Symmetry and Asymmetry

of dynamical laws in physics

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Dynamical Laws

- A dynamical law is a selection principle that characterises a subset
$$\mathbf{Sol} \subset \mathbf{Kin}\,. \tag{1}$$
Here **Kin** denotes the set of kinematically allowed trajectories and **Sol** of that of dynamically allowed "solutions" (J. Anderson 1967).

- As the word *trajectory* suggests, **Kin** is usually considered to be a set of mappings between two spaces X and Y, obeying certain properties P
$$\mathbf{Kin} := \{\mathsf{T} : \mathsf{X} \to \mathsf{Y} \mid \mathsf{T} \text{ satisfies } \mathsf{P}\}\,. \tag{2}$$
X might, e.g., be the affine real line (for time) and Y the affine real $3N$ space of N-particle configurations. Alternatively, X may be 4-dimensional spacetime and Y the total space of field values (i.e. **Kin** is the space of sections in the bundle Y with base X).

- Dynamical laws come in the form of *equations of motion*, which characterise **Sol** as zero-level set of a function (e.g., the gradient of an action)
$$\mathbf{EoM} : \mathbf{Kin} \to \mathsf{Z} \;\Rightarrow\; \mathbf{Sol} := \mathbf{EoM}^{-1}(0)\,. \tag{3}$$
The function **EoM** will generally depend (ex- or implicitly) on other parametric quantities Σ, like countably many numbers (masses, charges), or elements of other functions spaces (like external currents and other fields), which are themselves to be thought of as given (defining **EoM**), rather than to be solved for. Whether the Σ are eventually determined themselves by some equations of motion will not matter at this point.

Symmetries of dynamical laws

- Let G be a group and $\Phi : G \to \mathrm{Aut}(\mathbf{Kin}) \subseteq \mathrm{Bij}(\mathbf{Kin})$ be an *effective* action of G on the set $\mathbf{Kin}$ (i.e. Φ is an injective homomorphism). Then the pair (G, Φ) is called a *symmetry* of $\mathbf{EoM}$ iff $\mathbf{Sol} \subset \mathbf{Kin}$ is an invariant subset. That is,

$$\mathbf{EoM}[\Sigma; \Phi_g(T)] = 0 \;\Leftrightarrow\; \mathbf{EoM}[\Sigma; T] = 0\,. \tag{4}$$

  Note that Σ is *not* acted upon. $T' := \Phi_g(T)$ has to satisfy the *very same* equation of motion, not an *appropriately translated* one.

- Effectivity is imposed w.l.o.g and means that $\Phi_g = \mathrm{id} \Rightarrow g = e$ (injectivity of Φ). But particular Φ_g may have fixed points T in $\mathbf{Sol}$. We call

$$\mathrm{Stab}_{(G,\Phi)}(T) := \{ g \in G \mid \Phi_g(T) = T \} \subseteq G \tag{5}$$

  the symmetry group of the solution T induced by (G, Φ).

- Symmetry operations may or may not be thought of as connecting observationally distinguishable state of affairs, depending on dynamical response in the given setting. Responseless operations (e.g., zeros of momentum map) are considered gauge transformations, which form normal subgroup. The quotient group is then that of proper physical symmetries, e.g., "asymptotic symmetries" in theories with long-ranging fields (allowing for non-zero charges / momenta).

Some issues

- Note that in order for (G, Φ) to be called an a symmetry of the EoM we do not just require it to act on $\mathbf{Sol}$, but that this action extends to $\mathbf{Kin}$.
- Suppose next to an action (G, Φ) on $\mathbf{Kin}$ we also had an action (G, Ψ) on the set $\mathbf{Bac}$ of background structures Σ. Then, by choosing $(\mathbf{Bac}, \Psi)$ appropriately we may always achieve covariance

$$\mathbf{EoM}[\Psi_g(\Sigma); \Phi_g(T)] = 0 \ \Leftrightarrow \ \mathbf{EoM}[\Sigma; T] = 0 \qquad (6)$$

  without proper dynamical symmetries. The latter are given by the stabiliser sugbroup for Σ:

$$Stab_{(G,\Psi)}(\Sigma) := \{g \in G \mid \Psi_g(\Sigma) = \Sigma\} \subseteq G. \qquad (7)$$

- If we now enlarge $\mathbf{Kin}$ to $\mathbf{Kin}* := \mathbf{Bac} \times \mathbf{Kin}$ and extend $\mathbf{EoM}$ from $\mathbf{Kin}$ to $\mathbf{EoM}^*$ on $\mathbf{Kin}^*$ in such a way, that the projection of $\mathbf{Sol}^* \subset \mathbf{Kin}^*$ unto first factor is precisely Σ, then we turned the covariance group of $\mathbf{EoM}$ into the symmetry group of $\mathbf{EoM}^*$.
- This seems always possible, if we do not somehow distinguish “true” equations of motion (allowing for sufficiently many physical degrees of freedom) from fake ones, that just put certain external quantities onto fixed values. ($\rightarrow$ Example)

Example: Two formulations of vacuum electrodynamics in Minkowski space

1. Equation of Motion = vacuum Maxwell equations

$$\mathbf{EoM}[g;F]=0 \;\Leftrightarrow\; \begin{cases} dF=0 & \text{(no metric dependence)}\,, \\ d\star_g F=0 & \text{(metric dependence in } \star)\,. \end{cases} \tag{8}$$

   All diffeos are symmetries of first and covariance of second equation, but only (conformal) isometries of g are also symmetries of second equation.

2. This changes if we regard g as dynamical rather than as background variable and add the appropriate equation of motion, that simply enforces g to be flat.

$$\mathbf{EoM}[\emptyset;(g,F)]=0 \;\Leftrightarrow\; \begin{cases} dF=0\,, \\ d\star_g F=0\,, \\ \mathrm{Riem}(g)=0\,. \end{cases} \tag{9}$$

   Now the set of background structures is empty and all covariances are symmetries. The quotient-set of symmetry-equivalent solutions has not changed, but that does not yet imply physical equivalence to the first formulation. Such equivalence cannot be stated without assumptions on the observational indistinguishability of (certain) symmetry-related configurations.

How should G act on $\mathcal{M}ap(X, Y)$?

- Given G-actions Φ and Ψ on X and Y, respectively. Then G also acts on $X \times Y$ via $\Theta := \Psi \times \Psi$ and hence also on graphs of maps (trajectories) $T : X \mapsto Y$. This action is
$$\Theta_g T := \Psi_g \circ T \circ \Phi_g^{-1} \,. \tag{10}$$
This comprises all examples of, e.g., tensor fields over Minkowski space with G being the Poincaré group.

- There are examples where this is not general enough. The generalisation consists in letting the G-operation Ψ on the fibre Y depend on the base-point in X: For given action Φ on the base X, $\Psi : G \times X \to \mathcal{A}ut(Y)$ is required to satisfy
$$\Psi_{(g \cdot h, x)} = \Psi_{\left(g, \Phi_h(x)\right)} \circ \Psi_{(h,x)} \,. \tag{11}$$
Then
$$(\Theta_g T)(x) := \Psi_{\left(g, \Phi_g^{-1}(x)\right)} \Big(T\big(\Phi_g^{-1}(x)\big) \Big) \,, \tag{12}$$
too, defines an action of G on $\mathcal{M}ap(X, Y)$ generalising (10). This generalisation is needed in GR and also in classical mechanics ($\to$ example).

- Both actions are (ultra-)local in the sense that the Y-value of the Θ-transformed field at the Φ-transformed X-point depends only on the value of the untransformed field at the untransformed point. Many of the standard statements is crucially depend on this locality constraint, which is often made implicitly. ($\to$ examples).

Gal-action on trajectories in CM

- In CM (Classical Mechanics) have $X =$ affine $\mathbb{R}$ (time), $Y =$ affine $\mathbb{R}^{3N}$ (configuration space), and

$$\begin{aligned} G = Gal^{\uparrow}_{+} &\cong \left(\mathbb{R}_b \times \mathbb{R}^3_{\vec{a}}\right) \rtimes \left(\mathbb{R}^3_{\vec{v}} \rtimes SO(3)_D\right) \\ &\cong \left(\mathbb{R}_{\vec{a}} \times \mathbb{R}^3_{\vec{v}}\right) \rtimes \left(\mathbb{R}_b \times SO(3)_D\right). \end{aligned} \tag{13}$$

- The action Φ of G on time and the generalised action Ψ of G on configurations space are given by

$$\Phi_{(b,\vec{a},\vec{v},D)}(t) = t + b\,, \quad \Psi_{[(b,\vec{a},\vec{v},D),t]}(\vec{x}_a) = D\vec{x}_a + \vec{v}t + \vec{a}\,. \tag{14}$$

- Hence, according to (12), action of $Gal^{\uparrow}_{+}$ on $Map(\mathbb{R},\mathbb{R}^{3N})$ is given

$$\Theta_{(b,\vec{a},\vec{v},D)}[\vec{x}_a](t) = D\vec{x}_a(t-b) + \vec{v}(t-b) + \vec{a}\,. \tag{15}$$

  Theorem: $(Gal^{\uparrow}_{+},\Theta)$ is a symmetry of the Newtonian $\mathbf{EoM}$ if forces derive from $Gal^{\uparrow}_{+}$-invariant potential $V : \mathbb{R}^{3N} \to \mathbb{R}$.

- But there are other symmetries. For example, for N particles of masses m_a under mutual Newtonian gravitational attraction, the scaling actions of $G = \mathbb{R}_+$ on $X = \mathbb{R}$ and $Y = \mathbb{R}^{3N}$, given by $\Phi_s(t) = s^{3/2}t$ and $\Psi_s(\vec{x}_a) = s \cdot \vec{x}_a$, make $\mathbb{R}_+$ a symmetry group with action

$$\Theta_s[\vec{x}_a](t) = s \cdot \vec{x}(s^{-3/2}t)\,. \tag{16}$$

Locality example 1:
Symmetries of sourceless Maxwell equations

- Let $\vec{E} = (1/c)\times$(Electric field) and $\dot{X} := (1/c)\partial_t X$, then Maxwell's vacuum equations read

$$\dot{\vec{E}} = \vec{\nabla} \times \vec{B}\,, \qquad \vec{\nabla} \cdot \vec{E} = 0 \tag{17}$$

$$\dot{\vec{B}} = -\vec{\nabla} \times \vec{E}\,, \qquad \vec{\nabla} \cdot \vec{B} = 0\,. \tag{18}$$

These are invariant under Galilei *as well as* Poincaré group. The implementation in first case is, however, non-local: Given

$$t' = t\,, \qquad \vec{x}' = \vec{x} + \vec{v}t \qquad \text{Galilean boost} \tag{19a}$$

$$t' = t + \vec{v}\cdot\vec{x} + O(v^2)\,, \qquad \vec{x}' = \vec{x} + \vec{v}t + O(v^2) \qquad \text{Lorentzian boost} \tag{19b}$$

have respective symmetry-actions on $\vec{E}$ and $\vec{B}$ fields:

$$\begin{aligned} \vec{E}' &= \vec{E} - \vec{v}\times\vec{B} - (\vec{v}\cdot\vec{x})\vec{\nabla}\times\vec{B} &&+ O(v^2) \\ \vec{B}' &= \vec{B} + \vec{v}\times\vec{E} + (\vec{v}\cdot\vec{x})\vec{\nabla}\times\vec{E} &&+ O(v^2)\,, \end{aligned} \tag{20a}$$

$$\begin{aligned} \vec{E}' &= \vec{E} - \vec{v}\times\vec{B} &&+ O(v^2) \\ \vec{B}' &= \vec{B} + \vec{v}\times\vec{E} &&+ O(v^2)\,. \end{aligned} \tag{20b}$$

- Reading symmetry-groups off equations involves assumptions on action.

Locality example 2:
Proper Poincaré irreducibility

- In SR, all equations for free elementary fields are projection conditions onto (almost) irreducible subspaces of the proper orthochronous inhomogeneous Lorentz group $\mathrm{ILor}_{+}^{\uparrow}$, e.g., Weyl-, Dirac-, Maxwell-, Proca-, Rarita-Schwinger-, Dirac-Bargmann-, Pauli-Fierz-, etc., fields.
- From Mackey theory we know that proper-irreducible representations correspond to fields whose Fourier transform has support on positive- or negative-energy mass-shell (with values in irreducible rep. space of little group).
- Since projection operator corresponding to positive or negative energy-support restriction in Fourier space is non-local in space-time (it enlarges space-time support), it cannot be imposed as differential operator (unlike support restriction on union of pos. and neg. shells).
- If operators non-local in space-time (local in Fourier space) are admitted, fields forming proper irreducible representations of $\mathrm{Ilor}_{+}^{\uparrow}$ are possible and no $\mathrm{ILor}_{+}^{\uparrow}$ - invariant dynamics involves negative-mass states (antiparticles).
- In what sense does SR imply the existence of anti-particles?

No symmetry versus no implementability

- Recall that our definition of *symmetry* assumed a group G and an action Φ of G on $\mathbf{Kin}$, such that $\mathbf{Sol} \subset \mathbf{Kin}$ is invariant (as set) under (G, Φ). Acoordingly, stating that G "is no symmetry" might mean that, either

  1. given (G, Φ), $\mathbf{Sol} \subset \mathbf{Kin}$ is not invariant;

  2. given G, no "physically reasonable" action Φ of G on $\mathbf{Kin}$ can be defined.

- As an illustration, consider the question of parity-symmetry of the free equation for the massless neutrino field. It may be either described by a 2-component Weyl spinor ϕ^A satisfying the Weyl equation,

$$\partial_{AA'}\phi^A = 0\,, \tag{21}$$

  of by a four-component Dirac spinor $\psi = (\phi^A, \bar{\chi}_{A'})$ satisfying the Majorana condition $\bar{\chi}_{A'} = \bar{\phi}^{B'}\varepsilon_{B'A'}$ (so as to reduce the number of components to two):

$$\gamma^\mu \partial_\mu \psi := \sqrt{2}\begin{pmatrix} 0 & \partial^{AA'} \\ \partial_{A'A} & 0 \end{pmatrix}\begin{pmatrix} \phi^A \\ \bar{\phi}_{A'} \end{pmatrix} = 0\,. \tag{22}$$

No symmetry versus no implementability

- According to many text-books, the Weyl equation is not parity invariant, whereas the Dirac equation is. Indeed, the Dirac equation is symmetric under (inversion in the spacelike hyperplane perpendicular to timelike n):

$$\rho_n : x^\mu \mapsto -x^\mu + 2n^\mu (n_\nu x^\nu)\,, \tag{23}$$

acting on ψ as

$$\mathcal{P}_D : \psi \mapsto \psi^p := \eta\, n_\mu \gamma^\mu (\psi \circ \rho_n)\,, \tag{24}$$

where η is a complex number of unit modulus, called the *intrinsic parity* of the particular field ψ.

- On the other hand, there simply seems to be no implementation of parity transformations on Weyl spinors. The general proof is this: By definition, parity transformations commute with spatial rotations and turn (upon conjugation) boosts into their inverse. Hence parity is *never* implementable via complex-linear (sic!) transformations in any complex vector space that carries an irreducible representation of the Lorentz group which stays irreducibbe upon restriction to the subgroup of spatial rotations. But that is exactly the case for spinors of exclusively unprimed or primed indices.

No symmetry versus no implementability

- The foregoing sounds as if the Weyl- and Majorana theory (of the Neutrino) had different symmetry properties. Can that be? If so, in what sense?
- There is an obvious bijection between Weyl- and Majorana spinors, mapping solutions of (21) to solutions of (22), and vice versa:

$$\beta : \phi^A \mapsto \begin{pmatrix} \phi^A \\ \bar{\phi}_{A'} \end{pmatrix} . \qquad (25)$$

  Using this map we can pull-back the parity-action on Majorana spinors to Weyl spinors, thus turning into a symmetry of the Weyl equation:

$$\mathcal{P}_W := \beta \circ \mathcal{P}_D \circ \beta^{-1} : \phi^A \mapsto \eta\sqrt{2}\, n^{AA'}(\bar{\phi}_{A'} \circ \rho_n) . \qquad (26)$$

- This does *not* contradict the non-go result shown above, which assumed complex-linearity, whereas (26) is anti-linear. Note that on general Dirac spinors $\mathcal{P}_D$ *is* complex linear, but not on Majorana spinors (which are a priori reals) in the complex structure which would make β complex linear: $I : (\phi^A, \bar{\phi}_{A'}) \mapsto (i\phi^A, -i\bar{\phi}_{A'})$.
- So how do we (or did Pauli) known which implementation is allowed and which is not? The answer is: without considering interactions, we don't!

Conclusion / Outlook

- Notion of "dynamical symmetries" (in the sense used here) depends on choice of $\mathbf{Kin}$ and implementation Φ of G on $\mathbf{Kin}$. Prejudices concerning "physical" and "unphysical" Φ usually enter at this point.
- Locality assumptions referring to space-time seem crucial in all of modern field theory.
- Further refinement of this discussion considers symmetries with dynamical response ("proper physical") and those without ("gauge"), associated with characterisation of "true" degrees of freedom, e.g., via methods of phase-space reduction (constrained Hamiltonian systems, symplectic reduction, Dirac-Bergmann algorithm).
- A properly formulated theory should not leave ambiguous the local degrees of freedom. However, global ambiguities related to disconnected symmetry grous may arise if you follow strategy to reduce "exactly what's generated by first-class constraints".
- Such "gauge transformations" form normal subgroup within the group of all dynamical symmetries. The corresponding quotient group will, if carefully taken, often turn out finite dimensional, but not necessarily connected. The corresponding "asymptotic symmetries" might then be a countably infinite or finite extension of expected groups, e.g. $\mathbb{R}$ instead of $\mathrm{U}(1)$, or $S\mathrm{U}(2)$ instead of $SO(3)$. Associated theoretical phenomena are, e.g., "continuous charge sectors" (Dyon solutions in YM-Higgs theory) or "spin 1/2 from boson fields" (Skyrminos).