2

LIE GROUPS AND LIE ALGEBRAS

2.1 Lie groups

The most general definition of a Lie group G is, that it is a differentiable manifold with group structure, such that the multiplication map $G \times G \to G$, $(g, h) \mapsto gh$, and the inversion map $G \to G$, $g \mapsto g^{-1}$, are differentiable. But we shall not need this concept in full generality. In order to avoid more elaborate differential geometry, we will restrict attention to *matrix groups*.

Consider the set of all invertible $n \times n$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ stands either for the real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers. It is easily verified to form a group which we denote by $\mathsf{GL}(n, \mathbb{F})$, called the general linear group in n dimensions over the number field $\mathbb{F}$. The space of all $n \times n$ matrices, including non-invertible ones, with entries in $\mathbb{F}$ is denoted by $\mathtt{M}(n, \mathbb{F})$. It is an n^2-dimensional vector space over $\mathbb{F}$, isomorphic to $\mathbb{F}^{n^2}$. The determinant is a continuous function $\det : \mathtt{M}(n, \mathbb{F}) \to \mathbb{F}$ and $\mathsf{GL}(n, \mathbb{F}) = \det^{-1}(\mathbb{F} - \{0\})$, since a matrix is invertible iff its determinant is non zero. Hence $\mathsf{GL}(n, \mathbb{F})$ is an open subset of $\mathbb{F}^{n^2}$. Group multiplication and inversion are just given by the corresponding familiar matrix operations, which are differentiable functions $\mathbb{F}^{n^2} \times \mathbb{F}^{n^2} \to \mathbb{F}^{n^2}$ and $\mathbb{F}^{n^2} \to \mathbb{F}^{n^2}$ respectively.

2.1.1 *Examples of Lie groups*

$\mathsf{GL}(n, \mathbb{F})$ is our main example of a (matrix) Lie group. Any Lie group that we encounter will be a subgroups of some $\mathsf{GL}(n, \mathbb{F})$. The simplest such subgroup is

$$\mathsf{SL}(n, \mathbb{F}) := \{ g \in \mathsf{GL}(n, \mathbb{F}) \mid \det(g) = 1 \} \,, \tag{2.1}$$

the so called *special linear group* for the number field $\mathbb{F}$. Subgroups whose elements satisfy $\det(g) = 1$ are also called *unimodular*.

Choosing $\mathbb{F} = \mathbb{R}$, one has an important class of subgroups of $\mathsf{GL}(n, \mathbb{R})$, the so called *(pseudo) orthogonal groups*, defined by

$$\mathsf{O}(p, q) := \{ g \in \mathsf{GL}(n, \mathbb{R}) \mid g E^{(p,q)} g^{\top} = E^{(p,q)} \} \,, \tag{2.2}$$

where $E^{(p,q)}$ is the $n \times n$ diagonal matrix with the first p entries $+1$ and the remaining q entries -1 (clearly $p + q = n$):

$$E^{(p,q)} := \mathrm{diag}(\underbrace{1, \cdots, 1}_{p}, \underbrace{-1, \cdots, -1}_{q}) \,. \tag{2.3}$$

If either q or p are zero, the group is simply called orthogonal, otherwise pseudo-orthogonal. In this case one usually writes $\mathsf{O}(n)$ instead of $\mathsf{O}(n, 0)$ or $\mathsf{O}(0, n)$.

Taking the determinant of the defining relation in (2.4) shows that $\det(g) = \pm 1$ for $g \in \mathsf{O}(p,q)$. Those elements for which the determinant is $+1$ form a subgroup (of index two), and are called the *unimodular or special (pseudo) orthogonal groups*:

$$\mathsf{SO}(p,q) := \{g \in \mathsf{O}(p,q) \mid \det(g) = 1\}\,. \tag{2.4}$$

Here one usually writes $\mathsf{SO}(n)$ instead of $\mathsf{SO}(n,0)$ or $\mathsf{SO}(0,n)$.

Different tuples (p,q) and (p',q') with $n = p+q = p'+q'$ lead in general to non isomorphic Lie groups of the same dimension[7]. This is not the case over the field of complex numbers, where for any two (p,q) and (p',q') with fixed sum n there always exists a matrix $T \in \mathsf{GL}(n,\mathbb{C})$ such that $TE^{(p,q)}T^\top = E^{(p',q')}$; just take for T the diagonal matrix with an appropriate distribution of 1's and i's in the diagonal (in the real case this is excluded by 'Sylvester's law of inertia', see e.g. [7]). This means that over the complex numbers it would be meaningless to distinguish between the pseudo and proper orthogonal groups for fixed n, since they are all conjugate in $\mathsf{GL}(n,\mathbb{C})$ (hence, in particular, isomorphic). For $\mathbb{F} = \mathbb{C}$ one therefore just writes $O(n,\mathbb{C})$ for the subgroup of $\mathsf{GL}(n,\mathbb{C})$ defined through $gg^\top = E^{(n)}$, where $E^{(n)}$ is the unit $n \times n$ matrix. Note that the convention is such that $O(n)$ always means $O(n,\mathbb{R})$.

Next we mention the *(pseudo) unitary groups*, defined by

$$\mathsf{U}(p,q) := \{g \in \mathsf{GL}(n,\mathbb{C}) \mid gE^{(p,q)}g^\dagger = E^{(p,q)}\}\,, \tag{2.5}$$

where † denotes hermitian conjugation, i.e., transposition and complex conjugation. Here the terminology is entirely analogous to the orthogonal groups, i.e., we simply speak of unitary groups if $p = 0$ or $q = 0$, in which case we write $U(n)$ instead of $U(n,0)$ or $U(0,n)$, otherwise of pseudo unitary groups. Taking the determinant of the defining relation shows $|\det(g)| = 1$. The subgroups of matrices with unit determinant are the *unimodular or special (pseudo) unitary groups* :

$$\mathsf{SU}(p,q) := \{g \in \mathsf{U}(p,q) \mid \det(g) = 1\}\,. \tag{2.6}$$

Again we write $SU(n)$ instead of $SU(n,0)$ or $SU(0,n)$.

Finally, to complete the list of the so-called classical Lie groups, we mention the *symplectic groups*. Let $E^{(n)}$ be the unit $n \times n$ matrix, and $\hat{E}^{(2n)}$ the antisymmetric $2n \times 2n$ matrix

$$\hat{E}^{(2n)} := \begin{pmatrix} 0 & E^{(n)} \\ -E^{(n)} & 0 \end{pmatrix}\,, \tag{2.7}$$

we define $\mathsf{SP}(2n,\mathbb{F})$, the symplectic group in $2n$ dimensions over the field $\mathbb{F}$, by:

$$\mathsf{SP}(2n,\mathbb{F}) := \{g \in \mathsf{GL}(n,\mathbb{F}) \mid g\hat{E}^{(2n)}g^\top = \hat{E}^{(2n)}\} \tag{2.8}$$

[7] The dimension of a Lie group is the dimension of its underlying real manifold. Here it can be identified with n^2 minus the number of independent defining conditions, which are $\frac{1}{2}n(n+1)$ in (2.2), hence the dimension of $\mathsf{O}(p,q)$ and $\mathsf{SO}(p,q)$ is $\frac{1}{2}n(n-1)$.

We mention without proof that $\mathsf{SP}(2n, \mathbb{F}) \subset \mathsf{SL}(2n, \mathbb{F})$, which is not immediate from their definition.

Let us now make a few topological remarks (compare A.3). $\mathsf{GL}(n, \mathbb{F})$ and its subgroups are subsets of $\mathtt{M}(n, \mathbb{F}) \cong \mathbb{F}^{n^2}$ and are topologised by the subset topology. The maps of group multiplication and inversion are rational functions in the matrix entries and are clearly continuous (in fact, differentiable). Hence we deal, in particular, with topological groups. The subgroup $\mathsf{SL}(n, \mathbb{F})$ is a closed in $\mathtt{M}(n, \mathbb{F})$ (and also in $\mathsf{GL}(n, \mathbb{F})$) since it is the pre-image of $1 \in \mathbb{F}$ under the determinant function. Similarly, any other subgroup we mentioned is a closed subset of $\mathtt{M}(n, \mathbb{F})$ since its defining relation is $f(g) = 0$, and hence the group is given by $f^{-1}(0)$, with a continuous function $f : \mathtt{M}(n, \mathbb{F}) \to \mathtt{M}(n, \mathbb{F})$, which, for example, for $\mathsf{U}(p, q)$ takes the form $f(g) = gE^{(p,q)}g^\dagger - E^{(p,q)}$. Correspondingly, the unimodular groups (i.e. $\det(g) = 1$) are closed subgroups of $\mathsf{SL}(n, \mathbb{F})$. Finally we remark that $O(n), U(n)$ and their unimodular counterparts are compact. This follows immediately from the defining relations, e.g. from $gg^\dagger = E^{(n)}$ for $U(n)$, which in particular implies that the modulus of each matrix element is bounded above by 1. Hence $U(n)$ is a closed and bounded subset of $\mathtt{M}(n, \mathbb{F}) \cong \mathbb{F}^{n^2}$ and hence compact(compare A.3.6). This argument does not work for the pseudo orthogonal and unitary groups, since their defining relation contains $E^{(p,q)}$ with $p > 1$ and $q > 1$, which is not positive or negative definite, and hence we cannot conclude boundedness of the matrix elements. In fact, all these groups are non compact.

2.1.2 *The inhomogeneous linear groups*

The vector space $\mathbb{F}^n$ is an abelian group with respect to its additive structure. The usual linear action of $\mathsf{GL}(n, \mathbb{F})$ on $\mathbb{F}^n$ defines a homomorphism of $\mathsf{GL}(n, \mathbb{F})$ into the automorphism group of $\mathbb{F}^n$ with respect to which we can construct the semi-direct product $\mathbb{F}^n \rtimes \mathsf{GL}(n, \mathbb{F})$ (compare A.4.3). Multiplication and inversion are then given by (elements of $\mathsf{GL}(n, \mathbb{F})$ are now denoted by capital letters, elements of $\mathbb{F}^n$ by lower case letters):

$$(a', A')(a, A) = (a' + A'a, A'A)\,. \tag{2.9}$$

$$(a, A)^{-1} = (-A^{-1}a, A^{-1})\,. \tag{2.10}$$

This can be repeated verbatim for any subgroup G of $\mathsf{GL}(n, \mathbb{F})$). The corresponding semi-direct product is then called the corresponding inhomogeneous (or affine) group, and denoted by IG:

$$\mathsf{IG} := \mathbb{F}^n \rtimes \mathsf{G} \tag{2.11}$$

In this fashion one defines the groups $\mathsf{ISL}(n, \mathbb{F})$, $\mathsf{ISO}(p, q)$, etc.

The inhomogeneous group of some subgroup $\mathsf{G} \subseteq \mathsf{GL}(n, \mathbb{F})$ can be considered as a subgroup of $GL(n + 1, \mathbb{F})$, that is, there is an embedding (compare A.4.2) $\mathsf{IG} \to \mathsf{GL}(n + 1, \mathbb{F})$, given by:

$$\mathsf{IG} \ni (a, A) \longmapsto \begin{pmatrix} 1 & 0^\top \\ a & A \end{pmatrix} \tag{2.12}$$

where we used a $1 + n$-split matrix notation. One easily verifies that matrix multiplication and inversion in $\mathsf{GL}(n+1, \mathbb{F})$ reproduces (2.9) and (2.10). This means that the inhomogeneous groups are also matrix groups.

2.1.3 *Polar decomposition*

At this point we wish to mention an important way to uniquely decompose any matrix in $\mathsf{GL}(n, \mathbb{C})$ as product of a unitary and a positive-definite hermitian matrix. It is called the *polar decomposition*. This subsection is devoted to state and prove this result.

2.1.3.1 *Preliminaries: Maps of hermitean matrices* We start by recalling some useful properties of hermitean matrices and maps between them. The space of $n \times n$ hermitean matrices is a real vector space of dimension n^2 and will be denoted by $\mathtt{H}(n)$. Subject to a choice of basis, hermitean matrices may be identified with self-adjoint linear maps in an n-dimensional complex vector space with scalar product. In the following we will make use of such an identification and denote the matrix and the map by the same letter. Let now A be a self-adjoint map. Its eigenvalues, $\{\lambda\}$, are real and the eigenvectors are orthogonal if they belong to different eigenvalues. The subset $\{\lambda\} \subset \mathbb{R}$ is called the *spectrum* of A and will be denoted by $\mathcal{S}(A)$.

Let P_λ be the orthogonal projector onto the subspace spanned by eigenvectors of eigenvalue λ; then A can be written in the form

$$A = \sum_{\lambda \in \mathcal{S}(A)} \lambda P_\lambda \tag{2.13}$$

where

$$P_\lambda = P_\lambda^\dagger\,, \qquad \sum_{\lambda \in \mathcal{S}(A)} P_\lambda = E\,, \qquad P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_\lambda \text{ (no summation)}\,, \tag{2.14}$$

where $\delta_{\lambda\lambda'} = 1$ if $\lambda = \lambda'$ and zero otherwise. This is called the *spectral decomposition* of A. This decomposition is unique, which merely says that the set of eigenvalues, $\mathcal{S}(A)$, and the set of eigenspaces, $\{P_\lambda\}$ are uniquely determined by A.

The spectral decomposition is a useful tool to define and study maps of hermitean matrices in terms of maps of real numbers. Any map $f : \mathbb{R} \to \mathbb{R}$ defines a map $\hat{f} : \mathtt{H}(n) \to \mathtt{H}(n)$ via

$$\hat{f}(A) := \sum_\lambda f(\lambda) P_\lambda\,. \tag{2.15}$$

More generally, if the domain D of f is a proper subset of $\mathbb{R}$ (2.15) still defines a function $\hat{f}$ on the domain $\hat{D} = \{A \in \mathtt{H}(n) \mid \mathcal{S}(A) = D\}$. It turns out that many properties of f are inherited by $\hat{f}$, as we now show.

If f is injective on the domain D then $\hat{f}$ is injective on the domain $\hat{D}$. To see this, let $A, A' \in \hat{D}$ and assume $\hat{f}(A) = \hat{f}(A')$; then

$$\hat{f}(A) = \sum_{\lambda \in \mathcal{S}(A)} f(\lambda) P_\lambda = \sum_{\lambda \in \mathcal{S}(A')} f(\lambda) P'_\lambda = \hat{f}(A') \,. \tag{2.16}$$

Uniqueness of the spectral decomposition together with injectivity of f imply that the spectra and the families of projectors coincide so that $A = A'$.

Surjectivity of f is also inherited by $\hat{f}$. More precisely, suppose that $D \subseteq \mathrm{Im}(f)$, then $\hat{D} \subseteq \mathrm{Im}(\hat{f})$. To see this, pick an $A \in \hat{D}$, whose spectral projectors are P_λ, and define A' by

$$A' := \sum_{\lambda \in \mathcal{S}(A)} \mu_\lambda P_\lambda \tag{2.17}$$

for some (non-unique if f is not injective) $\mu_\lambda \in f^{-1}(\lambda)$. Then $f(A') = A$, as desired.

Finally we note that $\hat{f}$ is continuous if f is. With the foregoing this implies that if f is a homeomorphism between the domains D and D' in $\mathbb{R}$, then $\hat{f}$ is a homeomorphism between the domains $\hat{D}$ and $\hat{D}'$ in $\mathtt{H}(n)$. An example of particular interest to us is the exponential map, $\exp : \mathbb{R} \to \mathbb{R}_+$, which is a homeomorphism (the continuous inverse being the logarithm $\ln : \mathbb{R}_+ \to \mathbb{R}$). Hence we have a homeomorphism $\widehat{\exp} : \mathtt{H}(n) \to \mathtt{H}_+(n)$, where $\mathtt{H}_+(n)$ denotes the positive definite (i.e. all eigenvalues strictly positive) $n \times n$ hermitean matrices. In future we will save notation and simply drop the hat over matrix functions.

2.1.3.2 *The polar decomposition*

Theorem 2.1 *Let $X \in \mathsf{GL}(n, \mathbb{C})$; then there exits a unique $R \in \mathsf{U}(n)$ and a unique positive-definite hermitian matrix B (i.e. B's eigenvalues are all positive and $B = B^\dagger$) such that*

$$X = BR \,. \tag{2.18}$$

If $X \in \mathsf{GL}(n, \mathbb{R})$ then B is real, symmetric, and positive definite and R is real and orthogonal, i.e., $R \in \mathsf{O}(n)$.

Proof Define $A := XX^\dagger$, which is an element of $\mathtt{H}_+(n)$ (zero eigenvalues are excluded since $\det(X) \neq 0$). Let $B := \sqrt{A} \in \mathtt{H}_+(n)$ (note that the square-root-function is a homeomorphism of the positive real line) and define $R := B^{-1}X$. We have $R^\dagger = X^\dagger B^{-1} = X^{-1}B = R^{-1}$, where the first equality follows from hermiticity of B and the second from $B^2 = XX^\dagger$. Hence R is unitary and we have shown existence of a polar decomposition. To show uniqueness, assume there exist two decompositions: $X = B_1R_1 = B_2R_2$. Then $B_1 = B_2R_3$, where $R_3 := R_2R_1^{-1}$ is again unitary. Hermiticity of $B_{1,2}$ and unitarity of R_3 now imply $B_1^2 = B_1B_1^\dagger = B_2R_3R_3^\dagger B_2^\dagger = B_2^2$ and hence $B_1 = B_2$, since 'squaring' is an injective map on $\mathtt{H}_+(n)$. This in turn implies $R_1 = R_2$ and hence uniqueness. Finally, if X is real then B and consequently R are also real. □

Obviously the theorem could have also been formulated with the opposite order of the factors on the right hand side of (2.1), i.e., $X = RB$. Decomposing a given X according to both orders will generally result in the same unitary but a different (conjugate) hermitian factor. This follows from $B_1R_1 = R_1B_2$ where $B_2 := R_1^\dagger B_1 R_1$ is hermitian.

The polar decomposition defines maps

$$\mathrm{Pol}_{\mathbb{C}} : \quad \mathsf{GL}(n, \mathbb{C}) \quad \to \quad \mathtt{H}_+(n) \times \mathsf{U}(n)\,, \tag{2.19}$$

$$\mathrm{Pol}_{\mathbb{R}} : \quad \mathsf{GL}(n, \mathbb{R}) \quad \to \quad \mathtt{S}_+(n) \times \mathsf{O}(n)\,, \tag{2.20}$$

where $\mathtt{H}_+(n)$ and $\mathtt{S}_+(n)$ denote the spaces of positive definite hermitian matrices (over $\mathbb{C}$) and positive definite symmetric matrices (over $\mathbb{R}$). $\mathrm{Pol}_{\mathbb{R}}$ is just the restriction to $GL(n, \mathbb{R})$ of $\mathrm{Pol}_{\mathbb{C}}$, hence it is sufficient to state and prove the following result for $\mathrm{Pol}_{\mathbb{C}}$.

Theorem 2.2 *The maps* $\mathrm{Pol}_{\mathbb{C}}$ *and* $\mathrm{Pol}_{\mathbb{R}}$ *in (2.19) and (2.20) are homeomorphisms.*

Proof It is sufficient to give the proof for $\mathrm{Pol}_{\mathbb{C}}$; that for $\mathrm{Pol}_{\mathbb{R}}$ is entirely analogous. Bijectivity is obvious. The inverse of $\mathrm{Pol}_{\mathbb{C}}$ is $(B, R) \mapsto BR$ (matrix multiplication), which is clearly continuous. Hence it remains to show that $\mathrm{Pol}_{\mathbb{C}}$ is continuous. Let $\{X_i \in \mathsf{GL}(n, \mathbb{C}) \mid i \in \mathbb{N}\}$ be a sequence converging to X with $\mathrm{Pol}_{\mathbb{C}}(X_i) = (B_i, R_i)$ and $\mathrm{Pol}_{\mathbb{C}}(X) = (B, R)$. We need to show that $B_i \to B$ and $R_i \to R$ as $i \to \infty$. As $\mathsf{U}(n)$ is compact (cf. 2.1.1) the sequence $\{R_i \mid i \in \mathbb{N}\}$ has a subsequence $\{R_\alpha \mid \alpha \in I \subset \mathbb{N}\}$ converging to some $R' \in \mathsf{U}(n)$ (cf.A.3.6). Hence the sequence $\{B_\alpha := X_\alpha R_\alpha^{-1}\}$ of hermitean matrices converges to $B' := XR'^{-1}$ (using continuity of matrix multiplication). Since the set of hermitean matrices is closed (being $f^{-1}(0)$ of the continuous map $f : \mathtt{M}(n, \mathbb{C}) \to \mathtt{M}(n, \mathbb{C})$, $f(A) := A - A^\dagger$) we know that B' must also be hermitean. Therefore $X = B'R'$ is a polar decomposition whose uniqueness implies $R' = R$ and $B' = B$. Hence R is the unique accumulation point of $\{R_i\}$ so that $R_i \to R$ and $B_i \to B$ as $i \to \infty$, as desired. □

Clearly $\mathrm{Pol}_{\mathbb{R}}$ is just the restriction of $\mathrm{Pol}_{\mathbb{C}}$ to the subgroup $\mathsf{GL}(n, \mathbb{R})$ of $\mathsf{GL}(n, \mathbb{C})$. As it turned out, the image of this restriction is then just the intersection of the image of $\mathrm{Pol}_{\mathbb{C}}$ with $\mathsf{GL}(n, \mathbb{R})$. However, this will not be true for all subgroups G of $\mathsf{GL}(n, \mathbb{C})$. That is, we do not always have

$$\mathrm{Im}(\mathrm{Pol}_{\mathbb{C}}\big|_{\mathsf{G}}) \subseteq (\mathtt{H}_+(n) \cap \mathsf{G}) \times (\mathsf{U}(n) \cap \mathsf{G})\,. \tag{2.21}$$

This means that in general we cannot be sure that if we polar decompose an element of a subgroup that the resulting factors will again be members of that subgroup. However, it is useful to know that for a large class of subgroups this *is* the case and that these subgroups comprise $\mathsf{U}(p, q)$ and $\mathsf{O}(p, q)$ as well as their unimodular versions. More precisely, we have

Theorem 2.3 *Let* $\mathsf{G} \subset \mathsf{GL}(n, \mathbb{C})$ *be any of the subgroups* $\mathsf{U}(p,q)$, $\mathsf{SU}(p,q)$, $\mathsf{O}(p,q)$, *or* $\mathsf{SO}(p,q)$, *where* $p + q = n$. *Then we have a map*

$$Pol_{\mathsf{G}} := Pol_{\mathbb{C}}\big|_{\mathsf{G}} : \mathsf{G} \to (\mathtt{H}_+(n) \cap \mathsf{G}) \times (\mathsf{U}(n) \cap \mathsf{G}) \tag{2.22}$$

which is a homeomorphism onto its image (continuous and injective, but generally not surjective).

Proof We only need to show (2.21); the rest is a direct consequence of theorem 2.2. The case of unimodular groups is readily dealt with: for a polar decomposition $A = BR$ we have $|\det(A)| = \det(B)$ since $\det(B) \in \mathbb{R}_+$ and $|\det(R)| = 1$. $\det(A) = 1$ then implies $\det(B) = 1$ and $\det(R) = 1$. It remains to prove (2.21) for $\mathsf{G} = \mathsf{U}(p,q)$ (the case $\mathsf{G} = \mathsf{O}(p,q)$ then follows by restriction to real matrices). For this it will suffice to show that $A = BR \in \mathsf{U}(p,q)$ implies $B \in \mathsf{U}(p,q)$, since then $R = B^{-1}A \in \mathsf{U}(p,q)$. But B is defined by $B = \sqrt{AA^\dagger}$ and matrices in $\mathsf{U}(p,q)$ satisfy $AE^{(p,q)}A^\dagger = E^{(p,q)}$, by definition. Now $E^{(p,q)}$ is real symmetric and its own inverse and hence also in $\mathsf{U}(p,q)$. So $A \in \mathsf{U}(p,q)$ implies $A^\dagger \in \mathsf{U}(p,q)$ and therefore $AA^\dagger \in \mathsf{U}(p,q) \cap \mathtt{H}_+(n)$. But, generally, if some $C \in \mathsf{U}(p,q) \cap \mathtt{H}_+(n)$ then the unique square root $\sqrt{C}$ in $\mathtt{H}_+(n)$ will again lie in $\mathsf{U}(p,q)$. To see this we employ the exponential map $\exp : \mathtt{H}(n) \to \mathtt{H}_+(n)$ discussed above. There is a unique $X \in \mathtt{H}(n)$ so that $C = \exp(X)$. Using the defining relation for $\mathsf{U}(p,q)$ and injectivity of exp we find that $C \in \mathsf{U}(p,q)$ iff $E^{(p,q)}XE^{(p,q)} = -X$. But this relation is linear in X and hence also satisfied if we replace X with $X/2$. Hence $\sqrt{C} = \exp(X/2)$ is also in $\mathsf{U}(p,q)$. □

Being the intersection of two closed subgroups, $K := \mathsf{G} \cap \mathsf{U}(n)$ is again a closed subgroup of $\mathsf{GL}(n, \mathbb{C})$. Therefore it is also a closed subgroup of $\mathsf{U}(n)$ and hence compact since $\mathsf{U}(n)$ is compact (cf. A.3.6). In fact, it is a *maximal compact subgroup* of G, meaning that there is no strictly larger compact subgroup of G properly containing K. To see this by way of contradiction, assume there is a compact subgroup K' such that $K \subset K' \subset \mathsf{G}$. Then, by hypothesis, there exists an element $A \in K' - K$ whose polar decomposition is $A = BR$ with $B \neq E$. But $A \in K'$ and $R \in K \subset K'$ so that $AR^{-1} = B \in K'$ and therefore $B^m \in K'$ for all integers m. Since $B \neq E$ there exists an eigenvalue $\mathbb{R} \ni \lambda \neq 1$ of B with corresponding eigenvalue λ^m of B^m. Hence $\lambda^m \to \infty$ for $m \to \infty$ (if $\lambda > 1$) or for $m \to -\infty$ (if $\lambda < 1$). This implies that K' cannot be compact since it contains a sequence of matrices with unbounded eigenvalues (cf. A.3.6).

2.2 Lie algebras

To every Lie group one can uniquely associate an algebraic object, called a Lie algebra. Many operations on and between Lie groups have their unique correspondences as operations on and between Lie algebras. But as Lie algebras are linear spaces they are easier to handle and thus many problems concerning Lie groups can be solved by considering the corresponding problem of Lie algebras. This is why they are so useful.

An n-dimensional *Lie algebra* L over $\mathbb{F}$ is an n-dimensional vector space over $\mathbb{F}$ together with an map $L \times L \to L$ called the *Lie bracket*, which is denoted by a square bracket and which, for all $X, Y, Z \in L$ and $a \in \mathbb{F}$, satisfies the following conditions:

$$[X,Y] = -[Y,X] \qquad \text{(antisymmetry)} \tag{2.23}$$

$$[X, Y + aZ] = [X,Y] + a[X,Z] \qquad \text{(bilinearity)} \tag{2.24}$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \text{(Jacobi-identity)} \tag{2.25}$$

Note that Lie-algebras are anti-commutative and non-associative; associativity is replaced by the Jacobi-identity.

Given a basis $\{e_a\}$, $a = 1, \cdots, \dim(L)$, of L, we can write $[e_a, e_b] = C^c_{\ ab} e_c$ with coefficients $C^c_{\ ab} \in \mathbb{F}$, called the *structure constants* of L with respect to $\{e_a\}$. Equation (2.23) is equivalent to antisymmetry in the lower index pair, $C^c_{\ ab} = -C^c_{\ ba}$, and the Jacobi identity (2.25) is equivalent to

$$C^d_{ab} C^n_{cd} + C^d_{bc} C^n_{ad} + C^d_{ca} C^n_{bd} = 0\,, \tag{2.26}$$

which, using antisymmetry, is equivalent to the statement that $C^d_{ab} C^n_{cd}$ is totally antisymmetric in abc, i.e., $C^d_{[ab} C^n_{c]d} = 0$. Conversely, given an n-dimensional vector space V with basis $\{e_a\}$, any choice of $\frac{1}{2}n^2(n-1)$ numbers C^c_{ab} which satisfy (2.26) will make V into a Lie algebra through $[e_a, e_b] := C^c_{ab} e_c$ and linear extension.

Any associative algebra is automatically a Lie algebra by defining the Lie bracket to be the commutator (associativity is needed to ensure the Jacobi identity):

$$[A, B] := AB - BA\,. \tag{2.27}$$

This can be done, for example, with $\mathrm{End}(V)$, the endomorphisms of a vector space V (cf. A.5.2). The resulting Lie algebra is usually called $\mathfrak{gl}(V)$, or $\mathfrak{gl}(n, \mathbb{F})$ if it results from the concrete matrix algebra $\mathtt{M}(n, \mathbb{F})$ (see below). In fact, anticipating some terminology explained below, a deep theorem due to Ado implies that any finite dimensional Lie algebra has a faithful finite dimensional representation. This is the same as saying that any finite dimensional Lie algebra over $\mathbb{F}$ is isomorphic to a Lie subalgebra of $\mathtt{M}(n, \mathbb{F})$ for some n.

2.2.1 *General notions*

2.2.1.1 *Subalgebras and ideals* If L' and L'' are subsets of L, we shall write $[L', L'']$ for the linear span of all vectors of the form $[X', X'']$ with $X' \in L'$ and $X'' \in L''$. A linear subspace $L' \subseteq L$ is a *Lie subalgebra* (or simply 'subalgebra' if the Lie-context is obvious) iff $[L', L'] \subseteq L'$, and a *Lie ideal* (or simply 'ideal' if the Lie-context is obvious) iff $[L', L] \subseteq L'$. $\{0\}$ and L are the trivial ideals of L; an ideal is called non-trivial iff it is different from these two. The *center* of L, defined by $Z(L) := \{X \in L \mid [X, Y] = 0 \ \forall Y \in L\}$, is an obvious example of an ideal.

Having an ideal $I \subset L$, the quotient L/I (as vector spaces, cf. A.5.1) is again a Lie algebra with Lie bracket $[\langle X\rangle, \langle Y\rangle] := \langle [X,Y]\rangle$, where $\langle\cdot\rangle$ denote the equivalence class for the moment. It is a simple exercise to show that this is indeed a well defined Lie bracket.

Given two ideals I and I' of L, their intersection, $I \cap I'$, their sum, $I + I' := \mathrm{span}\{I \cup I'\}$, and their Lie bracket, $[I, I']$, are again ideals. The last assertion follows from the Jacobi identity which, written in our compact notation, implies: $[[I,I'],L] = [[L,I],I'] + [[I',L],I] = [I,I']$.

2.2.1.2 *Homomorphisms* A linear map $f : L \to L'$ between Lie algebras is a *Lie homomorphism* (or simply 'homomorphism' if the Lie-context is obvious) iff $f([X,Y] = [f(X), f(Y)]$ for all $X, Y \in L$. Here we denoted the Lie brackets in L and L' by the same symbol $[\cdot,\cdot]$, which should not give rise to confusion. Kernel and image of a Lie homomorphism are defined as for general linear maps (cf. A.5.2). Both are clearly Lie subalgebras and it is easy to see that the kernel is even an ideal. A homomorphism $f : L \to L'$ maps any subalgebra $K \subseteq L$ to a subalgebra $K' := f(K) \subseteq L'$. If K is an ideal in L, then K is surely an ideal in $f(L) \subseteq L'$ (the image of f) but not necessarily in L'. A bijective Lie homomorphism is called a *Lie isomorphism.* As usual, we have the (now Lie-) isomorphism $\mathrm{Im}(f) \cong L/\mathrm{Ker}(f)$.

2.2.1.3 *Direct sums, derivations, and semi-direct sums* Given two Lie algebras L and L'. An obvious way to make the vector space $L \oplus L'$ into a Lie algebra is to define $[(X,X'),(Y,Y')] := ([X,Y],[X',Y'])$, for all $X, Y \in L$ and $X', Y' \in L'$ and where we used the same symbol $[\cdot,\cdot]$ to denote the Lie brackets in the various spaces. With this Lie bracket $L \oplus L'$ is called the *direct sum* of the Lie algebras L and L'. The natural projections $\pi_L : L \oplus L' \to L$ and $\pi_{L'} : L \oplus L' \to L'$ are Lie homomorphisms whose kernels are the ideals $L \oplus \{0_{L'}\}$ and $\{0_L\} \oplus L'$ (we distinguishing the null vectors in L and L' by subscripts), which we may naturally identify with L and L' respectively.

A *derivation* of the Lie algebra L is a linear map $\varphi : L \to L$ which satisfies $\varphi([X,Y]) = [\varphi(X), Y] + [X, \varphi(Y)]$. The space of derivations of L is itself a Lie algebra, called $\mathfrak{Der}(L)$. Linearity is obvious and the Lie bracket is defined by $[\varphi,\varphi'] := \varphi \circ \varphi' - \varphi' \circ \varphi$ which clearly satisfies the Jacobi identity. One merely has to check that$[\varphi,\varphi']$ is again a derivation, which is also almost immediate. Hence we can consider $\mathfrak{Der}(L)$ as Lie subalgebra of $\mathfrak{gl}(L)$.

Given a Lie homomorphism $\sigma : L' \to \mathfrak{Der}(L)$, $X \mapsto \sigma_X$, we can generalize the construction above and turn the vector space $L \oplus L'$ into a Lie algebra through the definition

$$[(X,X'),(Y,Y')] := ([X,Y] + \sigma_{X'}(Y) - \sigma_{Y'}(X)\,,\,[X',Y'])\,. \tag{2.28}$$

Only the Jacobi identity needs to be checked, which is readily done if one uses the fact that, by linearity, it is sufficient to check it for the two cases where the first vector is of the form $X \oplus \{0_{L'}\}$ and the second of the form $\{0_L\} \oplus X'$

and vice versa. This Lie algebra is called the *semi direct sum* of L and L' with respect to σ, and denoted by $L \rtimes_\sigma L'$. It reduces to the direct sum if one chooses σ to be the constant map onto the trivial derivation which maps all of L to zero. Sometimes the reference to σ is suppressed if its choice is obvious. (Compare this to the notion of a semi direct product of groups explained in A.4.3)

$L \oplus \{0_{L'}\}$ and $\{0_L\} \oplus L'$ are still Lie subalgebras of $L \rtimes L'$, but now only the first is also an ideal, whereas the latter is an ideal iff σ is trivial. This is connected to the fact that now only the projection $\pi_{L'}$ but not π_L is a Lie homomorphism. Conversely, given a Lie algebra L'' and two subalgebras L and L' of which one, say L, is even an ideal. Given further that (i) $L'' = \text{span}\{L \cup L'\}$ and (ii) $L \cap L' = \{0\}$ (i.e. $L'' = L \oplus L'$ as vector spaces). Then it is easy to see that $L'' = L \rtimes_\sigma L'$, with $\sigma_{X'}(Y) = [X', Y]$. This is the analogue for Lie algebras of the statement made for groups at the end of A.4.3.

2.2.1.4 *Solvability* $L^{(1)} := [L, L]$ is called the *first derived subalgebra* of L. It is an ideal since the Lie bracket of ideals is again an ideal, as we already remarked. By the same token, any further member of the the *derived series* of second, third, etc. derived algebras, defined inductively by $L^{(n)} := [L^{(n-1)}, L^{(n-1)}]$, is an ideal. Now L is called *solvable* iff $L^{(n)} = \{0\}$ for some n. Note that this means that $L^{(n-1)}$ is abelian, so that every solvable Lie algebra L contains a nonzero abelian ideal, namely the last nonzero algebra in the derived series.

One immediately sees that subalgebras and homomorphic images of solvable Lie algebras are again solvable. Another very useful observation is the following: if $I \subset L$ is a solvable ideal such that the quotient L/I is solvable, then L itself must be solvable. Indeed, by hypothesis there exist integers n and m such that $(L/I)^{(n)} = \{0\}$ and $I^{(m)} = \{0\}$. If $\pi : L \to L/I$ denotes the canonical projection, we have $\pi(L^{(n)}) = (L/I)^{(n)} = \{0\}$ implying $L^{(n)} \subseteq I$ and hence $L^{(n+m)} \subseteq I^{(m)} = \{0\}$. Finally, it is not hard to see that if I, I' are solvable ideals of L, so is $I + I' = \text{span}\{I \cup I'\}$. To see this, first observe that there is a natural isomorphism $(I + I')/I' \cong I/I \cap I'$. As a homomorphic image of I the right side is solvable, hence the left side is, and by the result just proven $I + I'$ is solvable. This last result implies that we can uniquely associate a maximal solvable ideal to each Lie algebra, called the *radical* $R(L)$ of L.

There is a useful condition for $L \subseteq \mathfrak{gl}(V)$ to be solvable, known as *Cartan's criterion for solvability*: L is solvable iff $\text{trace}(XY) = 0$ for all $X \in L$ and all $Y \in [L, L]$. (See [10], 4.3, for a proof).

2.2.1.5 *Simplicity and semi-simplicity* A Lie algebra L is called *simple* iff it is at least two dimensional and has no nontrivial ideals. Simplicity clearly implies $L^{(1)} = [L, L] = L$, hence simplicity excludes solvability and, in particular, that the Lie algebra is abelian. The condition of having dimension at least two, or equivalently that $L^{(1)} \neq \{0\}$, is put to avoid having the (necessarily abelian) one-dimensional Lie algebra also among the list of simple Lie algebras, which would be inconvenient for many purposes. L is called *semi-simple* iff it has no solvable ideals other than $\{0\}$, which is equivalent to $R(L) = \{0\}$. It is also equivalent to saying that it has no abelian ideals other than $\{0\}$, since any nonzero solvable ideal must contain a nonzero abelian ideal, and any abelian ideal is trivially solvable. It follows that simplicity implies semi-simplicity, but not vice versa. We will explain the precise relation below. Note also that the direct sum $L = L' \oplus L''$ of two semi-simple Lie algebras L' and L'' is again semi-simple. Indeed, if there existed a non-trivial solvable ideal in L its projections into L' and L'' were again solvable ideals in L' and L'' of which at least one must be non-trivial, in contradiction to the assumed semi-simplicity of L' and L''. Finally we remark that the quotient $L/R(L)$ is semi-simple. To see this, assume $Q \subset L/R(L)$ is a non-zero solvable ideal. Then there exists an ideal $I \subseteq L$ such that $Q = I/R(L)$. But then I is solvable (since Q and $R(L)$ are solvable) and properly contains $R(L)$, which contradicts maximality of $R(L)$.

It is a non-trivial fact that for any Lie algebra L one can find a subalgebra M which is complementary and transversal to the radical $R(L)$, i.e., $\text{span}\{R(L) \cup M\} = L$ and $R(L) \cap M = \{0\}$ (see e.g. [16] for a proof). Such an M is called a *Levi subalgebra*. M is isomorphic to $L/R(L)$ and must, as we have just seen, consequently be semi-simple. Furthermore, from 2.2.1.3 we know that the two stated conditions imply that L is a semi-direct sum of the ideal $R(L)$ with M. Hence we see that any Lie algebra is the semi-direct sum of its radical, the maximal solvable ideal, and a semi-simple subalgebra. This is called a *Levi decomposition*.

2.2.1.6 *Representations* Let V be a vector space and L a Lie algebra over the common number field $\mathbb{F}$. A Lie homomorphism $L \to \mathfrak{gl}(V)$ is called a *representation* of L on V. The representation is called *faithful* iff the representation homomorphism is injective. Hence any representation of a simple Lie algebra is either trivial, i.e., maps everything to the zero endomorphism, or faithful. Of particular importance is the so-called *adjoint representation* $\text{ad} : L \to \mathfrak{gl}(L)$ of L on itself, given by

$$X \to \text{ad}_X , \quad \text{ad}_X(Y) := [X, Y] . \tag{2.29}$$

Indeed, (2.25) is equivalent to $\text{ad}_X \circ \text{ad}_Y - \text{ad}_Y \circ \text{ad}_X = \text{ad}_{[X,Y]}$ and hence implies the Lie homomorphism property. The kernel of ad is just the center $Z(L)$. The image ad_L of ad is a Lie subalgebra of $\mathfrak{gl}(L)$ called the *adjoint Lie algebra*, which is clearly isomorphic to $L/Z(L)$. With respect to a basis $\{e_a\}$ the matrix of ad_X (where $X = X^a e_a$) is given by $[\text{ad}_X]^a_b = X^c C^a_{cb}$.

2.2.1.7 *The Killing form* Using ad one defines a symmetric bilinear form on L, called the *Killing form*, given by

$$\kappa(X, Y) := \text{trace}(\text{ad}_X \circ \text{ad}_Y) . \tag{2.30}$$

Its components with respect to a basis are

$$\kappa_{ab} := \kappa(e_a, e_b) = C^n_{\ am} C^m_{\ bn} . \tag{2.31}$$

Besides being bilinear and symmetric, the Killing form also possesses an important property sometimes referred to as associativity, which reads

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])\,. \tag{2.32}$$

It simply follows from the identity $\mathrm{trace}([f,g]h) = \mathrm{trace}(f[g,h])$ for any three endomorphisms f, g, h of a finite dimensional vector space, applied to $f = \mathrm{ad}_X$, $g = \mathrm{ad}_Y$, $h = \mathrm{ad}_Z$, and the fact that ad is a Lie homomorphism. An interesting consequence is that the 3rd rank tensor $C_{abc} := \kappa_{ad} C^d_{bc}$ is totally antisymmetric.

The null space of κ, which is an invariant of L, is defined by $N(L) := \{X \in L \mid \kappa(X,Y) = 0\,\forall Y \in L\}$. It follows immediately from (2.32) that $N(L)$ is an ideal. In fact, (2.32) implies that if I is an ideal and $I^\perp := \{X \in L \mid \kappa(X,Y) = 0\,\forall Y \in I\}$, its κ-orthogonal complement, then $I^\perp$ is also an ideal. This follows from $\kappa([I^\perp, L], I) = \kappa(I^\perp, [L, I]) = \kappa(I^\perp, I) = \{0\}$.

We note that the Killing form κ_I of an ideal $I \subset L$ is just the restriction of the Killing form κ of L to I. This is generally false for Lie subalgebras which are not ideals. It follows from the general fact that for any $f \in \mathrm{End}(V)$, whose image lies in the subspace $W \subset V$, we have $\mathrm{trace}(f) = \mathrm{trace}(f|_W)$, where the trace on the right hand side taken in W. We apply this to $\mathrm{ad}_X \circ \mathrm{ad}_Y \in \mathrm{End}(L)$, whose image lies in I for $X \in I$, and get $\kappa(X,Y) = \mathrm{trace}(\mathrm{ad}_X \circ \mathrm{ad}_X) = \mathrm{trace}(\mathrm{ad}_X|_I \circ \mathrm{ad}_Y|_I) = \kappa_I(X,Y)$, for all $X, Y \in I$.

We can now give useful criteria for solvability and semi-simplicity of L in terms of κ. We start with solvability and consider the adjoint map $\mathrm{ad} : L \to \mathrm{ad}_L \subseteq \mathfrak{gl}(L)$. Since $\mathrm{ad}_L \cong L/Z(L)$ and $Z(L)$ is an abelian (hence solvable) ideal, we infer from 2.2.1.4 that L is solvable iff ad_L is solvable. Cartan's criterion for solvability applied to the latter shows that ad_L is solvable iff $0 = \mathrm{trace}(\mathrm{ad}_X \circ \mathrm{ad}_{[Y,Z]}) = \kappa(X,[Y,Z])$ for all $X, Y, Z \in L$. Hence L is solvable iff $\kappa(X,[Y,Z]) = 0$ for all $X, Y, Z \in L$, i.e., iff $L^{(1)} \subseteq L^\perp = N(L)$.

Next we turn to *Cartan's criterion for semi-simplicity*: A Lie algebra L is semi-simple iff its Killing form is non degenerate, that is, iff $N(L) = \{0\}$. In other words, the null space $N(L)$ is zero iff the radical $R(L)$ is zero. To show this, let us first suppose that L is not semi-simple. Then it contains an abelian ideal I and for $X \in I$ and $Y \in L$ we have $\kappa(X,Y) = \mathrm{trace}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = \mathrm{trace}(\mathrm{ad}_X|_I \circ \mathrm{ad}_Y|_I) = 0$, since $\mathrm{ad}_X|_I \equiv 0$ due to I being abelian. Hence $I \subseteq N(L) \neq \{0\}$. The converse follows from the trivial remark that $\kappa(X,[Y,Z]) = 0$ for each $X, Y, Z \in N(L)$, which immediately implies solvability of the ideal $N(L)$ by the criterion just proven, and hence $N(L) \subseteq R(L)$. This is still a totally general result. Assuming semi-simplicity now implies $N(L) = \{0\}$

Finally we mention the notion of compactness as applied to Lie algebras, where we limit attention to the semisimple case. A semisimple Lie algebra L is called *compact* iff its Killing form is negative definite. The reason for this expression lies in the fact, that Lie groups whose Lie algebras (see Sect. 2.3 below) are semisimple and compact are always compact, now in the proper topological sense. We refer to chapter II, § 6 of [9] for a detailed discussion.

2.2.1.8 *Unique decomposition of semi-simple algebras into simple ones* Let L be semi-simple and suppose it contains an ideal $I \subset L$. Then $\kappa([I^\perp, I], L) = \kappa(I^\perp, [I, L]) = \kappa(I^\perp, I) = \{0\}$ and hence $[I^\perp, I] = \{0\}$ since κ is non degenerate. This implies that $I \cap I^\perp \subset L$ is an abelian ideal and hence that $I \cap I^\perp = \{0\}$ since L is semi-simple. Therefore $L = I \oplus I^\perp$. Suppose I also contains a nontrivial ideal I'. Then, in the same fashion, $I = I' \oplus I'^\perp$, where I' and $I'^\perp$ are ideals of I. But being contained in I they have zero Lie brackets with $I^\perp$ and hence are also ideals of L. This means that we can continue breaking up ideals into direct sums of κ-orthogonal subideals of L. This procedure stops after a finite number of steps when all subideals so obtained are simple. Hence the semi-simple Lie algebra L is the direct sum of a finite number (say n) of κ-orthogonal simple ideals:

$$L = \bigoplus_{i=1}^{n} I_i \,. \tag{2.33}$$

Moreover, this decomposition is unique. To see this, assume $I \subseteq L$ is a simple ideal. We show that it must be one of the I_i. For this we note that $[I, L] \subseteq I$ must also be an ideal of L and hence of I. It is nonzero since otherwise I were a nonzero abelian ideal of L, in contradiction to semi-simplicity. Simplicity of I now implies $[I, L] = I$. On the other hand, (2.33) gives $I = [I, L] = [I, I_1] \oplus \cdots \oplus [I, I_n]$ and again simplicity of I implies that all but one summand, say $[I, I_i]$, are zero. Then $I = [I, I_i]$, implying $I \subset I_i$ and finally $I = I_i$ by simplicity of I_i.

Introducing (2.33) in $[L, L]$ and using $[I_i, I_i] = I_i$ and $[I_i, I_j] = \{0\}$ for $i \neq j$ one shows that $[L, L] = L$ for any semi-simple L. This relation is expressed in words by saying that L is *perfect*. Note that we have also shown that any ideal $I \subset L$ is given by some partial sum of (2.33). Hence I and L/I are also semi-simple. This means that ideals and homomorphic images of semi-simple algebras are again semi-simple.

2.2.1.9 *Complexification, realification, and reals forms* In the structure and representation theory it is mandatory to clearly distinguish between real and complex Lie algebras. There are various ways to connect these, which play a vital rôle in representation theory and which we therefore wish to explain in some detail. The following five points exactly parallel the corresponding discussion for vector spaces given in A.5.3).

1. A *complex structure* of a real Lie algebra L is a complex structure $J : L \to L$ of the underlying real vector space (cf. A.5.3) which satisfies $J([X, Y]) = [J(X), Y] = [X, J(Y)]$ for all $X, Y \in L$. L can then be considered as complex Lie algebra, $L_{\mathbb{C}}$, by defining complex scalar multiplication through $(a+ib)X := aX + bJ(X)$. Note that as sets and abelian groups (with respect to $+$) L and $L_{\mathbb{C}}$ coincide, but that $\dim_{\mathbb{C}} L_{\mathbb{C}} = \frac{1}{2} \dim_{\mathbb{R}} L$.
2. A *real structure* of a complex Lie algebra L is a real structure $C : L \to L$ of the underlying complex vector space (cf. A.5.3) which satisfies $C([X, Y]) = [C(X), C(Y)]$ for all $X, Y \in L$. The set of 'real vectors', $L_{\mathbb{R}} := \{X \in L \mid$

$C(X) = X\}$, then forms a real Lie algebra when scalar multiplication is restricted to $\mathbb{R}$.

3. The *complexification*, $L^{\mathbb{C}}$, of a real Lie algebra L is obtained by the complexification $(\mathbb{C} \otimes L)_{\mathbb{C}}$ of the underlying vector space (cf.A.5.3), made into a complex Lie algebra by defining the Lie bracket through $[z_1 \otimes X_1, z_2 \otimes X_2] := z_1 z_2 \otimes [X_1, X_2]$ and linear extension. Its complex dimension equals the real dimension of L. $L^{\mathbb{C}}$ inherits a natural real structure given by $C(z \otimes X) := \overline{z} \otimes X$ and antilinear extension, whose real vectors are just those in $1 \otimes L \subset L^{\mathbb{C}}$; hence $(L^{\mathbb{C}})_{\mathbb{R}} = 1 \otimes L \cong L$. Moreover, if originally L is the set of real elements in a complex Lie algebra L' with real structure C, i.e. $L = L'_{\mathbb{R}}$, then $L^{\mathbb{C}} = (L'_{\mathbb{R}})^{\mathbb{C}} \cong L'$.
4. The *realification*, $L^{\mathbb{R}}$, of a complex Lie algebra L is the realification of the underlying vector space (cf.A.5.3), made into a real Lie algebra by simply restricting the original $\mathbb{C}$-linear Lie bracket to an $\mathbb{R}$-linear one. The real dimension of $L^{\mathbb{R}}$ is twice the complex dimension of L. Clearly $L^{\mathbb{R}}$ has a natural complex structure given by $J(X) := iX$, so that $(L^{\mathbb{R}})_{\mathbb{C}} \equiv L$. Finally, if L arose via a complex structure on some real Lie algebra L', i.e. $L = L'_{\mathbb{C}}$, then $L^{\mathbb{R}} = (L'_{\mathbb{C}})^{\mathbb{R}} \equiv L'$.
5. Note that in the literature $\mathbb{C} \otimes L$ is sometimes referred to as the complexification of L, but that is an abuse of language: If L is real, then the symbol $\mathbb{C} \otimes L$ makes sense only if $\mathbb{C}$ is considered as two-dimensional real vector space and the tensor product is that of real vector spaces. Hence $\mathbb{C} \otimes L$ is a priori a real vector space. The complexification $L^{\mathbb{C}}$ is only obtained after we use the natural complex structure of $\mathbb{C} \otimes L$ to turn it into the complex vector space $(\mathbb{C} \otimes L)_{\mathbb{C}}$. Rather, the real vector space $\mathbb{C} \otimes L$ is identical to $L^{\mathbb{C}\mathbb{R}} := (L^{\mathbb{C}})^{\mathbb{R}}$, which we call the *complex double* of L.

Let us now discuss how the processes of complexification and realification preserve semi-simplicity and simplicity. We start with complexification first.

It is easy to see that $L^{\mathbb{C}}$ is semi-simple iff L is. Indeed, this follows immediately from $\kappa^{\mathbb{C}}(1 \otimes X, 1 \otimes y) = \kappa(X, Y)$, where $\kappa^{\mathbb{C}}$ and κ are the Killing forms of $L^{\mathbb{C}}$ and L respectively. Hence $\kappa^{\mathbb{C}}$ is non-degenerate iff κ is. Moreover, if I is a non-trivial ideal in L then $\mathbb{C} \otimes I$ is clearly non-trivial ideal in $L^{\mathbb{C}}$. Hence simplicity of $L^{\mathbb{C}}$ implies that of L. But, in general, the converse fails, like in the relevant example of the Lie algebra of the Lorentz group, which is simple, but whose complexification is not simple. At first one might think that this cannot happen, since if $I \subset L^{\mathbb{C}}$ is a non-trivial ideal then $I_{\mathbb{R}} := I \cap L$ is an ideal in L. However, the latter may turn out to be trivial.

As regards realification, it is again true that $L^{\mathbb{R}}$ is semi-simple iff L is. To show this, we calculate the Killing form $\kappa^{\mathbb{R}}$ of $L^{\mathbb{R}}$. Let $A + iB$ be the matrix of the map $\mathrm{ad}_X \circ \mathrm{ad}_Y$ with respect to a basis $\{e_1, \cdots, e_n\}$ of L, where A and B are real. Choosing $\{e_1, \cdots, e_n, J(e_1), \cdots, J(e_n)\}$ as basis for $L^{\mathbb{R}}$, and observing that J commutes with any ad_X and hence with $\mathrm{ad}_X \circ \mathrm{ad}_Y$, we see that the matrix of the latter map (now considered as $\mathbb{R}$-linear map of $L^{\mathbb{R}}$) has the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} . \tag{2.34}$$

This implies $\kappa^{\mathbb{R}}(X,Y) = 2\mathfrak{Re}(\kappa(X,Y))$ and hence that $\kappa^{\mathbb{R}}$ is non-degenerate iff κ is.

Moreover, in contrast to the case of complexification, it is true that the realification $L^{\mathbb{R}}$ is simple iff L is. First, if I is a non-trivial ideal in L then $I^{\mathbb{R}}$ is a non-trivial ideal in $L^{\mathbb{R}}$. Hence simplicity of $L^{\mathbb{R}}$ implies that of L. Conversly, suppose L is simple. Then L is clearly semi-simple. As just shown this implies that $L^{\mathbb{R}}$ is also semi-simple. Hence it uniquely decomposes into a direct sum of simple ideals I_i (cf. formula (2.33)). Pick some ideal, I_k say, and consider its image $I'_k := J(I_k)$ under the complex structure. It is easy to verify that I'_k is also an ideal of $L^{\mathbb{R}}$ (to prove that I'_k is a Lie subalgebra you already need to use that I_k is an ideal). Now, for $X, Y \in I_k$, $[J(X), J(Y)] = J([X, J(Y)]) = -[X,Y]$ shows that $[I'_k, I'_k] = [I_k, I_k] = I_k$, (since I_k is perfect, cf. 2.2.1.8) and hence that $I_k \subseteq I'_k$. But I'_k is an isomorphic image (via J) of I_k, so that $I'_k = J(I_k) = I_k$. This shows that all ideals I_i in the decomposition of $L^{\mathbb{R}}$ are invariant under the complex structure J and that consequently the complex Lie algebra L decomposes into the ideals $(I_i)_{\mathbb{C}}$. But L was assumed to be simple, hence there can be only one summand, and consequently $L^{\mathbb{R}}$ is also simple, as was to be shown.

Given a real Lie algebra L we can first complexify and then realify it to obtain another real Lie algebra, $L^{\mathbb{C}\mathbb{R}}$, of twice the dimension of L. Clearly this is just $\mathbb{C} \otimes L$, where $\mathbb{C}$ is considered as 2-dimensional real vector space and the tensor product is over $\mathbb{R}$. The natural real structure C of $L^{\mathbb{C}}$ now becomes a *linear* involution on $L^{\mathbb{C}\mathbb{R}}$ which still satisfies $C[X,Y] = [CX, CY]$. The maps $P_\pm := \frac{1}{2}(\mathrm{id} \pm C)$ are orthogonal projection operators (i.e. $P_\pm \circ P_\pm = P_\pm$ and $P_\pm \circ P_\mp = 0$) which satisfy $P_\pm \circ C = C \circ P_\pm = \pm P_\pm$. Their images are the eigenspaces of C corresponding to the eigenvalues ± 1. A simple computations shows

$$\begin{aligned} [P_\pm X, Y] &= P_\pm [X,Y] \pm [CX, P_- Y] \\ &= P_\mp [X,Y] \pm [CX, P_+ Y] \,, \end{aligned} \tag{2.35}$$

which respectively imply

$$\begin{aligned} [P_\pm X, P_+ Y] &= P_\pm [X,Y] \,, \\ [P_\pm X, P_- Y] &= P_\mp [X,Y] \,. \end{aligned} \tag{2.36}$$

Let us set $L_\pm := \mathrm{Im}(P_\pm) \subset L^{\mathbb{C}\mathbb{R}}$, so that $L^{\mathbb{C}\mathbb{R}} = L_+ \oplus L_-$ as vector spaces (not Lie algebras). Then (2.36) is equivalent to the following structure of Lie brackets

$$[L_+, L_+] \subseteq L_+ \,, \quad [L_+, L_-] \subseteq L_- \,, \quad [L_-, L_-] \subseteq L_+ \,. \tag{2.37}$$

Hence L_+ is a subalgebra isomorphic to L but not an ideal, wheras L_- is only a linear subspace (as such isomorphic to L) but not a subalgebra. This is precisely the structure of the Lie algebra of the Lorentz group, where L_+

corresponds to spatial rotations and L_- to boosts. The underlying mathematical explanation being that the Lie algebra of the Lorentz group arises from that of the group of spatial rotations by the procedure discussed here, i.e. by first taking the complexification and then the realification. Let us be slightly more concrete at this point: denote by $\{e_1, \cdots, e_n\}$ a basis of L, such that $[e_a, e_b] = C^c_{ab} e_c$, and by $\{e^{(+)}_1, \cdots, e^{(+)}_n, e^{(-)}_1, \cdots, e^{(-)}_n\}$ the associated basis of $L^{\mathbb{CR}}$, given by $e^{(+)}_k := 1 \otimes e_k$ and $e^{(-)}_k := i \otimes e_k$, which is adapted to the vector-space decomposition $L^{\mathbb{CR}} = L_+ \oplus L_-$, so that $\{e^{(+)}_a\}$ is a basis of L_+ and $\{e^{(-)}_a\}$ is a basis of L_-. Then (2.36) leads to

$$[e^{(+)}_a, e^{(+)}_b] = C^c_{ab}\, e^{(+)}_c\,, \quad [e^{(+)}_a, e^{(-)}_b] = C^c_{ab}\, e^{(-)}_c\,, \quad [e^{(-)}_a, e^{(-)}_b] = -C^c_{ab}\, e^{(+)}_c\,, \tag{2.38}$$

which concretises the structure (2.37). Since the construction described here is of some importance, we give it a special name:

Definition 2.4 *Let L be a real Lie algebra. The real Lie algebra $L^{\mathbb{CR}}$ is called the complex double of L.*

The complex double of a semi-simple Lie algebra is again semi simple, since both processes of complexification and realification preserve semi simplicity. This is not necessarily the case for simplicity, which, as discussed above, need not be preserved by complexification. Both cases arise. For example, suppose the real Lie algebra L has a complex structure, so that $L = (L_{\mathbb{C}})^{\mathbb{R}}$. Then $L^{\mathbb{C}} = (L_{\mathbb{C}})^{\mathbb{RC}} \cong L_{\mathbb{C}} \oplus L_{\mathbb{C}}$ as shown below; cf. equation (2.40). Hence $L^{\mathbb{CR}} \cong L \oplus L$ (as Lie algebras), so that the complex double of a Lie algebra with complex structure is never simple. On the other hand, the Lie algebra of the Lorentz group, which is simple, is the complex double of the Lie algebra of the group of spatial rotations.

Next we consider the composition of complexification and realification in the reversed order. For this let L now be a complex Lie algebra, which we first realify and then complexify to obtain $L^{\mathbb{RC}}$. The natural complex structure J of $L^{\mathbb{R}}$ becomes a $\mathbb{C}$-linear map in $L^{\mathbb{RC}}$, still satisfying $J \circ J = -\mathrm{id}$, given by $J(z \otimes X) := z \otimes J(X)$ for all $X \in L^{\mathbb{R}}$ and linear extension. Then $P_\pm := \frac{1}{2}(\mathrm{id} \mp iJ)$ are two orthogonal projection operators (i.e. $P_\pm \circ P_\pm = P_\pm$ and $P_\pm \circ P_\mp = 0$) on $L^{\mathbb{RC}}$ which satisfy $P_\pm \circ J = J \circ P_\pm = \pm i P_\pm$. Their images are the eigenspaces of J corresponding to the eigenvalues $\pm i$. From $J[X,Y] = [JX,Y] = [X,JY]$ it immediately follows that

$$[P_\pm X, Y] = [X, P_\pm Y] = P_\pm [X,Y]\,, \tag{2.39}$$

showing that $L_\pm := \mathrm{Im}(P_\pm) \subset L^{\mathbb{RC}}$ are both ideals, each isomorphic to L; hence we have the direct sum decomposition (now as Lie algebras, not just vector spaces):

$$L^{\mathbb{RC}} = L_+ \oplus L_-\,, \quad L_\pm \cong L\,. \tag{2.40}$$

Let us also state this more concretely in terms of an adapted basis: Denote by $\{e_1, \cdots, e_n\}$ a basis of L, such that $[e_a, e_b] = C^c_{ab} e_c$. Then $\{e_1, \cdots e_n, J(e_1), \cdots, J(e_n)\}$

is a basis of $L^{\mathbb{R}}$. An adapted basis for $L^{\mathbb{R}\mathbb{C}}$ is given by $\{e_1^{(+)}, \cdots, e_n^{(+)}, e_1^{(-)}, \cdots, e_n^{(-)}\}$, where

$$e_k^{(\pm)} := \tfrac{1}{2}(1 \otimes e_k \mp i \otimes J(e_k)) . \tag{2.41}$$

Here 'adapted' means that the basis vectors are eigenvectors of J: $J(e_k^{(+)}) = \pm i\, e_k^{(+)}$ so that $\text{span}\{e_1^{(\pm)}, \cdots, e_n^{(\pm)}\} = \text{Im}(P_\pm)$. An easy calculation then gives the concretisation of (2.40):

$$[e_a^{(+)}, e_b^{(+)}] = C_{ab}^c\, e_c^{(+)} , \quad [e_a^{(+)}, e_b^{(-)}] = 0 , \quad [e_a^{(-)}, e_b^{(-)}] = C_{ab}^c\, e_c^{(-)} . \tag{2.42}$$

Note also that $L^{\mathbb{R}\mathbb{C}}$, being the complexification of something ($L^{\mathbb{R}}$) real, has a natural real structure C, given by $C(z \otimes X) := \overline{z} \otimes X$. With respect to C the basis $\{e_1, \cdots e_n, J(e_1), \cdots, J(e_n)\}$ is real and $\{e_1^{(+)}, \cdots, e_n^{(+)}\}$ the complex conjugate (image under C) of $\{e_1^{(-)}, \cdots, e_n^{(-)}\}$. Hence L_+ and L_- are complex conjugate to each other. This can, of course, also be seen in a basis independent fashion. Indeed, the antilinear map C clearly commutes with J, which implies that $C \circ P_\pm = P_\mp \circ C$.

Finally we turn to the notion of real forms. Let L be a complex Lie algebra. Roughly speaking, a real form of L is a real Lie algebra whose complexification is (isomorphic to) L. More precisely, a *real form* L' of L is a subalgebra $L' \subset L^{\mathbb{R}}$ such that, as vector spaces, $L^{\mathbb{R}} = L' \oplus J(L')$. In general, a complex Lie algebra will have many, mutually non-isomorphic real forms. The coordinate description of this fact is as follows: Let L' and L'' be two real forms of L and let $\{e'_a\}$ and $\{e''_a\}$ be bases of L' and L'' with associated structure constants C'^c_{ab} and C''^c_{ab} respectively. That L' and L'' are not isomorphic (as real Lie algebras) is equivalent to saying that there are no invertible *real* matrices $\{A^a_b\}$ such that

$$C''^c_{ab} = A^c_l\, C'^l_{mn}\, A^{-1}{}^m_a\, A^{-1}{}^n_b . \tag{2.43}$$

However, that they are both reals forms of the same complex Lie algebra L is equivalent to the statement that (2.43) holds for some *complex* matrix $\{A^a_b\}$. Note that the structure constants C'^c_{ab} and C''^c_{ab} are real.

2.2.1.10 *The universal enveloping algebra* Let L be a Lie algebra over $\mathbb{F}$ and TL the tensor algebra of the vector space L (cf. A.6.4). Let I_S be the two sided ideal in TL generated by the set $S := \{X \otimes Y - Y \otimes X - [X, Y] \mid X, Y \in L\}$ (cf. A.6.3). The quotient $\text{Env}(L) := TL/I_S$ is an associative algebra over $\mathbb{F}$ and is called the *universal enveloping algebra* of the Lie algebra L. There is a natural embedding $j : L \to \text{Env}(L)$ given by $j := \pi \circ i$, where i is the natural embedding of L in TL and and $\pi : TL \to \text{Env}(L)$ is the natural projection. By construction one has $j([X, Y]) = j(X)j(Y) - j(Y)j(X)$. The centre (cf. A.6.3) of $\text{Env}(L)$ is called the *Casimir algebra* whose elements are called *Casimir elements*.

The important property of $\text{Env}(L)$ is that a Lie-algebra representation $\rho : L \to \mathfrak{gl}(V)$ induces a unique associative-algebra representation $R : \text{Env}(L) \to \text{End}(V)$ such that $\rho = R \circ i$. To see this go to the diagram (A.57) and apply it to the case at hand where $V = L$, $f = \rho$, and $A = \text{End}(V)$. This gives us an associative-algebra representation φ of TL in $\mathrm{End}(V)$ which satisfies $\varphi \circ i = \rho$. But ρ is a representation of a Lie algebra, not just a linear map of the underlying vector space. Hence φ satisfies $\varphi(X \otimes Y - Y \otimes X - [X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) - \rho([X,Y]) = 0$, which implies that I_S lies in the kernel of φ. Therefore there is a homomorphism $R : \mathrm{Env}(L) \to \mathrm{End}(V)$ so that $\varphi = R \circ \pi$ and hence $\rho = \varphi \circ i = D \circ j$. Uniqueness of φ (for given ρ) implies the uniqueness of R.

2.3 Lie algebras of Lie groups

We wish to associate a *real* (i.e. over $\mathbb{F} = \mathbb{R}$) Lie algebra to a Lie group $\mathsf{G} \subseteq \mathsf{GL}(n, \mathbb{F})$. For this we consider a differentiable curve $A(s) \in \mathsf{G}$, $s \in I \subseteq \mathbb{R}$ and $0 \in I$, such that $A(0) = E$. Its derivative at $s = 0$, denoted by $\dot{A} := \frac{d}{ds}\big|_{s=0} A(s)$, is called the 'tangent vector' of A at E. It is a matrix in $\mathtt{M}(n, \mathbb{F})$. The set of possible tangent vectors of G at E is called the 'tangent space' of G at E. We show that this tangent space is a Lie algebra over $\mathbb{R}$. Linearity is easy to see, since if X and Y are tangent vectors to $A(s)$ and $B(s)$ at E, $X + Y$ and aX $(a \in \mathbb{R})$ are tangent vectors at E to the curves $C(s) := A(s)B(s)$ and $\tilde{C}(s) := A(as)$ respectively. It remains to show that this real linear subspace of $\mathtt{M}(n, \mathbb{F})$ is a Lie algebra when the Lie bracket is defined to be the commutator. To do this, let X and Y again be tangent to $A(s)$ and $B(s)$ at E and define the new curve

$$D(s) := \begin{cases} A(\tau(s))B(\tau(s))A^{-1}(\tau(s))B^{-1}(\tau(s)) & \text{for } s \geq 0, \\ B(\tau(s))A(\tau(s))B^{-1}(\tau(s))A^{-1}(\tau(s)) & \text{for } s \leq 0, \end{cases} \tag{2.44}$$

with reparametrization $\tau(s) := \mathrm{sign}(s)\sqrt{|s|}$ and inverse $s(\tau) = \mathrm{sign}(\tau)\tau^2$. We have $D(0) = E$ and claim that D is differentiable at $s = 0$ with $\dot{D} = [X, Y]$. To see this, we first restrict to $s \geq 0$ and calculate the right derivative:

$$\begin{aligned} \dot{D} = \lim_{s \to 0} \frac{D(s) - E}{s} &= \lim_{s \to 0} \left\{ \frac{[A(\tau(s)), B(\tau(s))]A^{-1}(\tau(s))B^{-1}(\tau(s))}{s} \right\} \\ &= \lim_{\tau \to 0} \left\{ \left[\frac{A(\tau) - E}{\tau}, \frac{B(\tau) - E}{\tau} \right] A^{-1}(\tau)B^{-1}(\tau) \right\} \\ &= [X, Y] \,, \end{aligned} \tag{2.45}$$

where in the last step we used that the limit of a product is the product of the limits. For $s \leq 0$ the calculation works analogously leading to the same result for the left derivative. This proves the claim.

For $\mathsf{G} = \mathsf{GL}(n, \mathbb{F})$, any matrix $X \in \mathtt{M}(n, \mathbb{F})$ can be the tangent vector to a curve in E; take e.g. $A(s) = \exp(sX)$, where exp of a matrix is defined through its Taylor series. Hence the Lie algebra of $\mathsf{GL}(n, \mathbb{F})$ is given by $\mathtt{M}(n, \mathbb{F})$, considered as *real* vector space. It has real dimension $2n^2$ for $\mathbb{F} = \mathbb{C}$ and n^2 for $\mathbb{F} = \mathbb{R}$.

For the other Lie groups mentioned above, their Lie algebras are obtained by differentiation of the defining condition. For example, let $A(s)$ be a curve in $\mathsf{SL}(n, \mathbb{F})$ such that $A(0) = E$. The defining condition reads $\det(A(s)) = 1$, which upon differentiation (at $s = 0$) leads to $\mathrm{trace}(\dot{A}) = 0$. For a curve in $\mathsf{O}(p, q)$ the defining condition reads $A(s)E^{(p,q)}A^\top(s) = E^{(p,q)}$, whose differentiation leads to the condition that the matrix $E^{(p,q)}\dot{A}$ must be antisymmetric. Since this is equivalent to $E^{(p,q)}\dot{A}E^{(p,q)} = -\dot{A}$ (where $E^{(p,q)}$ squares to the identity) it already implies that $\dot{A}$ is trace free. Hence the Lie algebras of $\mathsf{O}(p,q)$ and $\mathsf{SO}(p,q)$ coincide. The same holds for the complex orthogonal groups. For a curve in $\mathsf{U}(p,q)$ we obtain the condition that $E^{(p,q)}\dot{A}$ must be antihermitean. This only implies a purely imaginary trace, so that for $\mathsf{SU}(p,q)$ we have the additional contraint $\mathrm{trace}(\dot{A}) = 0$. Finally, for $\mathsf{SP}(2n,\mathbb{F})$ we get that $\hat{E}^{(2n)}\dot{A}$ is symmetric. Here again tracelessness is implied. To see this, first note the antisymmetry of $\hat{E}^{(2n)}$ and that $(\hat{E}^{(2n)})$ squares to minus the identity; then taking the trace of $\hat{E}^{(2n)}X\hat{E}^{(2n)} = X^\top$ implies $\mathrm{trace}(X) = 0$. We summarize all this by the following list of Lie algebras, which we denote by the same letters as the corresponding group, but written in lower case letters:

$$\mathfrak{gl}(n,\mathbb{F}) = \mathtt{M}(n,\mathbb{F}) \tag{2.46}$$
$$\mathfrak{sl}(n,\mathbb{F}) = \{X \in \mathtt{M}(n,\mathbb{F}) \mid \mathrm{trace}(X) = 0\} \tag{2.47}$$
$$\mathfrak{o}(n,\mathbb{C}) = \{X \in \mathtt{M}(n,\mathbb{C}) \mid X = -X^\top\} \tag{2.48}$$
$$\mathfrak{so}(n,\mathbb{C}) = \mathfrak{o}(n,\mathbb{C}) \tag{2.49}$$
$$\mathfrak{o}(p,q) = \{X \in \mathtt{M}(n,\mathbb{R}) \mid E^{(p,q)}X = -(E^{(p,q)}X)^\top\} \tag{2.50}$$
$$\mathfrak{so}(p,q) = \mathfrak{o}(p,q) \tag{2.51}$$
$$\mathfrak{u}(p,q) = \{X \in \mathtt{M}(n,\mathbb{C}) \mid E^{(p,q)}X = -(E^{(p,q)}X)^\dagger\} \tag{2.52}$$
$$\mathfrak{su}(p,q) = \{X \in \mathtt{M}(n,\mathbb{C}) \mid E^{(p,q)}X = -(E^{(p,q)}X)^\dagger,\ \mathrm{trace}(X) = 0\} \tag{2.53}$$
$$\mathfrak{sp}(2n,\mathbb{F}) = \{X \in \mathtt{M}(n,\mathbb{F}) \mid \hat{E}^{(2n)}X = (\hat{E}^{(2n)}X)^\top\} \tag{2.54}$$

Clearly these Lie algebras can also be characterised in a coordinate independent fashion. As an example, let us consider $\mathfrak{o}(p,q)$. Let V be an $n = p+q$–dimensional real vector space with basis $\{e_a\}$ and dual basis $\{E^a\}$ of V^*. Let further $\omega = \omega_{ab}E^a \otimes E^b$ be a symmetric non-degenerate bilinear form, where $\{\omega_{ab}\} = E^{(p,q)}$. ω defines a isomorphism $\omega^\downarrow : V \to V^*$ ('index lowering'), as explained in Sect. A.5.7. We consider the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of V. The Lie algebra $\mathfrak{o}(p,q)$ corresponds to the subalgebra of endomorphisms which satisfy the relation $\omega(Xv,w) = -\omega(v,Xw)$ for all v,w in V. This can be written without v,w in terms of maps as follows:

$$\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}(V) \mid \omega^\downarrow \circ X = -X^\top \circ \omega^\downarrow\}\,, \tag{2.55}$$

which is the coordinate-free version of (2.50). Here $X^\top$ denotes the map transposed to X; cf. Sect. A.5.5. Analogously, coordinate-free definitions can be given for all the other Lie algebras listed above.

As already mentioned, all Lie algebras of Lie groups are a priori to be considered as *real* Lie algebras. However, those subalgebras of $\mathtt{M}(n,\mathbb{C})$ whose defining relations do not involve a complex conjugation of X have a complex structure given by multiplication by i. This simply follows from the reality of the matrices $E^{(p,q)}$ and $\hat{E}^{(2n)}$, so that iX satisfies the corresponding relation if X does.

Hence $\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{o}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(2n,\mathbb{C})$ have a natural complex structure and can therefore also be considered as complex Lie algebras, which according to our notations in Sect. 2.2.1.9 we denote by a subscript $\mathbb{C}$, like e.g. $\mathfrak{gl}(n,\mathbb{C})_{\mathbb{C}}$. But note that this does not make sense for $\mathfrak{u}(p,q)$ and $\mathfrak{su}(p,q)$.

Finally we note that the Lie algebra of a direct product $G' = H \times G$ of Lie groups is just given by the direct sum of the individual Lie algebras:

$$G' = H \times G \;\Rightarrow\; \mathfrak{g}' = \mathfrak{h} \oplus \mathfrak{g}\,. \tag{2.56}$$

This immediately follows from the definition of the direct product of groups (cf. Sect. A.4.3) and the definition of the Lie algebra of a Lie group, as given above. Slightly more complicated is the Lie algebra of the semi-direct product $G' = H \rtimes_\alpha G$, for some $\alpha : G \to \mathsf{Aut}(H)$ (cf. Sect. A.4.3), which turns out to be the semi-direct sum $\mathfrak{h} \rtimes_\sigma \mathfrak{g}$ (cf. (2.28), where the homomorphism $\sigma : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{h})$ derives from α. For the inhomogeneous groups, i.e. semi-direct products with vector-spaces, this structure is most easily determined directly: Let $\mathsf{G} \subseteq \mathsf{GL}(n,\mathbb{F})$ be any of the groups just considered, with Lie algebra $\mathfrak{g}$, and IG its inhomogeneous group as defined in (2.11). The lie algebra of the latter is then given by the set

$$\mathfrak{ig} = \{(a,X) \mid a \in \mathbb{F}^n,\, X \in \mathfrak{g}\}\,, \tag{2.57}$$

with Lie bracket being given by

$$[(a,X)\,,\,(b,Y)] = (X\cdot b - Y\cdot a\,,\,[X,Y])\,. \tag{2.58}$$

A convenient way to see this is via the embedding (2.12), which also embeds $\mathfrak{g}$ into $\mathtt{M}(n+1,\mathbb{F})$, so that

$$(a,X) \longmapsto \begin{pmatrix} 0 & 0^\top \\ a & X \end{pmatrix}\,. \tag{2.59}$$

Simple commutation then leads to (2.58). Equation (2.57) is clearly just a special case of (2.28) for $L = F^n$ (made into a Lie algebra by setting all Lie brackets to zero), $L' = \mathfrak{g} \subseteq \mathsf{GL}(n,\mathbb{F})$ and $\sigma_X(a) := X\cdot a$. Note that the endomorphisms of $\mathbb{F}^n$ are derivations of the abelian Lie algebra L made from $\mathbb{F}^n$.

2.3.1 *Simplicty*

In this subsection we wish to establish simplicity for many of the Lie algebras listed above. For this we recall that simplicity of a complex Lie algebra implies simplicity for any of its real forms (cf. 2.2.1.9). We have

Theorem 2.5 *The following complex Lie algebras are simple:*

1) $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ *for* $n \geq 2$,
2) $\mathfrak{so}(n,\mathbb{C})_{\mathbb{C}}$ *for* $n = 3$ *and* $n \geq 5$,
3) $\mathfrak{sp}(n,\mathbb{C})_{\mathbb{C}}$ *for* $n \geq 2$.

Proof We shall only prove parts 1) and 2). Part 3) is of less interest to us and only included for completeness. The proof of 3) works along the very same lines as those we present now.

1) We need to distinguish the cases $n = 2$ and $n \geq 2$. For $n = 2$ a convenient basis of traceless matrices is given by

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.60}$$

whose commutators are

$$[X^+, X^-] = H\,, \quad [H, X^\pm] = \pm 2X^\pm\,. \tag{2.61}$$

Suppose $C = x_+X^+ + x_-X^- + hH$ is an element of a non-zero ideal $I \subseteq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. If $x_\pm = 0$ then $H \in I$ then $[H, X^\pm] = \pm 2X^\pm \in I$; hence $I = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. If e.g. $x_- \neq 0$ then $[X^+,[X^+,H]] = -2x_-X^+$ implies $x^+ \in I$ and (2.61) imply $X^-, H \in I$. Similarly if $x_+ \neq 0$.
For $n \geq 2$ we introduce the n^2 matrices B_{ij}, $1 \leq i,j \leq n$, whose definition is that the entry of B_{ij} at the intersection of the i-th row and j-th column is 1 and 0 otherwise, i.e. its ab-component is $(B_{ij})_{ab} = \delta_{ia}\delta_{jb}$. A basis of $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ is given by the $n(n-1)$ matrices B_{ij} for $i \neq j$ and the $n-1$ matrices $B_i := B_{ii} - B_{nn}$ for $1 \leq i \leq n-1$. The product of two B-matrices is given by $B_{ij}B_{kl} = \delta_{jk}B_{il}$ and their commutator accordingly. In what follows, we shall disable the summation convention. Let now again $I \subseteq \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ be a non-zero ideal. Suppose $C = \sum_{ij} c_{ij}B_{ij} \in I$. We first assume that C is diagonal, i.e. $c_{ij} = 0$ for $i \neq i$. Since C is trace free the diagonal entries cannot all be identical, hence $c_{aa} \neq c_{bb}$ for some index pair a,b where $a \neq b$. Now, $[B_{ab}, C] = (c_{bb} - c_{aa})B_{ab}$ so that $B_{ab} \in I$. Then, for any i distinct from a and b (here we need $n \geq 3$) we have $[B_{ia}, B_{ab}] = B_{ib}$ so that $B_{ib} \in I$. Since for any pairwise distinct i,j,b we have $[B_{ib}, B_{bj}] = B_{ij}$, these results imply $B_{ij} \in I$ for any $i \neq j$. Now $[B_{ij}, B_{ji}] = B_{ii} - B_{jj}$ shows that I contains a basis of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ and hence that $I = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. Next we turn to the general case where C is not diagonal, so that $c_{ab} \neq 0$ for some pair of distinct indices a,b. Then, for any i distinct from a and b (here again we need $n \geq 3$), a short calculation gives $[B_{bi},[B_{ia},[B_{ba},C]]] = -c_{ab}B_{ba}$, so that $B_{ba} \in I$. Now the steps proceed as above, showing again that $I = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$.

2) It is easy to see that $\mathfrak{so}(3,\mathbb{C})_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$, so that only the cases $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ for $n \geq 5$ remain. A basis for $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ is given by the $n(n-1)/2$ antisymmetric matrices $K_{ij} := B_{ij} - B_{ji}$ for $1 \leq i < j \leq n$. Their commutation relations are

$$[K_{ij}, K_{kl}] = \delta_{jk}K_{il} + \delta_{il}K_{jk} - \delta_{ik}K_{jl} - \delta_{jl}K_{ik}\,. \tag{2.62}$$

Let $C = \sum_{i<j} c_{ij}K_{ij}$ be an element in a non-zero ideal $I \subseteq \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ so that $c_{ab} \neq 0$ for some pair of distinct indices a,b. Choose indices i,j,k such that a,b,i,j,k are pairwise distict (here we use $n \geq 5$). A slightly longer but straightforward calculation using (2.62) gives $[K_{ik},[K_{ji},[K_{bi},[K_{ak},C]]]] = c_{ab}K_{ij}$, so that $K_{ij} \in I$ for all i,j distinct from a and b. Then $[K_{ij}, K_{ja}] =$

K_{ia}, $[K_{ij}, K_{jb}] = K_{ib}$ and $[K_{ia}, K_{ib}] = -K_{ab}$ show that in fact all K_{ij} are in I and hence that $I = \mathfrak{so}(n,\mathbb{C})_{\mathbb{C}}$ □

Clearly $\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$. It is also easy to see, and will be shown explicitly in Sect. 3.6.4, that $\mathfrak{so}(p,q)^{\mathbb{C}} \cong \mathfrak{so}(n,\mathbb{C})_{\mathbb{C}}$ and $\mathfrak{su}(p,q)^{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$; cf. (3.85) and (3.88) respectively. Hence we have

Corollary 2.6 *The following families of real Lie algebras are simple*

1) $\mathfrak{sl}(n,\mathbb{R})$ *for* $n \geq 2$*,*
2) $\mathfrak{so}(p,q)$ *for* $p+q=3$ *or* $p+q \geq 5$*,*
3) $\mathfrak{su}(p,q)$ *for* $p+q \geq 2$*.*

The exceptional cases $p+q=4$ are interesiting in their own right and also physically, since $p=1, q=3$ corresponds to the Lorentz group in 3+1 spacetime dimensions. Let us therefore look at these examples in more detail. We shall use the coordinate free definition (2.55) for $\mathfrak{o}(p,q) = \mathfrak{so}(p,q)$. V is now a four-dimensional real vector space with basis $\{e_a\}$ and dual basis $\{E^a\}$ of V^*, where the latin indices range over $\{0,1,2,3\}$. Then $\omega = \omega_{ab} E^a \otimes E^b$, where $\{\omega_{ab}\} = E^{(p,q)}$ (cf. (2.3)).

In what follows we identify $\mathrm{End}(V)$ with $V \otimes V^*$ by their natural isomorphism. Writing $X = X^a{}_b e_a \otimes E^b$, (2.55) is equivalent to $X_{ab} = -X_{ba}$, where $X_{ab} := \omega_{ac} X^c{}_b$. Therefore a basis for $\mathfrak{so}(p,q)$ is given by the ten elements

$$M_{ab} = e_a \otimes E_b - e_b \otimes E_a\,, \tag{2.63}$$

where $E_a := \omega^{\downarrow}(e_a) = \omega_{ab} E^b$. Their Lie brackets are

$$[M_{ab}, M_{cd}] = \omega_{ad} M_{bc} + \omega_{bc} M_{ad} - \omega_{ac} M_{bd} - \omega_{bd} M_{ac}\,. \tag{2.64}$$

We define an isomorphism $\star$ of $\mathfrak{so}(p,q)$ by

$$\star M_{ab} := \tfrac{1}{2} \varepsilon_{abcd}\, \omega^{ci} \omega^{dj}\, M_{ij}\,, \tag{2.65}$$

where ε_{abcd} are the components of the totally antisymmetric tensor $E^0 \wedge E^1 \wedge E^2 \wedge E^3$ and where $\{\omega^{ab}\}$ is the inverse matrix of $\{\omega_{ab}\}$. Note that, up to signs, the map $\star$ merely permutes the basis elements M_{ab}. Note also that the existence of this isomorphism is particular to $p+q=4$. It is easy to see that $\star \circ \star = (-1)^q\, \mathrm{id}|_{\mathfrak{so}(p,q)}$, so that for odd q there are no real eigenvalues of $\star$. For even q, however, the eigenvalues are ± 1. The projectors onto the corresponding eigenspaces are $P_\pm = \frac{1}{2}(\mathrm{id} \pm \star)$. Moreover, a straightforward computation[8] shows

$$\star[\star A, B] = \star[A, \star B] = (-1)^q\, [A,B]\,, \tag{2.66}$$

or equivalently

$$[\star A, B] = [A, \star B] = \star[A,B]\,, \tag{2.67}$$

[8] Using $\varepsilon_{abcd}\varepsilon^{ijkl} = (-1)^q\, 4!\, \delta^i_{[a}\delta^j_b\delta^k_c\delta^l_{d]}$.

for all $A, B \in \mathfrak{so}(p,q)$. This implies that the projectors onto the eigenspaces of $\star$, $P_\pm = \frac{1}{2}(\mathrm{id} \pm \star)$, also satisfy

$$[P_\pm A, B] = [A, P_\pm B] = P_\pm [A, B]\,, \tag{2.68}$$

which shows that, for even q, the Lie algebra $\mathfrak{so}(p,q)$ is the direct sum of two ideals, given by the ranges of $P_\pm$. These ranges are isomorphic to $\mathfrak{so}(3)$ for $q = 0, 4$ and isomorphic to $\mathfrak{so}(1,2) = \mathfrak{so}(2,1)$ for $q = 2$. To see this, first note that (2.64) immediately implies that span$\{M_{\mu\nu}\}$ is a 3-dimensional subalgebra of $\mathfrak{so}(p,q)$, where greek indices range over $\{1,2,3\}$. This subalgebra has exactly the characterisation (2.55) with $\omega^\downarrow$ restricted to span$\{e_\mu\}$, which is just the definition of $\mathfrak{so}(3)$ if $q = 0$ or $q = 4$ and of $\mathfrak{so}(1,2)$ if $q = 2$. These subalgebras are isomorphic to the ranges of $P_\pm$. The isomorphisms are simply given by

$$M_{\alpha\beta} \to M_{ab} = \begin{cases} M_{\mu\nu} & \text{for } a = \mu \text{ and } b = \nu, \\ \pm\varepsilon_{0\mu\nu\lambda}\omega^{\nu\alpha}\omega^{\lambda\beta}M_{\alpha\beta} & \text{for } a = 0 \text{ and } b = \mu, \\ \pm\varepsilon_{\mu 0\nu\lambda}\omega^{\nu\alpha}\omega^{\lambda\beta}M_{\alpha\beta} & \text{for } a = \mu \text{ and } b = 0. \end{cases} \tag{2.69}$$

where the upper (+) sign is for the range of P_+ and the lower (-) sign for the range of P_-. Note that this simply uses the map $\star$ (2.65) to express the three elements $M_{0\mu} = -M_{\mu 0}$ through the three $M_{\mu\nu} = -M_{\nu\mu}$ so as to produce the required eigenvectors to $\star$. Hence we have shown that

$$\mathfrak{so}(4) \;\cong\; \mathfrak{so}(3) \oplus \mathfrak{so}(3)\,, \tag{2.70}$$

$$\mathfrak{so}(2,2) \;\cong\; \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)\,. \tag{2.71}$$

implying, in particular, that $\mathfrak{so}(4)$ and $\mathfrak{so}(2,2)$ are semi-simple (since the summands on the right-hand side are semi-simple by Corrolary 2.6) but not simple.

Finally there remains only the physically most relevant case $p = 1, q = 3$ (or, equivalently, $p = 3, q = 1$). We set

$$R_\alpha := \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}M_{\beta\gamma}\,, \quad B_\alpha := M_{\alpha 0}\,, \tag{2.72}$$

where $\varepsilon_{\alpha\beta\gamma}$ are the components of $E^1 \wedge E^2 \wedge E^3$. In terms of R_α, B_α (2.64) is equivalent to

$$[R_\alpha, R_\beta] = \varepsilon_{\alpha\beta\gamma}R_\gamma\,, \quad [R_\alpha, B_\beta] = \varepsilon_{\alpha\beta\gamma}B_\gamma\,, \quad [B_\alpha, B_\beta] = -\varepsilon_{\alpha\beta\gamma}R_\gamma\,. \tag{2.73}$$

Clearly, the Lie algebra span$\{R_1, R_2, R_3\}$ is isomorphic to $\mathfrak{so}(3)$. Comparison of (2.73) and (2.38) shows that

$$\mathfrak{so}(3)^{\mathbb{CR}} \cong \mathfrak{so}(1,3)\,. \tag{2.74}$$

Finally we note that

$$\mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R})\,. \tag{2.75}$$

as one immediately checks by first noting that for $p = 1, q = 2$ (2.64) boils down to (the index set is now $\{0, 1, 2\}$)

$$[M_{12}, M_{01}] = M_{02}\,, \quad [M_{12}, M_{02}] = -M_{01}\,, \quad [M_{01}, M_{02}] = -M_{12}\,. \tag{2.76}$$

Setting $X^{\pm} := M_{01} \pm M_{12}$ and $H := 2M_{02}$ this shows to be equivalent to (2.61), thereby proving (2.75). Since we obviously have $\mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}\mathbb{R}} \cong \mathfrak{sl}(2,\mathbb{C})$, (2.75) and (2.74) imply

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,2)^{\mathbb{C}\mathbb{R}} \cong \mathfrak{so}(3)^{\mathbb{C}\mathbb{R}} \cong \mathfrak{so}(1,3)\,. \tag{2.77}$$

In view of Theorem 2.5 this also shows simplicity of $\mathfrak{so}(1,3)$.

2.3.2 *Homomorphisms*

Let $\phi : \mathsf{G} \to \mathsf{G}'$ be a differentiable homomorphism of Lie groups and $A(s)$ be a curve with $A(0) = E$ and $\dot{A} = X$. We denote the derivative of ϕ at E by $\dot{\phi}$.[9] For the mapped curve $A' := \phi \circ A$ in G' we have $A'(0) = E'$ (identity matrix in G') and $\dot{A}' = \dot{\phi}(X)$ by the chain rule. Mapping the curve $D(s)$ defined in (2.44) leads to the analogously defined curve in terms of $A' := \phi \circ A$ and $B' := \phi \circ B$ whose tangent at E' can be calculated just as in (2.45). This shows that

$$\frac{d}{ds}\Big|_{s=0} (\phi \circ D)(s) = \dot{\phi}([X,Y]) = [\dot{\phi}(X), \dot{\phi}(Y)]\,, \tag{2.78}$$

which states that $\dot{\phi}$ is a Lie algebra homomorphism.

2.3.3 *The exponential map*

By $\exp : \mathtt{M}(n,\mathbb{F}) \to \mathtt{M}(n,\mathbb{F})$ we denote the so-called exponential map, which is defined by its usual Taylor series, which converges absolutely at any $X \in \mathtt{M}(n,\mathbb{F})$. If X lies in the Lie algebra $\mathfrak{g}$ of $\mathsf{G} \subseteq \mathsf{GL}(n,\mathbb{F})$ then $\exp(X)$ lies in $\in \mathsf{G}$. For the general and special linear groups this follows immediately from the general formula $\det(\exp(X)) = \exp(\operatorname{trace}(X))$. But it is also easy to check directly for the other groups. For example, for $X \in \mathfrak{u}(p,q)$, we have

$$\begin{aligned} \exp(X)E^{(p,q)}[\exp(X)]^{\dagger} &= \exp(X)\exp(E^{(p,q)}X^{\dagger}E^{(p,q)})E^{(p,q)} \\ &= E^{(p,q)}\,, \end{aligned} \tag{2.79}$$

where in the second step we used $E^{(p,q)} = (E^{(p,q)})^{-1}$ and in the last step $E^{(p,q)}X^{\dagger}E^{(p,q)} = -X$, which is equivalent to the defining relation in (2.52) due to the hermiticity of $E^{(p,q)}$. Hence we have shown that $\exp(X)$ fulfills the defining relation of $\mathsf{U}(p,q)$ if X satisfies those of $\mathfrak{u}(p,q)$. In a similar fashion, this result extends to all other Lie groups mentioned before.

We note some further properties of $\exp : \mathfrak{g} \to \mathsf{G}$: First we note that exp is infinitely differentiable (analytic, in fact) and that its first derivative at $0 \in \mathfrak{g}$,

[9]Usually the differential of the map ϕ is denoted by ϕ_* (or $d\phi$); then $\dot{\phi} := \phi_*(E)$. Since we shall only need the differential evaluated at E, and since no confusion with general derivatives of curves (also denoted by an overdot) should arise, we decided for this convenient shorthand.

$\exp_*(0) : \mathfrak{g} \to \mathfrak{g}$, is the identity map. Hence $0 \in \mathfrak{g}$ is a regular point and the inverse-function-theorem implies that there exist neighbourhoods U of $0 \in \mathfrak{g}$ and V of $E \in \mathsf{G}$ such that $\exp : U \to V$ is a diffeomorphism. Globally such a diffeomorphism cannot exist in general. Obviously exp cannot be injective in general. Consider e.g. the group $\mathsf{U}(1)$, which is just the circle of complex numbers of unit modulus, and its Lie algebra, $\mathfrak{u}(1) \cong i\mathbb{R}$, the line of purely imaginary numbers. Under exp this line is clearly wound infinitely many times around the circle.

Surjectivity also fails in general. It trivially fails for all those elements of the group which are not in the identity component (compare A.4.6), since any $\exp(X)$ is connected to the identity by the arc $t \mapsto \exp(tX)$. But exp may also fail to cover all of the identity component. This happens for example for the group $\mathsf{SL}(2,\mathbb{C})$, which is relevant in special relativity. To see this, consider the following element

$$A = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \quad \text{where} \quad 0 \neq a \in \mathbb{C}\,, \tag{2.80}$$

which is connected to the identity by the arc

$$A(t) = \begin{pmatrix} \exp(t\pi i) & ta \\ 0 & \exp(-t\pi i) \end{pmatrix}\,. \tag{2.81}$$

Assuming $\exp(X) = A$ with $X \in \mathfrak{sl}(2,\mathbb{C})$ it follows that the eigenvalues of X must be $\pm\lambda$ (since X is trace free) and satisfy $\exp(\lambda) = -1$ (since -1 is the double eigenvalue of A). In particular, X has two different eigenvalues and is therefore diagonalizable by some $T \in \mathsf{GL}(2,\mathbb{C})$. Writing $D = \mathrm{diag}(\lambda, -\lambda)$ we have $\exp(X) = T\exp(D)T^{-1} = T\mathrm{diag}(-1,-1)T^{-1} = -E \neq A$, a contradiction. This shows that no element of the form (2.80) lies in the image of the exponential map for $\mathsf{SL}(2,\mathbb{C})$.

However, its is true for any Lie group G that the elements in the image of exp *generate* the identity component G^0, meaning that any $g \in \mathsf{G}^0$ can be written as a finite product of the form

$$g = \exp(X_1)\exp(X_2)\cdots\exp(X_n)\,. \tag{2.82}$$

To see this we remark that elements of the form (2.82) clearly constitute a subgroup, G', which is contained in G^0 since replacing the X_i by tX_i gives an arc connecting g to the identity. G' certainly contains a whole neighbourhood, V, of the identity since such a V is already contained in the image of exp, as shown above. Hence also $gV \subset G'$ for $g \in G'$. But gV is a neighbourhood of g so that G' is open. Since any open subgroup of a topological group is also closed (cf. A.4.6) and since $G' \subseteq G^0$ with G^0 connected we must have $G' = G^0$ (cf. A.3.5). We remark that the argument just given in fact generally shows that any open neighbourhood of the identity element in a topological group generates the identity component, that is, any element in the identity component is a finite product of elements from the neighbourhood.

Very useful for the study of Lie groups is to consider their one-parameter subgroups. These are homomorphisms $A : \mathbb{R} \to \mathsf{G}$, with $\mathbb{R}$ considered as additive group, so that $A(t + t') = A(t) \cdot A(t')$ for all $t, t' \in \mathbb{R}$. The point we wish to make here is that A is uniquely determined by $X = \dot{A}$ and, in fact, necessarily given by $A(t) = \exp(tX)$ (which is obviously a one-parameter subgroup). To see this we first take $\frac{d}{dt'}|_{t'=0}$ of the equation expressing the homomorphism property and get $\dot{A}(t) = A(t) \cdot X$. Then we consider the curve $C(t) := A(t)\exp(-tX)$ for which $C(0) = E$. Differentiation gives $\dot{C}(t) = (\dot{A}(t) - A(t) \cdot X)\exp(-tX) = 0$ and therefore $C(t) \equiv E$, which proves the claim.

We can now show another important property of the exponential map concerning homomorphisms. For this let $\phi : \mathsf{G} \to \mathsf{G}'$ be a differentiable homomorphism of Lie groups. Then $A(t) := \phi(\exp(tX))$ is a one parameter subgroup of G' with tangent $\dot{A} = \dot{\phi}(X)$. The result above now implies $A(t) = \exp(t\dot{\phi}(X))$ and therefore $\phi(\exp(tX)) = \exp(t\dot{\phi}(X))$. Evaluation at $t = 1$ gives an equation valid for all $X \in \mathfrak{g}$; hence

$$\phi \circ \exp = \exp \circ \dot{\phi}\,. \tag{2.83}$$

We have seen above (2.3.2) that a homomorphism ϕ between Lie groups uniquely determines a homomorphisms $\dot{\phi}$ between the corresponding Lie algebras. What about the converse? Given two Lie groups G, G' with Lie algebras $\mathfrak{g}, \mathfrak{g}'$ and an algebra homomorphism $f : \mathfrak{g} \to \mathfrak{g}'$. Does a group homomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ exist such that $\dot{\phi} = f$? And, given its existence, is it unique? To answer the second (and easier) question first, we combine (2.83) with the fact that products of the form (2.82) generate the identity component. It immediately follows that if there exist two group homomorphisms ϕ_1, ϕ_2 for which $\dot{\phi}_1 = \dot{\phi}_2 = f$ one has $\phi_1\big|_{G^0} = \phi_2\big|_{G^0}$, that is, their restrictions to the identity component must coincide. This implies uniqueness for connected groups, but for groups with more than one connected component one generally cannot do better than this, since one may vary the group homomorphisms off the identity component. For example, a homomorphism $\phi_1 : \mathsf{O}(n) \to \mathsf{O}(n)$ may be changed to $\phi_2(g) := \det(g)\phi_1(g)$ where still $\dot{\phi}_1 = \dot{\phi}_2$.

Having seen that uniqueness is implied by the connectedness of G we mention that existence of ϕ is implied by another, more restrictive requirement on G, namely its simply-connectedness (cf. A.3.7; note that simply connectedness is defined to imply connectedness). The proof of the existence part, which is slightly outside the scope of this book, may be e.g. be found in [17], Thm. 3.27. Hence we can state the following

Theorem 2.7 *Let G and G' be Lie groups and $f : \mathfrak{g} \to \mathfrak{g}'$ a homomorphisms between their Lie algebras. If G is connected there is at most one homomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ for which $\dot{\phi} = f$. Φ exist if G is simply connected.*

In particular it follows that two simply connected Lie groups are isomorphic if their Lie algebras are isomorphic.

2.4 Universal covering groups

We have seen that every Lie group uniquely determines a Lie algebra. The converse is not quite true. Existence holds, that is, every abstract Lie algebra L is (isomorphic to) the Lie algebra $\mathfrak{g}$ of some Lie group G. This by no means trivial fact follows from the theorem of Ado, already mentioned at the beginning of section 2.2. But uniqueness may fail: non-isomorphic Lie groups may have isomorphic Lie algebras. A physically relevant and often cited example is given by the Lie groups $SO(3)$ and $SU(2)$. Let us look at this example first before we present some general and more abstract arguments.

2.4.1 *The relation between $SU(2)$ and $SO(3)$*

A general rotation in ordinary three-dimensional space by an angle α about an axis represented by the unit vector $\vec{n}$ is given by

$$\vec{x} \mapsto R(\alpha, \vec{n})\vec{x} := \vec{n}(\vec{n} \cdot \vec{x}) + (\vec{x} - \vec{n}(\vec{n} \cdot \vec{x}))\, \cos(\alpha) + (\vec{n} \times \vec{x})\, \sin(\alpha)\,, \tag{2.84}$$

which shows that the components of any $\mathsf{SO}(3)$ matrix can be written in the form

$$R_{ab} = n_a n_b + (\delta_{ab} - n_a n_b)\cos(\alpha) - \varepsilon_{abc} n_c \sin(\alpha) \tag{2.85}$$

for some normalized unit vector $\vec{n} = (n_1, n_2, n_3)$. Keeping $\vec{n}$ fixed and taking the derivative with respect to α at $\alpha = 0$ leads to the Lie-algebra element corresponding to an infinitesimal rotation about the $\vec{n}$ axis:

$$\vec{x} \mapsto I(\vec{n})\vec{x} := \frac{d}{d\alpha}\Big|_{\alpha=0} R(\alpha, \vec{n})\vec{x} = \vec{n} \times \vec{x}\,. \tag{2.86}$$

Their commutation relations are:

$$[I(\vec{n}), I(\vec{m})] = I(\vec{n} \times \vec{m})\,. \tag{2.87}$$

Let $\vec{e}_1 := (1,0,0)^\top$, $\vec{e}_2 := (0,1,0)^\top$, and $\vec{e}_3 := (0,0,1)^\top$ be the three standard basis vectors, (2.87) is equivalent to, writing $I_a := I(\vec{e}_a)$ etc.:

$$[I_a, I_b] = \varepsilon_{abc} I_c\,. \tag{2.88}$$

The Lie algebra of $SU(2)$ is given by the real vector space of all traceless anti-hermitean 2×2 matrices (cf. (2.53). A basis is e.g. given by $\{J_1, J_2, J_3\}$, where

$$J_a := -\tfrac{i}{2}\sigma_a \tag{2.89}$$

and the σ_i denote—as usual—the three Pauli matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.90}$$

By straightforward calculation one verifies the relations

$$\sigma_a \sigma_b = \delta_{ab} E^{(2)} + i\varepsilon_{abc}\sigma_c\,, \tag{2.91}$$

which imply

$$\mathrm{trace}(\sigma_a\sigma_b) = 2\delta_{ab}\,. \tag{2.92}$$

The commutation relations between the J_a follow from (2.91):

$$[J_a, J_b] = \varepsilon_{abc} J_c\,, \tag{2.93}$$

so that the map $\mathfrak{so}(3) \to \mathfrak{su}(2)$, given by $I_a \mapsto J_a$ plus linear extension, defines a Lie isomorphisms. However, the groups $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ are not isomorphic. Rather, there is a surjective homomorphism from $\mathsf{SU}(2)$ to $\mathsf{SO}(3)$ which fails to be injective since every image has two pre-images. Hence $\mathsf{SU}(2)$ is a double cover of $\mathsf{SO}(3)$, and since $\mathsf{SU}(2)$ is simply connected it is in fact a the universal cover.

Let us explain this in detail, since this will also be relevant for the Lorentz group. There is an obvious isomorphism of real vector spaces between $\mathbb{R}^3$ and the set of hermitean traceless elements in $\mathtt{M}(2,\mathbb{C})$, which forms a real vector space which we denote by $\mathtt{H}_{\mathrm{T}}(2,\mathbb{C})$ for the moment. The isomorphism is given by

$$\tau : \mathbb{R}^3 \to \mathtt{H}_{\mathrm{T}}(2,\mathbb{C}), \quad \vec{x} \mapsto \tau(\vec{x}) := \vec{x}\cdot\vec{\sigma}\,, \tag{2.94}$$

with inverse following from (2.92)

$$\tau^{-1} : \mathtt{H}_{\mathrm{T}}(2,\mathbb{C}) \to \mathbb{R}^3, \quad X \mapsto \tau^{-1}(X) := \tfrac{1}{2}\mathrm{trace}(X\vec{\sigma}),. \tag{2.95}$$

One readily checks that the square of the euclidean norm obeys

$$\|\quad\|^2 = -\det\circ\tau\,. \tag{2.96}$$

The group $\mathsf{SU}(2)$ acts linearly (cf. A.4.5) on $\mathtt{H}_{\mathrm{T}}(2,\mathbb{C})$) via conjugation:

$$\begin{aligned} \mathrm{Ad} : \mathsf{SU}(2)\times\mathtt{H}_{\mathrm{T}}(2,\mathbb{C}) &\to \mathtt{H}_{\mathrm{T}}(2,\mathbb{C})\,,\\ (A,X) &\mapsto \mathrm{Ad}_A(X) := AXA^{-1} = AXA^{\dagger}\,. \end{aligned} \tag{2.97}$$

Hence there is also a linear action of $\mathsf{SU}(2)$ on $\mathbb{R}^3$, given by

$$SU(2)\times\mathbb{R}^3 \to \mathbb{R}^3, \quad (A,\vec{x}) \mapsto \pi(A)\vec{x} := \tau^{-1}\circ\mathrm{Ad}_A\circ\tau(\vec{x})\,. \tag{2.98}$$

Equation (2.96) together with $\det\circ\mathrm{Ad}_A = \det$ (since $\det(A) = 1$ for all $A \in \mathsf{SU}(2)$) now immediately imply that $\pi(A)$ is norm preserving, so that the map $A \mapsto \pi(A) := \tau^{-1}\circ\mathrm{Ad}_A\circ\tau$ is a homomorphism from $\mathsf{SU}(2)$ to $\mathsf{O}(3)$. We show that it is in fact a homomorphism onto $\mathsf{SO}(3)$.

Matrices in $\mathsf{O}(3)$ have determinant equal to either plus or minus one (take the determinant of the defining equation in (2.50)). Those with determinant $+1$ are not in the same connected component (cf. A.3.5) as those with determinant -1. This follows from the continuity of the determinant function and the fact that its image, which is $+1, -1$, is disconnected (cf. A.3.5). On the other hand, a complex 2×2 matrix A is in $\mathsf{SU}(2)$ iff A has unit determinant and $A^\dagger$ is the inverse of A. Hence $\mathsf{SU}(2)$ can be equivalently characterized by:

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\} . \tag{2.99}$$

Decomposing $a = x_1 + ix_2$ and $b = x_3 + ix_4$, where all x_i are real, this shows that, topologically speaking, $\mathsf{SU}(2)$ is homeomorphic (cf. A.3.2) to the unit 3-sphere, here represented as unit 3-sphere in $\mathbb{R}^4$. In particular, it follows that $SU(2)$ is connected. Hence the image of the homomorphism $\pi : \mathsf{SU}(2) \to \mathsf{O}(3)$ must lie in a single connected component, namely the component which contains the identity, which is $\mathsf{SO}(3)$.

Hence we have a homomorphism

$$\pi : \mathsf{SU}(2) \to \mathsf{SO}(3), \quad A \mapsto \pi(A) := \tau^{-1} \circ \mathrm{Ad}_A \circ \tau . \tag{2.100}$$

In components this reads, writing R_{ab} for the matrix components of $\pi(A)$,

$$R_{ab} = \tfrac{1}{2}\mathrm{trace}(\sigma_a A \sigma_b A^\dagger) , \tag{2.101}$$

or equivalently (the equivalence being a consequence of (2.92)),

$$\sigma_a R_{ab} = A \sigma_b A^\dagger . \tag{2.102}$$

We have not yet shown surjecivity, i.e. that *any* special orthogonal matrix R_{ab} can be represented in the form (2.101). This can be easily verified explicitly by writing down an $\mathsf{SU}(2)$ matrix $A(\alpha, \vec{n})$, which, when inserted into the right hand side of (2.101), results in (2.85). In fact, the matrix $A(\alpha, \vec{n})$ is easy to guess: it is just the exponential of $\alpha\vec{n} \cdot \vec{J}$. The power-series expansion, which defines the exponential function, is readily evaluated if one uses the fact that any even power of $\vec{n} \cdot \vec{\sigma}$ equals the unit matrix, which in turn follows from (2.91); hence

$$A(\alpha, \vec{n}) := \exp\left(-\tfrac{i}{2}\alpha\, \vec{n} \cdot \vec{\sigma}\right) = \cos\left(\tfrac{\alpha}{2}\right) - i\vec{n} \cdot \vec{\sigma}\, \sin\left(\tfrac{\alpha}{2}\right) . \tag{2.103}$$

A direct computation of the right hand side of (2.102) with $A = A(\alpha, \vec{n})$, where (2.91) is systematically used to resolve products of Pauli matrices into linear combinations of the unit matrix and Pauli matrices, shows that R_{ab} is just given by (2.85).

Finally we show that if $\pi(A) = R$, then $\pi^{-1}(R) = \pm A$. Indeed, according to (2.102), the relation $\pi(A) = R = \pi(A')$ is equivalent to the statement that $A^{-1}A'$ commutes with all Pauli matrices and hence with all complex 2×2matrices (since the Pauli matrices together with the unit matrix span $\mathtt{M}(2, \mathbb{C})$). Hence $A^{-1}A'$ must be a complex multiple of the identity matrix (the centre of the associative algebra $\mathtt{M}(2, \mathbb{C})$ is generated by the unit matrix). But since $A^{-1}A' \in \mathsf{SU}(2)$ we must, in fact, have $A^{-1}A' = \pm\mathbf{1}$, which proves the assertion. We summarise our findings obtained so far in the following

Theorem 2.8 *The map $\pi : \mathsf{SU}(2) \to \mathsf{SO}(3)$, defined by (2.100) is a surjective 2-1 homomorphism of Lie groups. From its component version (2.101) it is clear that this map is continuous, in fact C^∞.*

We have seen above that topologically $\mathsf{SU}(2)$ can be identified with the unit 3-sphere in $\mathbb{R}^4$. Under this identification the self-map $A \mapsto -A$ on $\mathsf{SU}(2)$ corresponds to the antipodal map of the 3-sphere. Topologically the map $\pi : \mathsf{SU}(2) \to \mathsf{SO}(3)$ can therefore be considered as the identification map from S^3 to $\mathbb{R}\mathrm{P}^3$, where $\mathbb{R}\mathrm{P}^3$ is the 3-dimensional real-projective space, which can be thought of as quotient under antipodal identification on the 3-sphere S^3. Alternatively, it may be thought of as obtained from a closed solid 3-ball by identifying antipodal points on its boundary 2-sphere. This is because antipodal identification on S^3 allows us to cut off its, say, lower hemisphere, leaving us with the upper hemisphere—which topologically is an open 3-ball—without any identifications and the equator—which is a 2-sphere—with antipodal identifications. Note that (2.103) can then be thought of as coordinatisation of the 3-sphere by 3-dimensional spherical polar coordinates, where $\psi = \alpha/2 \in [0, \pi]$ is the polar angle and $\vec{n}$ (with $\vec{n} \cdot \vec{n} = 1$) parametrises 2-spheres, which degenerate to points for $\psi = 0$ and $\psi = \pi$ and become largest for the equatorial 2-sphere at $\psi = \pi/2$, i.e. $\alpha = \pi$. In these coordinates the antipodal map is given by $(\psi, \vec{n}) \mapsto (\pi - \alpha, -\vec{n})$ which identifies rotations by an angle α about the axis $\vec{n}$ with rotations about the oppositely orinted axis $-\vec{n}$ by an angle $2\pi - \alpha$, a well known identity in $\mathsf{SO}(3)$. More useful coordinates on $\mathsf{SU}(2)$ as well as on $\mathsf{SO}(3)$ are the *Euler angles*, (ψ, θ, φ), defined by

$$A(\psi, \theta, \varphi) := A(\psi, \vec{e}_3) A(\theta, \vec{e}_2) A(\varphi, \vec{e}_3) \,. \tag{2.104}$$

Using (2.103) we can evaluate the right-hand side:

$$A(\psi, \theta, \varphi) := \begin{pmatrix} e^{-i(\psi+\varphi)/2} \cos(\theta/2) \;, & -e^{-i(\psi-\varphi)/2} \sin(\theta/2) \\ e^{i(\psi-\varphi)/2} \sin(\theta/2) \;, & e^{i(\psi+\varphi)/2} \cos(\theta/2) \end{pmatrix} , \tag{2.105}$$

where ψ and φ are taken with periodic identifications modulo 4π and 2π respectively and $\theta \in [0, \pi]$. The antipodal map $A \mapsto -A$ is now simply given by $(\psi, \theta, \varphi) \mapsto (\psi + 2\pi\,, \theta, \varphi)$ so that the Euler angles also parametrise $\mathsf{SO}(3)$ when ψ is taken with periodic identifications modulo 2π. Explicit expressions for $R(\psi, \theta, \varphi) := \pi(A(\psi, \theta, \varphi))$ can be otained by either using (2.105) in (2.101) or, alternatively, by multiplying $R(\psi, \vec{e}_3) R(\theta, \vec{e}_2) R(\varphi, \vec{e}_3)$ since this is just the right-hand side of (2.104) after application of the homomorphism π.

$\mathsf{SO}(3)$ is the quotient group $\mathsf{SU}(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\mathbf{1}, -\mathbf{1}\}$ is the centre of $\mathsf{SU}(2)$. Note that already topologically $\mathsf{SO}(3)$ is a quotient- but not a subspace of $\mathsf{SU}(2)$: we cannot find a continuous map $i : \mathsf{SO}(3) \to \mathsf{SU}(2)$ such that $\pi \circ i = \mathrm{id}_{\mathsf{SO}(3)}$. The best one can do is to restrict to rotations by angles different from 180 degrees. Then, as discussed above, such rotations make up an open 3-ball which can either be considered as the upper or the lower hemisphere (without the equator) of the 3-sphere that makes up $\mathsf{SU}(2)$.

An explicit formula for the embedding of that subset of $\mathsf{SO}(3)$ into $\mathsf{SU}(2)$ can be obtained as follows. Multiply (2.102) with σ_b (summation over b) and apply the general formula (a proof of which will be given below)

$$\sigma_a M \sigma_a = 2\,\mathrm{trace}(M)\mathbf{1} - M \quad ,\text{(sumation over a=1,2,3)} \tag{2.106}$$

which is valid for any $M \in \mathtt{M}(2,\mathbb{C})$, to $M = A^\dagger$. One obtains

$$\sigma_a R_{ab} \sigma_b = 2\,\mathrm{trace}(A^\dagger) A - \mathbf{1}\,. \tag{2.107}$$

Taking the trace of this equation we get $2\delta_{ab}R_{ab} = 2\mathrm{trace}(R)$ on the left hand side, due to (2.92), and $2((\mathrm{trace}(A))^2 - 1)$ on the right hand side, due to the reality of traces in $\mathsf{SU}(2)$ (cf. (2.99)). This we can solve for trace(A) up to sign:

$$\mathrm{trace}(A) = \mathrm{trace}(A^\dagger) = \pm\sqrt{1 + \mathrm{trace}(R)}\,, \tag{2.108}$$

This can in turn be used to eliminate trace(A) in (2.107) and solve for A in terms of R:

$$A = \pm \frac{\mathbf{1} + \sigma_a R_{ab} \sigma_b}{2\sqrt{1 + \mathrm{trace}(R)}} \tag{2.109}$$

Since $\mathrm{trace}(R) = 1 + 2\cos(\alpha)$ (cf. (2.84), where α is the rotation angle, we see that the right-hand side of (2.109) exits for rotation angles different from 180 degrees. This are our two embeddings $i_\pm : S \rightarrow \mathsf{SU}(2)$ of the open and dense subset $S \subset \mathsf{SO}(3)$ of rotations with rotation angles different from 180 degrees, such that $\pi \circ i_\pm = \mathrm{id}_S$.

Let us finally prove (2.106): The Pauli matrices together with the unit matrix span $\mathtt{M}(2,\mathbb{C})$. Hence we can expand M in terms of them. Since the Pauli matrices are traceless, we have $M = M_b\sigma_b + \frac{1}{2}\mathrm{trace}(M)\mathbf{1}$. By (2.91) different Pauli matrices anticommute and each Pauli matrix squares to the identity matrix. Hence $\sigma_a\sigma_b\sigma_a = -\sigma_b$ and $\sigma_a\mathbf{1}\sigma_a = 3\mathbf{1}$ (summation over a!). This implies (2.106).

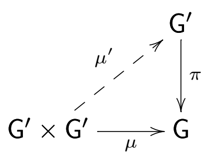
2.4.2 *The general construction*

In this subsection we wish to show that any universal covering space G' of a Lie group G (generally: topological group) can be given a group structure such that the covering map $\pi : \mathsf{G}' \rightarrow G$ is a group homomorphism. For this we shall merely need to make repeated use of the lifting property (A.1) of universal covering spaces explained in A.3.7.

We start by defining a map (we let elements of G' carry a prime)

$$\mu : \mathsf{G}' \times \mathsf{G}' \rightarrow \mathsf{G}, \quad (g', h') \mapsto \pi(g')\pi(h')^{-1}\,, \tag{2.110}$$

and observe that due to $\mathsf{G}' \times \mathsf{G}'$ being simply connected (since G' is, by definition) there exists a lift μ' of μ which makes the following diagram commute:



(2.111)

Moreover, choosing some $e' \in \pi^{-1}(e)$, where $e \in \mathsf{G}$ is the group identity, we can fix μ' uniquely by the specification $\mu'(e', e') = e'$, which makes sense since (2.110) requires $\mu(e', e') = e$.

By means of the map μ' we define the operations of inversion and group multiplication on G' as follows:

$$g'^{-1} := \mu'(e', g')\,, \tag{2.112a}$$

$$g'h' := \mu'(g', h'^{-1})\,, \tag{2.112b}$$

where (2.112a) is used in (2.112b) to define h'^{-1}. Then, using (2.110), it follows that

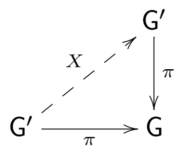
$$\pi(g'^{-1}) = \mu(e', g') = \pi(g')^{-1} \tag{2.113a}$$

$$\pi(g'h') = \mu(g', h'^{-1}) = \pi(g')\pi(h'^{-1})^{-1} = \pi(g')\pi(h')\,, \tag{2.113b}$$

where (2.113a) was already used in the last step of (2.113b). Hence $\pi : \mathsf{G}' \to \mathsf{G}$ is a group homomorphisms, provided (2.112) endows G' with a group structure. This we will now show by checking the group axioms Gr1-Gr3 as listed in A.4.

Gr1 The map $m : \mathsf{G}' \times \mathsf{G}' \to \mathsf{G}'$, $(g', h') \mapsto g'h'$, is defined in (2.112).

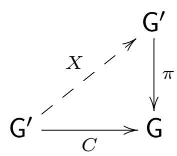
Gr2 We prove that e' is the group identity. Note that (2.112a) and the definition of μ' imply $e'^{-1} = \mu'(e', e') = e'$ and hence with (2.112b) $e'e' = e'$. Now consider the following three maps $\mathsf{G}' \to \mathsf{G}'$: the identity I, the left e'-multiplication $L(g') := e'g'$, and the right e'-multiplication $R(g'e')$. All these maps send e' to e' and satisfy $\pi \circ X = \pi$, where X stands for any of them. For example: $\pi \circ L(g') = \pi(e'g') = \alpha(e', g'^{-1}) = \pi(e')\pi(g'^{-1})^{-1} = \pi(g')$. Hence all three maps make the following diagram commute:



(2.114)

and satisfy $X(e') = e'$. This implies that all three maps coincide, i.e. are the identity map.

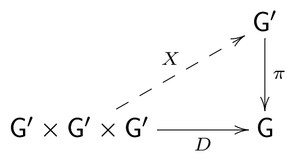
Gr3 We proceed similarly to prove that g'^{-1}, as defined by (2.112a), is indeed the inverse of g'. Consider three maps $\mathsf{G}' \to \mathsf{G}'$, given by: $C'(g') = e'$ (the constant map onto e'), $R(g') = g'g'^{-1}$, and $L(g') = g'^{-1}g'$. All maps send e' to e' and satisfy $\pi \circ X = C$, where $C : \mathsf{G}' \to G$ is the constant map onto e. For example, $\pi \circ L(g') = \pi(g'^{-1}g') = e$ due to (2.113b,(2.113a). Hence all three maps make the following diagram commute



(2.115)

and satisfy $X(e') = e'$. So they coincide, i.e. are all the constant map onto e'.

Gr4 To prove associativity we consider two maps $\mathsf{G}' \times \mathsf{G}' \times \mathsf{G}' \to \mathsf{G}'$ given by $L(g', h', k') = (g'h')k'$ and $L(g', h', k') = g'(h')k'$. Both send (e', e', e') to e'. A third map $D : \mathsf{G}' \times \mathsf{G}' \times \mathsf{G}' \to \mathsf{G}$ is defined by $D(g', h', k') = \pi(g')\pi(h')\pi(k')$. Equation (2.113b) implies $\pi \circ L = D = \pi \circ R$, so that both maps $X = L, R$ make the following diagram commute



(2.116)

and satisfy $X(e', e', e') = e'$. Hence the maps coincide and associativity holds.

This proves our general assertion. Note that $\pi^{-1}(e) \subset \mathsf{G}'$ is discrete and, being the kernel of the homomorphism π, necessarily a normal subgroup. Hence G' is a downward extension of G (cf. A.4.1) by a discrete group isomorphic to $\pi^{-1}(e)$. Moreover, $\pi^{-1}(e)$ is actually central, i.e. a subgroup within the centre (cf. A.4.1) of G. Let us prove this. We have to show that $h' \in \pi^{-1}(e)$ implies that $g'h' = h'g'$ for all $g' \in \mathsf{G}'$. Now, $\pi(g'h') = \mu(g'h'^{-1}) = \pi(g')\pi(h') = \pi(g')$, since h' lies in the kernel of π; similarly $\pi(h'g') = \pi(g')$. Define the maps $\mathsf{G}' \to \mathsf{G}'$ given by $g' \mapsto R(g') = g'h'$ and $g' \mapsto L(g') = h'g'$. As we have just seen, $\pi \circ X = \pi$ for $X = L$ or $X = R$. Moreover, both maps send e' to h'. Hence both maps make a diagram like (2.114) commute and coincide on e', which implies that these maps coincide, i.e. $h'g' = g'h'$ for all $g' \in \mathsf{G}'$. Repeating the argument for all $h' \in \pi^{-1}(e')$ proves the claim. Thus we learn that the universal cover group G' of G is a central downward extension of G by some discrete though not necessarily finite group.

3

ELEMENTS OF REPRESENTATION THEORY

3.1 Fundamentals

In this chapter we will review some basic facts of representation theory. The notion of a 'representation' applies to quite general algebraic objects, like groups, algebras, Lie algebras etc. Here we shall confine attention to Lie groups and their Lie algebras. For these, the notion of a representations of the object in a vector space V is given by homomorphisms

$$D \,:\, \mathsf{G} \,\rightarrow\, \mathsf{GL}(V)\,, \tag{3.1}$$

$$\rho \,:\, \mathfrak{g} \,\rightarrow\, \mathfrak{gl}(V) \quad , \tag{3.2}$$

where D and ρ are homomorphisms of groups and Lie algebras respectively. By a representation of G or $\mathfrak{g}$ in V we shall always understand the triple (G, D, V) or $(\mathfrak{g}, \rho, V)$ respectively, though often one just calls the maps D or ρ representations if G or $\mathfrak{g}$ and V are implicitly understood. As regards representations of Lie algebras, we shall be mostly interested in the cases where $\rho = \dot{D}$, i.e. where the Lie algebra homomorphisms is just the differential evaluated at the identity of the Lie group homomorphisms, as explained in 2.3.2.

For representation theory it is more appropriate to think of the object to be represented as an abstract one. In contrast, in 2.1.1 we introduced Lie groups as groups of matrices. This means that we defined Lie groups and their associated Lie algebras by one of their representations, the so called *defining representation*. A representation is called *faithful* iff the kernel of the homomorphism is trivial (cf. A.4.2 and A.5.2). Clearly, any faithful representation could have been used as definining representaion, but conventionally one selects one to be given this name, as is more or less apparent from the names of the groups themselves. For example, the Lie group $O(n)$ is defined as group of $n \times n$ orthogonal matrices, which selects an n-dimensional representation as defining one.

Let us now move straightaway to some basic definitions. We shall restrict all statements to Lie groups, the corresponding statements for Lie algebras are obtained by obvious changes. Two representations (G, D, V) and (G, D', V) are *equivalent* iff there exists $f \in \mathsf{GL}(V)$ sucht that

$$D'(g) = f \circ D(g) \circ f^{-1}\,, \quad \forall g \in \mathsf{G}\,. \tag{3.3}$$

Equivalence in this sense defines an equivalence relation on the set of representations of a group.

Let V be a complex vector space with real structure $C : V \to V$ (cf. A.5.3). Then the representation (G, D, V) is called *real with respect to* C iff (recall that $C = C^{-1}$)

$$C \circ D(g) \circ C = D(g)\,, \quad \forall g \in \mathsf{G}\,. \tag{3.4}$$

The matrix elements of each $D(g)$ with respect to a real basis of V are real numbers. One calls a representation (G, D, V) simply *real* iff it is equivalent to a real representation with respect to some C. If f is the eqivalence, so that $f \circ D \circ f^{-1}$ is C-real, then D is real with respect to the new real structure $C' := f^{-1} \circ C \circ f$. This means that we can find a basis with respect to which the matrix elements of all $D(g)$ are real.

Given representations (G, D_1, V_1) and (G, D_2, V_2) of the same group G, one can form the *direct-sum representation* $(\mathsf{G}, D_1 \oplus D_2, V_1 \oplus V_2)$ with $D_1 \oplus D_2(g)v_1 \oplus v_2 := D_1(g)v_1 \oplus D_2(g)v_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Similarly one can form the *tensor-product representation* $(\mathsf{G}, D_1 \otimes D_2, V_1 \otimes V_2)$, with $D_1 \otimes D_2(g)v_1 \otimes v_2 := D_1(g)v_1 \otimes D_2(g)v_2$ plus linear extension. To be distinguished from the tensor-product of two representations of the *same* group is the tensor-product of two representation of two *different* groups. Let (G_1, D_1, V_1) and (G_2, D_2, V_2) two representations, then their tensor product is defined as representation $(\mathsf{G}_1 \times \mathsf{G}_2, D_1 \otimes D_2, V_1 \otimes V_2)$, i.e. as representation of the direct-product group (cf. A.4.3) $G_1 \times \mathsf{G}_2$ with $D_1 \otimes D_2(g_1, g_2)v_1 \otimes v_2 := D_1(g_1)v_1 \otimes D_2(g_2)v_2$ plus linear extension. In particular, this notion should not be confused to the former in the case where G_1 and G_2 are isomorphic to the same group G. Then $D_1 \otimes D_2$ in the former case is the restriction of $D_1 \otimes D_2$ in the latter case to the 'diagonal' subgroup $\{(g, g) \mid g \in \mathsf{G}\}$ of $\mathsf{G} \times \mathsf{G}$, which is isomorphic to G.

Often one is interested in special classes of representations, where the images of the homomorphisms (3.1,3.2) are contained in certain subgroups or subalgebras of $\mathsf{GL}(V)$ and $\mathfrak{gl}(V)$ respectively. For example, if V is a complex vector space with inner product $\langle \cdot \mid \cdot \rangle$, one may require the $D(g)$ to satisfy

$$\langle D(g)v \mid D(g)w \rangle = \langle v \mid w \rangle \tag{3.5}$$

for all $g \in \mathsf{G}$ and all $v, w \in V$. This is equivalent to (with some slight abuse of notation we write $D^\dagger(g)$ for $[D(g)]^\dagger$ and $D^{-1}(g)$ for $[D(g)]^{-1}$)

$$D^\dagger(g) = D(g^{-1}) = D^{-1}(g) \tag{3.6}$$

for all $g \in \mathsf{G}$. Such representations are called *unitary* (and *orthogonal* if V is real). Recall that the antilinear operation $\dagger : \mathrm{End}(V) \to \mathrm{End}(V)$ is defined by the inner product through

$$\langle Tv \mid w \rangle =: \langle v \mid T^\dagger w \rangle\,, \quad \forall v, w \in V\,. \tag{3.7}$$

Hence unitary representations are homomorphisms (3.1) whose image is contained in

$$\mathsf{U}(V) := \{T \in \mathsf{GL}(V) \mid T^\dagger = T^{-1}\}\,, \tag{3.8}$$

the group of unitary (with respect to $\langle\cdot \mid \cdot\rangle$) isomorphisms of V. For the Lie-algebra representation the corresponding statement is that the homomorphism (3.2) has image contained in the Lie subalgebra (cf. 2.52)

$$\mathfrak{u}(V) := \{T \in \mathrm{End}(V) \mid T^\dagger = -T\} \tag{3.9}$$

of anti-hermitean endomorphisms of V. These form obviously a *real* Lie algebra, since real linear combinations as well as the commutator of anti-hermitean endomorphisms are again anti hermitean. Hence, for unitary representations, we write instead of (3.1,3.2)

$$D : \mathsf{G} \quad \to \mathsf{U}(V)\,, \tag{3.10}$$

$$\rho : \mathfrak{g}(V) \to \mathfrak{u}(V)\;. \tag{3.11}$$

3.2 Reducibility, irreducibility, and full reducibility

Let (G, D, V) be a representation; A subspace $V' \subseteq V$ is said to be an *invariant subspace* for this represenation iff $D(g)v \in V'$ for all $v \in V'$ and all $g \in G$. For this one also writes $D(g)V' \subseteq V'$ for all $g \in G$. The representation (G, D, V) is *reducible* iff there exists a non-trivial invariant subspace, i.e. a subspace $V' \subseteq V$ different from $\{0\}$ and V. If there is no such invariant subspace the representation is called *irreducible*. In the reducible case the subspace V' is sometimes said to *reduce* the given representation. Then the homomorphism D uniquely defines a homomorphism $D' : \mathsf{G} \to \mathsf{GL}(V')$ by restricting each $D(g)$ to V'. The triple (G, D', V') defines a *sub-representation* of (G, D, V). If $i : V' \to V$ is the (injective) embedding map, D' is characterised by $i \circ D'(g) = D(g) \circ i$ for all $g \in \mathsf{G}$. One says that the embedding i intertwines D' with D (the general definition of an intertwining map will be given below). If $V' \subset V$ is an invariant subspace, each linear map $D(g)$ also defines a linear map $D''(g)$ on the quotient $V'' = V/V'$ such that $D'' : \mathsf{G} \to \mathsf{GL}(V'')$ is also a homomorphism. The ensuing representation (G, D'', V'') is called a *quotient-representation* of (G, D, V). If $\pi : V \to V/V'$ is the (surjective) projection map, D'' is characterized by $\pi \circ D(g) = D''(g) \circ \pi$ for all $g \in \mathsf{G}$. This means that π intertwines D with D''.

If $\mathsf{H} \subset \mathsf{G}$ is a subgroup and (G, D, V) a representation, the restriction $D|_\mathsf{H}$ of the homomorphism $D : \mathsf{G} \to \mathsf{GL}(V)$ to H defines a representation $(\mathsf{H}, D|_\mathsf{H}, V)$ which usually is simply referred to as a *restriction* or, more precisely, H-*restriction* of (G, D, V). Clearly, irreduciblity of a restriction implies irreducibility of the original representation, whereas a restriction of an irreducible represenation will generally be reducible. The 'smaller' the subgroup $\mathsf{H} \subset \mathsf{G}$ the more likely it is that the H-restriction of an irreducible D becomes reducible.

The representation (G, D, V) is called *fully reducible* iff for any invariant subspace $V' \subset V$ there is a complementary invariant subspace, i.e. another invariant subspace $V'' \subset V$ such that $V = V' \oplus V''$. In this case one also writes $D = D' \oplus D''$, where D' and D'' are the sub-representations of G in V' and V''. Note that the existence of such a V'' is by no means guaranteed. For example, consider the following two-dimensional reducible representation of the additive abelian group of real numbers:

$$\mathbb{R} \ni g \mapsto D(g) := \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} . \tag{3.12}$$

The one-dimensional subspace $V' = \text{span}\{(1,0)^\top\}$ of eigenvectors (eigenvalue 1) for each $D(g)$ is clearly invariant, but there is no complementary invariant subspace since no other eigenvectors exist (the $D(g)$ are not diagonalizable for $g \neq 1$).

If $V' \subset V$ reduces (G, D, V) and if $V'' \subset V$ is complementary to V' but not necessarily invariant, we can write each $D(g)$ into triangular form with respect to the decomposition $V = V' \oplus V''$:

$$D(g) = \begin{pmatrix} D'(g) & K(g) \\ 0 & D''(g) \end{pmatrix} . \tag{3.13}$$

Here D' is the sub-representation in V' and D'' the quotient-representation in V/V', where we identify V/V' with V''. With this identification in mind $K(g)$ should be thought of as a linear map $V/V' \to V'$. Let $g = g_1 g_2$, then the homomorphism property of D implies

$$D'(g_1 g_2) = D'(g_1) \circ D'(g_2) , \tag{3.14}$$

$$D''(g_1 g_2) = D''(g_1) \circ D''(g_2) , \tag{3.15}$$

$$K(g_1 g_2) = D'(g_1) \circ K(g_2) + K(g_1) \circ D''(g_2) . \tag{3.16}$$

Equations (3.14) and (3.15) just state the obvious homomorphism properties for the sub- and quotient-representations D' and D'' respectively. Note that there is no canonical way to identify the quotient space V/V' with a particular subspace V''. For example, given a basis $\{e_1, \cdots, e_n\} = \{e_i\}$ of V', then any completion by $\{e_{n+1}, \cdots, e_m\} = \{e_\mu\}$ to a basis of V defines a complementary subspace $V'' = \text{span}\{e_\mu\}$ isomorphic to V/V'. Any other choice of such a complementary subspace can be realized by a change of basis,

$$e_i \mapsto \tilde{e}_i := e_i , \tag{3.17}$$

$$e_\mu \mapsto \tilde{e}_\mu := e_\mu + X^i_\mu \, e_i , \tag{3.18}$$

such that $V'' \mapsto \tilde{V}'' := \text{span}\{\tilde{e}_\mu\}$. The coefficients X^i_μ should be thought of as matrix elements of a linear map $X : V/V' \to V'$ with respect to the bases $\{e_\mu\}$ of V/V'—here being identified with V''—and $\{e_i\}$ of V', such that $X(e_\mu) = X^i_\mu e_i$. Writing the representation D into triangular form with respect to the new decomposition $V = V' \oplus \tilde{V}''$ changes (3.13) by an equivalence transformation

$$\begin{pmatrix} D'(g) & K(g) \\ 0 & D''(g) \end{pmatrix} \mapsto \begin{pmatrix} \text{id}_{V'} & -X \\ 0 & \text{id}_{V''} \end{pmatrix} \begin{pmatrix} D'(g) & K(g) \\ 0 & D''(g) \end{pmatrix} \begin{pmatrix} \text{id}_{V'} & X \\ 0 & \text{id}_{V''} \end{pmatrix} \tag{3.19}$$

which leaves D' and D'' unchanged and redefines K according to

$$K(g) \mapsto \tilde{K}(g) = K(g) + D'(g) \circ X - X \circ D''(g). \tag{3.20}$$

The representation is fully reducible iff among these equivalences there is one for which $\tilde{K}(g) = 0$ for all $g \in \mathsf{G}$. Hence we have

Lemma 3.1 *A reducible representation* (G, D, V) *with sub-representation* (G, D', V') *and quotient-representation* (G, D'', V'')*, where* $V'' := V/V'$*, is fully reducible iff for a given triangular decomposition (3.13) there exists a linear map* $X : V'' \to V'$ *such that*

$$K(g) = X \circ D''(g) - D'(g) \circ X\,. \tag{3.21}$$

Unitary representations are always fully reducible; for if $V' \subset V$ is an invariant subspace, its orthogonal complement

$$V'_\perp := \{v \in V \mid \langle v \mid v' \rangle = 0 \ \forall v' \in V'\} \tag{3.22}$$

is also invariant, as is easily seen from (3.6) and (3.6). There is a very important application of this fact of which we shall make essential use of later on:

Theorem 3.2 *Representations of compact groups are always fully reducible.*

Proof The idea is to show that, for any given representation (G, D, V) with G compact there always is an inner product with respect to which this representation is unitary. Indeed, let $(\cdot \mid \cdot)$ be any inner product on V, we define a new inner product by simply integrating the old one over all of G:

$$\langle v \mid w \rangle := \int_{\mathsf{G}} (D(g)v \mid D(g)w)\, d\mu(g)\,. \tag{3.23}$$

For finite-dimensional representaions D the integrand is a bounded function on G for any pair of vectors v and w. Hence the compactness assumption implies the existence (i.e. convergence) of this integral. We will show that there exists a measure $d\mu(g)$ on G which is invariant under left multiplications, i.e. under the maps $L_h : \mathsf{G} \to \mathsf{G}$, $g \mapsto g' := L_h(g) := hg$. This will then immediately prove the theorem, since it implies

$$\begin{aligned}
\langle D(h)v \mid D(h)w \rangle &= \int_{\mathsf{G}} (D(hg)v \mid D(hg)w)\, d\mu(g) \\
&= \int_{\mathsf{G}} (D(g')v \mid D(g')w)\, d\mu(L_{h^{-1}}g') \\
&= \int_{\mathsf{G}} (D(g')v \mid D(g')w)\, d\mu(g') = \langle v \mid w \rangle\,.
\end{aligned} \tag{3.24}$$

In order to prove existence of such measure, we will need some elementary concepts form the differential geometry of Lie groups. We first remark that, in this context, a ‘measure’ on G is a function $g \mapsto \omega(g)$, where $\omega(g)$ is a non-vanishing

$n := \dim(\mathsf{G})$ form (cf. A.6.5) over the vector space $T_g(\mathsf{G})$ of tangent vectors at $g \in \mathsf{G}$, which may be represented as tangent vectors to differentiable curves through g. As introduced in (cf. 2.3), the Lie algebra $\mathfrak{g}$ is just $T_e(\mathsf{G})$. Given a differentiable invertible map $\phi : \mathsf{G} \to \mathsf{G}$ (not necessarily a homomorphism), we obtain a map of the set of curves through g to the set of curves through $\phi(g)$, just by taking the composition of the curve with ϕ, which, upon differentiation at g, gives an isomorphism $\phi_*|_g : T_g(\mathsf{G}) \to T_{\phi(g)}(\mathsf{G})$. The dual map is $\phi^*|_g : T^*_{\phi(g)}(\mathsf{G}) \to T^*_e(\mathsf{G})$, where $T^*_g(\mathsf{G})$ is the vector space dual to $T_g(\mathsf{G})$. Setting $\phi = L_{g^{-1}}$ we obtain an isomorphism $L^*_{g^{-1}} : T^*_e(\mathsf{G}) \to T^*_g(\mathsf{G})$, which extends naturally to the n-fold exterior products of the cotangent spaces. Hence, given any n-form ω on $T_e(\mathsf{G})$, we obtain an n-form $\omega(g) := L^*_{g^{-1}}|_g d\mu(e)$ on each $T_g(\mathsf{G})$ which defines a function $g \mapsto \omega(g)$ that is, by definition, left invariant in the following sense: $L^*_{h^{-1}}\omega(g) = \omega(hg)$. Going back to the measure-theoretic language, the left translation by $h \in \mathsf{G}$ of $d\mu$ is just $d\mu \to d\mu \circ L_{h^{-1}}$, which in terms of forms translates to $\omega \to L^*_{h^{-1}}\omega$ by the 'change-of-variables-formula'.[10] □

So we have seen that there exist Lie groups, like the compact ones, for which *all* representations are fully reducible. In the sequel we shall therefore simply call a Lie group or a Lie algebra *fully reducible* iff all their representastions are fully reducible.

3.3 Schur's Lemma and some of its consequences

Let (G, D_1, V_1) and (G, D_2, V_2) be two representations of G and $f : V_1 \to V_2$ a linear map. f is called an *intertwining map* or simply an *intertwiner* (of the first with the second representation) iff for all $g \in \mathsf{G}$

$$f \circ D_1(g) = D_2(g) \circ f\,. \tag{3.26}$$

The set of all intertwiners of D_1 with D_2 obviously forms a linear subspace of $\mathrm{Lin}(V_1, V_2)$, which we denote by $\mathrm{Lin}_{\mathsf{G}}(V_1, V_2)$ (this notation does not refer to D_1 and D_2 which are assumed given). If $V = V_1 = V_2$ the set of intertwiners is a subalgebra of $\mathrm{End}(V)$ (cf. A.5.2), denoted by $\mathrm{End}_{\mathsf{G}}(V)$. Note that an intertwiner f is an equivalence iff it is an isomorphism (cf. (3.3).

For better readability we now state and prove the main result in form of three lemmas:

Lemma 3.3 *Let f be an intertwiner between (G, D_1, V_1) and (G, D_2, V_2). Then 1) $\mathrm{Ker}(f) \subseteq V_1$ and 2) $\mathrm{Im}(f) \subseteq V_2$ are invariant subspaces.*

[10] For example, in local coordinates $\{x^\mu\}$, let the measure $d\mu$ be given by the n-form $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$, for some positive real-valued function f. Then

$$\begin{aligned} \phi^*\omega &= (f \circ \phi)\, d(x^1 \circ \phi) \wedge \cdots \wedge d(x^n \circ \phi) \\ &= (f \circ \phi) \left(\tfrac{\partial(x^1 \circ \phi \circ x^{-1})}{\partial x^i} dx^i\right) \wedge \cdots \wedge \left(\tfrac{\partial(x^n \circ \phi \circ x^{-1})}{\partial x^k} dx^k\right) = \tfrac{f \circ \phi}{f}\ \det(\phi_*)\ \omega\,, \end{aligned} \tag{3.25}$$

where ϕ_* here stands for the Jacobi-Matrix of the map ϕ with respect to $\{x^\mu\}$. In the special case where f is constant, i.e. $d\mu$ is the translation-invariant measure for the coordinates $\{x^\mu\}$ (their 'Lebesgue measure'), (3.25) reduces to the standard formula for integration in $\mathbb{R}^n$.

Proof If $v \in \mathrm{Ker}(f)$ then, for all $g \in \mathsf{G}$, $f(D_1(g)v) = D_2(g)f(v) = 0$ and hence $D_1(g)v \in \mathrm{Ker}(f)$, which proves the first statement. If $w = f(v) \in \mathrm{Im}(f)$ then, for all $g \in \mathsf{G}$, $D_2(g)w = D_2(g)f(v) = f(D_1(g)v) \in \mathrm{Im}(f)$, which proves the second statement. □

Lemma 3.4 *Let f be an intertwiner between (G, D_1, V_1) and (G, D_2, V_2). Then 1) D_1 irreducible implies either f is injective or $f \equiv 0$ and 2) D_2 irreducible implies either f is surjective or $f \equiv 0$.*

Proof By the previous lemma, D_1 irreducible implies hat $\mathrm{Ker}(f)$ is either V_1 or $\{0\}$ (zero vector in V_1), which proves the first statement. Likewise D_2 irreducible implies that $\mathrm{Im}(f)$ is either V_2 or $\{0\}$ (zero vector in V_2), which proves the second statement. □

Lemma 3.5. (Schur) *Let f be an intertwiner between irreducible representations (G, D_1, V_1) and (G, D_2, V_2). Part 1: f is either an isomorphism or $f \equiv 0$. Part 2: If $V_1 = V_2 = V$ and $D_1 = D_2 = D$ and if $f \not\equiv 0$ has a non-zero eigenvector—which is always true if V is complex—then $f = a\,\mathrm{id}_V$ for some $a \in \mathbb{F} - \{0\}$.*

Proof Part 1 is a direct consequence of the previous Lemma. To prove Part 2 we observe that an eigenspace of f is invariant under D, since $f(v) = av$ for $a \in \mathbb{F}$ implies $f(D(g)v) = D(g)f(v) = aD(g)v$. Irreducibility of f implies the eigenspace to be all of V and hence $f = a\,\mathrm{id}_V$. □

Two immediate and important implications of Schur's Lemma for the case $\mathbb{F} = \mathbb{C}$ are the following:

Proposition 3.6 *Intertwiners for irreducible representations on complex vector spaces are uniquely determined up to a complex multiple.*

Proof Let (G, D_1, V_1) and (G, D_2, V_2) be irreducible representations and $f_{1,2} : V_1 \to V_2$ two intertwiners, so that for all $g \in \mathsf{G}$

$$f_1 \circ D_1(g) \;=\; D_2(g) \circ f_1 \,, \tag{3.27}$$

$$f_2 \circ D_1(g) \;=\; D_2(g) \circ f_2 \,. \tag{3.28}$$

If at least one of the maps $f_{1,2}$ is identically zero the claim is trivially true, so assume $f_{1,2} \not\equiv 0$. Then Lemma 3.4 and the irreducibility of $D_{1,2}$ imply that $f_{1,2}$ are isomorphisms and hence invertible. Let $f := f_2^{-1} \circ f_1$ then (3.27,3.28) imply

$$D_1(g) = f \circ D_1(g) \circ f^{-1} \,. \tag{3.29}$$

Since we work over the complex numbers f has at least one eigenvalue. Schur's Lemma then implies $f = a\,\mathrm{id}_{V_1}$, i.e. $f_1 = a\, f_2$ for $a \in \mathbb{C} - \{0\}$. □

Proposition 3.7 *Let (G, D, V) be an irreducible representation of an abelian group G on the complex vector space V, then dim$(V) = 1$.*

Proof Let $h \in \mathsf{G}$ be some element different from the identity. Then, since $hg = gh$, $D(h) \circ D(g) = D(g) \circ D(h)$ so that $f := D(h)$, being non-zero, has a non-zero eigenvector. Part 2 of Schur's Lemma implies $f = D(h) = a(h)\,\mathrm{id}_V$ where $a : \mathsf{G} \to \mathbb{C} - \{0\}$ is some function. This implies that any subspace of V is invariant, which is compatible with the assumed irreducibility iff V is one dimensional. □

We stress that this result is not true for real vector spaces. For example, the defining representation of $\mathsf{SO}(2)$ is given by

$$[\varphi] \mapsto \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}, \tag{3.30}$$

where $[\varphi] \in \mathbb{R}/2\pi\mathbb{Z} = [0, 2\pi)$ labels the rotation angle which faithfully parametrises the group. This representation is clearly irreducible and two-dimensional.

3.4 Decomposing fully reducible representations into irreducibles

The significance of *full* reducibility lies in the fact that such representations are 'almost' uniquely determined by their irreducible subrepresentations, in a sense made precise below. This means that the representation theory of fully reducible Lie groups or algebras is essentially exhausted by the classification of their irreducible representations (up to equivalence). We have already seen that compact Lie groups are fully reducible. But, in fact, any semi-simple Lie algebra and corresponding Lie group is fully reducible. In particular, this holds for the Lorentz group, as we will see later on.

For the rest of this section we restrict attention to *finite*-dimensional representations over *complex* vector spaces. Hence by 'representation' we shall always mean 'finite dimensional representation' and by 'vector space' always 'complex vector space'!

We begin with an almost trivial statement, that for easier reference we formulate as

Lemma 3.8 *Let* (G, D, V) *and* (G, D', V') *be representations.*

1) *There exists an injective intertwiner* $f : V \to V'$ *of* D *with* D' *iff* D' *contains a sub-representation equivalent to* D.
2) *There exists an surjective intertwiner* $f : V \to V'$ *of* D *with* D' *iff* D *has a quotient-representation equivalent to* D'.

Proof Part 1: Let $f : V \to V'$ be an injective intertwiner and $\tilde{V} := \mathrm{Im}(f) \subset V'$. Then f is an equivalence of (G, D, V) with $(\mathsf{G}, \tilde{D}, \tilde{V})$, where $\tilde{D}$ is the sub-representation (cf. Lemma 3.3). Conversely, let $(\mathsf{G}, \tilde{D}, \tilde{V})$ be a sub-representation of (G, D', V') equivalent to (G, D, V) with equivalence map $f' : V \to \tilde{V}$. Let $i : \tilde{V} \to V'$ be the embedding (intertwining $\tilde{D}$ with D'), then $f := i \circ f' : V \to V'$ is an injective intertwiner of D with D'. Part 2: Let $f : V \to V'$ be surjective intertwiner and $\tilde{V} := V/\mathrm{Ker}(f)$. Then f is equivalence between $(G, \tilde{D}, \tilde{V})$ and (G, D', V'), where $\tilde{D}$ is the quotient-representation (cf. Lemma 3.3). Conversely, let $(\mathsf{G},\tilde{D},\tilde{V})$ be a quotient-representation of (G,D,V) equivalent to (G,D',V') with equivalence map $f':\tilde{V}\rightarrow V'$. Let $\pi:V\rightarrow\tilde{V}$ be the projection (intertwining D with $\tilde{D}$), then $f:=f'\circ\pi:V\rightarrow V'$ is a surjective intertwiner of D with D'. □

Let us now consider the situation where a representation (G,D,V) decomposes as direct sum of subrepresentations (G,D_j,V_j), where $j\in J=\{1,\cdots,n\}$ for some $n\in\mathbb{N}$, such that

$$V=\bigoplus_{j\in J}V_j\,. \tag{3.31}$$

This decompostion defines n projection maps $\pi_j:V\rightarrow V_j$, where π_j projects parallel to the complement $\oplus_{J\ni k\neq j}V_k$ of V_j and intertwines D with D_j, i.e. $\pi_j\circ D(g)=D_j(g)\circ\pi_j$ for all $a\in\mathsf{G}$. Conversely, the embedding map ('inclusion') $i_j:V_j\rightarrow V$ intertwines D_j with D, i.e. $i_j\circ D_j(g)=D(g)\circ i_j$ for all $g\in\mathsf{G}$. The maps $P_j:=i_j\circ\pi_j:V\rightarrow V$ commute with all $D(g)$, $P_j\circ D(g)=D(g)\circ P_j$ for all $j\in J$ and $g\in\mathsf{G}$, and satisfy the usual 'projector identities' (which would not make sense if written in terms of π_j instead of P_j due to mismatches of ranges and domains):

$$P_j\circ P_k\;=\;\delta_{jk}\,P_k\quad(\text{no }k\text{-summation})\,, \tag{3.32}$$

$$\bigoplus_{j\in J}P_j\;=\;\mathrm{id}_V\,. \tag{3.33}$$

Now, let (G,D',V') be an irreducible subrepresentation of D, which we geometrically characterise by an inclusion map $i':V'\rightarrow V$ intertwining D' with D. Then the map

$$\pi'_j:=\pi_j\circ i'\;:\;V'\rightarrow V_j \tag{3.34}$$

intertwines D' with D_j and is either zero (i.e. the constant map onto the zero vector) or injective, due to the first part of Lemma 3.4. Let $J'\subseteq J$ be the set of indices whose assigned maps are not zero. J' is clearly not empty, as follows e.g. from (3.33), and Part 1 of Lemma 3.8 implies that each D_j for $j\in J'$ contains a subrepresentation equivalent to D'. Geometrically speaking this means the following: if J' has just one element, say j', then $V'\subseteq V_{j'}$ and $V'\cap V_j=\{0\}$ for all $j\in J-\{j'\}$. If, however, J' has more than one element then $V'\subset\oplus_{j\in J'}V_j$ and $V'\cap V_j=\{0\}$ for all $j\in J$, i.e. V' lies 'skew' to *all* V_j.

Let us now specialize to the case where all subrepresentations D_j, $j\in J$, are irreducible. This does not imply a loss of generality if D is fully reducible, because any fully reducible representation can clearly be written as direct sum of irreducibles. The second part of Lemma 3.4 then implies that the maps π'_j (cf. (3.34) are isomorphisms for $j\in J'$ and hence equivalences. In particular, all D_j for $j\in J'$ are mutually equivalent. In the extreme case where all D_j for $j\in J$ are pairwise inequivalent there is therefore a unique $j'\in J$ such that $\mathrm{Im}(\pi'_{j'})=V_{j'}$ and $\mathrm{Im}(\pi'_j)=\{0\}$ for $J\ni j\neq j'$. Hence, in this case, where D' is equivalent to precisely one of the D_j, the image in V of the embedding $i':V'\rightarrow V$ is uniquely determined.

Next we turn to the other extreme case, where all D_j for $j \in J$ are mutually equivalent. That is, D is a representation of n mutually equivalent irreducible representations. Such representations are given a special name:

Definition 3.9 *Let (G, D, V) be a representation which is a direct sum of mutually equivalent representations (G, D_j, V_j) where $j \in \{1, \cdots, n\}$. If (G, D', V') is equivalent to each (G, D_j, V_j) then (G, D, V) is called an isotypic representation of type (G, D', V') and multiplicity n.*

In contrast to the case above, for isotypic representations of multiplicity $n > 1$ the image $i'(V') \subset V$ is far from being uniquely determined. In fact, there are many more irreducible sub-representations of D equivalent to the D_j then just the D_j themselves. To arrive at a classification, let $f'_j : V' \to V_j$ be the equivalence maps so that $f'_j \circ D'(g) = D_j(g) \circ f'_j$ for all $g \in \mathsf{G}$. We wish to consider V' as a *sub*-representation of D. In order to do this we need to specify an embedding $i' : V' \to V$ which intertwines D' with D, i.e. $i' \circ D'(g) = D(g) \circ i'$ for all $g \in \mathsf{G}$. Having done this, we can use i' to construct the maps π'_j as in (3.34) which are also intertwiners between D' and D_j. Since these representations are assumed irreducible, Proposition 3.6 implies

$$\pi_j \circ i' = a_j \, f_j \quad (\text{no } j\text{-summation}) \tag{3.35}$$

for some $a_j \in \mathbb{C}$, not all of which are zero. Composing the maps on both sides to the left with i_j and summing over j, using (3.33), gives

$$i' = \sum_{j \in J} a_j \, (i_j \circ f_j) \, . \tag{3.36}$$

Note that two n-tuples $(a_1, \cdots, a_n)$ and $(\tilde{a}_1, \cdots, \tilde{a}_n)$ define the same subspace $i'(V') \subset V$ iff they are proportional, i.e. iff $a_j = \lambda \tilde{a}_j$ for $\lambda \in \mathbb{C} - \{0\}$. Proportionality in this sense defines an equivalence relation on $\mathbb{C}^n - \{0\}$, the set of non-zero complex n-tuples. The set of equivalence classes is known as $\mathbb{C}\mathrm{P}^{n-1}$, the complex projective space of (complex) dimension $n-1$.[11] Conversely, given an embedding defined through (3.36) for some non-zero n-tuple $(a_1, \cdots, a_n)$, we obtain an equivalence between D' and D_k for each $k \in J$ for which $a_k \neq 0$. Indeed, composing both sides of (3.36) with π_k and using $\pi_k \circ i_j = \delta_{kj} \mathrm{id}_{V_j}$ gives $\pi'_k := \pi_k \circ i' = a_k f_k$ (no k-summation). Hence we have shown that the set of subspaces in V whose sub-representation is equivalent to all D_j is in bijective correspondence to $\mathbb{C}\mathrm{P}^{n-1}$.

Up to equivalence, isotypic representations can be put into a convenient form by using an isomorphism between the n-fold direct sum $\oplus_{j \in J} V_j$ and the tensor product $\mathbb{C}^n \otimes V'$, where V' is isomorphic to each V_j. Let, as before, $f_j : V' \to V_j$

[11] $\mathbb{C}\mathrm{P}^n$ is a compact complex manifold of dimension n. As real manifold it has dimension $2n$. For example, one may show quite easily that $\mathbb{C}\mathrm{P}^1$ is homeomorphic to the 2-sphere, which is a two-dimensional real and one-dimensional complex manifold.

be a collection of n isomorphisms and let $\{e_1, \cdots, e_n\}$ denote the standard basis of $\mathbb{C}^n$. Then we define an isomorphism $I : \mathbb{C}^n \otimes V' \to \oplus_{j\in J} V_j = V$ by

$$I(e_j \otimes v) := (i_j \circ f_j)(v) \tag{3.37}$$

plus linear extension. The embedding (3.36) is then just given by the image $I(V'(a))$, where $a = \sum_j a_j e_j \in \mathbb{C}^n$ and $V'(a) := \{a \otimes v \mid v \in V'\} \subset \mathbb{C}^n \otimes V'$ (linear subspace). Hence, using I, we may identify V with $\mathbb{C}^n \otimes V'$ such that any irreducible subspace of V is identical to some $V'(a)$ for $a \in \mathbb{C}^n - \{0\}$ and where $V'(a) = V'(b)$ iff $b = \lambda a$ for some $\lambda \in \mathbb{C} - \{0\}$. This we summarise as

Lemma 3.10 *Let (G, D, V) be an isotypic representation of type (G, D', V') and multiplicity n. Then D is equivalent to $(\mathsf{G}, \mathrm{id}_{\mathbb{C}^n} \otimes D', \mathbb{C}^n \otimes V')$*

We can now precisely state the uniqueness properties of decompositions of fully reducible representations. Let us first say it in words: A fully reducible representation decomposes uniquely, up to equivalence, into isotypic representations of mutually inequivalent types. For each specification of an equivalence class of an irreducible sub-representation which occurs with multiplicity n, there is a $\mathbb{C}\mathrm{P}^{n-1}$ worth of different subspaces with sub-representations in that equivalence class. Stated more formally we have

Theorem 3.11 *Let (G, D, V) be a fully reducible representation. Then there is a unique set $\{(\mathsf{G}, D_k, V_k) \mid k = 1, \cdots n\}$ of mutually inequivalent irreducible representations and a unique set $\{m(k) \in \mathbb{N} \mid k = 1, \cdots n\}$ of multiplicities such that the following equivalence ($\cong$) holds:*

$$(\mathsf{G}, D, V) \cong \left(\mathsf{G}\ ,\ \bigoplus_{k=1}^{n} \mathrm{id}_{\mathbb{C}^{m(k)}} \otimes D_k\ ,\ \bigoplus_{k=1}^{n} \mathbb{C}^{m(k)} \otimes V_k \right). \tag{3.38}$$

Irreducible sub-representations in the equivalence class of (G, D_k, V_k) are precisely given by the sub-representations on $V_k(a) := \{a \otimes v \mid v \in V_k\}$ for some $a \in \mathbb{C}^{m(k)} - \{0\}$.

As an application, let us state and prove a useful result concerning irreducible representations of direct-product groups:

Proposition 3.12 *Let $G = \mathsf{G}' \times \mathsf{G}''$, then a representation of (G, D, V) is irreducible iff it is equivalent to the tensor product of irreducible representations (G', D', V') and (G'', D'', V''). The equivalence class of (G, D, V) determines the equivalence classes of (G', D', V') and (G'', D'', V''), and vice versa.*

Proof To save notation we identify G' and G'' with the subgroups $\{(g', e'') \mid g' \in \mathsf{G}'\} \subset \mathsf{G}' \times \mathsf{G}''$ and $\{(e', g'') \mid g'' \in \mathsf{G}''\} \subset \mathsf{G}' \times \mathsf{G}''$ of G respectively, where e' and e'' are the neutral elements of G' and G'' respectively. Let (G, D, V) be irreducible. We consider the restricted representation (G'', D'', V), where $D'' := D|_{\mathsf{G}''}$. This will generally now be reducible. Let $(\mathsf{G}'', D''_0, V_0)$ be an irreducible subrepresentation. Then, for any $g'_1 \in \mathsf{G}'$, the subspace $V_1 := D(g'_1)V_0$ of V will also carry an irreducible representation $(\mathsf{G}'', D''_1, V_1)$. This is because the maps $D(g')$ and

$D(g'')$ commute for all $g' \in \mathsf{G}'$ and all $g'' \in \mathsf{G}''$, so that $D(g'_1) : V_0 \to V_1$ defines an equivalence between $(\mathsf{G}'', D''_0, V_0)$ and $(\mathsf{G}'', D''_1, V_1)$, i.e. $D(g'_1) \circ D''_0(g'') = D''_1(g'') \circ D(g'_1)$ for all $g'' \in \mathsf{G}''$, where $D(g'_1)$ is clearly invertible. Since the intersection $V_0 \cap V_1$ then also reduces G'', and since both representations are already irreducible, we must have either $V_0 = V_1$ or $V_0 \cap V_1 = \{0\}$. We assume we have chosen g'_1 such that the latter holds. Note that such a g'_1 certainly exists, since otherwise $V_0 \subset V$ would be G invariant, contrary to the assumption that (G, D, V) is irreducible. Now we can continue in this fashion, picking $g'_2, g'_3, \cdots$ and defining at each stage $V_i := D(g'_i)V_0$, so that $V_i \cap \oplus_{j<i} V_j = \{0\}$ for all $i \leq n$. The process terminates after a finite number, n, of steps where $\oplus_{i=1}^n V_i = V$. This shows that (G'', D'', V) is an isotypic representation of type $(\mathsf{G}'', D''_0, V_0)$ and multiplicity n (cf. Definition 3.9). Theorem 3.11 now implies the equivalence $(\mathsf{G}'', D'', V) \cong (\mathsf{G}'', \mathrm{id}_{\mathbb{C}^n} \otimes D''_0, \mathbb{C}^n \otimes V_0)$. Now consider the restricted representation (G', D', V), where $D' := D|_{\mathsf{G}'}$. Since D' and D'' commute, Schur's Lemma implies that under *this* equivalence we have $(\mathsf{G}', D', V) \cong (\mathsf{G}', D'_0 \otimes \mathrm{id}_{V_0}, \mathbb{C}^n \otimes V_0)$ for some representation D'_0 on $\mathbb{C}^n$. But the latter must be irreducible, for if there was an G'-invariant subspace $W \subset \mathbb{C}^n$ then $W \otimes V_0 \subset V$ would be G-invariant, in violation of the irreducibility of D. Hence, to sum up, we have shown that there are irreducible representations (G', D', V') with $V' \cong \mathbb{C}^n$ and (G'', D'', V'') with $V'' \cong V_0$ such that

$$(\mathsf{G}' \times \mathsf{G}'' \, , \, D \, , \, V) \cong (\mathsf{G}' \times \mathsf{G}'' \, , \, D' \otimes D'' \, , \, V' \otimes V'') \, . \tag{3.39}$$

Coversely, given two irreducible representations (G', D', V') and (G'', D'', V''), then it is easy to see that their tensor-product is an irreducible representation of $\mathsf{G} = \mathsf{G}' \times \mathsf{G}''$. □

Proposition 3.13 *Let* $\mathsf{G} = \mathsf{G}' \times \mathsf{G}''$*, then* G *is fully reducible if* G' *and* G'' *are fully reducible*

Proof We adopt the notation from the proof above. Let (G, D, V) be a reducible representation and (G', D', V), (G'', D'', V) their restrictions to G' and G'' reprectively. Since D'' is fully reducible, it can be written in the form (3.38):

$$(\mathsf{G}'', D'', V) \cong \left(\mathsf{G}'' \, , \, \bigoplus_{k=1}^{n} \mathrm{id}_{\mathbb{C}^{m(k)}} \otimes D''_k \, , \, \bigoplus_{k=1}^{n} \mathbb{C}^{m(k)} \otimes V''_k \right) . \tag{3.40}$$

Since D' and D'' commute, Theorem 3.11 further implies that $D'(g')$ leaves each isotypic component $\mathbb{C}^{m(k)} \otimes V''_k$ invariant. Moreover, since each D''_k is irreducible, $D'(g')$ restricted to each isotypic component is of the form $D'_k(g') \otimes \mathrm{id}_{V''_k}$ for some possibly reducible representation $(\mathsf{G}', D'_k, \mathbb{C}^{m(k)})$. Full reducibility of each D'_k now allows to write it as direct sum of irreducibles. Hence D can be written as direct sum of tensor products of irreducible representations of D' and D'', that is, by Proposition 3.12, as direct sum of irreducible representations of D, which shows fully reducibility of D. □

3.5 Upward extensions by $\mathbb{Z}_2$

Let G be a group and $\mathsf{H} \subset \mathsf{G}$ a subgroup of index two (cf. A.4.1). H is necessarily normal, since left and right cosets each partition G into two sets, one of which is H (coset of the identity), so that the other coset must in each case be the complement $\bar{\mathsf{H}} := \mathsf{G} - \mathsf{H}$. Hence left and right cosets coincide and H is normal. Note that $\bar{\mathsf{H}} \subset \mathsf{G}$ is merely a subset and not a subgroup. If $\bar{h} \in \bar{\mathsf{H}}$ then $\bar{h}^{-1} \in \bar{\mathsf{H}}$ (since $\bar{h}^{-1} \in \mathsf{H}$ would imply $h \in \mathsf{H}$ due to H being a group) and for $h \in \mathsf{H}$ one has $h\bar{h} \in \bar{\mathsf{H}}$ and $\bar{h}h \in \bar{\mathsf{H}}$ (since e.g. $h\bar{h} = h' \in \mathsf{H}$ would imply $\bar{h} = h^{-1}h' \in \mathsf{H}$ due to H being a group). Hence the product of a finite number of elements in G is in H iff the number of elements taken from $\bar{\mathsf{H}}$ is even. These rules merely state the homomorphism property for the projection map $\pi : \mathsf{G} \to \mathsf{G}/\mathsf{H} \cong \mathbb{Z}_2$, given by

$$\pi(g) = \begin{cases} 0 & \text{for } g \in H\,, \\ 1 & \text{for } g \in \bar{H}\,. \end{cases} \tag{3.41}$$

So we see that G is an upward extension of H by $\mathbb{Z}_2$ (cf. A.4.1). In the previous subsection 2.4.1 we have seen that *downward* extensions by $\mathbb{Z}_2$ play a rôle in the transition form certain Lie groups to their universal cover groups, like e.g. in the transition from $\mathsf{SO}(3)$ to $\mathsf{SU}(2)$ or, similarly, in the transition form the Lorentz group $SO(1,3)_0$ (the identity component of $\mathsf{SO}(1,3)$) to $\mathsf{SL}(2,\mathbb{C})$ that we will discuss later on. Now we consider *upward* extensions of $\mathbb{Z}_2$, which also play a rôle in our subsequent discussions, namely when we adjoin new transformations, like e.g. the parity transformation (space reflection).

Then it will be of interest to know whether representations of the new group will still be fully reducible. More precisely we ask: given that all representations of H are fully reducible, is it true that all representations of its upward $\mathbb{Z}_2$-extension, G, are also fully reducible? The affirmative answer is given by the following

Theorem 3.14 *Let G be an upward extension of H by $\mathbb{Z}_2$ and (G, D, V) a reducible representations with invariant subspace $V' \subset V$, so that $(\mathsf{H}, D|_{\mathsf{H}}, V)$ is fully reducible. Then (G, D, V) is also fully reducible.*

Proof Since $D|_{\mathsf{H}}$ is fully reducible we can choose a complement $V'' \subset V$ to V' so that $K(h) = 0$ for all $h \in \mathsf{H}$. The proof will then show that we can always redefine $V'' \mapsto \tilde{V}''$ with associated change $K \mapsto \tilde{K}$ according to (3.20), such that $\tilde{K}(g) = 0$ for all $g \in \mathsf{G}$. We proceed by evaluating (3.16) for several cases:

- Let $\bar{h}_{1,2} \in \bar{\mathsf{H}}$ so that $g = \bar{h}_1\bar{h}_2 \in \mathsf{H}$, then (3.16) and $K(g) = 0$ imply

$$D'(\bar{h}_1^{-1}) \circ K(\bar{h}_1) = -K(\bar{h}_2) \circ D''(\bar{h}_2^{-1})\,. \tag{3.42}$$

  Since left and right hand side depend on different variables this is equivalent to the constancy of each side. In particular we have

$$\tfrac{1}{2}\,K(\bar{h}) \circ D''(\bar{h}^{-1}) = X \in \mathrm{Lin}(V/V', V') \tag{3.43}$$

  for *any* $\bar{h} \in \bar{\mathsf{H}}$, where X is $\bar{h}$-independent.

- Since $gg^{-1} \in \mathsf{H}$ we trivially have $K(gg^{-1}) = 0$ for all $g \in \mathsf{G}$, which in (3.16) leads to
$$K(g^{-1}) = -D'(g^{-1}) \circ K(g) \circ D''(g^{-1})\,. \tag{3.44}$$
- Now we choose $h \in \mathsf{H}$ and $\bar{h} \in \bar{\mathsf{H}}$ so that $\bar{h}h\bar{h}^{-1} \in \mathsf{H}$ and hence $K(\bar{h}h\bar{h}^{-1}) = 0$. Using (3.16) twice to resolve the triple product $\bar{h}h\bar{h}^{-1} \in \mathsf{H}$ and also $K(h) = 0$ leads to
$$K(\bar{h}) \circ D''(h) = D'(\bar{h}h\bar{h}^{-1}) \circ K(\bar{h})\,. \tag{3.45}$$
An equivalent form is obtained by replacing $h \to \bar{h}^{-1}h\bar{h}$:
$$D'(h) \circ K(\bar{h}) = K(\bar{h}) \circ D''(\bar{h}^{-1}h\bar{h}) \tag{3.46}$$

Now we show: there exists an $X \in \mathrm{Lin}(V/V', V')$ such that $\tilde{K}(g) = 0$ for all $g \in \mathsf{G}$, where $\tilde{K}(g)$ is given in terms of $K(g)$ and X by (3.20). In fact, X is just given by (3.43). To verify this, set $X = \frac{1}{2}K(h_*)D''(h_*^{-1})$ for some $\bar{h}_* \in \bar{\mathsf{H}}$ and evalute $\tilde{K}(g)$. First we choose $g = h \in \mathsf{H}$ and obtain with $K(h) = 0$
$$\begin{aligned}\tilde{K}(h) &= \tfrac{1}{2}\big(D'(h) \circ K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}) - K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}) \circ D''(h)\big)\\ &= \tfrac{1}{2}\big(K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}h) - K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}h)\big) = 0\,,\end{aligned} \tag{3.47}$$
where we used (3.46)in the transition from the first to the second line.

Next we choose $g = \bar{h} \in \bar{\mathsf{H}}$:
$$\begin{aligned}\tilde{K}(\bar{h}) &= K(\bar{h}) + \tfrac{1}{2}\Big(D'(\tilde{h}) \circ K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}) - K(\tilde{h}) \circ D''(\bar{h}_*^{-1}\bar{h})\Big)\\ &= K(\bar{h}) - \tfrac{1}{2}\big(K(\bar{h}) + K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}\bar{h})\big)\\ &= \tfrac{1}{2}\big(K(\bar{h}) - K(\bar{h}_*) \circ D''(\bar{h}_*^{-1}\bar{h})\big)\\ &= \tfrac{1}{2}\big(K(\bar{h}) \circ D''(\bar{h}^{-1}) - K(\bar{h}_*) \circ D''(\bar{h}_*^{-1})\big) \circ D''(\bar{h})\,,\\ &= 0\end{aligned} \tag{3.48}$$
where we used (3.42) in the transition from the first to the second and (3.43) in the transition to the last line. □

Having established the fact that G inherits full reducibility from H, we now turn to some results concerning irreducible representations. More precisely, we wish to know to what degree an irreducible representation of G is determined by its H-restriction. As one might expect, this depends on whether the H-restriction stays irreducible or becomes reducible. We deal with the irreducible case first.

Theorem 3.15 *Let G be an upward extension of H by $\mathbb{Z}_2$ and (G, D, V) an irreducible representation whose restriction $(\mathsf{H}, D|_{\mathsf{H}}, V)$ is also irreducible. The vector space V is assumed complex. Then the only other representation $(\mathsf{G}, \tilde{D}, V)$ whose restriction $(\mathsf{H}, \tilde{D}|_{\mathsf{H}}, V)$ is equivalent to $(\mathsf{H}, D|_{\mathsf{H}}, V)$ must be equivalent to*
$$\hat{D}(g) = \begin{cases} D(g) & \text{for } g \in \mathsf{H}\,,\\ -D(g) & \text{for } g \in \bar{\mathsf{H}}\,.\end{cases} \tag{3.49}$$
D and $\hat{D}$ are inequivalent representations.

Proof Let us first assume that D and $\tilde{D}$ be such that $D|_\mathsf{H} = \tilde{D}|_\mathsf{H}$, leaving the case or mere equivalence until the end. Let $h \in \mathsf{H}$ and $\bar{h} \in \bar{\mathsf{H}}$ so hat $\bar{h}h\bar{h}^{-1} \in \mathsf{H}$. Hence, by hypothesis, $D(\bar{h}h\bar{h}^{-1}) = \tilde{D}(\bar{h}h\bar{h}^{-1})$, which is easily seen to be equivalent to the condition that $\tilde{D}(\bar{h}^{-1}) \circ D(\bar{h})$ commutes with $D(h)$ for all $h \in \mathsf{H}$ and all $\bar{h} \in \bar{\mathsf{H}}$. Since $D|_\mathsf{H}$ is irreducible, Schur's Lemma together with $\mathbb{F} = \mathbb{C}$ implies

$$D(\bar{h}) = \lambda(\bar{h})\tilde{D}(\bar{h}) \tag{3.50}$$

for some function $\lambda : \bar{\mathsf{H}} \to \mathbb{C} - \{0\}$. Replacing $\bar{h} \to h\bar{h}$ and using the homomorphism property of D and $\tilde{D}$ together with $D(h) = \tilde{D}(h)$ shows $\lambda(h\bar{h}) = \lambda(\bar{h})$ for all $h \in \mathsf{H}$. Since any element in $\bar{\mathsf{H}}$ can be written in the form $h\bar{h}$ for some h and fixed $\bar{h}$ this implies constancy of λ; let its value be also denoted by λ. Then, writing down (3.50) with $\bar{h}$ replaced by $\bar{h}^{-1}$ and comparison with the inversion of (3.50) finally shows $\lambda = \pm 1$. The value $\lambda = +1$ corresponds to $\tilde{D} = D$ and $\lambda = -1$ to $\tilde{D} = \hat{D}$. Note that $\hat{D}$ is indeed a representation, as follows e.g. immediately from the remark that $\hat{D}$, as defined by (3.49), is just $\hat{D}(g) = (-1)^{\pi(g)}D(g)$, where $\pi : \mathsf{G} \to \mathsf{G}/\mathsf{H} = \mathbb{Z}_2$ is the projection homomorphism. To show the inequivalence of $\hat{D}$ and D by way of contradiction, assume $f \circ \hat{D}(g) \circ f^{-1} = D(g)$. Choosing $g \in \mathsf{H}$ implies that f commutes with the irreducible representation $D|_\mathsf{H}$ so that Schur's Lemma gives $f = a\,\mathrm{id}_V$. Now choosing $g \in \bar{\mathsf{H}}$ yields $D(g) = -D(g)$, a contradiction. Finally we relax the initial assumption of equality to mere equivalence of the H-restrictions, i.e. we only assume $\tilde{D}|_\mathsf{H} = f \circ D|_\mathsf{H} \circ f^{-1}$. Then all agruments given above hold unchanged when $\tilde{D}$ is replaced by $f \circ \tilde{D} \circ f^{-1}$. Hence $\tilde{D}$ turns out to be either equal to $f \circ D \circ f^{-1}$ or to $f \circ \hat{D} \circ f^{-1}$. □

Now we consider the other case, where the restriction to D to H is reducible

Theorem 3.16 *Let* G *be an upward extension of* H *by* $\mathbb{Z}_2$ *and* (G, D, V) *an irreducible representation whose restriction* $(\mathsf{H}, D|_\mathsf{H}, V)$ *is reducible. The vector space* V *is assumed complex. Then* $(\mathsf{H}, D|_\mathsf{H}, V) = (\mathsf{H}, D' \oplus D'', V' \oplus V'')$, *where* (H, D', V') *and* (H, D'', V'') *are inequivalent irreducible representations of equal dimension. The equivalence class of* (G, D, V) *is uniquely determined by the equivalence classes of* (H, D', V') *and* (H, D'', V'').

Proof *Part A: Showing* $D|_\mathsf{H} = D' \oplus D''$.
Let $V' \subset V$ irreducible subspace for $D|_\mathsf{H}$. Let $V'' := D(\bar{h}_*)V'$ for some $\bar{h}_* \in \bar{\mathsf{H}}$. V'' is independent of the choice of $\bar{h}_*$, since any other $\bar{h} \in \bar{\mathsf{H}}$ can be written in the form $\bar{h} = \bar{h}_* h$ for some (uniquely determined) $h \in \mathsf{H}$. (Hence we may also use $\bar{h}_*^{-1}$ instead of $\bar{h}_*$, showing that $V' = D(\bar{h}_*)V''$. This we shall use in Part B.) V'' is also $D|_\mathsf{H}$-invariant, since $D(\mathsf{H})V'' = D(\mathsf{H})D(\bar{h})V' = D(\bar{h})D(\bar{h}^{-1}\mathsf{H}\bar{h})V' = D(\bar{h})V' = V''$, using normality of H. Hence $V' \cap V''$ is also $D|_\mathsf{H}$-invariant. Irreducibility of $D|_\mathsf{H}$ then implies $V' \cap V'' = \{0\}$, the only other possibility $V' \cap V'' = V'$, i.e. $V'' \subseteq V'$, being excluded since it would imply that V' reduced G, contrary to our initial assumption. But now $V' \oplus V''$ reduces G and irreducibility of D hence implies $V' \oplus V'' = V$. This shows $(\mathsf{H}, D|_\mathsf{H}, V) = (\mathsf{H}, D' \oplus D'', V' \oplus V'')$.

Part B: Showing inequivalence of D' and D''.
Suppose, by way of contradiction, that $f : V' \to V''$ is an equivalence of D' and D''. Then $D|_{\mathsf{H}}$ is an isotypic representation of type D' and multiplicity 2 and all subspaces $V(a) := \{a_1 v + a_2 f(v) \mid v \in V'\}$ give mutually equivalent irreducible representations of H for any non-vanishing tuple $a := (a_1, a_2) \in \mathbb{C}^2$ (cf. the discussion following equation 3.36). We show that then there exists a tuple (a_1, a_2) such that $V(a_1, a_2)$ in fact reduces G, contrary to the assumed irreducibility of D. To do this, we consider the maps

$$U \ : V' \to V'' \,, \quad U \ := D(\bar{h}_*)|_{V'} \tag{3.51}$$

$$W : V'' \to V' \ , \quad W := D(\bar{h}_*)|_{V''} \tag{3.52}$$

together with their inverse maps. In terms of them, the action of $D(\bar{h}_*)$ on $V(a)$ is given by:

$$D(\bar{h}_*)(a_1 v + a_2\, f(v))) = a_1\, U(v) + a_2 W \circ f(v)\,. \tag{3.53}$$

Writing down the intertwining property $f \circ D'(h) = D''(h) \circ f$ with $\bar{h}_*^{-1} h \bar{h}_*$ replacing h gives

$$f \circ (U^{-1} \circ D''(h) \circ U) = (W^{-1} \circ D'(h) \circ W) \circ f\,. \tag{3.54}$$

Using the intertwining property once more to express D'' in terms of D', this may be equivalently rewritten as saying that $W \circ f \circ U^{-1} \circ f$ commutes with $D'(h)$ for all $h \in \mathsf{H}$. Irreducibility of D' and Schur's Lemma then imply

$$U = \lambda\, (f \circ W \circ f) \tag{3.55}$$

for some $\lambda \in \mathbb{C} - \{0\}$. Note that U, W and hence λ depend on the choice on $\bar{h}_*$. Inserting (3.55) into (3.53) gives

$$D(\bar{h}_*)(a_1 v + a_2\, f(v))) = a_2 v' + a_1 \lambda f(v')\,, \tag{3.56}$$

where $v' := W \circ f(v)$. This shows that $V(a)$ is invariant under $D(\bar{h}_*)$ iff (a_1, a_2) are chosen such that $a_2/a_1 = \lambda a_1/a_2$, i.e. iff $(a_2/a_1)^2 = \lambda$, which always possesses two different solutions in the space of non-vanishing tuples (a_1, a_2) modulo proportionality. Hence there are in fact two subspaces $V(a) \subset V$ which are invariant under H and $D(\bar{h}_*)$ and hence under G, which contradicts the irreducibility of (G, D, V).

Part C: Showing that the equivalence class of D is uniquely determined by those of D' and D''.
It suffices to show that (G, D, V) and $(\mathsf{G}, \tilde{D}, V)$ must be equivalent if they both restrict to $(\mathsf{H}, D' \oplus D'', V' \oplus V'')$. From this, equivalence of (G, D, V) and $(\mathsf{G}, \tilde{D}, V)$ then immediately follows, given that their restrictions are $(\mathsf{H}, D' \oplus D'', V' \oplus V'')$ and $(\mathsf{H}, \tilde{D}' \oplus \tilde{D}'', \tilde{V}' \oplus \tilde{V}'')$ respectively, with (H, D', V') and (H, D'', V'')

merely equivalent to $(\mathsf{H}, \tilde{D}', \tilde{V}')$ and $(\mathsf{H}, \tilde{D}'', \tilde{V}'')$ respectively. To prove the first statement, we define analogously to (3.51) and (3.52) the maps

$$\tilde{U} \ : V' \to V'' \,, \quad \tilde{U} \ := \ \tilde{D}(\bar{h}_*)|_{V'} \,, \tag{3.57}$$

$$\tilde{W} : V'' \to V' \ , \quad \tilde{W} \ := \ \tilde{D}(\bar{h}_*)|_{V''} \,. \tag{3.58}$$

By hypothesis, we have for all $h \in \mathsf{H}$

$$D(h)|_{V'} \ = \tilde{D}(h)|_{V'} \ = D'(h) \,, \tag{3.59}$$

$$D(h)|_{V''} \ = \tilde{D}(h)|_{V''} \ = D''(h) \,. \tag{3.60}$$

Replacing h by $\bar{h}_*^{-1} h \bar{h}_*$, these two equations can be written in the form

$$(U \circ \tilde{U}^{-1}) \circ D''(h) \ = D''(h) \circ (U \circ \tilde{U}^{-1}) \,, \tag{3.61}$$

$$(W \circ \tilde{W}^{-1}) \circ D'(h) \ = \ D'(h) \circ (W \circ \tilde{W}^{-1}) \,, \tag{3.62}$$

valid for all $h \in \mathsf{H}$. Irreducibility of D' and D'' and Schur's Lemma imply

$$\tilde{U} = a\, U \,, \quad \tilde{W} = b\, W \,, \tag{3.63}$$

for some $a, b \in \mathbb{C} - \{0\}$. Now, even though the maps $U, W, \tilde{U}, \tilde{W}$ depend on the choice of $\bar{h}_*$ the numbers a, b do not depend on it. To see this, replace $\bar{h}_*$ with $h_* \bar{h}_*$ for some $h_* \in \mathsf{H}$ and define maps U', W' and $\tilde{U}', \tilde{W}'$ as in (3.51,3.52) and (3.57,3.58) repectively by using $h_* \bar{h}_*$ instead of $\bar{h}_*$. We then obtain

$$\tilde{U}' = a'\, U' \,, \quad \tilde{W}' = b'\, W' \,, \tag{3.64}$$

for some $a', b' \in \mathbb{C} - \{0\}$. But we have

$$U' \ := \ D(h_*)|_{V''} \circ U \,, \quad W' \ := \ D(h_*)|_{V'} \circ W \,, \tag{3.65}$$

$$\tilde{U}' \ := \ \tilde{D}(h_*)|_{V''} \circ \tilde{U} \,, \quad \tilde{W}' \ := \ \tilde{D}(h_*)|_{V'} \circ \tilde{W} \,, \tag{3.66}$$

where $D(h_*) = \tilde{D}(h_*)$, since the restrictions of D and $\tilde{D}$ to H coincide by hypothesis. Hence (3.63) and (3.64) imply $a = a'$ and $b = b'$, showing that a, b are universal. In particular, choosing $h_* = \bar{h}_*^{-2}$, so that $U' = W^{-1}$, $W' = U^{-1}$, $\tilde{U}' = \tilde{W}^{-1}$, and $\tilde{W}' = \tilde{U}^{-1}$, this leads to $b = a^{-1}$. But this implies the equivalence of D and $\tilde{D}$ as follows: let

$$f : V' \oplus V'' \to V' \oplus V'' \,, \quad f := a\, \mathrm{id}_{V'} \oplus \mathrm{id}_{V''} \tag{3.67}$$

and $v = (v', v'') \in V' \oplus V''$; then

$$D(\bar{h}_*)v = (Wv'' \,, Uv') \,, \quad \tilde{D}(\bar{h}_*)v = (\tilde{W}v'' \,, \tilde{U}v') \,, \tag{3.68}$$

so that

$$\begin{aligned} f^{-1} \circ D(\bar{h}_*) \circ f &= f^{-1} \circ D(\bar{h}_*)\,(Wv''\,,\, a\,Uv') \\ &= f^{-1}\,(Wv''\,,\, a\,Uv') \\ &= (b\,Wv''\,,\, a\,Uv') \\ &= \left(\tilde{W}v''\,,\, \tilde{U}v'\right) \\ &= \tilde{D}(\bar{h}_*)v\,, \end{aligned} \tag{3.69}$$

where we used $a^{-1} = b$ on the right of the third equality sign. Since (3.69) is valid for all $v \in V' \oplus V''$, we infer $\tilde{D}(\bar{h}_*) = f^{-1} \circ D(\bar{h}_*) \circ f$. But, trivially, we also have $\tilde{D}(h) = f^{-1} \circ D(h) \circ f$ since $D|_{\mathsf{H}} = D' \oplus D''$. Hence we also have $\tilde{D}(h\bar{h}_*) = f^{-1} \circ D(h\bar{h}_*) \circ f$ for all $h \in \mathsf{H}$ or, in other words, $\tilde{D}(g) = f^{-1} \circ D(g) \circ f$ for all $g \in \mathsf{G}$. This finally shows equivalence of D and $\tilde{D}$ and finishes the proof of the theorem. □

3.6 Weyl's unitarian trick

In this section we wish to prove a result due to Hermann Weyl (cf. [18]), known as 'unitarian trick', which establishes bijections between the sets of finite-dimensional complex representations of three related objects: a Lie group, its Lie algebra, and its complexified Lie algebra. Its use will lie in the fact that it establishes tight correspondences between representations of Lie groups which have isomorphic complexified Lie algebras, but may themeselves be very different. This often allows to reduce the representation theory of one class of groups to that of another one, which may have been already developed. This is precisely what happens for the Lorentz group, as we shall see below. All vector spaces will be complex and finite dimensional.

3.6.1 *Correspondences between representations of simply connected Lie groups and their Lie algebras: Part 1*

Lemma 3.17 *Let (G, D_1, V_1) and (G, D_2, V_2) be representations of the Lie group G with associtated representations $(\mathfrak{g}, \dot{D}_1, V_1)$ and $(\mathfrak{g}, \dot{D}_2, V_2)$ of its Lie algebra. Then*

i) every D_1-invariant subspace of V_1 is also $\dot{D}_1$-invariant;

ii) if D_1 is equivalent to D_2 then $\dot{D}_1$ is equivalent to $\dot{D}_2$;

iii) if G is connected, the converse of i) and ii) holds.

Proof We shall employ the exponential map (cf. Sect. 2.3.3).

i) Let $V_1' \subseteq V_1$ be D_1-invariant, then $D_1(\exp(tX))V_1' \subset V_1$ for all $X \in \mathfrak{g}$ and all $t \in \mathbb{R}$. Taking the t-derivative at $t = 0$ gives $\dot{D}_1(X)V_1' \subseteq V_1$ for all $X \in \mathfrak{g}$.

ii) Let $f : V_1 \to V_2$ the equivalence map, i.e. $D_2(g) = f \circ D_1(g) \circ f^{-1}$ for all $g \in \mathsf{G}$. Then $f \circ D_1(\exp(tX)) \circ f^{-1} = D_2(\exp(tX))$ for all $X \in \mathfrak{g}$ and all $t \in \mathbb{R}$. Taking the t-derivative at $t = 0$ gives $f \circ \dot{D}_1(X) \circ f^{-1} = \dot{D}_2(X)$ for all $X \in \mathfrak{g}$.

iii) converse of i): Let $V_1' \subseteq V_1$ be $\dot{D}_1$-invariant, then, using (2.83), $D_1(\exp(X))V_1' = \exp(\dot{D}_1(X))V_1' = \sum_n \frac{1}{n!}[\dot{D}_1(X)]^n V_1' \subseteq V_1$ for all $X \in \mathfrak{g}$ and hence

$$D_1(\exp(X_1)\cdots\exp(X_n))V_1' \subset V_1\,, \tag{3.70}$$

for all $X_1, \cdots, X_n \in \mathfrak{g}$. If G is connected, this and (2.82) imply $D_1(g)V_1' \subset V_1$ for all $g \in \mathsf{G}$.

iii) converse of ii): Let $\dot{D}_1$ and $\dot{D}_2$ be equivalent, i.e. $\dot{D}_2(X) = f \circ \dot{D}_1(X) \circ f^{-1}$ for all $X \in \mathfrak{g}$, then $D_2(\exp(X)) = \exp(\dot{D}_2(X)) = \exp(f \circ \dot{D}_1(X) \circ f^{-1}) = f \circ \exp(\dot{D}_1(X)) \circ f^{-1} = f \circ D_1(\exp(X)) \circ f^{-1}$ for all $X \in \mathfrak{g}$. Hence

$$D_2(\exp(X_1)\cdots\exp(X_n)) = f \circ D_1(\exp(X_1)\cdots\exp(X_n)) \circ f^{-1} \tag{3.71}$$

for all $X_1, \cdots, X_n \in \mathfrak{g}$. If G is connected, this and (2.82) imply $D_2(g) = f \circ D_1(\mathsf{G}) \circ f^{-1}$ for all $g \in \mathsf{G}$. □

As a direct consequence of this lemma we can, without further proof, assert the following

Proposition 3.18 *Let G be a connected Lie group. Then*

i) a representation (G, D, V) is reducible or fully reducible respectively iff this is true for $(\mathfrak{g}, \dot{D}, V)$.

ii) two representations (G, D_1, V_1) and (G, D_2, V_2) are equivalent iff this is true for $(\mathfrak{g}, \dot{D}_1, V_1)$ and $(\mathfrak{g}, \dot{D}_2, V_2)$.

We now introduce the following notation: Let $\mathrm{Rep}(\mathsf{G}, V)$ denote the set of all representations of G in V and similarly $\mathrm{Rep}(\mathfrak{g}, V)$ the set of all representations of $\mathfrak{g}$ in V. Then we have the "dot"-map

$$\Delta\,:\,\mathrm{Rep}(\mathsf{G}, V) \to \mathrm{Rep}(\mathfrak{g}, V)\,,\quad D \mapsto \Delta(D) := \dot{D} \tag{3.72}$$

Proposition 3.19 *The map Δ is i) injective if G is connected and ii) bijective if G is simply connected.*

Proof Assume $\dot{D}_1 = \dot{D}_2$, then $D_1(\exp(X)) = \exp(\dot{D}_1(X)) = \exp(\dot{D}_2(X)) = D_2(\exp(X))$ for all $X \in \mathfrak{g}$. Hence

$$D_1(\exp(X_1)\cdots\exp(X_n)) = D_2(\exp(X_1)\cdots\exp(X_n)) \tag{3.73}$$

for all $X_1, \cdots, X_n \in \mathfrak{g}$. Connectedness and (2.82) now imply $D_1 = D_2$, which proves part i). To prove part ii) we need to show surjectivity. This follows directly from Theorem 2.7. □

Remark Part ii) of Proposition 3.18 implies that for connected G the map Δ projects to a map $\tilde{\Delta}$ between the sets of equivalence classes of representations. That is, if $\pi_{\mathsf{G}} : \mathrm{Rep}(\mathsf{G}, V) \to [\mathrm{Rep}(\mathsf{G}, V)]$ and $\pi_{\mathfrak{g}} : \mathrm{Rep}(\mathfrak{g}, V) \to [\mathrm{Rep}(\mathfrak{g}, V)]$ are the projection maps that assign to each representation its equivalence class, then $\pi_{\mathfrak{g}} \circ \Delta = \tilde{\Delta} \circ \pi_{\mathsf{G}}$. Proposition 3.19 could then be phrased in terms of $\tilde{\Delta}$, saying that $\tilde{\Delta}$ is injectice and bijective if G is simply connected.

3.6.2 *Correspondences between representations of real and complex Lie algebras*

Let L be a real Lie algebra with representation $\rho : L \to \mathfrak{gl}(V)$ in a complex vector space V. We consider the complexification $L^{\mathbb{C}}$ (cf. Sect. 2.2.1.9). Then we obtain a representation $\rho_{\mathbb{C}} : L^{\mathbb{C}} \to \mathfrak{gl}(V)$ by setting

$$\rho_{\mathbb{C}}(z \otimes X) := z\rho(X) \tag{3.74}$$

and $\mathbb{R}$-linear extension; clearly $\rho_{\mathbb{C}}|_L = \rho$. Conversely, let $\rho : L^{\mathbb{C}} \to \mathfrak{gl}(V)$ be a ($\mathbb{C}$-linear) representation, then $(\rho|_L)_{\mathbb{C}} = \rho$, since $(\rho|_L)_{\mathbb{C}}(z \otimes X) = z(\rho|_L)(X) = z\rho(1 \otimes X) = \rho(z \otimes X)$ for all $z \otimes X \in L^{\mathbb{C}}$, implying the claim due to $\mathbb{C}$-linearity of both sides. For the further discussion it will be helpful to introduce a little refinement into our notation:

Definition 3.20 *If L is a real Lie algebra and V a complex vector space, then $Rep_{\mathbb{R}}(L,V)$ denotes the set of all $\mathbb{R}$-linear representations. If L is a complex Lie algebra, then $Rep_{\mathbb{C}}(L,V)$ denotes the set of all $\mathbb{C}$-linear representations.*

We then have the following maps, of which the discussion above shows that they are inverse to each other:

$$\Gamma \;\;:\;\; \mathrm{Rep}_{\mathbb{R}}(L,V) \;\to \mathrm{Rep}_{\mathbb{C}}(L^{\mathbb{C}},V)\,, \quad \rho \mapsto \rho_{\mathbb{C}}\;, \tag{3.75}$$

$$\Gamma^{-1} : \;\; \mathrm{Rep}_{\mathbb{C}}(L^{\mathbb{C}},V) \to \mathrm{Rep}_{\mathbb{R}}(L,V)\;\;, \quad \rho \mapsto \rho|_L\,. \tag{3.76}$$

Indeed, representations related by these maps have the same reducibility and equivalence properties. This is expressed by the following

Lemma 3.21 *Regarding representation $\rho \in Rep_{\mathbb{R}}(L,V)$ and their images $\rho_{\mathbb{C}} \in Rep_{\mathbb{C}}(L^{\mathbb{C}},V)$ under (3.75), the following statements hold:*

- *i) The map (3.75) is a bijection with inverse (3.76).*
- *ii) $V' \subseteq V$ is invariant under ρ iff it is invariant under $\rho_{\mathbb{C}}$*
- *iii) ρ is (fully) reducible iff $\rho_{\mathbb{C}}$ is (fully) reducible.*
- *iv) ρ_1 and ρ_2 are equivalent iff $\rho_{1\mathbb{C}}$ and $\rho_{2\mathbb{C}}$ are equivalent.*

Proof We have already seen that (3.76) is the inverse of (3.75), which also implies bijectivity and hence proves i). The other parts are proven as follows:

- ii) Let $V' \subseteq V$ be a (complex!) (ρ, L)-invariant subspace, then for any $v \in V'$ and all $z \in \mathbb{C}$ and all $X \in L$ we have $\rho_{\mathbb{C}}(z \otimes X)v = z\rho(X)(v) = \rho(X)(zv) \in V'$, so that V' is also $(\rho_{\mathbb{C}}, L^{\mathbb{C}})$-invariant. Conversely, let $V' \subseteq V$ be a $(\rho_{\mathbb{C}}, L^{\mathbb{C}})$-invariant subspace, then it is trivially (ρ, L)-invariant, since for any $v \in V'$ we have $\rho(X)(v) = \rho_{\mathbb{C}}(1 \otimes X)v \in V'$ for all $X \in L$.
- iii) Is a direct consequence of part ii).
- iv) Let ρ_1 and ρ_2 be equivalent; since the vector space V is complex, this means that there exists a $\mathbb{C}$-linear bijection $f : V \to V$, so that $\rho_2(X) = f \circ \rho_1(X) \circ f^{-1}$. Then, for all $z \in \mathbb{C}$ and all $X \in L$ we have $\rho_{2\mathbb{C}}(z \otimes X) = z\rho_2(X) = zf \circ \rho_1(X) \circ f^{-1} = f \circ z\rho_1(X) \circ f^{-1} = f \circ \rho_{1\mathbb{C}}(z \otimes X) \circ f^{-1}$. Complex linearity of both sides implies $\rho_{2\mathbb{C}} = f \circ \rho_{1\mathbb{C}} \circ f^{-1}$. Conversely,

let $\rho_{1\mathbb{C}}$ and $\rho_{2\mathbb{C}}$ be equivalent. Then $\rho_{2\mathbb{C}}(z \otimes X) = f \circ \rho_{1\mathbb{C}}(z \otimes X) \circ f^{-1}$ for all $z \in \mathbb{C}$ and all $X \in L$, so that, in particular, $\rho_2(X) = \rho_{2\mathbb{C}}(1 \otimes X) = f \circ \rho_{1\mathbb{C}}(1 \otimes X) \circ f^{-1} = f \circ \rho_1(X) \circ f^{-1}$ for all $X \in L$. □

This immediately imples

Theorem 3.22 *Let the real Lie algebras L_1 and L_2 be real forms of the complex Lie algebra L, i.e. $L_1^{\mathbb{C}} \cong L \cong L_2^{\mathbb{C}}$. Then, for any finite-dimensional complex vector space V, there is a bijection $\mathfrak{f} : Rep_{\mathbb{R}}(L_1, V) \to Rep_{\mathbb{R}}(L_2, V)$ such that for $\rho_1, \rho_1' \in Rep(L_1, V)$:*

i) $V' \subseteq V$ is ρ_1-invariant iff it is $\rho_2 := \mathfrak{f}(\rho_1)$-invariant,

ii) ρ_1 and ρ_1' are equivalent iff $\rho_2 := \mathfrak{f}(\rho_1)$ and $\rho_2' := \mathfrak{f}(\rho_1')$ are equivalent.

The value of this insight lies in the fact that amongst the various real forms of a given complex Lie algebra there are often some which are 'nice' as far as representation theory is concerned. For semi-simple Lie algebras one such nice property is compactness (cf. Sect. 2.2.1.7) for reasons that will become clear in the next subsection. In this regard the following result is most useful:

Theorem 3.23 *Any complex semisimple Lie algebra has a compact real form.*

For a proof we refer to Theorem 6.3 in [9]. Since complexification preserves semisimplicity (cf. Sect. 2.2.1.9) this implies, that for any semisimple real Lie algebra L_1 we can find a compact semisimple real Lie algebra L_2 such that $L_{1\mathbb{C}} = L_2^{\mathbb{C}}$.

Let us finally remark on the comparison between $\mathrm{Rep}_{\mathbb{C}}(L, V)$ and $\mathrm{Rep}_{\mathbb{R}}(L^{\mathbb{R}}, V)$, where L is now a complex Lie algebra. It is clear that

$$\mathrm{Rep}_{\mathbb{C}}(L, V) = \{\rho \in \mathrm{Rep}_{\mathbb{R}}(L^{\mathbb{R}}, V) \mid \rho \circ J = i\rho\}\,, \tag{3.77}$$

where J is the natural complex structure of $L^{\mathbb{R}}$. This defines an embedding $\iota : \mathrm{Rep}_{\mathbb{C}}(L, V) \to \mathrm{Rep}_{\mathbb{R}}(L^{\mathbb{R}}, V)$, which combines with the bijection $\Gamma : \mathrm{Rep}_{\mathbb{R}}(L^{\mathbb{R}}, V) \to \mathrm{Rep}_{\mathbb{C}}(L^{\mathbb{R}\mathbb{C}}, V)$ to an embedding $\iota' := \Gamma \circ \iota : \mathrm{Rep}_{\mathbb{C}}(L, V) \to \mathrm{Rep}_{\mathbb{C}}(L^{\mathbb{R}\mathbb{C}}, V)$. On the other hand, we also know that $L^{\mathbb{R}\mathbb{C}} = L_+ \oplus L_-$ where $L_\pm \cong L$ are the eigenspaces for the eigenvalues $\pm i$ of the natural complex-structure map J of $L^{\mathbb{R}}$ linearly extended to $L^{\mathbb{R}\mathbb{C}}$; compare (2.40). Identifying $\mathrm{Rep}_{\mathbb{C}}(L, V)$ with its image under ι', equation (3.77) can then be expressed in the form

$$\mathrm{Rep}_{\mathbb{C}}(L, V) = \{\rho \in \mathrm{Rep}_{\mathbb{C}}(L^{\mathbb{R}\mathbb{C}} = L_+ \oplus L_-, V) \mid \ \rho|_{L_-} = 0\}\,, \tag{3.78}$$

that precisely characterises which of the representations in $\mathrm{Rep}_{\mathbb{R}}(L^{\mathbb{R}}, V)$ correspond to representations in $\mathrm{Rep}_{\mathbb{C}}(L, V)$.

3.6.3 *Correspondences between representations of simply connected Lie groups and their Lie algebras: Part 2*

As an common result from Propositions 3.18, 3.19 and Lemma 3.21 let us note, that the following composition of maps,

$$
\begin{array}{ccc}
\mathrm{Rep}(\mathsf{G}, V) & \xrightarrow{\Gamma\circ\Delta} & \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, V) \\
& {\scriptstyle\Delta}\searrow \quad \nearrow{\scriptstyle\Gamma} & \\
& \mathrm{Rep}_{\mathbb{R}}(\mathfrak{g}, V) &
\end{array}
\tag{3.79}
$$

is a bijection for simply connected G which, moreover, respects all reducibility and equivalenc properties. This means that, as far as finite-dimensional complex representations are concerned, the representation theory of a simply connected Lie group is fully determined by that of its complexified Lie algebra. In particular it imples the group version of Theorem 3.22.

Theorem 3.24 *Let G_1 and G_2 be simply-connected Lie groups whose complexified Lie algebras $\mathfrak{g}_1^{\mathbb{C}}$ and ${\mathfrak{g}_2}^{\mathbb{C}}$ are isomorphic. Then, for any finite-dimensional complex vector space V, there is a bijection $F : Rep(\mathsf{G}_1, V) \to Rep(\mathsf{G}_2, V)$ such that for $D_1, D_1' \in Rep(\mathsf{G}_1, V)$*

i) $V' \subseteq V$ is D_1-invariant iff it is $D_2 := F(D_1)$-invariant,

ii) D_1 and D_1' are equivalent iff $D_2 := F(D_1)$ and $D_2' := F(D_1')$ are equivalent.

Note that $\mathfrak{g}_1$ and $\mathfrak{g}'$ are real forms of $\mathfrak{g}^{\mathbb{C}}$. Hence simply-connected Lie groups whose Lie algebras are both real forms of the same complex Lie algebra have the same (finite dimensional) represenation theory. Moreover, Theorem 3.23 and the ensuing discussion tells us that for any semisimple real Lie algebra $\mathfrak{g}_1$ we can find a compact semisimple real Lie algebra $\mathfrak{g}_2$ such that $\mathfrak{g}_1^{\mathbb{C}} = \mathfrak{g}_2^{\mathbb{C}}$. The universal-cover group G_2 with Lie algebra $\mathfrak{g}'$ must then be compact (cf. Sect. 2.2.1.7) and its representations therefore fully reducible, according to Theorem 3.2. Hence we have

Corollary 3.25 *Complex representations of semi-simple Lie algebras and their corresponding Lie groups are fully reducible.*

We may consider the realification $\mathfrak{g}^{\mathbb{C}\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}\mathbb{C}}$ as the Lie algebra of a new Lie group. (As we have repeatedly emphasised, the Lie algebra of a Lie group is a priori to be understood as real.) We make the following extension of Definition 2.4:

Definition 3.26 *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. The unique simply connected Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}\mathbb{R}}$ is called the complex double of G and denoted by $2\mathsf{G}$.*

It is interesting to compare $\mathrm{Rep}(2G, V)$ with $\mathrm{Rep}(G, V)$. For the first we have the string of bijections (preserving reducibilities and equivalences)

$$\mathrm{Rep}(2\mathsf{G}, V) \leftrightarrow \mathrm{Rep}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}\mathbb{R}}, V) \leftrightarrow \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}\mathbb{R}\mathbb{C}}, V) \leftrightarrow \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}_+^{\mathbb{C}} \oplus \mathfrak{g}_-^{\mathbb{C}}, V)\,. \tag{3.80}$$

and for the second, using (3.78):

$$\mathrm{Rep}(\mathsf{G}, V) \leftrightarrow \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}, V) \leftrightarrow \{\rho \in \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}_+^{\mathbb{C}} \oplus \mathfrak{g}_-^{\mathbb{C}}, V) \mid \ \rho|_{\mathfrak{g}_-^{\mathbb{C}}} = 0\}\,. \tag{3.81}$$

3.6.4 *Examples*

As first example we consider the Lie algebra $\mathfrak{so}(p,q)$. Its defining relation is given by equation (2.50). Its complexification is then simply given by allowing the matrices X to be complex:

$$\mathfrak{so}(p,q)^{\mathbb{C}} = \{X \in \mathtt{M}(n,\mathbb{C}) \mid E^{(p,q)}X = -(E^{(p,q)}X)^{\top}\}\,. \tag{3.82}$$

Note that the defining relation is $\mathbb{C}$-linear in X, hence it is the same for $\mathfrak{so}(p,q)$ and $\mathfrak{so}(p,q)^{\mathbb{C}}$. We claim that $\mathfrak{so}(p,q)^{\mathbb{C}}$ is isomorphic to

$$\mathfrak{so}(n,\mathbb{C})_{\mathbb{C}} = \{X \in \mathtt{M}(n,\mathbb{C}) \mid X = -X^{\top}\}\,, \tag{3.83}$$

where $n = p+q$ and where we put the subscript $\mathbb{C}$ on $\mathfrak{so}(n,\mathbb{C})$ to indicate that we regard the a priori real Lie algebra $\mathfrak{so}(n,\mathbb{C})$ as complex (due to its complex structure given by multiplicating the matrices by i). Indeed, an isomorphism $\mathfrak{so}(p,q)^{\mathbb{C}} \to \mathfrak{so}(n,\mathbb{C})$ is given by $X \mapsto X' := AXA^{-1}$, where

$$A := \mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{i,\cdots,i}_{q})\,. \tag{3.84}$$

This obviously defines an isomorphism of vector spaces and X' satisfies the defining relations in (3.83) iff X satisfies the defining relations in (3.82), simply because conjugation of $E^{(p,q)}$ by A results in the identity matrix. Hence for all non-negative integers p,q, where $p+q=n$, we have an isomorphism of complex Lie algebras

$$\mathfrak{so}(p,q)^{\mathbb{C}} \cong \mathfrak{so}(n,\mathbb{C})_{\mathbb{C}}\,. \tag{3.85}$$

Since the right hand side depends only on the sum $n = p+q$, this implies that any two $\mathfrak{so}(p,q)$ for fixed $n = p+q$ are real forms of the same complex Lie algeba and therefore have the same complex finite-dimensional representations according to Theorem 3.22. The compact real form, whose existence is ensured by Theorem 3.23 for $n > 2$ ($\mathfrak{so}(2)$ is not semi-simple), is obtained by setting $p = n, q = 0$ (or $p = 0$ and $q = n$), resulting in $\mathfrak{so}(n)$.

As second example we consider the Lie algebra $\mathfrak{su}(p,q)$, where $n = p+q \geq 2$. This time, the first part of its defining relations (cf. equation (2.53)),

$$E^{(p,q)}X = -(E^{(p,q)}X)^{\dagger}\,, \quad \mathrm{trace}(X=0)\,, \tag{3.86}$$

is not $\mathbb{C}$-linear in X because of the complex conjugation in $\dagger$, and therefore does not survive complexification. Tracelessness remains of course. Those X which still do satisfy (3.86) form the real subalgebra $\mathfrak{su}(p,q) \subset \mathfrak{su}(p,q)^{\mathbb{C}}$. Conjugation by the complex matrix A (3.84) now defines an isomorphism $\mathfrak{su}(p,q)^{\mathbb{C}} \to \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ for $n = p+q$ (again we put the subscript $\mathbb{C}$ on the right hand side since $\mathfrak{sl}(n,\mathbb{C})$ is a priori to be taken as real Lie algebra). Indeed, conjugating those elements which satisfy (3.86) results in all traceless anti-hermitian $n \times n$ matrices, i.e. in the real subalgebra $\mathfrak{su}(n) \subset \mathfrak{sl}(n,\mathbb{C})$, whose span over $\mathbb{C}$ is the space of all complex $n \times n$ traceless matrices, i.e. $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$. The last statement follows e.g. from the identity

$$A \equiv \underbrace{\tfrac{1}{2}(A - A^\dagger)}_{\text{anti-herm.}} + i \underbrace{\tfrac{1}{2i}(A + A^\dagger)}_{\text{anti-herm.}} . \tag{3.87}$$

Hence

$$\mathfrak{su}(p,q)^{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}} . \tag{3.88}$$

Again the right hand side only depends on the sum $n = p + q$ so that all $\mathfrak{su}(p,q)$ are real forms of the same complex Lie algebra $\mathfrak{sl}(n,\mathbb{C})_{\mathbb{C}}$ and hence have the same complex finite-dimensional representations according to Theorem 3.22. Moreover, all Lie algebras $\mathfrak{su}(p,q)$ are semisimple. The compact real form is obtained by setting $p = n, q = 0$ (or $p = 0$ and $q = n$), resulting in $\mathfrak{su}(n)$. The corresponding simply connected Lie group is $\mathsf{SU}(n)$.

Taking the realification of (3.88) shows that $\mathfrak{sl}(n,\mathbb{C})$ is $\mathfrak{su}(p,q)^{\mathbb{C}\mathbb{R}}$, i.e. the complex double of $\mathfrak{su}(p,q)$ (cf. Definition 2.4). Taking the composition of realification and complexification of (3.88) gives, using (2.40),

$$\mathfrak{sl}(n,\mathbb{C})^{\mathbb{C}} \cong \mathfrak{su}(p,q)^{\mathbb{C}\mathbb{R}\mathbb{C}} \cong \mathfrak{su}(n)^{\mathbb{C}\mathbb{R}\mathbb{C}} = \mathfrak{su}(n)^{\mathbb{C}} \oplus \mathfrak{su}(n)^{\mathbb{C}} \tag{3.89}$$

where the two $\mathfrak{su}(n)^{\mathbb{C}}$ on the right hand side considered as subalgebras of $\mathfrak{sl}(n,\mathbb{C})^{\mathbb{C}}$, are complex conjugate to each other (with respect to the natural real structure that $\mathfrak{sl}(n,\mathbb{C})^{\mathbb{C}}$ obtains by being the complexification of something real). In particular, $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{su}(n)^{\mathbb{C}} \oplus \mathfrak{su}(n)$ are both real forms of the same complex Lie algebra.