Sheet 7 : Solutions

Problem 1

Let N and Q be groups and

$$\alpha : Q \to \mathrm{Aut}(N)$$
$$q \mapsto \alpha_q \qquad (7.1.1)$$

be a homomorphism. That means that

$$\alpha_{q_1} \circ \alpha_{q_2} = \alpha_{q_1 q_2} \qquad (7.1.2)$$

and $$\alpha_q(n_1 n_2) = \alpha_q(n_1)\,\alpha_q(n_2) \qquad (7.1.3)$$

We denote by e_N and e_Q the identity elements of N and Q.

The semi-direct product $H \rtimes_\alpha Q$ of H with G relative to α is defined to be the set $H \times G$ with the following group multiplication

1) $$(n_1, q_1)(n_2, q_2) = (n_1\, \alpha_{q_1}(n_2),\, q_1 q_2) \qquad (7.1.4)$$

The identity element in $H \rtimes_\alpha Q$ is (e_H, e_Q) since, according to (7.1.4)

$$(e_H, e_Q)(n, q) = (e_H\, \alpha_{e_Q}(n),\, e_Q q)$$
$$= (n, q) \qquad (7.1.5)$$

since $\alpha_{e_Q} = id|_H$, which follows from

(7.1.2) for $q_1 = q_2 = e_Q$ (as for any automorphism). Likewise

$$(n,q)(e_H, e_Q) = (n \cdot \alpha_q(e_H), q e_Q)$$
$$= (n, q) \qquad (7.1.6)$$

since $\alpha_q(e_H) = e_H$ for any $q \in Q$, which follows from (7.1.3) for $n_1 = n_2 = e_H$.

The inverse element of (n,q) is

$(n,q)^{-1} = (\alpha_{q^{-1}}(n^{-1}), q^{-1})$, since

$$(n,q)^{-1}(n,q) = (\alpha_{q^{-1}}(n^{-1}), q^{-1})(n,q)$$
$$= (\alpha_{q^{-1}}(n^{-1}) \cdot \alpha_{q^{-1}}(n), q^{-1}q)$$
$$= (\alpha_{q^{-1}}(n^{-1}n), e_Q) = (e_H, e_G) \qquad (7.1.7)$$

$$(n,q)(n,q)^{-1} = (n,q)(\alpha_{q^{-1}}(n^{-1}), q^{-1})$$
$$= (n\, \alpha_q(\alpha_{q^{-1}}(n^{-1})), q q^{-1})$$
$$= (n\, \alpha_{qq^{-1}}(n^{-1}), e_Q)$$
$$= (e_H, e_Q). \qquad (7.1.8)$$

For associativity we consider (n_1, q_1), (n_2, q_2), and (n_3, q_3) in $N \times_\alpha Q$ and compute

$$[(n_1, q_1)(n_2, q_2)](n_3, q_3) =: X \qquad (7.1.10)$$

and $$(n_1, q_1)[(n_2, q_2)(n_3, q_3)] =: y \qquad (7.1.11)$$

and then compare

$$X = (n_1 \alpha_{q_1}(n_2), q_1 q_2)(n_3, q_3)$$

$$= (n_1 \alpha_{q_1}(n_2) \alpha_{q_1 q_2}(n_3), q_1 q_2 q_3)$$

$$= n_1 \alpha_{q_1}(n_2 \alpha_{q_2}(n_3)), q_1 q_2 q_3) \qquad (7.1.12)$$

Here we used

$$\alpha_{q_1}(n_2) \alpha_{q_1 q_2}(n_3)$$

$$= \alpha_{q_1}(n_2) \alpha_{q_1}(\alpha_{q_2}(n_3))$$

$$= \alpha_{q_1}(n_1 \alpha_{q_2}(n_3)) \qquad (7.1.13)$$

$$y = (n_1, q_1)[n_2 \alpha_{q_2}(n_3), q_2 q_3]$$

$$= (n_1 \alpha_{q_1}(n_2 \alpha_{q_2}(n_3)), q_1 q_2 q_3) \qquad (7.1.14)$$

showing equality

2)
$$N' := N \times \{e_Q\} = \{(n, e_Q) : n \in N\}$$
$$Q' := \{e_N\} \times Q = \{(e_N, q) : q \in Q\} \qquad (7.1.15)$$

They are indeed subgroups:

$$(n_1, e_Q)(n_2, e_Q) = (n_1 \alpha_{e_Q}(n_2), e_Q)$$
$$= (n_1 n_2, e_Q) \qquad (7.1.16)$$

and $(e_H, e_Q) \in N'$;

$$(e_H, q_1)(e_H, q_2) = (e_H \alpha_{q_1}(e_H), q_1 q_2)$$
$$= (e_H, q_1 q_2) \qquad (7.1.17)$$

and $(e_H, e_Q) \in Q'$

Each subgroup is invariant under conjugation with its own element. Moreover each $(n, q) \in H \rtimes_\alpha Q$ is a product of $(n, e_Q) \in N'$ and $(e_N, q) \in Q'$

$$(n, e_Q)(e_H, q) = (n \alpha_{e_Q}(e_H), e_Q q)$$
$$= (n, q) \qquad (7.1.18)$$

or

$$(e_H, q)(\alpha^{-1}_q(n), e_Q) = (e_H \alpha_q(\alpha^{-1}_q(n)), q)$$
$$= (n, q) \qquad (7.1.19)$$

Hence we only need to look at the conjugations of N' with elements of Q' and vice versa.

- $(e_H, q)(n, e_G)(e_H, q)^{-1}$

$$= (e_H\, \alpha_q(n), q)(\underbrace{\alpha_q^{-1}(e_H)}_{e_H}, q^{-1})$$

$$= (\alpha_q(n)\, \alpha_q(e_H), q q^{-1})$$

$$= (\alpha_q(n), e_Q) \qquad (7.1.20)$$

- $(n, e_Q)(e_H, q)(n, e_Q)^{-1}$

$$= (n, q)(\alpha_{e_Q}^{-1}(n^{-1}), e_Q)$$

$$= (n, q)(n^{-1}, e_G)$$

$$= (n\, \alpha_q(n^{-1}), q)$$

$$= (n\, (\alpha_q(n))^{-1}, q) \qquad (7.1.21)$$

which is only $\in Q'$ for all n iff

$$n = \alpha_q(n) \quad \text{for all } n$$

$$\Leftrightarrow \alpha_q = id|_N \qquad (7.1.22)$$

and this must be true for any $q \in Q$. Which is the case iff $\ker(\alpha) = G$, i.e. α is trivial.

3)
$$Q'_n := (n, e_Q)\, Q'\, (n, e_Q)^{-1} \tag{7.1.23}$$

For $(e_H, q) \in Q'$ get

$$(n, e_Q)(e_H, q)(n^{-1}, e_Q)$$
$$= (n, q)(n^{-1}, e_Q)$$
$$= (n\, \alpha_q(n^{-1}), q)$$
$$= (n[\alpha_q(n)]^{-1}, q) \tag{7.1.24}$$

Hence $(n_1, e_Q)(e_N, q_1)(n_1, e_Q)^{-1}$

$$= (n_2, e_Q)(e_N, q_2)(n_2, e_Q) \tag{7.1.25}$$

iff $\quad q_1 = q_2 = q$ $\quad$ (7.1.26)

and $\quad n_1\, \alpha_q(n_1^{-1}) = n_2\, \alpha_q(n_2^{-1})$ $\quad$ (7.1.27a)

or $\quad \alpha_q(n_1^{-1} n_2) = n_1^{-1} n_2$ $\quad$ (7.1.27b)

$$\Rightarrow Q'_{n_1} \cap Q'_{n_2} = \left\{ \bigcup_{q \in Q} (n_1[\alpha_q(n_1)]^{-1}, q) \right\}$$
$$\cap \left\{ \bigcup_{q \in Q} (n_2[\alpha_q(n_2)]^{-1}, q) \right\}$$

$$= \{ (n_1(\alpha_q(n_1))^{-1}, q) : q \in Q,\ \alpha_q(n_1^{-1} n_2) = n_1^{-1} n_2 \} \tag{7.1.28}$$

Take euclidean group $E_3 = \mathbb{R}^3 \rtimes SO(3)$ as example. The multiplication law is

$$(\vec{a}_1, D_1)(\vec{a}_2, D_2) = (a_1 + D_1 a_2, D_1 D_2) \quad (7.1.29)$$

The rotation subgroup :

$$SO(3)' = \{(\vec{0}, D) : D \in SO(3)\} \quad (7.1.30)$$

transforms under conjugation with translation $\vec{a}$

$$(\vec{a}, E_3)(\vec{0}, D)(-\vec{a}, E_3)$$

$$= (\vec{a}, D)(-\vec{a}, E_3) = (\vec{a} - D\vec{a}, D) \quad (7.1.31)$$

The subset

$$[SO(3)]_{\vec{a}} := \{(\vec{a} - D\vec{a}, D) : D \in SO(3)\}$$
$$\subset E_3 \quad (7.1.32)$$

is a subgroup (clearly, it is the conjugate to $SO(3)'$):

$$(\vec{a} - D_1\vec{a}, D_1)(\vec{a} - D_2\vec{a}, D_2)$$
$$= (\vec{a} - D_1\vec{a} + D_1\vec{a} - D_1D_2\vec{a}, D_1D_2)$$
$$= (\vec{a} - D_1D_2\vec{a}, D_1D_2), \quad (7.1.33)$$

and it also contains $(\vec{0}, E_3)$.

Note that $[SO(3)]_{\vec{a}}$ consists of all rotations leaving the point $\vec{a}$ fixed. The intersection

$$[SO(3)]_{\vec{a}_1} \cap [SO(3)]_{\vec{a}_2} \qquad (7.1.34)$$

therefore consists of all rotations leaving $\vec{a}_1$ as well as $\vec{a}_2$ fixed. This is the set of all rotations that fix all points on the straight line connecting $\vec{a}_1$ and $\vec{a}_2$, i.e. with axis along $\vec{a}_2 - \vec{a}_1$.

Since in the case $\mathbb{R}^3 \rtimes_d SO(3)$

$$d_D(\vec{a}) = D\vec{a} \qquad (7.1.35)$$

(7.1.27) reads

$$D(\vec{a}_2 - \vec{a}_1) = \vec{a}_2 - \vec{a}_2 \qquad (7.1.36)$$

Which is precisely saying that D must be a rotation about the axis $\vec{a}_2 - \vec{a}_1$. Hence

$$[SO(3)]_{\vec{a}_1} \cap [SO(3)]_{\vec{a}_2} \cong SO(2) \qquad (7.1.37)$$

= abelian subgroup of rotations around axis $\vec{a}_2 - \vec{a}_1$.

4) Let G, N and Q be groups and embeddings (injective homomorphisms)

$$i : N \hookrightarrow G, \qquad (7.1.38a)$$
$$j : Q \hookrightarrow G, \qquad (7.1.38b)$$

where

$$N' := i(N) \subset G \quad \text{(image of } i\text{)} \qquad (7.1.39a)$$

$$Q' := j(Q) \subset G \quad \text{(image of } j\text{)} \qquad (7.1.39b)$$

Assume

1.) $N' \subset G$ is normal

2.) $N' \cdot Q' = \{n' \cdot q' : n' \in N', q' \in Q'\} = G$

3.) $N' \cap Q' = \{e_G\}$

group identity of G.

Consider the map

$$\phi : N \times Q \to G$$

$$(n, q) \mapsto \phi(n,q) := i(n)\, j(q) = n' \cdot q' \qquad (7.1.40)$$

where
$$\left.\begin{aligned} n' &:= i(n) \in N' \subset G \\ q' &:= j(q) \in Q' \subset G \end{aligned}\right\} \qquad (7.1.41)$$

Condition immediately implies that the map ϕ is surjective

Let us take the product in G of two images

$$\phi(n_1, q_1)\, \phi(n_2, q_2)$$

$$= i(n_1)\, j(q_1)\, i(n_2)\, j(q_2)$$

$$= i(n_1)\, j(q_1)\, i(n_2)\, [j(q_1)]^{-1}\, j(q_1 q_2) \qquad (7.1.42)$$

Since N' is normal in G

$$j(q_1)\, i(n_2)\, [j(q_1)]^{-1} \in N' \qquad (7.1.43)$$

Hence there is an element $n \in N$, depending on q_1 and n_2, the image $i(n)$ of which is that element. We set $n := \alpha_{q_1}(n_2)$

$$i(\alpha_{q_1}(n_2)) = j(q_1)\, i(n_2)\, [j(q_1)]^{-1} \qquad (7.1.44)$$

Note that on its image N', i has an inverse, $i_{N'}^{-1} : N' \to N$, which is a homomorphism. Hence we can define

$$\alpha_q(n) := i_{N'}^{-1}\left[\, j(q)\, i(n)\, (j(q))^{-1}\right] \qquad (7.1.45)$$

For fixed q, $n \to \alpha_q(n)$ defines an element of $\mathrm{Aut}(N)$: Indeed, conjugation in N' by elements of Q' define homomorphisms of N' to itself which are injective ($q' n' (q')^{-1} = e_{H'} \Rightarrow n' = (q')^{-1} e_{H'} q' = e_{H'}$) and surjective ($q' n' (q')^{-1} = n \Rightarrow n' = (q')^{-1} n \, q'$). Hence α_q is indeed $\in \mathrm{Aut}(N)$. Moreover, the map

$$Q \ni q \mapsto \alpha_q \in \mathrm{Aut}(N) \qquad (7.1.46)$$

is an homomorphism since j is a homomorphism

$$\alpha_{q_1} \circ \alpha_{q_2}(n) =$$

$$\alpha_{q_1}\left(i^{-1}_{N'} \left[j(q_2)\, i(n) [j(q_2)]^{-1} \right] \right)$$

$$= i^{-1}_{N'} \left[j(q_1) j(q_2)\, i(n) (j(q_2))^{-1} (j(q_1))^{-1} \right]$$

$$= i^{-1}_{N'} \left[j(q_1 q_2)\, i(n) (j(q_1 q_2))^{-1} \right]$$

$$= \alpha_{q_1 q_2}(n). \qquad (7.1.47)$$

Using that α, the map ϕ from (7.1.40) becomes a homomorphism from $N \rtimes_\alpha Q$ to G, of which we already know that it is surjective. Injectivity now follows from condition 3), i.e. $N' \cap Q' = \{e_G\}$, for if

$$\phi(n, q) = i(n) \cdot j(q) = e_G \qquad (7.1.48)$$

$$\Leftrightarrow \quad N' \ni i(n) = j(q^{-1}) \in Q' \qquad (7.1.49)$$

and hence $i(n) = j(q) = e_G$

$$\Leftrightarrow \quad n = e_H, \; q \in e_Q \tag{4.1.50}$$

All this proves that G is isomorphic to $N \rtimes_\alpha Q$ where $\alpha_q \in \mathrm{Aut}(N)$ is such that

$$i\,(\alpha_q(n)) = j(q)\, i(n)\, (j(q))^{-1} . \tag{4.1.51}$$

or

$$\alpha_q(n) = i_{N'}^{-1} \left[j(q)\, i(n) (j(q))^{-1} \right] \tag{4.1.52}$$

5) Suppose

$$\pi : G \to Q \tag{4.1.53}$$

is a projection homomorphism, i.e. surjective and

$$\pi \circ \pi = \pi \tag{4.1.54}$$

Then $\ker(\pi) =: N$ (4.1.55)

is a normal subgroup.

Now, $N \cap Q = \{e\}$ (4.1.56)

Proof: Let $g \in N \cap Q$, then

$$\left.\begin{array}{lll} & \pi(g) = e & g \in N \\ \text{and} & g = \pi(h) & g \in Q = \mathrm{Im}(\pi) \end{array}\right\} \tag{4.1.57}$$

$$\Rightarrow \pi(\pi(h)) = \pi(h) = e \Rightarrow h \in \ker(\pi) \tag{4.1.58}$$

and hence $g = \pi(h) = e$ (4.1.59)

Moreover

$$N \cdot Q = G \tag{4.1.60}$$

Proof. Let $g \in G$

let $q := \pi(g) \in Q$

and define

$$n := g\, q^{-1}. \tag{4.1.61}$$

Then

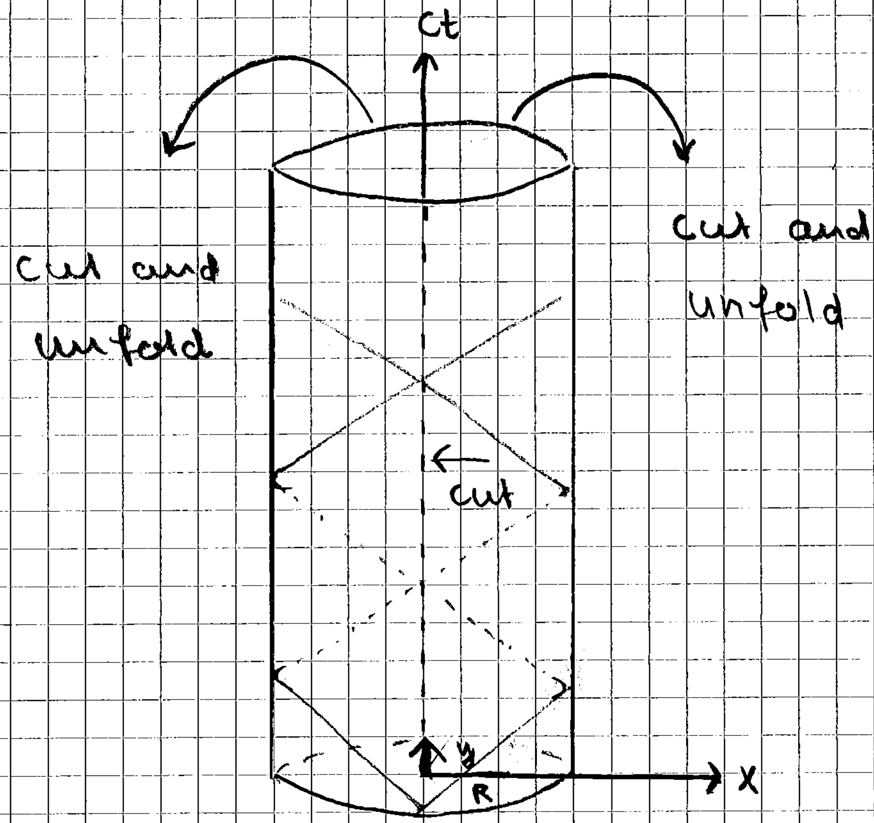
$$\begin{aligned} \pi(n) &= \pi(g)\,(\pi(q))^{-1} \\ &= q\,(\pi(\pi(g)))^{-1} \\ &= q\,(\pi(g))^{-1} \\ &= q \cdot q^{-1} = e \; ; \end{aligned} \tag{4.1.62}$$

hence $h \in N$ (4.1.63)

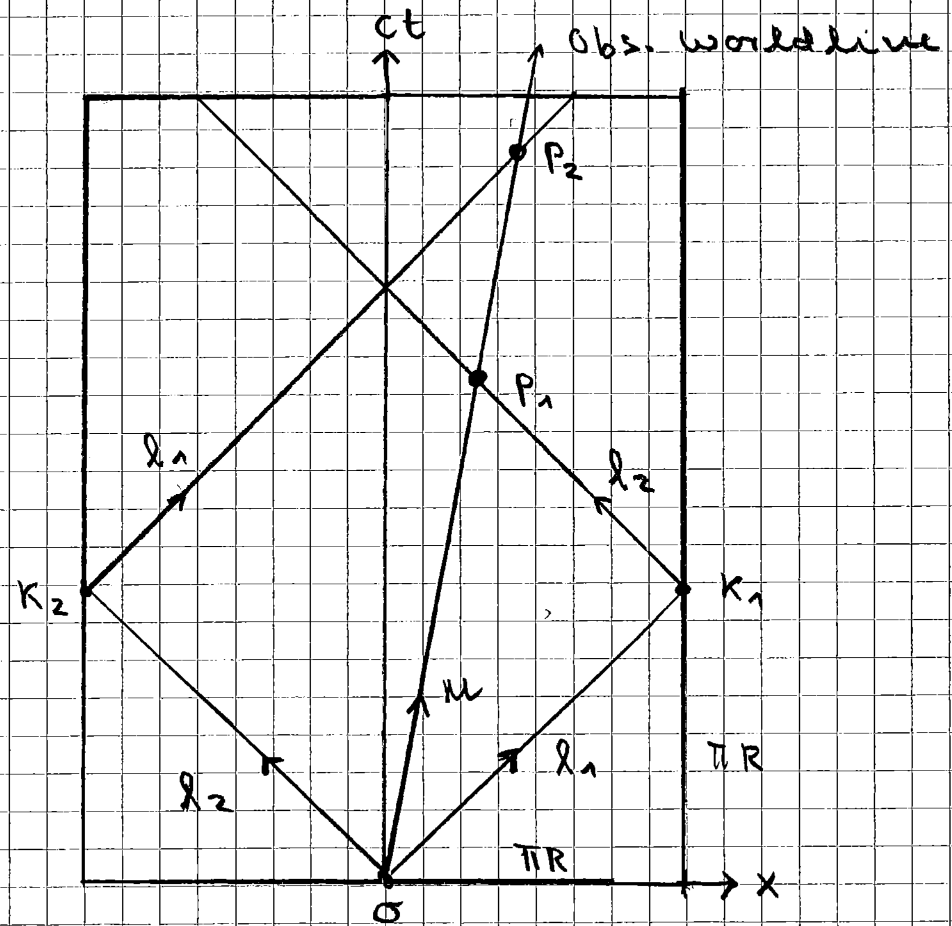
and $g = n \cdot q$ (4.1.64)

We have thus shown that N and Q are subgroups of G with N normal and $N \cap Q = \{e\}$ and $N \cdot Q = G$. Part 4) now proves the claim.

Problem 2.



(4.2.1)



(4.2.2)

The metric on the two-dimensional sheet in (ct, x) - coordinates is

$$\eta = c\, dt^2 - dx^2 \qquad (4.2.3)$$

The unit vector in the worldline direction of the observer is

$$u = (u^0, u^1)$$

$$u^2 = (u^0)^2 - (u^1)^2 = 1 \qquad (4.2.4)$$

and $\beta = \frac{v}{c} = \frac{u^1}{u^0}$ (4.2.5)

hence $(u^0)^2 - \beta^2 (u^0)^2 = 1$

$\rightsquigarrow \quad u^0 = \gamma = (1-\beta^2)^{-1/2}$

$\rightsquigarrow \quad u = \gamma(1, \beta)$ (4.2.6)

For the lightlike directions l_1 and l_2 have

$$l_1 = (1, 1) \qquad (4.2.7a)$$
$$l_2 = (1, -1) \qquad (4.2.7b)$$

The light signal in the direction of running, i.e. pos. x-direction, is tangent to l_1, starts at O, travels to K_1, re-emerges at K_2 and continues until it meets the observer's world line at p_2. The light signal in the opposite direction, i.e. negative x-direction, is tangent to ℓ_2, starts at O, travels to K_2, re-emerges at K_1 and continues until it meets the observer's world-line at p_1.

We want to calculate the Minkowski distance of the segment $\overline{p_1 p_2}$. For this we calculate the distances

$$\left.\begin{aligned} S_1 &:= |\overrightarrow{OP_1}|_\eta \\ S_2 &:= |\overrightarrow{OP_2}|_\eta \end{aligned}\right\} \text{Minkowski lengths} \quad (4.2.8)$$

and take the difference

1.) S_1 is obtained as follows:

We have $\quad \overrightarrow{OK_1} = \pi R \, \ell_1 \qquad (4.2.9)$

and $$\overrightarrow{OP_1} = \overrightarrow{OK_1} + \overrightarrow{K_1P_1} \qquad (4.2.10)$$

$$\Leftrightarrow \quad S_1 u = \pi R \ell_1 + \overrightarrow{K_1P_1}$$

$$\Leftrightarrow \quad S_1 u - \pi R \ell_1 = \overrightarrow{K_1P_1} \qquad (4.2.11)$$

As $\overrightarrow{K_1P_1} \sim \ell_2$ = lightlike, we have
$$(S_1 u - \pi R \ell_1)^2 = 0 \qquad (4.2.12)$$

Mink. square

Now,

$$u^2 = 1 \quad (4.2.13)$$

$$u \cdot l_1 = \gamma(1,\beta)\cdot(1,1) = \gamma(1-\beta) \quad (4.2.14)$$

$$l_1^2 = 0 \quad (4.2.15)$$

hence

$$S_1^2 - 2S_1 \pi R\, \gamma(1-\beta) = 0 \quad (4.2.16)$$

Neglecting the trivial solution $S_1 = 0$, get

$$S_1 = 2\pi R\, \gamma(1-\beta) \quad (4.2.17)$$

2.) In the same fashion we compute S_2:

Have $\overrightarrow{OK_2} = \pi R\, l_2$ (4.2.18)

and $\overrightarrow{OP_2} = \overrightarrow{OK_2} + \overrightarrow{K_2P_2}$ (4.2.19)

$\Leftrightarrow$ $S_2 u = \pi R\, l_2 + \overrightarrow{K_2P_2}$ (4.2.20)

$\Leftrightarrow$ $S_2 u - \pi R\, l_2 = \overrightarrow{K_2P_2}$ (4.2.21)

As $\overrightarrow{K_2P_2} \sim l_1$ is lightlike we have

$$(S_2 u - \pi R\, l_2)^2 = 0 \quad (4.2.22)$$

The only difference to the first case is that now

$$u \cdot \lambda_2 = \gamma(1, \beta)(1, -1) = \gamma(1+\beta) \qquad (4.2.23)$$

hence

$$S_2^2 - 2 S_2 \pi R \gamma (1+\beta) = 0 \qquad (4.2.24)$$

$$\Rightarrow \quad S_2 = 2\pi R \gamma (1+\beta) \qquad (4.2.25)$$

Therefore, the length $|\overrightarrow{P_1 P_2}|_\eta$ is

$$\begin{aligned} S_2 - S_1 &= 4\pi R \beta \gamma \\ &= 4\pi R \gamma \frac{R\Omega}{c} \\ &= 4 \cdot (\pi R^2) \frac{\Omega}{c} \gamma \qquad (4.2.26) \end{aligned}$$

Proper time τ is $\frac{1}{c} \times$ (proper length):

$$\begin{aligned} \tau_2 - \tau_2 &= \frac{1}{c}(S_2 - S_1) \\ &= 4 \cdot (\pi R^2) \frac{\Omega}{c^2} \gamma \qquad (4.2.27) \end{aligned}$$

The difference of inertial coord. time t is $\Delta t = \gamma^{-1} \Delta S$

$$\Rightarrow \quad t_2 - t_1 = 4 \cdot \underbrace{(\pi R^2)}_{=: A = \text{ area of disc of radius } R} \frac{\Omega}{c^2} \qquad (4.2.28)$$

The difference in travel times for clock- and anti-clockwise light paths translates into a phase difference

$$\Delta \phi = 2\pi \nu \cdot \Delta t$$
$$= 2\pi \frac{c}{\lambda} \cdot 4 \cdot A \cdot \frac{\Omega}{c^2}$$
$$= 8\pi \cdot A \cdot \frac{\Omega}{\lambda c} \quad \text{(Sagnac formula)} \quad (4.2.29)$$

Note that to leading order it does not matter whether we take Δt or $\Delta \tau$ in the expression of $\Delta \phi$. The difference is a factor of γ.

If we try to synchronise clocks along the circle $x^2 + y^2 = R^2$ in such a way that the velocity of light with respect to them is the same in both directions for the running observer the hypersurfaces (here lines) of constant time would be Minkowski-perpendicular to the observer's vector field u

Simultaneity hypersurface with respect to circling observer are these straight lines in the (ct-x) plane which are Minkowski perpendicular to the observer direction

$$u = \gamma(1, \beta) \quad , \quad u^2 = 1 \qquad (4.2.30)$$

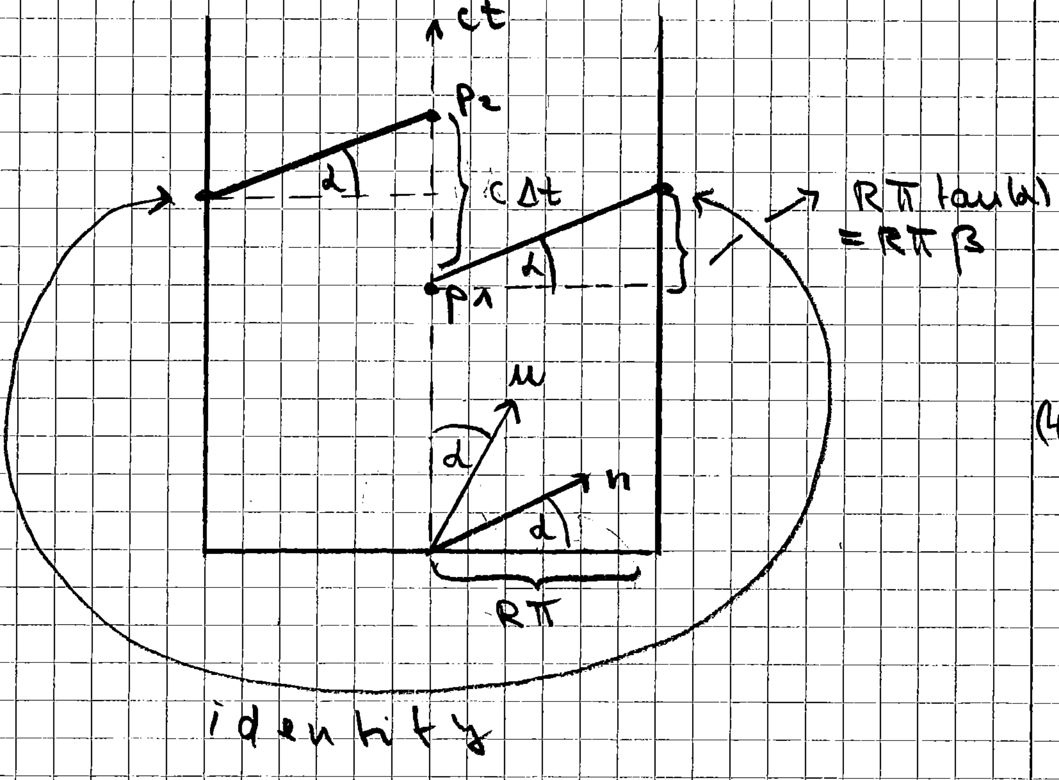
= timelike normal vector

The spacelike normal vector perpendicular to u is

$$n = \gamma(\beta, 1) \quad , \quad n^2 = -1 \qquad (4.2.31)$$

$$n \cdot u = 0 \qquad (4.2.32)$$

But note: The integral lines of n are not closed on the cylinder $x^2 + y^2 = R^2$ in spacetime



(4.2.33)

If the clocks are synchronised along the tilted line perpendicular to u there will be a time gap of

$$c\,\Delta t = 2 \cdot R\pi \tan(\alpha)$$

$$= 2\pi \cdot R \cdot \beta$$

$$= 2\pi \cdot R \cdot \frac{\Omega R}{c}$$

$$= 2 \cdot (\pi R^2) \frac{\Omega}{c}$$

$$= \frac{1}{2} \; c\,\Delta t\big|_{\text{sagnac}} \tag{4.2.34}$$

$\Rightarrow$ you cannot consistently synchronise the clocks along the circle in such a way that the velocity of light appears constant in both directions for the rotating observer