Sheet 8: Solutions

Problem 1

$$p_1 + p_2 = p_1' + p_2' \tag{8.1.1}$$

p_1 and p_1' are the photon's 4-momenta before and after collision, hence

$$p_1^2 = p_1'^2 = 0 \tag{8.1.2}$$

p_2 and p_2' are the massive particles 4-momenta before and after collision, hence

$$p_2^2 = p_2'^2 = (mc)^2 \tag{8.1.3}$$

Squaring (8.1.1) gives

$$\cancel{p_1^2} + 2\, p_1 \cdot p_2 + \cancel{p_2^2} = \cancel{p_1'^2} + 2\, p_1' \cdot p_2' + \cancel{p_2'^2}$$

$$\leadsto \quad p_1 \cdot p_2 = p_1' \cdot p_2' \tag{8.1.4}$$

Multiply (8.1.1) with p_1'

$$p_1 \cdot p_1' + p_2 \cdot p_1' = p_1' \cdot p_2' \tag{8.1.5}$$

or, using (8.1.4)

$$p_1 \cdot p_1' + p_2 \cdot p_1' = p_1 \cdot p_2 \tag{8.1.6}$$

Now,

$$p_1 = \left(\frac{E_1}{c}, \vec{p}_1\right) = \frac{\hbar\omega}{c}(1, \vec{n}) \qquad (8.1.7)$$

$\vec{n}$ = unit vector in the direction of travel of incoming photon

$$p_1' = \left(\frac{E_1'}{c}, \vec{p}_1'\right) = \frac{\hbar\omega'}{c}(1, \vec{n}') \qquad (8.1.8)$$

$\vec{n}'$ = unit vector in the direction of travel of outgoing photon

$$p_2 = \left(\frac{E_2}{c}, \vec{p}_2\right) = (mc, \vec{0}) \qquad (8.1.9)$$

the particle is initially at rest

$$p_2' = \left(\frac{E_2'}{c}, \vec{p}_2'\right) \qquad (8.1.10)$$

Note that p_2' does not appear in (8.1.6)

$$p_1 \cdot p_1' = \frac{\hbar^2 \omega\omega'}{c^2}(1 - \cos\varphi) \qquad (8.1.11)$$

where $\vec{n} \cdot \vec{n}' = \cos\varphi$ (8.1.12)

$$p_2 \cdot p_1' = m\hbar\omega' \qquad (8.1.13)$$

$$p_1 \cdot p_2 = m\hbar\omega \qquad (8.1.14)$$

Insert (8.1.11, 13, 14) into (8.1.6) and get

$$\hbar^2 \frac{\omega\omega'}{c^2}(1-\cos\varphi) + m\hbar\omega' = m\hbar\omega \qquad (8.1.15)$$

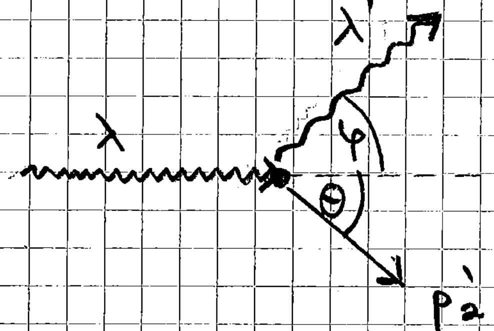
Division by $m\hbar\omega\omega'$ gives

$$\frac{\hbar}{mc^2}(1-\cos\varphi) = \frac{1}{\omega'} - \frac{1}{\omega}$$

or with $\frac{1}{\omega'} = \frac{1}{2\pi}\frac{1}{\nu'} = \frac{\lambda'}{2\pi c}$

$$\frac{1}{\omega} = \frac{1}{2\pi}\frac{1}{\nu} = \frac{\lambda}{2\pi c}$$

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\varphi) \qquad (8.1.16)$$



Compton scattering

An explicit solution to to vector equation (8.1.1) with (8.1.7 - 10) would be as follows

$$\frac{\hbar\omega}{c}(1,\vec{n}) + (mc,\vec{0})$$

$$= \frac{\hbar\omega'}{c}(1,\vec{n}') + \left(\frac{E_2'}{c}, p_2'\right) \qquad (8.1.17)$$

$\Leftrightarrow$

$$\frac{\hbar\omega}{c} + mc = \frac{\hbar\omega'}{c} + \frac{E_2'}{c} \qquad (8.1.18)$$

$$\frac{\hbar\omega}{c}\vec{n} = \frac{\hbar\omega'}{c}\vec{n}' + \vec{p}_2' \qquad (8.1.19)$$

We rewrite this in terms of the photon's initial and final energy

$$E_\gamma = \hbar\omega, \quad E_\gamma' = \hbar\omega' \qquad (8.1.20)$$

the particle's final kinetic energy

$$\Delta E = E_2' - mc^2 \qquad (8.1.21)$$

(we do not use a prime because the initial kinetic energy is zero)

and the particle's final momentum

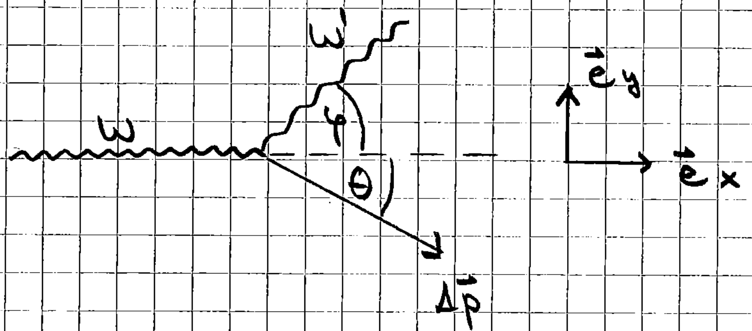
$$\Delta\vec{p} = \vec{p}_2' \qquad (8.1.22)$$

We decompose the vector equation (8.1.19) into the part $\parallel$ and $\perp$ to $\vec{n}$

$$\vec{n}' = \cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y \qquad (8.1.23)$$

$$\vec{n} = e_x \qquad 8.1.24$$

$$\Delta\vec{p} = \Delta p\,(\cos\theta\, \vec{e}_x - \sin\theta\, \vec{e}_y) \qquad (8.1.25)$$



(8.1.26)

Then (8.1.18) and (8.1.19) become

$$E_\gamma = E'_\gamma + \Delta E \qquad (8.1.27)$$

$$E_\gamma \vec{e}_x = E'_\gamma (\cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y) + c\Delta p\,(\cos\theta\, \vec{e}_x - \sin\theta\, \vec{e}_y)$$

$$E_\gamma = E'_\gamma \cos\varphi + c\,\Delta p \cos\theta \qquad (8.1.28)$$

$$0 = E'_\gamma \sin\varphi - c\,\Delta p \sin\theta \qquad (8.1.29)$$

(8.1.27–29) are three equations relating the five quantities E_γ, E'_γ, Δp, θ and φ. As a result, we should be able to express any quantity as function of two others. (8.1.16) is but one example, relating E_γ, E'_γ and φ

Problem 2

We identify $V \otimes V^* \cong \mathrm{End}(V)$ with $V \otimes V$ via the isomorphism

$$\begin{aligned} id_V \otimes \eta_\uparrow &: V \otimes V^* \to V \otimes V \\ id_V \otimes \eta_\downarrow &: V \otimes V \to V \otimes V^* \end{aligned} \qquad (8.2.1)$$

so that

$u \otimes v \in V \otimes V$ acts on $w \in V$ via

$$(u \otimes v)(w) = (id_V \otimes \eta_\downarrow)(u \otimes v)(w)$$

$$= [u \otimes \eta_\downarrow(v)](w)$$

$$= u \cdot \eta(v, w)$$

We abbreviate this by writing

$$(u \otimes v)(w) = (v \cdot w)\, u \qquad (8.2.2)$$

1.) $\quad u \wedge n := u \otimes n - n \otimes u$

$$\eta(v, (u \wedge n)(w))$$

$$= (v \cdot u)(n \cdot w) - (v \cdot n)(u \cdot w)$$

$$= -\eta((u \wedge n)(v), w)$$

We write this as

$$(u \wedge n)^{+} = -(u \wedge n), \qquad + = \text{conjugate map w.r.t. } \eta \qquad (8.2.3)$$

2.) Note that for

$$(u \wedge n)^k := \underbrace{(u \wedge n) \circ \cdots \circ (u \wedge n)}_{\text{composition of } k \text{ maps}}$$

$$\left[(u \wedge n)^k\right]^\dagger = \left[(u \wedge n)^\dagger\right]^k$$

e.g. for $u \wedge n =: M$

$$\eta(v, M^k w) = \eta(v, M M^{k-1} w)$$

$$= \eta(M^\dagger v, M^{k-1} w) = \eta(M^2 v, M^{k-2} w)$$

$$= \cdots = \eta((M^\dagger)^k v, w). \qquad (8.2.4)$$

Also, $\dagger : \mathrm{End}(V) \to \mathrm{End}(V)$ is linear. Hence

$$[\exp(M)]^k = \exp(M^\dagger)$$

$$\Rightarrow [\exp(u \wedge n)]^\dagger = \exp((u \wedge n)^\dagger)$$

$$= \exp(-u \wedge n)$$

$$= [\exp(u \wedge n)]^{-1} \qquad (8.2.5)$$

But this is the definition of $O(\eta, V)$

$= \{ M \in \mathrm{End}(V) : \eta(Mv, Mw) = \eta(v, w) \}$

$= \{ M \in \mathrm{End}(V) : M^\dagger = M^{-1} \}$.

Hence

$$M := \exp(u \wedge n) \in O(V, \eta)$$

But, in fact, M lies in the identity component, since the continuous 1-parameter curve

$$M(s) := \exp((1-s)\, u \wedge n)$$

$$s \in [0, 1]$$

connects $\exp(u \wedge n)$ for $s = 0$ with $\mathrm{id}|_V$ for $s = 1$ within $O(V, \eta)$. Therefore $\exp(u \wedge n)$ is an element of the proper orthochronous Lorentz group.

3.) Let now u and n be such that

$$u^2 = -n^2 = 1, \quad u \cdot n = 0 \tag{8.2.6}$$

$$\rightsquigarrow \quad (u \wedge n)^2 = (u \otimes n - n \otimes u) \circ (u \otimes n - n \otimes u)$$

$$= -(n \cdot n)\, u \otimes u - (u \cdot u)\, n \otimes n$$

$$= u \otimes u - n \otimes n = P_{(u,n)} \tag{8.2.7}$$

This is the η-orthogonal projector onto Span $\{u, n\}$. To prove that, we note

$$(u\otimes u - n\otimes n)(u) = u$$
$$(u\otimes u - n\otimes n)(n) = n \qquad (8.2.8)$$
$$(u\otimes u - n\otimes n)(W) = 0 \quad \forall W \perp \mathrm{Span}\{u,n\}$$

and also

$$(u\otimes u - n\otimes n)\circ(u\otimes u - n\otimes n)$$
$$= u\otimes u - n\otimes n \qquad (8.2.9)$$

Hence any even power of $u\wedge n$ – except the zeroth – is proportional to $P_{(u,n)}$ and any odd power is proportional to $u\wedge n$. Note that $(u\wedge n)\circ P_{(u,n)} = P_{(u,n)}\, u\wedge n = u\wedge n$ because $u\wedge n$ only acts on the $P_{(u,n)}$ projected part.

Now, let $\rho\in\mathbb{R}$; then

$$\exp(-\rho\, u\wedge n) = \sum_{k=0}^{\infty} \frac{(-\rho)^k}{k!}(u\wedge n)^k$$

$$= \sum_{k=\text{even}} \cdots + \sum_{k=\text{odd}} \cdots$$

$$= \underbrace{\mathrm{id}|_V + \sum_{k=1}^{\infty}\frac{\rho^{2k}}{(2k)!}P_{(u,n)}}_{\underbrace{\mathrm{id}|_V - P_{(u,n)}} + \sum_{k=0}^{\infty}\frac{\rho^{2k}}{(2k)!}P_{(u,n)}} + \sum_{k=0}^{\infty}\frac{(-\rho)^{2k+1}}{(2k+1)!}(u\wedge n)$$

$$= \mathrm{id}_V + (\cosh(\rho)-1)P_{(u,n)} - \sinh(\rho)\, u\wedge n \qquad (8.2.10)$$

$\Rightarrow$

$$B(u,n,\varrho) := \exp(-\varrho\; u \wedge n)$$

$$= \mathrm{id}_V + [\cosh(\varrho) - 1]\, P_{(u,n)} - \sinh(\varrho)\; u \wedge n \tag{8.2.11}$$

This map is obviously the identity on the orthogonal complement of $\mathrm{Span}\{u,n\}$, i.e.

$$B(u,n,\varrho)\, w = w \qquad \forall\, w \in \{u,n\}^{\perp}$$

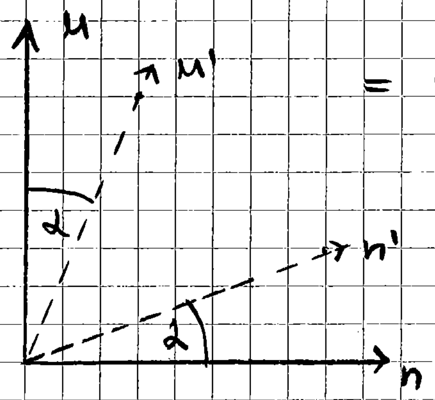
and

$$B(u,n,\varrho)\, u = u + (\cosh(\varrho) - 1)\, u + \sinh(\varrho)\, n \tag{8.2.12}$$

$$= \cosh(\varrho)\, u + \sinh(\varrho)\, n$$

$$B(u,n,\varrho)\, n = n + (\cosh(\varrho) - 1)\, n + \sinh(\varrho)\, u$$

$$= \cosh(\varrho)\, n + \sinh(\varrho)\, u \tag{8.2.13}$$



Lorentz-Boost in plane $\mathrm{Span}\{u,n\}$.

$$\tan(\alpha) = \beta = \tanh(\varrho) \tag{8.2.14}$$

4) If $u' = a u + b n$
$n' = c u + d n$

with $u'^2 = -n'^2 = 1$, $u' \cdot n' = 0$ (8.2.15)
then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in O(1,1) \tag{8.2.16}$$

$$\Rightarrow \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc = \pm 1 \tag{8.2.17}$$

with $+1$ if orientation preserving

But $u' \wedge n' = (ad - bc)\, u \wedge n = u \wedge n$
for orientation preserving changes
$(u, n) \to (u', n')$.

Let $l_\pm := \frac{1}{\sqrt{2}} (u \pm n)$ (8.2.18)

so that $l_+^2 = l_-^2 = 0$, $l_+ \cdot l_- = 1$, (8.2.19)
then

$$l_+ \wedge l_- = \frac{1}{2} (u + n) \wedge (u - n)$$

$$= - u \wedge n \tag{8.2.20}$$

hence the same boost may be written as

$$B(\rho) = \exp(\rho\, \underbrace{l_+ \wedge l_-}_{\text{lightlike}}) \tag{8.2.21}$$

5.)

$$\text{If } u^2 = n^2 = -1, \quad u \cdot n = 0 \tag{8.2.22}$$

We have

$$(u \wedge n)^2 = (u \otimes n - n \otimes u) \circ (u \otimes n - n \otimes u)$$

$$= u \otimes u + n \otimes n$$

$$= -P_{(u,n)} \tag{8.2.23}$$

since
$$(u \otimes u + n \otimes n)(u) = -u$$
$$(u \otimes u + n \otimes n)(n) = -n$$
$$(u \otimes u + n \otimes n)(W) = 0, \quad W \in \{u,n\}^\perp$$

and $(u \otimes u + n \otimes n)^2 = -u \otimes u - n \otimes n$.

The same calculation as before, taking into account the (−)-sign, gives

$$\exp(-\varrho\, u \wedge n) = \sum_{k=0}^{\infty} \frac{(-\varrho)^k}{k!} (u \wedge n)^k$$

$$= \sum_{k=\text{even}} \cdots + \sum_{k=\text{odd}} \cdots$$

$$= id|_V + \sum_{k=0}^{\infty} (-1)^n \frac{\varrho^{2n}}{(2n)!} P_{(u,n)}$$
$$+ \sum_{k=0}^{\infty} (-1)^k \frac{(-\varrho)^{2k+1}}{(2k+1)!} (u \wedge n)$$

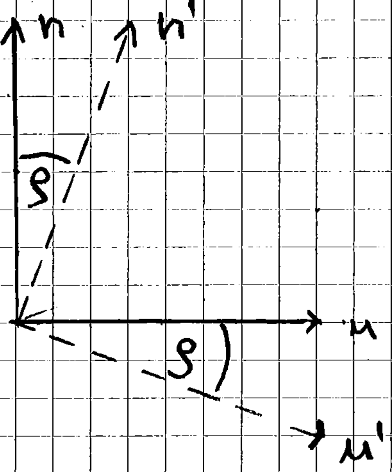
$$= id|_V - P_{(u,n)} + \cos(\varrho) P_{(u,n)} - \sin(\varrho)(u \wedge n) \tag{8.2.24}$$

Again this is the identity on $\{u, n\}^\perp$ and

$$\exp(-\varrho\, u \wedge n)$$
$$= \mathrm{id}_V + (\cos(\varrho) - 1)\, P(u,n) - \sin(\varrho)\, u \wedge n \quad (8.2.25)$$

applied to u and n give

$$\exp(-\varrho\, u \wedge n)(u)$$
$$= \cos(\varrho)\, u - \sin(\varrho)\, n$$
$$\exp(-\varrho\, u \wedge n)(n)$$
$$= \cos(\varrho)\, n + \sin(\varrho)\, u \quad (8.2.26)$$



6) Let now u be lightlike,
n spacelike normalized and orthogonal to u

$$u^2 = 0 \,, \quad n^2 = -1 \,, \quad u \cdot n = 0 \tag{8.2.27}$$

$$(u \wedge n)^2 = (u \otimes n - n \otimes u)(u \otimes n - n \otimes u)$$
$$= u \otimes u \tag{8.2.28}$$

Hence all higher powers of $u \wedge n$ vanish since u is orthogonal to both, u and n

$\Rightarrow \exp(-\rho \, u \wedge n)$

$$= \mathrm{id}_V - \rho \, u \wedge n + \frac{\rho^2}{2} \, u \otimes u \tag{8.2.29}$$

Hence

$$\exp(-\rho \, u \wedge n)(u) = u$$
$$\exp(-\rho \, u \wedge n)(n) = n + \rho u \tag{8.2.30}$$

The interpretation of this case is easier to grasp if we write

$$u = e_0 + e_1 \,, \quad n = e_2 \tag{8.2.31}$$

with $e_0 \cdot e_0 = 1$, $e_1 \cdot e_1 = e_2 \cdot e_2 = -1$
and $e_1 \cdot e_2 = 0$ (8.2.32)

Then

$$\exp(-\varsigma(u \wedge n)) =$$

$$\exp(-\varsigma(e_0 + e_1) \wedge e_2)$$

$$= \exp(-\varsigma(\underbrace{e_0 \wedge e_2}_{\text{boost in 2-direction}} + \underbrace{e_1 \wedge e_2}_{\text{rotation about 3-axis}}))$$

$$= \mathrm{id} - \varsigma(e_0 + e_1) \wedge e_2 + \frac{\varsigma^2}{2}(e_0 + e_1) \otimes (e_0 + e_1)$$

$$= \mathrm{id} - \varsigma(e_0 \wedge e_2) - \varsigma\, e_1 \wedge e_2$$

$$+ \frac{\varsigma^2}{2}(e_0 \otimes e_0 + e_1 \otimes e_1)$$

$$+ \frac{\varsigma^2}{2}(e_0 \otimes e_1 + e_1 \otimes e_0) \qquad (8.2.33)$$

$$=: L$$

Then

$$L(e_0) = e_0 + \varsigma e_2 + \frac{\varsigma^2}{2} e_0 + \frac{\varsigma^2}{2} e_1$$

$$= \left(1 + \frac{\varsigma^2}{2}\right) e_0 + \frac{\varsigma^2}{2} e_1 + \varsigma e_2 \qquad (8.2.34)$$

$$L(e_1) = e_1 - \varsigma e_2 - \frac{\varsigma^2}{2} e_1 - \frac{\varsigma^2}{2} e_0$$

$$= -\frac{\varsigma^2}{2} e_0 + \left(1 - \frac{\varsigma^2}{2}\right) e_1 - \varsigma e_2 \qquad (8.2.35)$$

$$L(e_2) = e_2 + \varsigma e_0 + \varsigma e_1$$

$$= \varsigma e_0 + \varsigma e_1 + e_2 \qquad (8.2.36)$$

Writing, as usual,

$$L(e_a) = L^b{}_a e_b$$

the matrix $L^b{}_a$ is

$$\{L^b{}_a\} = \begin{pmatrix} 1 + \frac{\varsigma^2}{2} & -\varsigma^2/2 & \varsigma \\ \varsigma^2/2 & 1 - \frac{\varsigma^2}{2} & \varsigma \\ \varsigma & -\varsigma & 1 \end{pmatrix} \qquad (8.2.37)$$

One verifies that the column-vectors are normalized and perpendicular

$$\eta_{ab} L^a{}_c L^b{}_d = \eta_{cd} . \qquad (8.2.38)$$