Mapping-class groups in canonical (quantum) gravity

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Initial value formulation

- The 10 Einstein equations

$$G(g) := Ric(g) - \tfrac{1}{2} g\, S(g) \;=\; \kappa\, T$$

correspond to 6 (under-determined hyperbolic) evolution equations and 4 (under-determined) elliptic constraints for the initial data:

$$\begin{aligned} & \mathrm{div}_g G(g) \equiv 0 \qquad \text{(Bianchi id.)} \\ \Leftrightarrow \quad & \partial_t G^{t\nu} \equiv -\partial_k G^{k\nu} - \Gamma^{\mu}_{\mu\sigma} G^{\sigma\nu} - \Gamma^{\nu}_{\mu\sigma} G^{\mu\sigma} \\ \Rightarrow \quad & G(\perp, \perp) \text{ and } G(\perp, \parallel) \text{ contain no 2nd time derivatives.} \end{aligned}$$

- Initial value problem:
  - Pick 3-d manifold M,
  - Riemannian metric h on M,
  - and symmetric 2nd rank tensor field K, such that

$$\begin{aligned} |K|^2_h - (\mathrm{Trace}_h K)^2 - S(h) \;&=\; -2\kappa\, T(\perp, \perp) \;=\; -2\rho \\ \mathrm{div}_h(K - h\, \mathrm{Trace}_h K) \;&=\; \kappa\, T(\perp, \parallel) \;=\; J \end{aligned}$$

  - Integrate evolution equations and get solution to Einstein's equations: $g(x,t) = -dt^2 + h(x,t)$

No topological obstructions

Observation (D. Witt 1986) *Every closed 3-manifold Σ admits smooth initial data. Every $\Sigma - \{p_1, \cdots, p_n\}$ admits smooth initial data which are asymptotically flat (complete) in each of the n ends.*

- For closed Σ this is a corollary of a theorem of Kazdan & Warner (1975), that for $f \in C^\infty(\Sigma)$ with $f \ngeq 0$ there exists smooth metric h such that $S(h) = f$. Indeed, let $\rho_0 := max_\Sigma(\rho)$, then the following pair (h, K) is immediately seen to satisfy the constraints for some $\rho \geq 0$ and $J = 0$:

$$\begin{aligned} f &:= \rho - (\rho_0 + \varepsilon) < 0 && \text{(some } \varepsilon > 0) \\ h &\text{ such that } \quad S(h) = f && \text{(by K \& W)} \\ K &= h\,\sqrt{(\rho_0 + \varepsilon)/6} \end{aligned}$$

- For $\Sigma - \{p_1, \cdots, p_n\}$ we can achieve exact Schwarzschild data for each end by glueing construction (follows also from Corvino).
- Note that there is a strong topological obstruction to maximal ($\text{Trace}_h K = 0$) data, due to Gromow & Lawson's (1983) theorem, since then h must satisfy $S(h) \geq 0$.

Time symmetric initial data

- Consider (relevant) special case

$$\begin{aligned} T &= 0 && \text{pure gravity} \\ K &= 0 && \text{time-symmetry} \\ h &= \phi^4\,\delta && \text{conformally flat} \end{aligned}$$

  Then, constraints are equivalent to $\Delta_\delta \phi = 0$.

- Consider spherical-inversion map on $M = \mathbb{R}^3 - \{0\}$:

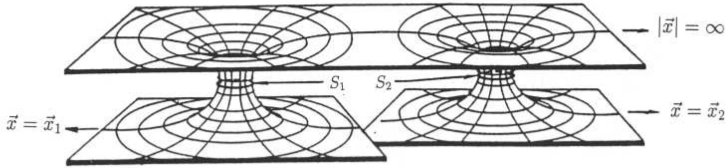
$$\begin{aligned} I_1 : \ (r,\theta,\varphi) &\mapsto \left(\tfrac{a^2}{r}\,,\,\theta\,,\,\varphi\right) \\ I_2 : \ (r,\theta,\varphi) &\mapsto \left(\tfrac{a^2}{r}\,,\,\pi-\theta\,,\,\varphi+\pi\right) \end{aligned}$$

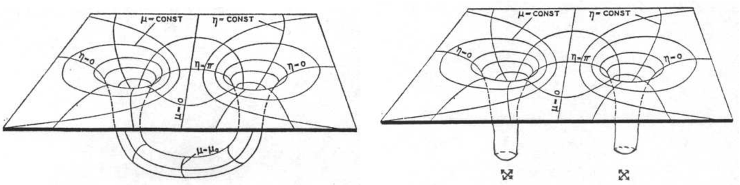
  and on functions $f : M \to \mathbb{R}$, $J_{1,2} : f \mapsto \frac{a}{r}(f \circ I_{1,2})$.
  Then we have

$$\begin{aligned} \Delta_\delta \circ J_{1,2} &= \left(\tfrac{a}{r}\right)^4 J_{1,2} \circ \Delta_\delta \\ I^*_{1,2}(\phi^4\delta) &= \big(J_{1,2}(\phi)\big)^4\,\delta \end{aligned}$$

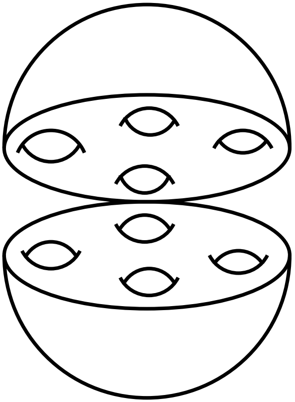
  This can be used for ‘plumbing’ a large variety of locally inversion-symmetric metrics.

Topologies for two BHs





Euclidean Bianchi Models



- Can cut S^4 into two isometric halves along minimal $\Sigma = S^3/D_8^*$ that has twice the volume of equatorial S^3.
- For fixed 3-volume, the (euclidean) action for Σ is $2^{-2/3}$ that for equatorial S^3.
- Construction: View S^4 as symmetric 3x3 matrices A with tr(A)=0 and tr(A^2)=1. Consider orbits of adjoint $SO(3)$ action.

Topological Structure of Configuration Space

- In physics we are interested in asymptotically flat 3-d Riemannian manifolds with one end. Consider one-point (point = ∞) compactification M.
- Relevant 'symmetry' groups are

$$\begin{aligned} D_\infty(M) &= \{\phi \in Diff^+(M) \mid \phi(\infty) = \infty\} \\ D_F(M) &= \{\phi \in D_\infty(M) \mid T\phi|_\infty = id\} \end{aligned}$$

- Here we wish to study some topological structures of $Q := Riem(M)/D_F(M)$. To this end, consider D_F-principal-bundle with contractible total space:

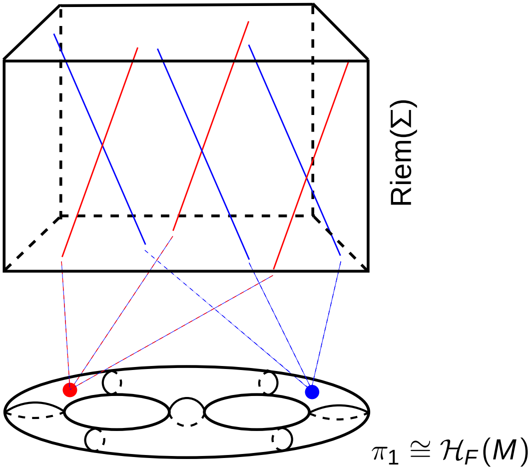
$$D_F(M) \overset{i}{\hookrightarrow} Riem(M) \overset{p}{\twoheadrightarrow} Q$$

$$\begin{aligned} \text{so that} \quad \pi_n(Q) &\cong \pi_{n-1}(D_F(M)) \quad \text{for} \quad n > 1 \\ \text{and} \quad \pi_1(Q) &\cong \pi_0(D_F(M)) := D_F(M)/D_F^0(M)\,. \end{aligned}$$

$\Rightarrow$ In particular, we are interested in the following groups of mapping classes (**homeotopy groups**):

$$\begin{aligned} \mathcal{H}_\infty(M) &:= D_\infty(M)/D_\infty^0(M)\,, \\ \mathcal{H}_F(M) &:= D_F(M)/D_F^0(M)\,. \end{aligned}$$

Superspace Topology



Invariance Properties of $\mathcal{H}$-Groups

- Consider 'lens spaces' $L(p,q) =: S^3/\sim$, where $q<p$ are coprime integers, with $S^3 = \{|z_1|^2+|z_1|^2 = 1 \mid (z_1,z_2)\in\mathbb{C}^2\}$ and $(z_1,z_2)\sim(z_1',z_2') \Leftrightarrow z_1' = \exp(2\pi i/p)z_1$ and $z_2' = \exp(2\pi iq/p)z_2$.
- Have homotopy ($\simeq$) and topological ($\cong$) equivalence properties

$$\begin{aligned} L(p,q) &\simeq L(p,q') \Leftrightarrow qq' = \pm n^2 \pmod p \\ L(p,q) &\cong L(p,q') \Leftrightarrow qq' = \pm 1 \pmod p \text{ or } q' = \pm q \pmod p \end{aligned}$$

- For lens spaces $\mathcal{H}^+,\mathcal{H}_\infty,\mathcal{H}_F$ coincide. For $p>2$ they are

$$\mathcal{H}_F(L(p,q)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } q^2 = 1 \pmod p \text{ and } q\neq\pm1 \pmod p \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

- For example, $L(15,1)\simeq L(15,4)$ and $L(15,1)\ncong L(15,4)$, but

$$\mathcal{H}_F(L(15,1)) = \mathbb{Z}_2 \neq \mathbb{Z}_2\times\mathbb{Z}_2 = \mathcal{H}_F(L(15,4))$$

Fibrations of $D = \mathit{Diff}$ by D_∞ and D_F

$$\begin{array}{ccc} D_\infty(M) & \overset{i}{\hookrightarrow} & D(M) \\ & & \downarrow p \\ & & M \end{array} \qquad \begin{array}{ccc} D_F(M) & \overset{\tilde{i}}{\hookrightarrow} & D(M) \\ & & \downarrow \tilde{p} \\ & & FM \end{array}$$

$$p(\phi) := \phi(\infty) \qquad \tilde{p}(\phi) := T\phi(f_\infty)$$

$$\cdots \longrightarrow \pi_2 M \longrightarrow \pi_1 D_\infty \longrightarrow \pi_1 D \xrightarrow{p_*} \pi_1 M \xrightarrow{\partial_*} \pi_0 D_\infty \xrightarrow{i_*} \pi_0 D \longrightarrow 1$$

$$\cdots \longrightarrow \pi_2 FM \longrightarrow \pi_1 D_F \longrightarrow \pi_1 D \xrightarrow{\tilde{p}_*} \pi_1 FM \xrightarrow{\tilde{\partial}_*} \pi_0 D_F \xrightarrow{\tilde{i}_*} \pi_0 D \longrightarrow 1$$

General Strategy

- Starting point is the map

$$h_\infty : \mathcal{H}_\infty(M) \to Aut(\pi_1 M)$$
$$[\phi] \mapsto ([\gamma] \mapsto [\phi \circ \gamma])\,.$$

- One e.g. has

$$\begin{array}{ccccccccc}
1 & \to & \pi_1 M/Im(p_*) & \xrightarrow{\partial_*} & \mathcal{H}_\infty(M) & \xrightarrow{p_*} & \mathcal{H}(M) & \to & 1 \\
 & & \downarrow q & & \downarrow h_\infty & & \downarrow h & & \\
1 & \to & Inn(\pi_1 M) & \to & Aut(\pi_1 M) & \to & Out(\pi_1 M) & \to & 1
\end{array}$$

  and uses it to gain information about the image and the kernel of h_∞. One has $Im(p_*) \in Z(\pi_1 M)$.

- Very often, an important input is the *HI-property*: 'homotopy $\Rightarrow$ isotopy'. We have the following

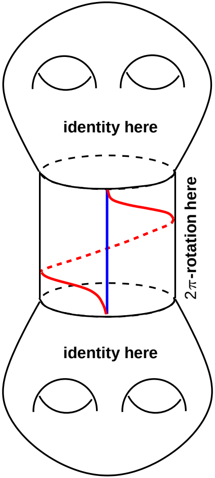
  Theorem: *Let M be prime with HI-property, then h_∞ is injective and q is an isomorphism. If M is Haken, then h_∞ is surjective onto $Aut^+(\pi_1 M)$.*

Spinorial Manifolds

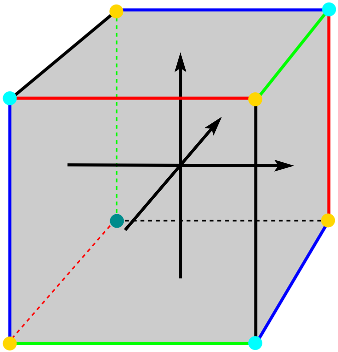
$$\begin{array}{ccc} D_F(M) & \overset{i}{\hookrightarrow} & D_\infty(M) \\ & & \big\downarrow p \\ & & GL_3^+(\mathbb{R}) \end{array}$$

where $p(\phi) := T\phi|_\infty$.

- **Either** $\mathcal{H}_F(M) \cong \mathcal{H}_\infty(M)$
- **or** $\mathcal{H}_F(M)$ is a $\mathbb{Z}_2$-extension of $\mathcal{H}_\infty(M)$. In this case M is called ‘spinorial’.



My Favourite: S^3/D_8^*

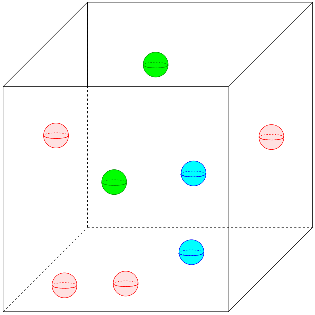


$$D_8^* = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$$

$$\begin{aligned} Inn(D_8^*) &= D_4 \\ Aut(D_8^*) &= O = \mathcal{H}_\infty(S^3/D_8^*) \\ Out(D_8^*) &= P_3 = \mathcal{H}(S^3/D_8^*) \end{aligned}$$

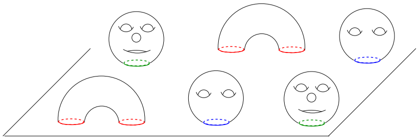
$$\mathcal{H}_F(S^3/D_8^*) = O^*$$

Connected Sums



⇐ intrinsic view

⇓ extrinsic view

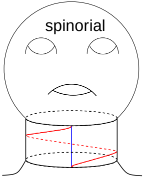


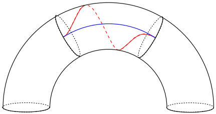
Physical Approach

- Let M be the connected sum of prime manifolds from n distinct oriented homeomorphism classes and multiplicities m_i, $i = 1, \cdots, n$.
- In case of 'inpenetrable' particles, the physical symmetry group would be

$$\prod_{i=1}^{n} G_i^{m_i} \rtimes \prod_{i=1}^{n} S_{m_i}$$

- But $\mathcal{H}_F(M)$ in addition contains 'slides' (see picture on next viewgraph).
- If all primes are of HI-type (and there are no 'fake' spheres), then $ker(h_\infty)$ is generated by 'obvious' Dehn-twists; see below:



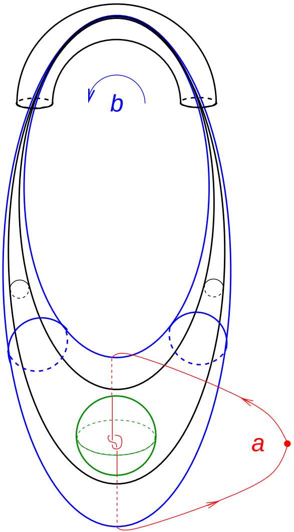


Slides

- Internal diffeos and permutations generate 'particle group'. They respect 'inside' and 'outside'.
- Slides mix inside and outside and generate conjugations $a \mapsto bab^{-1}$.
- See picture: Have $S = id$ within black and outside blue torus. Inbetween have

$$S : (\rho, \theta, \varphi) \mapsto (\rho, \theta, \varphi + \beta(\rho)\, 2\pi)$$

- Finite presentations of $\mathcal{H}_F(M)$ are obtained through those of $Aut(*_i G_i)$ (if the $Aut(G_i)$ are finitely presented).



Some Results

- Let $M = \#_n S^1 \times S^2$, then $\pi_1 M \cong F_n$ and $\mathcal{H}_F(M) \cong Aut(F_n)$. The latter has known presentation with 4 generators. Its quotient with respect to normal closure of slides is $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by (Dehn-) twist and 'flips' (exchanging handle-ends).
- For any homomorphic image B of $\mathcal{H}_F$, the following statements are equivalent:
  1) B is abelian, 2) all slides are contained in the kernel,
  3) exchange = flip, .i.e. there is a 'flip-statistics-correlation'.
- Let $M = \#_n \mathbb{R}P^3$, then $\pi_1 M \cong *_n \mathbb{Z}_2$ with likewise known presentation with 3 generators (see next sheet). Its quotient wrt. normal closure of slides is S_n ('particle group').
- Some generalisations thereof (using Fouxe-Rabinovich):
  - If the prime decomposition of M contains at least three 'handles' ($S^1 \times S^2$), then the normal closure of slides is perfect subgroup.
  - If the prime decomposition of M contains no handle, then $\mathcal{H}_F$ is a semi-direct product of the 'particle group' with the normal closure of slides.

Example: Presentation of $\mathcal{H}_F(\#_n\mathbb{RP}^3)$

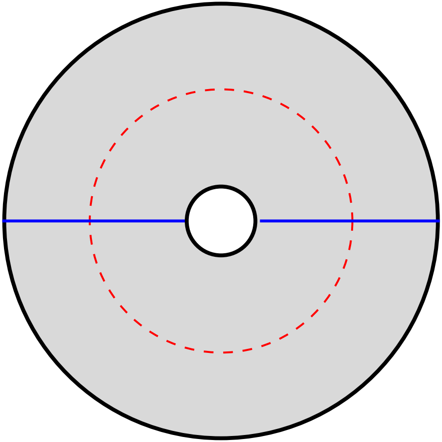
Generators:

P	$: [g_1, g_2, g_3, \cdots, g_N] \mapsto [g_2, g_1, g_3, \cdots, g_N]$		pair-exchange
Q	$: [g_1, g_2, g_3, \cdots, g_N] \mapsto [g_2, g_3, g_4, \cdots, g_N, g_1]$		cyclic-exchange
U	$: [g_1, g_2, g_3, \cdots, g_N] \mapsto [g_2^{-1} g_1 g_2, g_2, g_3, \cdots, g_N]$		slide

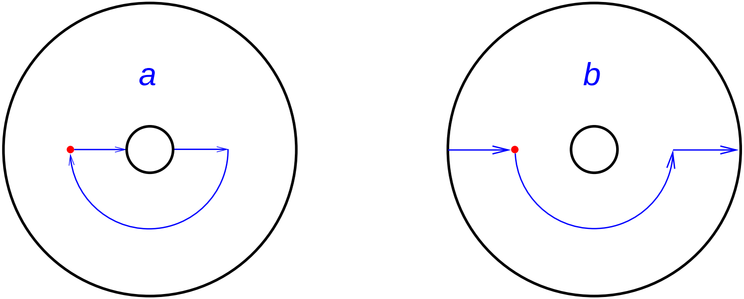
Relations:

1.	$P^2 = U^2$	$= 1$	
2.	$(QP)^{N-1} = Q^N$	$= 1$	
3.	$[P, Q^{-i} P Q^i]$	$= 1$	for $2 \leq i \leq N/2$
4.	$[U, Q^{-2} P Q^2]$	$= 1$	for $N > 3$
5.	$[U, QPQ^{-1}PQ]$	$= 1$	
6.	$[U, Q^{-2} U Q^2]$	$= 1$	
7.	$[U, Q^{-1} PUPQ]$	$= 1$	
8.	$Q^{-1}UQUQ^{-1}UQ = PQ^{-1}UQPUPQ^{-1}UQP$		for $N \geq 3$

The Simplest Case: $\mathbb{RP}^3 \# \mathbb{RP}^3$

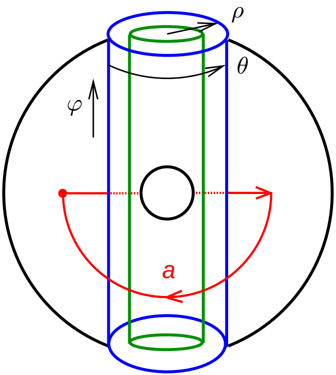


The Fundamental Group



$$\underbrace{\langle a,b \mid a^2 = 1 = b^2\rangle}_{\mathbb{Z}_2 * \mathbb{Z}_2} = \underbrace{\langle a,c \mid a^2 = 1,\ aca^{-1} = c^{-1}\rangle}_{\mathbb{Z}_2 \ltimes \mathbb{Z}}, \quad c := ab$$

Generators of o.p. MCG I.



$(\rho, \theta, \varphi) \sim$
$(\rho\,,\ \theta + \pi\,,\ \varphi + 2\pi)$

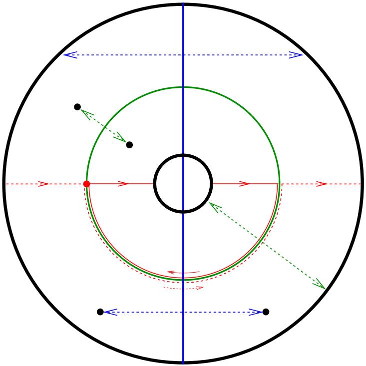
- Let β be smooth step function from 0 (for ρ larger ρ_{blue}) to 1 (for ρ smaller ρ_{green}).
- Define Diff $: (\rho, \theta, \varphi) \mapsto$

$$\big(\rho\,,\ \theta + \pi\beta(\rho)\,,\ \varphi + 2\pi\beta(\rho)\big)$$

- This represents S in $Aut(\mathbb{Z}_2 * \mathbb{Z}_2)$:

$$S : (a, b) \mapsto (bab^{-1}\,,\ b)$$

Generators of o.p. MCG II.



$E_1 :=$ reflection at blue plane

$$\{a, b\} \mapsto \{a^{-1}, b^{-1}\} = \{a, b\}$$

$E_2 :=$ inversion at green sphere

$$\{a, b\} \mapsto \{b^{-1}, a^{-1}\} = \{b, a\}$$

$$\Rightarrow E := E_1 \circ E_2 \in Aut(\mathbb{Z}_2 * \mathbb{Z}_2)$$

$$\{a, b\} \mapsto \{b, a\}$$

Linear Irreps. of o.p. MPG

- The group of mapping classes is given by

$$\begin{aligned} MCG &\cong Aut(\mathbb{Z}_2 * \mathbb{Z}_2) \\ &\cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle E, S \mid E^2, S^2 \rangle \end{aligned}$$

$\Rightarrow$ $ES + SE \subset$ centre of group algebra. Hence $\{1, E, S, ES\}$ generate algebra of irreducible representing operators.

$\Rightarrow$ Linear irreducible representations are at most 2-dimensional. They are are: $E \mapsto \pm 1$, $S \mapsto \pm 1$ and, for $0 < \theta < \pi$,

$$E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

$\Rightarrow$ There are two 'statistics sectors', which get 'mixed' by S; the 'mixing angle' is θ.

Residual Finiteness

- A group G is *RF* iff the complement of any $1 \neq g \in G$ contains a co-finite normal subgroup. The group is then said to be approximated by finite groups. E.g., finite dim. reps. separate the group. Also, for finitely presented groups, *RF* implies a soluble word problem.
- *RF* is preserved under taking subgroups (but not quotients) and free products. It is preserved under taking *Aut* if G is finitely generated.
- If G is finitely generated and contains a *RF* subgroup of finite index, then G is *RF*. Again, this is not true for quotients.
- MCG is extension of subgroup of $Aut(*\pi_1(primes))$. Need to check *type* of extension!

⇒ MCG of many (possibly all) 3-manifolds is *RF*. This is e.g. easy to prove for all non-spinorial ones (connected sums of handles and lens-spaces), since then extension splits (semi-direct product).