \gamma \vec{\beta}^T \\ \gamma \vec{\beta} & E_3 + (\gamma-1)\vec{n} \otimes \vec{n}^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & , & \gamma \vec{\beta}^T \\ \gamma D \vec{\beta} & , & D + (\gamma-1)(D\vec{n}) \otimes \vec{n}^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & , & \gamma (D\vec{\beta})^T \\ \gamma D \vec{\beta} & , & E_3 + (\gamma-1)\, D\vec{n} \otimes (D\vec{n})^T \end{pmatrix} \tag{5.1.4}$$

Since $\gamma = (1 - \vec{\beta}^2)^{-1/2} = (1 - (D\vec{\beta})^2)^{-1/2}$ (5.1.5)
this equals $B(D\vec{\beta})$

2.)

$$L(\vec{\beta}_1, D_1)\, L(\beta_2, D_2)$$

$$= B(\beta_1)\, R[D_1)\; B(\vec{\beta}_2)\, R(D_2)$$

$$= B(\vec{\beta}_1)\, \underbrace{R(D_1) B(\vec{\beta}_2) R(D_1^{-1})}_{B(D_1\vec{\beta}_2)}\, R(D_1)\, R(D_2)$$

$$= B(\vec{\beta}_1)\, B(D_1\vec{\beta}_2)\; R(D_1 D_2)$$

$$= B(\vec{\beta}_1 * D_1\vec{\beta}_2)\; R(T[\vec{\beta}_1, \vec{\beta}_2])\, R(D_1 D_2)$$

$$= B(\vec{\beta}_1 * D_1\vec{\beta}_2)\; R(T[\vec{\beta}_1, D_1\vec{\beta}_2] D_1 D_2) \quad (5.1.6)$$

, which proves (4)

Now, if this is to equal E_4, i.e. if $L(\beta_2, D_2) = [L(\beta_1, D_1)]^{-1}$, then

$$\beta_1 * D_1\vec{\beta}_2 = \vec{0} \Leftrightarrow D_1\vec{\beta}_2 = -\vec{\beta}_1, \quad (5.1.7)$$

or $$\vec{\beta}_2 = -D_1^{-1}\vec{\beta}_1, \quad (5.1.8)$$

implying $$T[\beta_1, D_1\vec{\beta}_2] = T[\vec{\beta}_1, -\vec{\beta}_1] = E_3, \quad (5.1.9)$$

and $$T[\beta_1, D_1\vec{\beta}_2]\, D_1 D_2 = E_3 \quad (5.1.10)$$

which, using (5.1.9), implies

$$D_2 = D_1^{-1} \quad (5.1.11)$$

Hence $$[L(\beta_1, D_1)]^{-1} = L(-D_1^{-1}\vec{\beta}_1, D_1^{-1}) \quad (5.1.12)$$

Comparison with Galilei-transformations

$$G(\vec{V}, D) = B(V)\, R(D) \qquad (5.1.13a)$$

$$G(V) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{V} & E_3 \end{pmatrix} \qquad (5.1.13b)$$

$$R(D) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & D \end{pmatrix} \qquad (5.1.13c)$$

leading to

$$G(\vec{V}_1, D_1)\, G(\vec{V}_2, D_2) = G(\vec{V}_1 + D_1 \vec{V}_2, D_1 D_2)$$

$$[G(V, D)]^{-1} = G(-D^{-1}\vec{V}, D^{-1}) \qquad (5.1.14)$$

Shows that the operation of taking the inverse is literally the same on the parameters

$$(V, D)^{-1} = (-D^{-1}\vec{V}, D^{-1}) \qquad (5.1.15)$$

$$(\vec{\beta}, D)^{-1} = (-D^{-1}\vec{\beta}, D^{-1}) \qquad (5.1.16)$$

except that $\vec{\beta} \in B_1(\mathbb{R}^3) \subset \mathbb{R}^3$ is a proper subset of the range of $\vec{V}$.

For multiplication there are differences:

$$(\vec{V}_1, D_1)(\vec{V}_2, D_2) = (\vec{V}_1 + D_1\vec{V}_2 \, , \, D_1 D_2) \qquad (5.1.17a)$$

$$(\vec{\beta}_1, D_1)(\vec{\beta}_2, D_2) = (\underset{\substack{\uparrow \\ \text{replaces "+"}}}{\beta_1 * D_1\vec{\beta}_2}, \underbrace{T[\vec{\beta}_1, D_1\vec{\beta}_2]}_{\text{replaces "}E_3\text{"}} D_1 D_2) \qquad (5.1.17b)$$

In the Galilei group pure boosts, i.e. transformations of the form $G(\vec{V}, E_3)$ form a subgroup

$$G(\vec{V}_1, E_3)\, G(V_2, E_3) = G(\vec{V}_1 + \vec{V}_2 \, ; E_3)$$

whereas in the Lorentz group they do not:

$$L(\vec{\beta}_1, E_3)\, L(\vec{\beta}_2, E_3) = L(\vec{\beta}_1 * \beta_2, \underbrace{T[\vec{\beta}_1, \beta_2]}_{\neq E_3 \text{ in general}}) \qquad (5.1.18)$$

The subgroup of pure boosts is invariant in the Galilei case

$$G(\vec{V}_1, E_3)\, G(\vec{V}_2, E_3)\, [G(\vec{V}_1, E_3)]^{-1}$$

$$= G(\vec{V}_2, E_3) \qquad (5.1.19)$$

$$R(D_1)\; G(\vec{V}_2, E_3)\; R(D_1^{-1})$$

$$= G(D_1\vec{V}_2, E_3) \qquad (5.1.20)$$

hence it is a normal subgroup.

In case of the Lorentz group the subset of pure boosts is still invariant under conjugation with pure rotations

$$R(D_1)\, B(\vec{\beta}_2)\, R(D_1^{-1}) = B(D_1 \vec{\beta}_2) \qquad (5.1.21)$$

but generally not under conjugation with pure boosts

$$B(\vec{\beta}_1)\, B(\vec{\beta}_2)\, B(-\vec{\beta}_1)$$

$$= B(\vec{\beta}_1 * \vec{\beta}_2)\, R(T[\beta_1, \vec{\beta}_2])\, B(-\beta_1)$$

$$= B(\vec{\beta}_1 * \beta_2)\, B(-T[\vec{\beta}_1, \vec{\beta}_2]\vec{\beta}_1)\, R(T[\vec{\beta}_1, \vec{\beta}_2])$$

$$= B((\beta_1 * \beta_2) * (-T[\beta_1, \beta_2]\beta_2))$$

$$\cdot R(T[\beta_1 * \beta_2, -T[\beta_1, \beta_2]\beta_1)$$

$$\cdot R(T[\vec{\beta}_1, \vec{\beta}_2])$$

$$= B((\beta_1 * \beta_2) * (-T[\beta_1, \beta_2]\beta_2))$$

$$\cdot R(\underbrace{T[\beta_1 * \beta_2, -T[\vec{\beta}_1, \vec{\beta}_2]\vec{\beta}_1] \cdot T[\vec{\beta}_1, \vec{\beta}_2]}_{(*)}) \qquad (5.1.22)$$

unless $(*) = E_3$, this is not a pure boost

Pure rotations clearly form subgroups in both cases

Problem 2

$$B(\vec{\beta}_1)\, B(\vec{\beta}_2) = B(\vec{\beta}_1 * \vec{\beta}_2)\, R(T[\vec{\beta}_1, \vec{\beta}_2]) \quad (5.2.1)$$

Conjugate that equation with $R(D)$.

- The left-hand side is

$$R(D)\, B(\vec{\beta}_1) \underbrace{R(D^{-1})\, R(D)}_{E_3 \text{ inserted}} B(\vec{\beta}_2)\, R(D^{-1})$$

$$= B(D\vec{\beta}_1)\, B(D\vec{\beta}_2) \quad (5.2.2)$$

Polar decomposing this gives

$$B(D\vec{\beta}_1 * D\vec{\beta}_2)\, R(T[D\vec{\beta}_1, D\vec{\beta}_2]). \quad (5.2.3)$$

- The right-hand side is

$$R(D)\, B(\vec{\beta}_1 * \vec{\beta}_2) \underbrace{R(D^{-1})\, R(D)}_{E_3 \text{ inserted}} R(T[\beta_1, \beta_2])\, R(D^{-1})$$

$$= B(D(\beta_1 * \beta_2))\, R(D\, T[\beta_1, \beta_2]\, D^{-1}) \quad (5.2.4)$$

which is of polar-decomposed form.
Equality of (5.2.3) and (5.2.4) gives, by uniqueness of polar decomposition, also using $B(\vec{\beta}) = E_4 \Leftrightarrow \vec{\beta} = \vec{0}$ and $R(D) = E_4 \Leftrightarrow D = E_3$:

$$D\vec{\beta}_1 * D\vec{\beta}_2 = D\,(\vec{\beta}_1 * \vec{\beta}_2) \tag{5.2.5}$$

$$T[D\vec{\beta}_1, D\vec{\beta}_2] = D\,T[\vec{\beta}_1, \vec{\beta}_2]\,D^{-1} \tag{5.2.6}$$

Problem 3

$$\cos\Theta = 1 - \frac{(\gamma_1 - 1)(\gamma_2 - 1)\sin^2\varphi}{1+\gamma} \qquad (5.3.1)$$

$$\gamma = \gamma_1 \gamma_2 (1 + \vec{\beta}_1 \cdot \vec{\beta}_2)$$

$$= \gamma_1 \gamma_2 (1 + \beta_1 \beta_2 \cos\varphi) \qquad (5.3.2)$$

$$\varphi = \sphericalangle(\vec{\beta}_1, \vec{\beta}_2)$$

Also

$$\beta_a = (1 - \gamma_a^{-2})^{1/2}$$

$$\Rightarrow \quad \gamma_a \beta_a = (\gamma_a^2 - 1)^{1/2}$$

$$\Rightarrow \quad \gamma = \gamma_1 \gamma_2 + (\gamma_1^2 - 1)^{1/2} (\gamma_2^2 - 1)^{1/2} \cos\varphi \qquad (5.3.3)$$

hence

$$\frac{\gamma - \gamma_1 \gamma_2}{(\gamma_1^2 - 1)^{1/2} (\gamma_2^2 - 1)^{1/2}} = \cos\varphi \qquad (5.3.4)$$

$$\leadsto \quad \sin^2\varphi = 1 - \cos^2\varphi = 1 - \frac{(\gamma - \gamma_1 \gamma_2)^2}{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}$$

$$= \frac{(\gamma_1^2 - 1)(\gamma_2^2 - 1) - (\gamma - \gamma_1 \gamma_2)^2}{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}$$

$$= \frac{\cancel{\gamma_1^2 \gamma_2^2} + 1 - \gamma_1^2 - \gamma_2^2 - \gamma^2 - \cancel{\gamma_1^2 \gamma_2^2} + 2\gamma\gamma_1\gamma_2}{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}$$

$$= \frac{1-\gamma^2-\gamma_1^2-\gamma_2^2+2\gamma\gamma_1\gamma_2}{(\gamma_1^2-1)(\gamma_2^2-1)} \tag{5.3.5}$$

$$\rightsquigarrow \frac{(\gamma_1-1)(\gamma_2-1)}{\gamma+1}\sin^2\varphi$$

$$= \frac{1-\gamma^2-\gamma_1^2-\gamma_2^2+2\gamma\gamma_1\gamma_2}{(\gamma+1)(\gamma_1+1)(\gamma_2-1)} \tag{5.3.6}$$

Inserting this in (5.3.1) gives

$$\cos\theta = 1-\frac{1-\gamma^2-\gamma_1^2-\gamma_2^2+2\gamma\gamma_1\gamma_2}{(\gamma+1)(\gamma_1+1)(\gamma_2+1)}$$

$$= 2-\frac{1-\gamma^2-\gamma_1^2-\gamma_2^2+2\gamma\gamma_1\gamma_2}{(\gamma+1)(\gamma_1+1)(\gamma_2+1)}-1$$

$$= \frac{2(\gamma+1)(\gamma_1+1)(\gamma_2+1)-1+\gamma^2+\gamma_1^2+\gamma_2^2-2\gamma\gamma_1\gamma_2}{(\gamma+1)(\gamma_1+1)(\gamma_2+1)}-1 \tag{5.3.7}$$

The numerator is

$$2(1+\gamma+\gamma_1+\gamma_2+\gamma\gamma_1+\gamma\gamma_2+\gamma_1\gamma_2+\gamma\gamma_1\gamma_2)$$

$$-1+\gamma^2+\gamma_1^2+\gamma_2^2-2\gamma\gamma_1\gamma_2$$

$$= 1+2\gamma+2\gamma_1+2\gamma_2+2\gamma\gamma_1+2\gamma\gamma_2+2\gamma_1\gamma_2$$

$$+\gamma^2+\gamma_1^2+\gamma_2^2 = (1+\gamma+\gamma_1+\gamma_2)^2$$

$$\Rightarrow \cos\theta = \frac{(1+\gamma+\gamma_1+\gamma_2)^2}{(1+\gamma)(1+\gamma_1)(1+\gamma_2)}-1 \tag{5.3.8}$$

The stationary points of $\cos\theta$ as a function of φ can either be determined from expression (5.3.1) or (5.3.8). Let us first start with (5.3.1)

$$\frac{d}{d\varphi}\cos\theta = 0 \iff$$

$$\frac{d}{d\varphi}\left(\frac{\sin^2\varphi}{\gamma+1}\right) = 0 \qquad (5.3.9)$$

$$\iff \frac{2\sin\varphi\cos\varphi}{\gamma+1} - \frac{\sin^2\varphi}{(\gamma+1)^2}\frac{d\gamma}{d\varphi} = 0 \qquad (5.3.10)$$

1. solution: $\sin\varphi = 0$

$$\iff \varphi = 0 \text{ or } \varphi = \pi. \qquad (5.3.11)$$

2. solution: $\sin\varphi \neq 0$

$$\Rightarrow 2(\gamma+1)\cos\varphi - \sin\varphi\frac{d\gamma}{d\varphi} = 0 \qquad (5.3.12)$$

Now, from (5.3.3)

$$\frac{d\gamma}{d\varphi} = -(\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}\sin\varphi \qquad (5.3.13)$$

Inserting this as well as the expression (5.3.3) for γ into (5.3.12) gives

$$2\left(\gamma_1\gamma_2 + [\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2}\cos\varphi + 1\right)\cos\varphi$$
$$+\sin^2\varphi\,[\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2} = 0 \qquad (5.3.14)$$

$$2(\gamma_1\gamma_2+1)\cos\varphi$$

$$+\ [\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2}\cos^2\varphi+1=0 \qquad (5.3.15)$$

$$\Leftrightarrow\quad y^2+2\,\frac{(\gamma_1\gamma_2+1)}{[\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2}}\,y+1=0 \qquad (5.3.16)$$

where $y=\cos\varphi$ (5.3.17)

$$\rightsquigarrow\quad y_{1,2}=-\frac{\gamma_1\gamma_2+1}{[\cdot\cdot]^{1/2}[\cdot\cdot]^{1/2}}$$

$$\pm\left\{\frac{(\gamma_1\gamma_2+1)^2-[\gamma_1^2-1][\gamma_2^2-1]}{[\gamma_1^2-1]\ [\gamma_2^2-1]}\right\}^{1/2} \qquad (5.3.18)$$

The term under the square-root is

$$\frac{\cancel{\gamma_1^2\gamma_2^2}+2\gamma_1\gamma_2+\cancel{1}-\cancel{1}+\gamma_1^2+\gamma_2^2-\cancel{\gamma_1^2\gamma_2^2}}{[\gamma_1^2-1][\gamma_2^2-1]}$$

$$=\frac{(\gamma_1+\gamma_2)^2}{[\gamma_1^2+1][\gamma_2^2-1]} \qquad (5.3.19)$$

hence

$$y_{1,2}=\frac{-(\gamma_1\gamma_2+1)\pm(\gamma_1+\gamma_2)}{[\gamma_1^2-1]^{1/2}\ [\gamma_2^2-1]^{1/2}}$$

upper sign: $$y_1=-\frac{(\gamma_1-1)(\gamma_2-1)}{[\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2}}$$

$$=-\left[\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}\right]^{1/2} \qquad (5.3.20)$$

lower sign : $y_2 = -\frac{(\gamma_1+1)(\gamma_2+1)}{[\gamma_1^2-1]^{1/2}[\gamma_2^2-1]^{1/2}}$

$$= -\left[\frac{(\gamma_1+1)(\gamma_2+1)}{(\gamma_1-1)(\gamma_2-1)}\right]^{1/2} \qquad (5.3.21)$$

But $|y_2| > 1$ and hence cannot correspond to any $\cos\varphi$. Therefore only y_1 is relevant :

$$\cos\varphi_* = -\left[\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}\right]^{1/2} \qquad (5.3.22)$$

That $\cos\varphi_* < 0$ means that either $\varphi_* \in (\pi/2, \pi)$ or $\in (-\pi, -\pi/2)$

The value of $\cos\theta$ at φ_* is calculated as follows :

$$\sin^2\varphi_* = 1 - \cos^2\varphi_* = 1 - \frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}$$

$$= \frac{(\gamma_1+1)(\gamma_2+1) - (\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}$$

$$= \frac{2(\gamma_1+\gamma_2)}{(\gamma_1+1)(\gamma_2+1)} \qquad (5.3.23)$$

Hence

$$\gamma_* = \gamma_1\gamma_2 + (\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}\cos\varphi_*$$

$$= \gamma_1\gamma_2 - (\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}\left[\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}\right]^{1/2}$$

$$= \gamma_1\gamma_2 - (\gamma_1-1)(\gamma_2-1) = \gamma_1+\gamma_2-1 \qquad (5.3.24)$$

$$\Rightarrow \gamma_* + 1 = \gamma_1 + \gamma_2 \tag{5.3.25}$$

Inserting (5.3.23) and (5.3.25) into (5.3.1) gives

$$\cos\Theta_* = 1 - \frac{(\gamma_1 - 1)(\gamma_2 - 1)}{\gamma_* + 1} \sin^2\varphi_*$$

$$= 1 - \frac{(\gamma_1 - 1)(\gamma_2 - 1)}{\cancel{(\gamma_1 + \gamma_2)}} \, 2 \, \frac{\cancel{(\gamma_1 + \gamma_2)}}{(\gamma_1 + 1)(\gamma_2 + 1)}$$

$$= 1 - 2 \, \frac{(\gamma_1 - 1)(\gamma_2 - 1)}{(\gamma_1 + 1)(\gamma_2 + 1)} \tag{5.3.26}$$

$$= 1 - 2\cos^2\varphi_* = -\cos(2\varphi_*)$$
$$= \cos(\pi - 2\varphi_*) \tag{5.3.27}$$

From this we can only infer that $|\Theta_*| = 2\varphi_* - \pi$ and if $\varphi_* \in (\frac{\pi}{2}, \pi)$ that $|\Theta_*| \in (0, \pi)$. But from the special example of perpendicular equal-magnitude velocities (Sheet 4, Problem 3) we know that Θ is oriented oppositely to φ. Hence, if $\varphi_* > 0$,

$$\Theta_* = 2\varphi_* - \pi \in (-\pi, 0) \tag{5.3.28}$$

Now, $|\Theta_*| > \pi/2$ if $\cos\Theta_* < 0$. According to (5.3.26) this is the case iff

$$(\gamma_1 - 1)(\gamma_2 - 1) > \frac{1}{2} (\gamma_1 + 1)(\gamma_2 + 1) \tag{5.3.29}$$

Let us specialise to $\beta_1 = \beta_2 = \beta$ and accordingly $\gamma_1 = \gamma_2 = \gamma$. Then (5.3.29) reads

$$(\gamma - 1)^2 > \frac{1}{2}(\gamma + 1)^2 \quad , \text{or}$$

$$\gamma^2 - 2\gamma + 1 > \frac{1}{2}\gamma^2 + \gamma + \frac{1}{2} \text{, or}$$

$$\gamma^2 - 6\gamma + 1 > 0 \tag{5.3.30}$$

The parabola $P(\gamma) = \gamma^2 - 6\gamma + 1$ has zeros at

$$\gamma_{1,2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

$$\gamma_1 = 3 + 2\sqrt{2} = 5.83 \tag{5.3.31a}$$

$$\gamma_2 = 3 - 2\sqrt{2} = 0.17 \tag{5.3.31b}$$

Since our γ is per definition > 1 only the first root matters. Hence we see that $\gamma^2 - 6\gamma + 1 > 0$ iff

$$\gamma > 3 + 2\sqrt{2} \cong 5.84 \tag{5.3.32}$$

corresponding to

$$\beta = (1 - \gamma^{-2})^{1/2} > 0.985 \tag{5.3.33}$$

As the calculation of φ_m and Θ_m look rather complicated, let us try the same calculation starting from (5.3.8) rather than (5.3.1). In (5.3.8) the only φ-dependence is through

$$\gamma = \gamma_1 \gamma_2 + (\gamma_1^2 - 1)^{1/2} (\gamma_2^2 - 1)^{1/2} \cos\varphi \qquad (5.3.34)$$

$$\Rightarrow \gamma' := \frac{d\gamma}{d\varphi} = -(\gamma_1^2 - 1)^{1/2} (\gamma_2^2 - 1)^{1/2} \sin\varphi \qquad (5.3.35)$$

$$\Rightarrow \frac{d\cos\Theta}{d\varphi} = \frac{d\cos\Theta}{d\gamma} \gamma' = 0 \qquad (5.3.36)$$

$$\Rightarrow 1.\ \text{solution} \quad \gamma' = 0 \Leftrightarrow \varphi = 0, \pi \qquad (5.3.37)$$

$$2.\ \text{solution} \quad d\cos\Theta / d\gamma = 0 \qquad (5.3.38)$$

Now,

$$\frac{d\cos\Theta}{d\gamma} = 0 \Leftrightarrow \frac{d}{d\gamma} \frac{(1+\gamma+\gamma_1+\gamma_2)^2}{(1+\gamma)} = 0 \qquad (5.3.39)$$

$$\Leftrightarrow \frac{2(1+\gamma+\gamma_1+\gamma_2)}{1+\gamma} - \frac{(1+\gamma+\gamma_1+\gamma_2)^2}{(1+\gamma)^2} = 0 \qquad (5.3.40)$$

$$\Leftrightarrow [2(1+\gamma) - (1+\gamma+\gamma_1+\gamma_2)](1+\gamma+\gamma_1+\gamma_2) = 0$$

$$\Leftrightarrow \gamma + 1 = \gamma_1 + \gamma_2 = \gamma_m + 1 \qquad (5.3.41)$$

Which is (5.3.25), but much quicker arrived at!

The value of $\cos\theta$ at that γ is

$$\cos\theta = \frac{(1+\gamma_m+\gamma_2+\gamma_2)^2}{(1+\gamma_1)(1+\gamma_1)(1+\gamma_2)} - 1$$

$$= \frac{4(\gamma_1+\gamma_2)^2}{(\gamma_1+\gamma_2)(1+\gamma_1)(1+\gamma_2)} - 1$$

$$= \frac{4(\gamma_1+\gamma_2) - (\gamma_1+1)(\gamma_2+1)}{(\gamma_1+1)(\gamma_2+1)}$$

$$= 1 - \frac{4(\gamma_1+\gamma_2) - 2(\gamma_1+1)(\gamma_2+1)}{(\gamma_1+1)(\gamma_2+1)}$$

$$= 1 - 2\,\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)} \qquad (5.3.42)$$

which is just (5.3.26). The value of φ_m would now be calculated via

$$\gamma_m = \gamma_1\gamma_2 + (\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}\cos\varphi_m \qquad (5.3.43)$$

$$\rightsquigarrow \cos\varphi_m = \frac{\gamma_m - \gamma_1\gamma_2}{(\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}}$$

$$= \frac{\gamma_1+\gamma_2-1-\gamma_1\gamma_2}{(\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}}$$

$$= -\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1^2-1)^{1/2}(\gamma_2^2-1)^{1/2}} = -\left[\frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1+1)(\gamma_2+1)}\right]^{1/2}. \qquad (5.3.44)$$

So, starting from (5.3.8) is a little easier.