Energy-Momentum Tensors
and
Motion in Special Relativity

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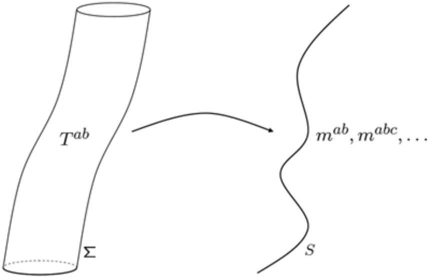
What is motion?



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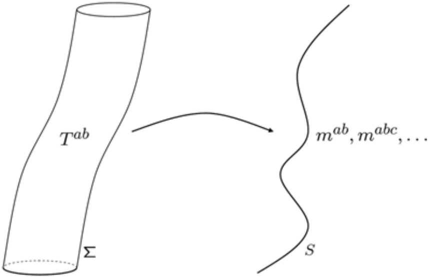

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What centres?

Maurice Henry Lecorney Pryce

(1913-2003)

"*The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles*", Proc. Roy Soc. (London) 1948

- Motivated by Fokker ("Relativitätstheorie", 1929) and Born & Fuchs ("The Mass Centre in Relativity", Nature 1940), Pryce elaborates on the notion of *mass–centre* and discusses 6 definitions (a)–(f) for it.
- As Born & Fuchs point out, the "right" weights for the spatial mass–centre depend not only on the rest masses, but on the dynamical masses and hence on other momenta. This will generally mess up canonical commutation relations.
- Newton & Wigner 1949, Wightman 1962, Mackey-Theory.

General starting point

- Let there be given a spacetime (M, g) and a symmetric and conserved energy-momentum tensor

$$T = T_{ab}\, dx^a \otimes dx^b\,, \quad T_{ab} = T_{ba}\,, \quad \nabla^a T_{ab} = 0 \tag{1}$$

of "spatially compact support"

$$\begin{aligned} &\mathrm{supp}(T) \cap \Sigma = \text{compact} \\ &\Sigma \text{ spacelike and ending at } i_0 \end{aligned} \tag{2}$$

- This may be weakened to allow for sufficiently rapid fall-off in "spacelike directions". Note that this needs careful phrasing for matter models involving radiating fields.

General question

- In this conference we are interested in the problem of how to associate a piecewise C^2 timelike line in the convex hull of $\mathrm{supp}(T)$.
- A related problem is, as we will see, how to associate conserved quantities like energy, momentum, and angular momentum.
- In a Special Relativistic setting these quantities are best defined as (Hamiltonian) generators of Poincaré transformations. In this case we have a **momentum map**.
- This talk intends to raise awareness for the structures implicitly used in the SR context. On the other hand, in GR, some of these structures will be missing, most prominently, of course, a globally acting group of isometries of (M, g). All workarounds then have to face the questions of **existence** and **uniqueness**.

Global charges

- A related problem to that of motion is that of global charges (conserved quantities). Often one encounters expressions like

$$\begin{aligned} p_a &= \int_\Sigma T_{ab} n^b \, \mathrm{d}\mu \\ J_{ab} &= \int_\Sigma \big(x_a T_{bc} - x_b T_{ac}\big) n^c \, \mathrm{d}\mu \end{aligned} \tag{3}$$

  even though *prima facie* they make no sense.

- What is the habitat of "momenta"? What is the meaning of global "energy-momentum" transforms as a four vector? In what vector space and under what group/action?

- **Let's see in the most trivial example ...**

Spacetime symmetries 1

- Let (M, g) spacetime and H a finite dimensional Lie group acting as isometries
$$\Phi : H \times M \to M\,, \quad \Phi_h^* g = g$$
- We assume a left action
$$\Phi(h_1 h_2, m) = \Phi\big((h_1, \Phi(h_2, m)\big)\,, \quad \Rightarrow \Phi(h, m) =: h \cdot m$$
- This induces vector fields V on M, one for each $X \in \mathfrak{h}$
$$V_X(m) = \frac{d}{ds}\Big|_{s=0} \exp(sX) \cdot m$$

Spacetime symmetries 2

- Hence there is a map

$$V : \mathfrak{h} \to Vec(M)\,, \quad X \mapsto V_X$$

which is not only linear, but also satisfies other structurally important properties:

$$(\Phi_h)_*(V_X(m)) = X_{\mathrm{Ad}_h(X)}(h \cdot m) \quad \text{Ad equivariance} \qquad (4)$$

$$[V_X, V_Y] = -V_{[X,Y]} \quad \text{anti-homomorphism} \qquad (5)$$

- Indeed

$$(\Phi_h)_*(V_X(m)) = \frac{d}{ds}\Big|_{s=0}(h\,\exp(sX)\,h^{-1}\,h\cdot m) = \frac{d}{ds}\Big|_{s=0}(\exp(s\,Ad_h(X))h\cdot m)$$

and the "anti" is due to the "left"-action.

Conserved currents

- Since H acts by isometries, all V_X are Killing fields

$$L_{V_X} g = 0$$

- Hence have linear map J from the Lie-algebra $\mathfrak{h}$ to the linear space of conserved currents (co-closed one forms)

$$J_X = V_X^a T_{ab} \, dx^b \,, \quad \delta J_X = -\nabla_a J_X^a = 0$$

- Alternatively, to the closed 3-forms

$$\star i_{V_X} T = V_X^a T_{ab} \, \eta^{bc} \, \varepsilon_{cdef} \, \tfrac{1}{3!} dx^d \wedge dx^e \wedge dx^f$$

where

$$d \star i_{V_X} T = 0 \,.$$

Global "momenta"

- Given the quadruple of input data $d := \big((M,g);(H,\Phi);\Sigma;\,T\big)$, get element in $\mathfrak{h}*$ by

$$\mathfrak{M}_d : X \mapsto \mathfrak{M}[T,\Sigma](X) := \int_\Sigma \star J_X[T] \tag{6}$$

- For given (M,g) and (H,Φ), and sufficiently restricted family of space-like hypersurfaces (ending at i_0) together with sufficiently well behaved T (spatially compact support) this is independent of Σ.
- Considering T as function of fields F, have

$$\mathfrak{M} : F \to \mathfrak{M}(F) \in \mathfrak{h}^* \qquad \text{("momentum map")} \tag{7}$$

- The image of the momentum map lies in the dual of the symmetry Lie-algebra. This is the habitat of global charges = "momenta".
- How does G act on these momenta?

Momenta and co-adjoint G-action

- For each field configuration F and each $X \in \mathfrak{h}$ have momentum

$$\mathfrak{M}[F,\Sigma](X) = \int_\Sigma \star \circ i_{V_X} \circ T\ [F]$$

- The fields F carry a representation D of H. The momentum of $D_h(F)$ is

$$\begin{aligned}
\mathfrak{M}[D_h(F),\Sigma](X) &\overset{1}{=} \int_\Sigma \star \circ i_{V_X} \circ T \circ D_h\ [F] \\
&\overset{2}{=} \int_\Sigma \star \circ i_{V_X} \circ \Phi_{h*} \circ T\ [F] \\
&\overset{3}{=} \int_\Sigma \star \circ \Phi_{h*} \circ i_{\Phi^{-1}_{h*} V_X} \circ T\ [F] \\
&\overset{4}{=} \int_\Sigma \Phi^*_{h^{-1}} \left(\star \circ i_{\Phi^{-1}_{h*} V_X} \circ T\ [F]\right) \\
&\overset{5}{=} \int_{\Phi_{h^{-1}}(\Sigma)} \left(\star \circ i_{V_{\mathrm{Ad}_{h^{-1}}(X)}} \circ T\ [F]\right) \\
&\overset{6}{=} \mathfrak{M}[F,\Phi_{h^{-1}}(\Sigma)](\mathrm{Ad}_h^{-1}(X)) \\
&\overset{7}{=} \mathfrak{M}[F,\Sigma](\mathrm{Ad}_h^{-1}(X)) \equiv \mathrm{Ad}_h^*(\mathfrak{M})[F,\Sigma](X)
\end{aligned} \tag{8}$$

- Space of momenta carries co-adjoint representation.

H = Poincaré group (Poin)

- Let M 4-dim. be real affine space and V corresponding vector space Lorentz inner product η. Have

$$\begin{aligned} \mathrm{Lor} &:= \{h \in GL(V) \mid \eta(hv, hw) = \eta(v, w)\,, \quad v, w \in V\} \\ \mathfrak{lor} &:= \{X \in \mathrm{End}(V) \mid \eta(Xv, w) = -\eta(v, Xw)\,, \quad v, w \in V\} \end{aligned}$$

- We consider the case

$$H = V \rtimes \mathrm{Lor} =: \mathrm{Poin} \Rightarrow \mathfrak{h} = V \rtimes \mathfrak{lor} =: \mathfrak{poin}$$

- We may identify as vector spaces

$$\mathfrak{poin} \equiv V \oplus \bigwedge^2 V \equiv \mathfrak{poin}^*$$

with Lie-product on $\mathfrak{poin}$ given by

$$\begin{aligned} &[t^a, t^b] = 0\,, \qquad [t^a, M^{bc}] = -\eta^{ab} t^c + \eta^{ac} t^b \\ &[M^{ab}, M^{cd}] = \eta^{ac} M^{bd} + \eta^{bd} M^{ac} - \eta^{ad} M^{bc} - \eta^{bc} M^{ad} \end{aligned}$$

Adjoint and co-adjoint representation of Poin

- With foregoing identification, the pairing $\mathfrak{poin}^* \times \mathfrak{poin} \to \mathbb{R}$ is given as η–inner product

$$(p, J) \times (t, M) \to p^a t^b \eta_{ab} + \tfrac{1}{2} J^{ab} M^{cd} \eta_{ac} \eta_{bd}$$

- The adjoint and co-adjoint representation of Poin on $\mathfrak{poin}$ and $\mathfrak{poin}^*$ are then given, respectively, by

$$\begin{aligned} \mathrm{Ad}_{(a,L)}(t, M) &= \big(Lt - (L \otimes LM)a\,,\; L \otimes LM\big) \\ \mathrm{Ad}^*_{(a,L)}(p, J) &= \big(Lp\,,\; L \otimes LJ - a \wedge Lp\big) \end{aligned} \tag{9}$$

- In particular, for pure translations

$$\begin{aligned} \mathrm{Ad}_{(a,1)}(t, M) &= \big(t - Ma\,,\; M\big) \\ \mathrm{Ad}^*_{(a,1)}(p, J) &= \big(p\,,\; J - a \wedge p\big) \end{aligned} \tag{10}$$

Connection with standard way of writing

- To implement an action of $V \rtimes \mathrm{Lor}$ on M (affine space!) one needs to specify an origin $z \in M$. The Killing vector-fields V_X for $X = (t, M)$ are then given in terms of the privileged *affine coordinates* by

$$V_X^z = t^a\,\partial/\partial x^a + \tfrac{1}{2} M^{ab}\big[(x-z)_a\,\partial/\partial x^b - [(x-z)_b\,\partial/\partial x^a\big]$$

- Only the M-dependent part of momentum depends on z: Have

$$\mathfrak{M}\left(X = (t, M)\right) = \eta_{ab} t^a p^b + \tfrac{1}{2}\eta_{ac}\eta_{bd} M^{ab} J^{cd}(z)$$

where

$$p^a = \int_\Sigma T^{ab} n_b\, d\mu$$

and

$$J^{ab}(z) = \int_\Sigma \big[(x-z)^a T^{bc} - (x-z)^b T^{ac}\big] n_c\, d\mu$$

- Note

$$J(z+a) = J(z) - a \wedge p \qquad \text{(co-adjoint representation)}$$

Supplementary conditions

- A supplementary condition puts a restriction on $J(z)$, the solution of which is a timelike line of possible z.
- If $u \in V$ is a unit timelike vector characterising an inertial frame of reference, we consider the supplementary condition

$$J(z+a)\cdot u = 0 \Leftrightarrow J(z)\cdot u - a(p\cdot u) + p(a\cdot u) = 0$$

- This is equivalent to linear inhomogeneous equation for a

$$\underbrace{\left[\mathrm{Id} - \frac{p\otimes u}{p\cdot u}\right]}_{\pi} \cdot a = \frac{J(z)\cdot u}{p\cdot u}$$

  where π is the projector onto $u^\perp := \{v \in V \mid v\cdot u = 0\}$ parallel to p.
- Hence solution space is one-dimensional (timelike worldline)

$$a(\lambda) = \frac{J(z)\cdot u}{p\cdot u} + \lambda p\,, \qquad \lambda \in \mathbb{R} \tag{11}$$

- Dependence of $a(\lambda)$ on z is clear. Replacing z by $z' = z + b$ results in translated worldline $a'(\lambda) = a(\lambda) + b$.
- But how does $a(\lambda)$ depend on u for fixed z?

Worldline dependence on u

- For any $u \in V_1 = \{v \in V \mid v \cdot v = 1\}$ equation (11) yields a line $a(\lambda)$ parallel to p. As u varies over the 3-dimensional hyperbola V_1 ("mass shell") we obtain a sheaf of geodesics in M.
- In that sheaf one line $a = a_*$ is distinguished: that for $u = p$. We call it the centre of inertia and the corresponding angular momentum

$$J(z + a_*) = S$$

  the *Spin*
- It can be shown by elementary geometric means that the spatial diameter of this sheaf is isotropic with respect to the centre of inertia and has a radius of

$$R_M = \frac{\|S\|}{\|p\|} = \frac{\|S\|}{M_0 c}, \qquad \text{Møller 1949} \tag{12}$$

Examples for Møller radii: Spin 1/2

- Spin 1/2 particle has $\|S\| = \hbar/2$ and thus

$$R_M = \frac{\hbar}{2M_0 c} = \frac{1}{4\pi}\frac{h}{M_0 c} = \frac{1}{4\pi}\lambda_C$$

- An electrically charged spin 1/2 particle has a classical charge-radius $R_{\text{classical}}$ determined by

$$\frac{e^2}{8\pi\varepsilon_0 R_{\text{classical}}} = M_o c^2$$

  This gives

$$R_M = R_{\text{classical}}/\alpha \approx 137 R_{\text{classical}}$$

Examples for Møller radii: Proton

- Experimentally determined "charge radius" of Proton is (2010 CODATA)

$$R_{\text{charge}}^{(\text{Proton})} = 0.87 \cdot 10^{-15}\,\text{m}$$

- It Compton wavelength is

$$\lambda_{\text{Proton}} = 1.32 \cdot 10^{-15}$$

- The Møller radius is

$$R_M^{(\text{Proton})} = \frac{\lambda_{\text{Proton}}}{4\pi} = 1.05 \cdot 10^{-15}\,\text{m} \approx \frac{1}{8} \cdot R_{\text{charge}}^{(\text{Proton})}$$

Examples for Møller radii: Classical bodies

- A homogeneous rigid body of mass M and Radius R, spinning at angular frequency ω, has spin angular-momentum equal to

$$S = \frac{2}{5}\,MR^2\,\omega$$

- Hence the ratio of its Møller radius to its geometric radius is

$$\frac{R_M}{R} = \frac{S}{McR} = \frac{2}{5}\left(\frac{R\omega}{c}\right)$$

- This gives

$$R_M^{(\mathrm{Earth})} = 4\,\mathrm{m}\,, \quad R_M^{(\mathrm{Moon})} = 1.1\,\mathrm{cm}\,,$$

and for the 716 Hz Pulsar *PSR J1748-2446ad*, for which $R\omega/c \approx 0.24$,

$$\left(\frac{R_M}{R}\right)_{\mathrm{Pulsar}} \approx 0.1$$

Localisation and position observables

- Foliate spacetime by spacelike hyperplanes Σ_s, $s \in \mathbb{R}$, orthogonal to $n \in V_1$

$$\Sigma_s = \{x \in M \mid (x - z) \cdot n = s\}$$

Define *centre-of-mass* by on Σ_s by $z + q$, where

$$\begin{aligned} q^a[\Sigma_s] &= \frac{1}{p \cdot n} \int_{\Sigma_s} (x - z)^a \, T^{bc} n_b n_c \, d\mu \\ &= \frac{1}{p \cdot n} \int_{\Sigma_s} \left(2(x - z)^{[a} T^{b]c} + (x - z)^b T^{ac}\right) n_b n_c \, d\mu \\ &= \frac{1}{p \cdot n} \left(s p^a + M^{ab} n_b\right) \end{aligned}$$

Define spin angular-momentum w.r.t. q as

$$S = M - q \wedge p = M - \frac{p \wedge i_n M}{p \cdot n}$$

so that (suppl. condition)

$$i_n S = 0$$

Poisson structure

- Assume Poisson structure

$$\{x^a, x^b\} = 0\,, \quad \{x^a, p^b\} = \delta^{ab}\,, \quad \{p^a, p^b\} = 0$$

- Induces Poisson structure for q and p (Born & Infeld 1935)

$$\{q^a, q^b\} = -\frac{S^{ab}}{(p\cdot n)^2}\,, \quad \{q^a, p^b\} = \pi^{ab}\,, \quad \{p^a, p^b\} = 0 \qquad (13)$$

  where $\pi^{ab} = \eta^{ab} - p^a n^b/(p\cdot n)$

- In order to arrive at (13) one assumes $\{x^a, n^b\} = 0 = \{p^a, n^b\}$, i.e. independence of n on x and p. This changes for, e.g., $n \propto p$

- Newton-Wigner localisation is such that $\{q^a, q^b\} = 0$.

Summary

- Talking about *motion* presupposes a notion of *position*.
- Position becomes ambiguous in the transition from Galilean to Poincaré relativity.
- There are obvious reasons for this in case of extended objects, but ambiguities continue to exist for point particles. This transcends the realm of classical physics and relates to the infamous localisation problem in RQFT.
- It has been the idea behind this talk to make explicit those structures that give meaning to notions of localisation in the context of SR and which will either not exist or fail uniqueness in the context of GR.
- This (hopefully) helps to distinguish the generic difficulties of the gravitational case from those merely inherited by SR.

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