Global Expansion
versus
Local Kinematics and Dynamics

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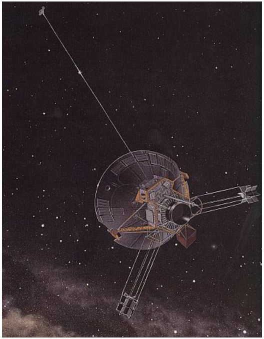
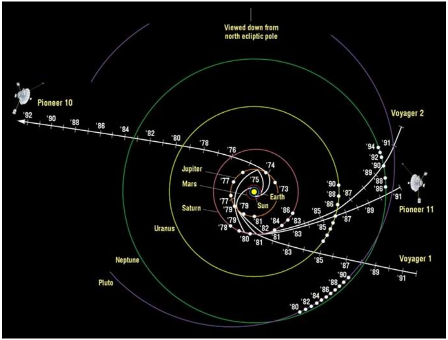
Hermann Weyl (1885 - 1955)

- "First of all, we observe *that in General Relativity the notion of relative motion of two bodies is as meaningless as the notion of absolute motion applied to a single one*."
  (Space-Time-Matter, 1923)

- "We keep the dualism of guidance and force; *but guidance is a physical field of state* (like the electromagnetic) *interacting with matter. Gravitation is part of the guidance, it's not a force*."
  (Massenträgheit und Kosmos, 1924)

Motivation and Irritation: The “Pioneer Anomaly”





$$a_{anom} = (8.6 \pm 1.34) \times 10^{-10}\,\mathrm{m}\cdot\mathrm{s}^{-2}$$

$$H_0 c \approx (72\,\mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1})\cdot(3\times 10^{5}\,\mathrm{km}\cdot\mathrm{s}^{-1}) \approx 7\times 10^{-10}\,\mathrm{m}\cdot\mathrm{s}^{-2}$$

Newtonian theory

- Suppose local inertial systems mutually expand from each other according to Hubble's law:

$$\dot{R}(t) = H(t)R(t), \quad \text{where} \quad H(t) := \dot{a}(t)/a(t) \tag{1}$$

- Taking into account $\dot{H} = (\ddot{a}/a) - H^2$, the Hubble acceleration is

$$\ddot{R} = \dot{H}R + H\dot{R} = (\ddot{a}/a)\,R = -qH^2\,R \tag{2}$$

  where $q := -\ddot{a}a/\dot{a}^2$ is the usual deceleration parameter

- The Newtonian force is proportional to the acceleration relative to these local inertial frames, so that the Newtonian equations of motion in an expanding universe is obtained from the usual one by replacing

$$\ddot{\vec{x}} \mapsto \ddot{\vec{x}} - (\ddot{a}/a)\,\vec{x} \tag{3}$$

Newtonian theory (contd.)

- Applied to the motion of a single particle in a radial $1/r^2$ force-field gives (setting $(\ddot{a}/a)_{\mathrm{now}} = -q_0 H_0^2 =: A$)

$$\tfrac{1}{2}\dot{r}^2 + U(r) = E \qquad r^2\dot{\varphi} = L \tag{4}$$

with

$$U(r) = \frac{L^2}{2r^2} - \frac{C}{r} - \frac{A}{2}\,r^2 \tag{5}$$

- The critical radius where the attracting force ($C > 0$) is balanced by the cosmological repulsion (for $A > 0$) is

$$r_c = \left[\frac{C}{A}\right]^{1/3} = \begin{cases} \left[R_s R_H^2/(2|q_0)\right]^{1/3} & \approx\ (M/M_\odot)^{1/3}\,400\,\mathrm{ly} \\ \left[R_q R_H^2/(2|q_0)\right]^{1/3} & \approx\ (q/e)^{2/3}\,30\,\mathrm{AU} \end{cases} \tag{6}$$

Here the upper and lower equality holds in case of gravitational and electrical attraction, where $R_s = 2GM/c^2$ and $R_q = 2q^2/mc^2$ respectively. Stable circular orbits exist for $r < (1/4)^{1/3} r_c \approx 0.63\, r_c$.

General Relativity

- In Newtonian theory we can (mathematically) add the potential of an isolated source to any other solution and obtain a new solution (with the same boundary conditions at spatial infinity) in which a new source has been (physically) "added".
- Due to the non-linear nature of Einstein's equations this is not possible in GR. There is no unambiguous way to compose a new solution from two old ones. Correspondingly, there is no *obvious* way (if any) to state the simultaneous physical presence of two systems characterised by individual solutions. In particular, there is no *obvious* way to add a local source to a cosmological background.
- Two strategies have been mainly employed: to *join* and to *alloy (or amalgamate)* solutions.

Joined Solutions

- Given two spacetimes $(M_\pm, g_\pm)$ with boundaries $\Gamma_\pm$ and diffeomorphism $\phi : \Gamma_+ \to \Gamma_-$, we can form $M := M_+ \cup_\phi M_-$ and require **Darmois' junction conditions**

$$g_+\big|_{T\Gamma_+} = \phi^* g_-\big|_{T\Gamma_-}\,, \quad K_+ = \phi^* K_-\,. \tag{7}$$

  which state the continuity of the $(\perp, \|)$ and $(\perp, \perp)$ components of the Einstein tensor across Γ and hence, via Einstein's equation, the continuity across Γ of the matter's current densities $\perp S$ for energy and momentum.
- Joining spherically symmetric spacetimes along a curve $\gamma \subset B$ of $\mathrm{SO}(3)$-orbits (2-spheres) requires continuity of:
  1. γ's arc length and curvature;
  2. the areal radius R;
  3. the Misner-Sharp energy.

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Intermezzo: Hawking mass and Misner-Sharp energy

- In case of spherical symmetry, the **Hawking mass** (defined for any closed spacelike 2d-submanifold S)

$$M_H(S) := \sqrt{\frac{\mathrm{Area}(S)}{16\pi}} \left(1 + \frac{1}{2\pi}\int_S \theta^+\theta^- \, d\mu_S\right) \tag{8}$$

  equals the Misner-Sharp energy

$$E := -\tfrac{1}{2}\, R^3\, \mathbf{Sec}(g)\big|_{TS} = \tfrac{1}{2} R\big(1 + g(dR, dR)\big)\,, \tag{9}$$

  if S is a $SO(3)$ orbit of spherical symmetry.
- E can be considered as positive real-valued function on B, i.e. of time and radius.
- Corresponding to **Riem** $=$ **Ricci** $+$ **Weyl**, E decomposes as

$$E = E_R + E_W\,. \tag{10}$$

  E_W corresponds to the mass/energy of a black-hole.

The Swiss-Cheese Universe

- Join Schwarzschild-DeSitter spacetime

$$
\begin{aligned}
g_{SdS} &= V(R)\,dT^2 - \big(V(R)\big)^{-1}\,dR^2 - R^2\,g_F \\
V(R) &= 1 - (2m/R) - \tfrac{1}{3}\Lambda R^2
\end{aligned}
\qquad (11)
$$

to FLRW Universe

$$
g_{FLRW} = dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\,g_F\right)\,. \qquad (12)
$$

- Have $R_{SdS} = R$ and $R_{FLRW} = a(t)r$. At gluing radius have $R = a(t)r$ and

$$
E_{SdS} = m + \tfrac{4\pi}{3}R^3_{SdS}\rho_\Lambda = E_{FLRW} = \tfrac{4\pi}{3}R^3_{FLRW}(\rho + \rho_\Lambda) \qquad (13)
$$

$$
\Leftrightarrow\quad m = \tfrac{4\pi}{3}R^3\rho \quad\Leftrightarrow\quad \boxed{r(t) = \left(\frac{m}{(4\pi/3)a^3(t)\rho(t)}\right)^{1/3}} \qquad (14)
$$

Einstein-Straus

- Einstein & Straus 1945, Schücking 1954:

$$R_V(t) = \left(\frac{3m}{4\pi\rho(t)}\right)^{1/3} \tag{15}$$

Note: R_v is an areal radius, whose definition is that the proper surface-area of the $SO(3)$ orbit is $4\pi R_v^2$. Hence $(4\pi/3)R_v^3$ is the proper volume if $g^{(3)}$ is flat, but smaller/larger if $g^{(3)}$ is of positive/negative curvature.

- Applied to spatially flat universe of critical density $\rho_{\mathrm{crit}} := \frac{3H_0}{8\pi G}$, get

$$R_v = (R_S\, R_H^2)^{1/3} \approx (m/m_\odot)^{1/3}\, 400\,\mathrm{ly}\,, \tag{16a}$$

$$R_S := (2Gm/c^2) \approx (m/m_\odot)\, 3\,\mathrm{km}\,, \tag{16b}$$

$$R_H := (c/H_0) \approx 4\,\mathrm{Gpc} \approx 1.3\times 10^{23}\,\mathrm{km}\,. \tag{16c}$$

R_S and R_H are the **Schwarzschild radius** for the mass M and the **Hubble radius** respectively.

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GR: Alloying solutions

- Close to inhomogeneity solutions should approximate exterior Schwarzschild metric (possibly filled in by some interior solution with matter)

$$g^{(4)} = \left[\frac{1 - m/2r}{1 + m/2r}\right]^2 dt^2 - \left[1 + \frac{m}{2r}\right]^4 g^{(3)}_{\mathrm{flat}} \qquad (17)$$

  and far out it should approximate FLRW universe

$$g^{(4)} = T^2 - a^2(t)\, g^{(3)}_{\mathrm{cc}} \qquad (18)$$

- The question is: how to combine these two exact solutions in order to get new exact solution describing a compact object 'immersed' in cosmological background?

McVittie's Ansatz

- In 1933, McVittie suggested the following simple Ansatz in his paper: 'The mass-particle in an expanding universe':

$$g^{(4)} = \left[\frac{1 - m(t)/2r}{1 + m(t)/2r}\right]^2 dt^2 - a^2(t)\left[1 + \frac{m(t)}{2r}\right]^4 g^{(3)}_{\mathrm{cc}} \qquad (19)$$

  where $m(t)$ and $a(t)$ are two positive functions of time t.
- In recent years, solutions within the reach of this Ansatz have often been used to study the impact of cosmological expansion on local inhomogeneities.
- In the following we shall investigate some consequences of this Ansatz. For simplicity we will set

$$g^{(3)}_{\mathrm{cc}} = g^{(3)}_{\mathrm{flat}} = dr^2 + r^2 g^{(2)}_{S^2_1} \qquad (20)$$

- Solutions in this class have first been considered by McVittie in 1933 and have recently again been proposed and analysed.

General solutions in McVittie's class

- We keep McVittie's Ansatz and wish to generalise the matter model so as to allow for arbitrary energy infall (D.G& Carrera 2010).
- It turns out that this is possible only if radial mass currents as well as non-vanishing heat currents are present, i.e. if

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + p\,(u^\mu u^\nu - g^{\mu\nu}) + (u^\mu q^\nu + q^\mu u^\nu) \tag{21}$$

with

$$\begin{aligned} u &= \cosh(\chi)\, e_t + \sinh(\chi)\, e_r \\ q &= |q|\,\big(\sinh(\chi)\, e_t + \cosh(\chi)\, e_r\big) \end{aligned} \tag{22}$$

- For general χ and q^μ have a radial energy current

$$J = -T(e_t, e_r) = \underbrace{q\,(1 + 2\,\sinh^2\chi)}_{J_h} + \underbrace{(\rho + p)\,\sinh\chi\,\cosh\,\chi}_{J_m} \tag{23}$$

whose components are constrained to satisfy (here for small χ)

$$J = J_m + J_h \approx J_m/2 \approx -J_h \tag{24}$$

- The t-rate of energy accretion by central mass is given by

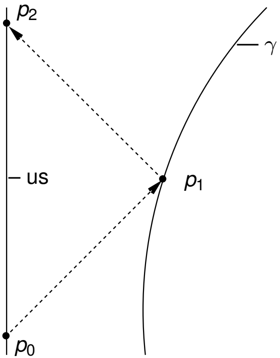
$$(am)^{\cdot} = (-4\pi R^2\, J)\, g_{tt} \tag{25}$$

Doppler Tracking

- **Doppler Tracking** is a common method of tracking the position of vehicles in space by measuring the Doppler shift of exchanged electromagnetic signals. If the geometry of space is time dependent, this will clearly influence such measurements. What does it mean to *map out* a trajectory.
- The following is based on work with Matteo Carrera (CQG 26, 2006).

DT in Minkowski space I

- At the event $p_0 = (t_0, \vec{x}_0)$ a radio signal of frequency ω_0 is emitted towards the spacecraft and received by it at event $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1.
- In case of simple **reflection**, a returning radio signal is emitted at $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1 and received by us at event $p_2 = (t_2, \vec{x}_2)$ with frequency ω_2.
- Note: All frequencies refer to those measured by observers that are locally co-moving with the given worldlines ($\gamma =$ worldline of spacecraft)



DT in Minkowski space II

- Given a lightlike vector k (wave vector) and observers u, v at the same spacetime point. The observed frequencies are

$$\omega_v(k) := g(v,k) \qquad \omega_u(k) := g(u,k) \tag{26}$$

whose ratio is given by

$$\begin{aligned}\frac{\omega_v(k)}{\omega_u(k)} = \frac{g(v,k)}{g(u,k)} &= \frac{g(P_u^{\parallel} v + P_u^{\perp} v, k)}{g(u,k)} \\ &= g(u,v)\left[1 + \frac{g(v, P_u^{\perp} k)}{g(u,v)g(u,k)}\right] = g(u,v)\big[1 - \beta_u^{\hat{k}}(v)\big]\end{aligned} \tag{27}$$

where the spacelike unit vector $\hat{k} := P_u^{\perp} k / \| P_u^{\perp} k \|$ defines the direction of k in the rest system of u.

DT in Minkowski space III

- We wish to take the differential quotient of $\omega(t_2)/\omega(t_0)$ with respect to t_2, assuming a constant function ω_0. We get

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{(1+\beta(t_1))^2}\frac{dt_1}{dt_2} \tag{28}$$

- If we are resting at the origin of the coordinate system, with respect to which r is the radial distance to the spacecraft, we have $t_2 - t_1 = r(t_1)/c$ and therefore

$$1 - \frac{dt_1}{dt_2} = \frac{1}{c}\frac{dr(t_1)}{dt_1}\frac{dt_1}{dt_2} \iff \frac{dt_1}{dt_2} = \frac{1}{1+\beta(t_1)} \tag{29}$$

- Hence (28) becomes ($\dot{\beta} \equiv \alpha$)

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{(1+\beta(t_1))^3} \approx -2\alpha(t_1)\Big(1 - 3\beta(t_1) + \cdots\Big) \tag{30}$$

DT in general I

- In a general spacetime (M, g) [we use signature $(+,-,-,-)$ for g] there is no privileged (e.g. inertial) global reference frame by means of which we may introduce kinematical variables that characterise worldlines (different ones collectively). Hence 'appropriate' fiducial observer-fields need to be introduced.
- An **observer** at the event p is a future pointing unit timelike vector. An **observer field** is a field of observers. Any observer u at p gives rise to an orthogonal split of the tangent space at p, $T_p(M) = T_p^{\|}(M) \oplus T_p^{\perp}(M)$, where

$$T_p^{\|}(M) := \mathrm{Span}\{u\}\,, \quad T_p^{\perp}(M) := \{v \in T_p(M) \mid g(v,u) = 0\} \tag{31}$$

The associated projection operators are given by

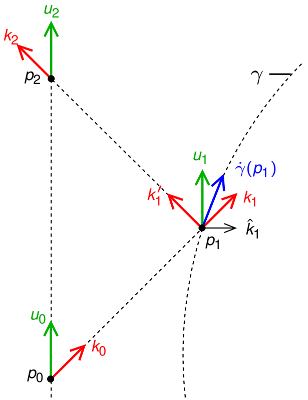
$$P_u^{\|} : T_p \to T_p^{\|}(M)\,, \quad v \mapsto P_u^{\|}(v) := u\, g(u,v) \tag{32a}$$

$$P_u^{\perp} : T_p \to T_p^{\perp}(M)\,, \quad v \mapsto P_u^{\perp}(v) := v - u\, g(u,v) \tag{32b}$$

DT in general II

- Let u be an observer **field** along one integral line of which we are moving. As before, γ is the worldline of the spacecraft. The field u is defined in a neighbourhood of γ.
- The wave-vector k_0 emitted at p_0 suffers three changes:
  1. propagation from p_0 to p_1: $k_0 \to k_1$
  2. reflection at p_1: $k_1 \to k_1'$
  3. propagation from p_1 to p_2: $k_1' \to k_2$
- We are interested in

$$\frac{\omega_2}{\omega_0} = \frac{g(u_2, k_2)}{g(u_0, k_0)} = \left[\frac{\omega_2}{\omega_1'}\right] \left[\frac{\omega_1'}{\omega_1}\right] \left[\frac{\omega_1}{\omega_0}\right]$$



DT in FLRW/McVittie

- An *exact* formula for $\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2}$ can be derived for FLRW universes.
- For purely radial motion and keeping only quadratic terms in β, linear terms in $H\Delta t$, and also mixed terms $\beta H\Delta t$, we get:

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} \approx -\frac{2}{c}\left\{c\alpha\big(1-3\beta-3H\Delta t\big)+Hc\beta\right\} =: -2\,a_*/c \qquad (33)$$

  where a_* is the naive Doppler-tracking acceleration. Hence in this approximation there are two modifications due to cosmic expansion:

  1. a downscaling of acceleration by $(1-3H\Delta t)$
     $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-12}$
  2. a constant contribution $Hc\beta$ in velocity direction
     $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-7}$

  This can be generalised to McVittie, where an extra term $(m_0 c/R^2)\Delta\tau$ appears in $c\alpha(\cdots)$.

Other coordinates I

- Instead of the standard co-moving radial coordinate r in FLRW models on may employ the cosmologically simultaneous geodesic distance r_* (here flat case):

$$(t\,,\,r)\mapsto(t_*\,,\,r_*):=(t\,,a(t)r) \tag{34}$$

so that the new field of geodesically equidistant observers $r_* = const.$ is

$$u_* = \frac{1}{\|\partial/\partial t_*\|}\frac{\partial}{\partial t_*} \quad \text{where} \quad \frac{\partial}{\partial t_*} = \frac{\partial}{\partial t} - H(t)r\frac{\partial}{\partial r} \tag{35}$$

- Since u_* is not geodesic (inward accelerated) get additional cosmological acceleration $(\ddot{a}/a)r_*$ in radial direction in Newtonian equation of motion. More general, for geodesics in McVittie spacetime, we obtain to leading order

$$\alpha_{u_*}(\gamma) \approx \left(\frac{\ddot{a}}{a}r_* - \frac{m_0}{r_*^2}\right)\vec{e}_r \circ \gamma \tag{36}$$

Other coordinates II

In (t_*, r_*) coordinates, the flat FLRW metric assumes the form

$$g = c^2\left\{1-(Hr_*/c)^2\right\}\underbrace{\left\{dt_* + \frac{Hr_*/c^2}{1-(Hr_*/c)^2}\,dr_*\right\}}_{\theta=\text{simultaneity 1-form}}^2 - \underbrace{\left\{\frac{dr_*^2}{1-(Hr_*/c)^2} + r_*^2\,d\Omega^2\right\}}_{h=\text{spatial radar metric}}$$

- Radar distance (measured by h) and Einstein simultaneity ($\theta = 0$) are given by

$$l_* = (c/H)\sin^{-1}(Hr_*/c) \approx r_*\{1 + \tfrac{1}{6}(Hr_*/c)^2 + \mathcal{O}(3)\} \quad (37)$$

$$\Delta t_* = (1/2H)\ln\big(1-(Hr_*/c)^2\big) \approx (r_*/c)\{-\tfrac{1}{2}(Hr_*/c) + \mathcal{O}(2)\} \quad (38)$$

- Mapping out a trajectory $l_*(t_*)$ in terms of **radar distance of Einstein-simultaneous events** hence means to write

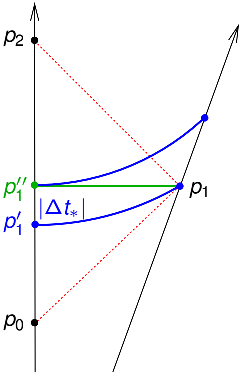
$$l_*(t_*) := (c/H)\sin^{-1}\big(r_*(t_* + \Delta t_*)H/c\big) \approx r_* - \tfrac{1}{2}(v/c)(Hc)(r_*/c)^2 + \cdots$$

which in leading order gives

$$\ddot{l}_* \approx \ddot{r}_* - (Hc)(v/c)^3 + \cdots \quad (39)$$

Cosmological versus Einstein Simultaneity

- The reflection-event p_1 lies on the same hypersurface of constant cosmological time $t = t_*$ as the event p_1' on our worldline. However, our eigentime at p_1' is not the mean of our eigentimes at p_0 and p_2. Rather, this is true for the event p_1'', which is hence Einstein-simultaneous with p_1 and which lies to the future of p_1' by the amount $|\Delta t_*|$, given by (38).



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Summary

- Derivation of exact double-Doppler-formula for FLRW spacetimes.
- Derivation of approximate double-Doppler-formula for McVittie spacetime.

$\Rightarrow$ There exist no Pioneer-like anomalies due to cosmic expansion.

$\Rightarrow$ Kinematical effects consistently estimated, which e.g. lead to Hc–term at $(v/c)^3$–suppressed level.

THANK YOU!