Sheet 9: Solutions

Problem 1

$$p_A + p_B = p_C \tag{9.1.1}$$

$$\curvearrowright \quad p_A^2 + p_B^2 + 2\, p_A \cdot p_B = p_C^2 \tag{9.1.2}$$

$$(m_A c)^2 + (m_B c)^2 + 2\, p_A \cdot p_B = p_C^2 \tag{9.1.3}$$

Since (p_A, p_B) span a timelike plane (if they are linearly independent), we have (Sheet 6, formula (2a))

$$(p_A \cdot p_B)^2 \geq p_A^2\, p_B^2 = (m_A c)^2 (m_B c)^2 \tag{9.1.4}$$

and since p_A and p_B are both future oriented $p_A \cdot p_B \geq 0$; hence

$$p_A \cdot p_B \geq (m_A c)(m_B c) \tag{9.1.5}$$

Inserting this into (9.1.3) gives

$$\begin{aligned} p_C^2 &= (m_A c)^2 + (m_B c)^2 + 2\, p_A \cdot p_B \\ &\geq (m_A c + m_B c)^2 \\ &> 0 \quad \text{if at least } m_A \text{ or } m_B \neq 0. \end{aligned}$$

$\Rightarrow$ p_C is timelike if $(m_A + m_B) > 0$

Problem 2

The total four momentum is

$$p = p_1 + p_2 \tag{9.2.1}$$

The energy in the centre-of-mass (henceforth abbreviated cm) frame is

$$E_{cm} = c\, u_{cm} \cdot p \tag{9.2.2}$$

with
$$u_{cm} = \frac{p_1 + p_2}{\sqrt{(p_1+p_2)^2}} \tag{9.2.3}$$

$$\leadsto E_{cm} = c \frac{(p_1+p_2)^2}{\sqrt{(p_1+p_2)^2}} = c\sqrt{(p_1+p_2)^2}$$

$$= c\left[(m_1 c)^2 + (m_2 c)^2 + 2\, p_1 \cdot p_2\right]^{1/2} \tag{9.2.4}$$

And in the laboratory frame, where the second particle is at rest

$$E_l = c\, u_l \cdot p \tag{9.2.4}$$

with
$$u_l = \frac{p_2}{\sqrt{p_2^2}} = \frac{p_2}{m_2 c} \tag{9.2.5}$$

$$\leadsto E_l = c\,(p_1 + p_2)\frac{p_2}{m_2 c}$$

$$= \frac{(m_2 c)^2 + p_1 \cdot p_2}{m_2} \tag{9.2.6}$$

We solve (9.2.6) for $p_1 \cdot p_2$:

$$p_1 \cdot p_2 = E_\ell m_2 - (m_2 c)^2 \tag{9.2.7}$$

and insert this into (9.2.4):

$$E_{cm} = c\left[(m_1 c)^2 - (m_2 c)^2 + 2 E_\ell m_2\right]^{1/2} \tag{9.2.8}$$

We observe that for $E_\ell \gg$ (any of the masses) $\times\, c^2$ $E_{cm} \sim \sqrt{E_\ell}$, i.e. one needs exceedingly high laboratory energies in order to reach high cm-energies. The inverse of (9.2.8) is

$$E_\ell = \frac{E_{cm}^2 - (m_1 c^2)^2 + (m_2 c^2)^2}{2 m_2} \tag{9.2.9}$$

showing that E_ℓ grows quadratically with E_{cm} for large energies (larger than $m_1 c^2$ and $m_2 c^2$).

Problem 3

$$P_0 \longrightarrow P_1 + P_2 \tag{9.3.1}$$

$$p_1 + p_2 = p_0 \tag{9.3.2}$$

The energy of P_1 in the rest frame of P_0 is

$$E_1 = c\, u_0 \cdot p_1 \tag{9.3.3}$$

with
$$u_0 = \frac{p_0}{\sqrt{(p_0)^2}} = \frac{p_1 + p_2}{m_0 c} \tag{9.3.4}$$

$$\rightsquigarrow \quad E_1 = c \, \frac{p_1 \cdot (p_1 + p_2)}{m_0 c}$$

$$= \frac{(m_1 c)^2 + p_1 \cdot p_2}{m_0} \tag{9.3.5}$$

$p_1 \cdot p_2$ we get from

$$p_0^2 = (m_0 c)^2 = (p_1 + p_2)^2$$
$$= (m_1 c)^2 + (m_2 c)^2 + 2\, p_1 \cdot p_2$$

$$\Rightarrow \quad p_1 p_2 = \frac{1}{2}\left[m_0^2 - m_1^2 - m_2^2 \right] c^2 \tag{9.3.6}$$

$$\rightsquigarrow \quad E_1 = c^2 \, \frac{m_0^2 + m_1^2 - m_2^2}{2 m_0} \tag{9.3.7}$$

From

$$p_1^2 = \frac{E_1^2}{c^2} - \|\vec{p}_1\|^2 = (m_1 c)^2$$

get

$$\|\vec{p}_1\| = \left[\frac{E_1^2}{c^2} - (m_1 c)^2\right]^{1/2} \qquad (9.3.8)$$

Now

$$\frac{E_1^2}{c^4} - m_1^2 = \frac{(m_0^2 + m_1^2 - m_2^2)^2}{4 m_0^2} - m_1^2$$

$$= \frac{m_0^4 + m_1^4 + m_2^4 + 2(m_0^2 m_1^2 - m_0^2 m_2^2 - m_1^2 m_2^2 - 2 m_1^2 m_0^2)}{4 m_0^2}$$

$$= \frac{m_0^4 + m_1^4 + m_2^2 - 2(m_0^2 m_1^2 + m_0^2 m_2^2 + m_1^2 m_2^2)}{4 m_0^2}$$

$$= \frac{[m_0^2 - (m_1 + m_2)^2][m_0^2 - (m_1 - m_2)^2]}{4 m_0^2} \qquad (9.3.9)$$

[check : $[m_0^2 - (m_1 + m_2)^2][m_0^2 - (m_1 - m_2)^2]$

$$= m_0^4 - m_0^2[(m_1 + m_2)^2 + (m_1 - m_2)^2]$$

$$+ (m_1^2 - m_2^2)^2$$

$$= m_0^4 - 2 m_0^2 (m_1^2 + m_2^2)$$

$$+ m_1^4 + m_2^4 - 2 m_1^2 m_2^2$$

$$= m_0^4 + m_1^4 + m_2^4 - 2(m_0^2 m_1^2 + m_0^2 m_2^2 + m_1^2 m_2^2)] \qquad (9.3.10)$$

Hence

$$\|\vec{p}_1\| = c\frac{\sqrt{[m_0^2-(m_1+m_2)^2][m_0^2-(m_1-m_2)^2]}}{2m_0} \qquad (9.3.11)$$

Selecting particle 2 rather than 1 we get the corresponding equations for E_2 and $\|\vec{p}_2\|$ by exchanging the indices 1 and 2 in (9.3.7) and (9.3.11).

In the rest frame of the original particle P_0 have

$$p_0 = \left(\frac{E_0}{c}, \vec{0}\right) \qquad (9.3.12)$$

and

$$p_{1,2} = \left(\frac{E_{1,2}}{c}, \vec{p}_{1,2}\right) \qquad (9.3.13)$$

with

$$\vec{p}_1 + \vec{p}_2 = 0 \qquad (9.3.14)$$

Hence

$$\|\vec{p}_1\| = \|\vec{p}_2\| \qquad (9.3.15)$$

explaining why (9.3.11) is symmetric under $1 \leftrightarrow 2$. (9.3.7) is not symmetric since for equal magnitude of momenta the energies need not be the same unless $m_1 = m_2$:

$$E_{1,2} = \Big[\underbrace{(m_{1,2}c^2)^2}_{\text{not the same}} + \underbrace{\|\vec{p}_{1,2}\|^2}_{\text{same}}\Big]^{1/2} \qquad (9.3.16)$$

Problem 4

E_1 in (9.3.7) gives the energy of particle 1 in the rest frame of particle 0. The corresponding velocity β_{10} follows from

$$E_1 = \frac{m_1 c^2}{\sqrt{1-\beta_{10}^2}} \tag{9.4.1}$$

hence

$$\beta_{10} = \frac{\sqrt{E_1^2 - (m_1 c^2)^2}}{E_1} = \frac{c\,\|\vec{p}_1\|}{E_1} \tag{9.4.2}$$

From (9.3.11) and (9.3.7) get

$$\beta_{10} = \frac{\sqrt{[m_0^2-(m_1+m_2)^2][m_0^2-(m_1-m_2)^2]}}{m_0^2+m_1^2-m_2^2} \tag{9.4.3}$$

Again for particle 2 we just need to exchange the indices 1 and 2.

For the relative velocity between P_1 and P_2 seen from the rest frame of P_2 we have

$$\gamma_{12} = \frac{p_1 \cdot p_2}{(m_1 c)(m_2 c)} = \frac{1}{\sqrt{1-\beta_{12}^2}} \tag{9.4.4}$$

$$\rightsquigarrow \quad \beta_{12} = \frac{\sqrt{(p_1\cdot p_2)^2 - (m_1 c)^2 (m_2 c)^2}}{(p_1 \cdot p_2)} \tag{9.4.5}$$

The scalar product $p_1 \cdot p_2$ is eliminated via (9.3.6)

$$p_1 \cdot p_2 = \tfrac{1}{2}\left[m_0^2 - m_1^2 - m_2^2\right] c^2 . \qquad (9.4.6)$$

Then the term under the square-root in (9.4.5) becomes

$$\frac{c^2}{2}\Big[m_0^4 + m_1^4 + m_2^4 - 2\left(m_0^2 m_1^2 + m_0^2 m_2^2 - m_1^2 m_2^2\right) - 4\, m_1^2\, m_2^2\Big]$$

$$= \frac{c^2}{2}\Big[m_0^4 + m_1^4 + m_2^2 - 2\left(m_0^2 m_1^2 + m_0^2 m_2^2 + m_1^2 m_2^2\right)\Big]$$

$$= \frac{c^2}{2}\left[m_0^2 - (m_1 + m_2)^2\right]\left[m_0^2 - (m_1 - m_2)^2\right] \qquad (9.4.7)$$

like before in (9.3.9)

Inserting (9.4.6) and (9.4.7) into (9.4.5) gives

$$\beta_{12} = \frac{\sqrt{\left[m_0^2 - (m_1 + m_2)^2\right]\left[m_0^2 - (m_1 - m_2)^2\right]}}{m_0^2 - m_1^2 - m_2^2} \qquad (9.4.8)$$

Problem 5

$$L(\vec{x},\dot{\vec{x}}) = -mc^2\,(1-\dot{\vec{x}}^2/c^2)^{1/2} + eE(\vec{x}\cdot\vec{n}) \quad (9.5.1)$$

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \gamma m \dot{\vec{x}} \quad , \quad \gamma := \left(1-\frac{\dot{\vec{x}}^2}{c^2}\right)^{-1/2} \quad (9.5.2)$$

$$\frac{\partial L}{\partial \vec{x}} = eE\vec{n} \quad (9.5.3)$$

$\Rightarrow$ Euler-Lagrange equations

$$\frac{d}{dt}(\gamma\dot{\vec{x}}) = \frac{e}{m}E\vec{n} \quad (9.5.4)$$

Integration $\int_0^t dt'$ with $\dot{\vec{x}}(0)=0$ gives

$$\gamma\dot{\vec{x}}(t) = \frac{e}{m}E\vec{n}\,t \quad (9.5.5)$$

This means that $\dot{\vec{x}} \parallel \vec{n}$ and if we choose $\vec{n}=\vec{e}_x$ we can replace this by

$$\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} = \frac{e}{m}Et \quad (9.5.6)$$

Since $\dot{\vec{x}}$ will always be in $\vec{e}_x$ direction so that motion takes place along x-axis. Now, (9.5.6) can be solved for $\dot{x}(t)$:

$$\dot{x}^2 = \left(1 - \frac{\dot{x}^2}{c^2}\right) \alpha^2 t^2 \qquad (9.5.7)$$

$$\text{with} \quad \alpha := \frac{e}{m} E \qquad (9.5.8)$$

$$\rightsquigarrow \quad \dot{x}^2 \left[1 + \left(\frac{\alpha t}{c}\right)^2\right] = \alpha^2 t^2$$

$$\dot{x}(t) = \frac{\alpha t}{\sqrt{1 + \left(\frac{\alpha t}{c}\right)^2}} \qquad (9.5.9)$$

where we have chosen the positive root since with $\dot{\vec{x}}(0) = \vec{0}$ and $\vec{E} = E\vec{e}_x$ the motion takes place in $\vec{e}_x$ direction, i.e. $\dot{\vec{x}} = \dot{x}\,\vec{e}_x$ with $\dot{x} \geq 0$.

One more integration $\int_0^t dt'$ of (9.5.9) with $x(0) = 0$ leads to

$$x(t) = \int_0^t dt' \frac{\alpha t'}{\sqrt{1 + \left(\frac{\alpha t'}{c}\right)^2}}$$

$$= \frac{c^2}{\alpha} \int_0^{\frac{\alpha t}{c}} dy \frac{y}{\sqrt{1 + y^2}}$$

$$\left(\text{with } y := \frac{\alpha t'}{c}\right)$$

$$= \frac{c^2}{\alpha} \left\{ \sqrt{1 + \left(\frac{\alpha t}{c}\right)^2} - 1 \right\} \qquad (9.5.10)$$

or

$$\left(\frac{\alpha}{c^2}\, x + 1\right)^2 = 1 + \left(\frac{\alpha t}{c}\right)^2 \qquad (9.5.11)$$

$$\Leftrightarrow \quad \left(x + \frac{c^2}{\alpha}\right)^2 - (ct)^2 = \left(\frac{c^2}{\alpha}\right)^2$$

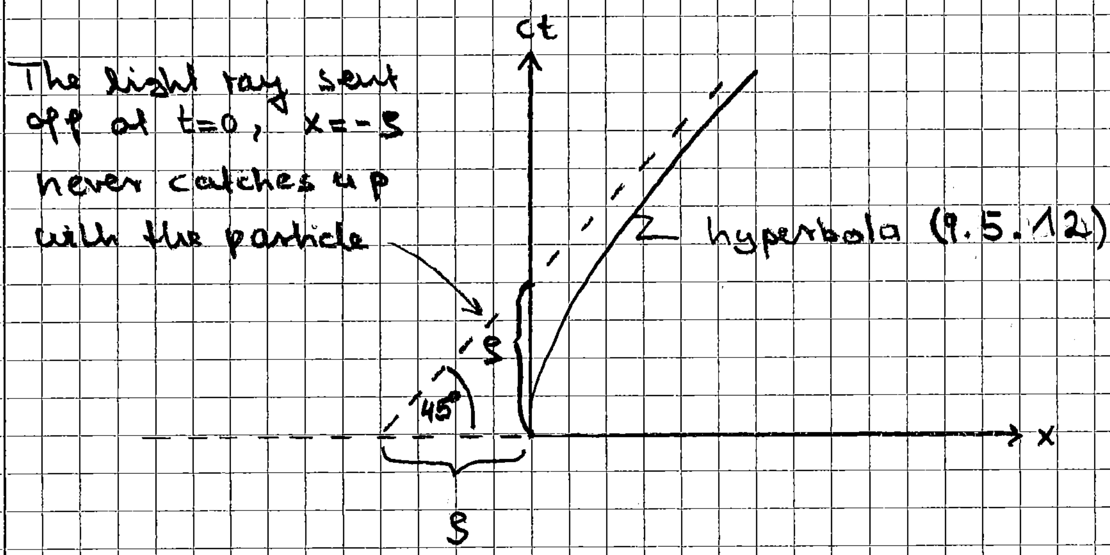
$$\Leftrightarrow \quad (x + \varrho)^2 - (ct)^2 = \varrho^2 \qquad (9.5.12)$$

where $\varrho = \frac{c^2}{\alpha} = \frac{mc^2}{eE}$ (9.5.13)

is a parameter with a physical dimension of a length, the interpretation of which is the distance such that

$$\underbrace{\varrho e E}_{\substack{\text{work done by the electric field } E \\ \text{on the charge } e \text{ over distance } \varrho}} = mc^2 \qquad (9.5.14)$$

So the work done over distance ϱ is just the rest energy mc^2.



(9.5.14)

The world-line of the particle is a timelike hyperbola in the (ct, x)-plane that has the form (9.5.12) for $x \geq 0$ and $t \geq 0$, or

$$ct = \sqrt{(x+\varrho)^2 - \varrho^2}$$

$$= x\sqrt{\left(1+\frac{\varrho}{x}\right)^2 - \left(\frac{\varrho}{x}\right)^2}$$

$$= x\sqrt{1 + 2\frac{\varrho}{x}}$$

$$= x\left(1 + \frac{\varrho}{x} + \mathcal{O}((\varrho/x)^2)\right)$$

$$= x + \varrho + \varrho\,\mathcal{O}(\varrho/x) \qquad (9.5.15)$$

Hence the hyperbola is asymtotic to the straight line $ct = x + \varrho$ for large x. This straight line is the worldline of a photon sent off at $t = 0$, $x = -\varrho$ in the positive x-direction. The photon is chasing the particle but never catches up with it.