Sheet 6: Solutions

Problem 1

$$\eta : V \times V \to \mathbb{R} \qquad \mathrm{sign}(\eta) = (1, \underbrace{-1, \cdots, -1}_{(n-1)}) \qquad (6.1.1)$$

$w \in V - \{0\}$

$$w^{\perp} := \{ v \in V : \eta(v, w) = 0 \} \qquad (6.1.2)$$

Consider linear map

$$\phi : V \to \mathbb{R} \quad , \quad v \mapsto \eta(v, w) \qquad (6.1.3)$$

1) From

$$\dim(\ker(\phi)) + \dim(\mathrm{im}(\phi)) = \dim(V) \qquad (6.1.4)$$

We know that

$$\dim\{w^{\perp}\} = n - 1 \qquad (6.1.5)$$

Now,

- $w \in w^{\perp} \Rightarrow \eta(w, w) = 0$
  $\Rightarrow w$ is lightlike $\qquad (6.1.6)$

- w is lightlike $\Leftrightarrow \eta(w, w) = 0$
  $\Rightarrow w \in w^{\perp}$ $\qquad (6.1.7)$

Hence:

$$w \in w^{\perp} \iff w \text{ is lightlike} \qquad (6.1.8)$$

2) If $\eta(W,W) \neq 0$ then $\bar{\eta} := \eta|_{W^\perp}$ is non-generate.

Proof by contradiction: Suppose there exists a $u \in W^\perp$ s.t.

$$\eta(u, v) = 0 \quad \forall v \in W^\perp \tag{6.1.9}$$

Then u is $\perp$ to W and also $\perp$ to $W^\perp$. Hence u can only be non-zero if W and $W^\perp$ are linearly dependent, i.e. iff $W \in W^\perp$, which is, as we have seen in 1), the case iff W is lightlike. But that is excluded by hypothesis.

If W is timelike, i.e. $\eta(W,W) = 1$, W cannot be contained in any maximal subspace restricted to which η is negative definite; hence $\bar{\eta}$ is negative definite in this case.
If W is spacelike, i.e. $\eta(W,W) = -1$, W is contained in some maximal subspace of V restricted to which η is negative definite; hence $\bar{\eta}$ is of signature $(1, n-2)$ in this case.

Generally, if $\eta(W,W) \neq 0$ we can assume w.l.o.g. that $\eta(W,W) = \pm 1$.

We can choose an orthonormal basis $\{e_0, e_1, \dots, e_{n-1}\}$ with

$$\eta_{ab} := \eta(e_a, e_b) = \mathrm{diag}(1, -1, \dots, -1) \qquad (6.1.10)$$

where $W = e_0$ if $\eta(W, W) = 1$ (6.1.11a)

or $W = e_1$ if $\eta(W, W) = -1$. (6.1.11b)

If $\{\theta^0, \theta^1, \dots, \theta^{n-1}\}$ is dual to $\{e_0, \dots e_n\}$, i.e.

$$\theta^a(e_b) = \delta^a{}_b \qquad (6.1.12)$$

then

$$\eta = \theta^0 \otimes \theta^0 - \sum_{a=1}^{n-1} \theta^a \otimes \theta^a$$

$$= \eta_{ab}\, \theta^a \otimes \theta^b \qquad (6.1.13)$$

and

$$\bar{\eta} = \eta|_{W^\perp} = \begin{cases} -\sum_{i=1}^{n-1} \theta^a \otimes \theta^a & \text{if } W \text{ timelike} \\ \theta^0 \otimes \theta^0 - \sum_{i=2}^{n-1} \theta^a \otimes \theta^a & \text{if } W \text{ spacelike} \end{cases} \qquad (6.1.14)$$

3.)

a.) Span$\{V, W\}$ timelike

Choose basis $\{e_0, e_1\}$ with dual bases $\{\theta^0, \theta^1\}$ such that

$$\eta = \theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1 \qquad (6.1.15)$$

or $\eta_{ab} = \eta(e_a, e_b) = \mathrm{diag}(1, -1)$

Then

$$V = V^0 e_0 + V^1 e_1$$
$$W = W^0 e_0 + W^1 e_1 \qquad (6.1.16)$$

and

$$V^2 = (V^0)^2 - (V^1)^2$$
$$W^2 = (W^0)^2 - (W^1)^2 \qquad (6.1.17)$$

$$(V \cdot W) = V^0 W^0 - V^1 W^1 \qquad (6.1.18)$$

Then

$$V^2 W^2 - (V \cdot W)^2 =$$

$$((V^0)^2 - (V^1)^2)((W^0)^2 - (W^1)^2) - (V^0 W^0 - V^1 W^1)^2$$

$$= -(V^0)^2 (W^1)^2 - (W^0)^2 (V^1)^2 + 2 V^0 W^0 V^1 W^1$$

$$= -(V^0 W^1 - V^1 W^0)^2 \leq 0 \qquad (6.1.19)$$

with equality iff $\det\begin{pmatrix} V^0 & W^0 \\ V^1 & W^1 \end{pmatrix} = 0$, i.e. iff V and W are linearly dependent.

b) Span{V, W} spacelike ⇒

$$\eta = -\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 \tag{6.1.20}$$

or $\eta(e_a, e_b) = \mathrm{diag}(-1, -1)$ (6.1.21)

$$V = V^1 e_1 + V^2 e_2$$
$$W = W^1 e_1 + W^2 e_2 \tag{6.1.22}$$

$$V^2 = -(V^1)^2 - (V^2)^2 \tag{6.1.23}$$

$$W^2 = -(W^1)^2 - (W^2)^2 \tag{6.1.24}$$

$$(V \cdot W) = -V^1 W^1 - V^2 W^2 \tag{6.1.25}$$

$$V^2 \cdot W^2 - (V \cdot W)^2 =$$

$$((V^1)^2 + (V^2)^2)((W^1)^2 + (W^2)^2)$$
$$- (V^1 W^1 + V^2 W^2)^2$$

$$= (V^1)^2 (W^2)^2 + (V^2)^2 (W^1)^2 - 2 V^1 W^1 V^2 W^2$$

$$\geq (V^1 W^2 - V^2 W^1)^2 \tag{6.1.26}$$

with equality iff $\det \begin{pmatrix} V^1 & W^1 \\ V^2 & W^2 \end{pmatrix} = 0$

$= 0$, i.e. iff V an W

are linearly dependent. This is of course just the ordinary Cauchy-Schwarz inequality, that holds for pos. def. and neg. def. metrics alike.

e.) Span $\{V, W\}$ lightlike $\Rightarrow$

$\exists$ basis $\{e_0, e_1\}$ with $e_0 =$ lightlike and e_1 spacelike and e_0, e_1 orthogonal

$$\eta_{ab} = \eta(e_0, e_1) = \mathrm{diag}(0, -1) \qquad (6.1.27)$$

$$V = V^0 e_0 + V^1 e_1$$
$$W = W^0 e_0 + W^1 e_1 \qquad (6.1.28)$$

$$V^2 = -(V^1)^2 \qquad (6.1.29)$$

$$W^2 = -(W^1)^2 \qquad (6.1.30)$$

$$V \cdot W = -V^1 W^1 \qquad (6.1.31)$$

$$\Rightarrow V^2 W^2 - (V \cdot W)^2 = -(V^1)^2 (W^1)^2 + (V^1 W^1)^2$$
$$= 0. \qquad (6.1.32)$$

Problem 2

Let $f : V \to V$ be such that

$$\eta(f(v), f(w)) = \eta(v, w) \qquad (6.2.1)$$

for all $v, w \in V$ and where

$$\eta : V \times V \to \mathbb{R} \qquad (6.2.2)$$

is non degenerate. Assume f to be surjective; no other conditions on f are assumed!

Consider the function

$$I = I(a, b, u, v, w)$$
$$= \eta(a f(u) + b f(v) - f(au + bv), w) \qquad (6.2.3)$$

Since η is non-degenerate, we can deduce linearity of f, i.e.

$$f(au + bv) = a f(u) + b f(v) \qquad (6.2.4)$$

if $I = 0 \;\; \forall \; w$.

Now, since f is surjective any $w \in V$ can be written in the form

$$w = f(w') \qquad (6.2.5)$$

for some w'

But then, since φ preserves η,

$$I = \eta(a\varphi(u) + b\varphi(v) - \varphi(au+bv), \varphi(w'))$$

$$= a\,\eta(\varphi(u), \varphi(w'))$$

$$+ b\,\eta(\varphi(v), \varphi(w'))$$

$$- \eta(\varphi(au+bv)), \varphi(w'))$$

$$= a\,\eta(u, w') + b\,\eta(v, w')$$

$$- \eta(au+bv, w')$$

$$= 0 \quad \text{by bilinearity of } \eta. \qquad (6.2.6)$$