Energy-momentum tensors and space-time symmetries
– Laue's theorem in particular –

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Energy-Momentum Tensor: As usual

- Let (M, g) be a spacetime (4-dimensional globally hyperbolic Lorentzian manifold). We consider sections

$$T = \tfrac{1}{2} T^{\mu\nu} e_\mu \vee e_\nu \in \mathcal{S}(TM \vee TM) \tag{1a}$$

for some g-orthonormal frame $\{e_0, e_1, e_2, e_3\}$, with e_0 timelike and all other e_a spacelike.

- The components in time/space decomposition have the following physical interpretation:

$$\{T^{\mu\nu}\} = \begin{pmatrix} W & cG^n \\ S^m/c & M^{mn} \end{pmatrix} \tag{1b}$$

$$\begin{aligned} W &= \text{energy density} \\ S^m &= [\text{energy-current density}]^m \\ G^n &= [\text{momentum density}]^n \\ M^{mn} &= [\text{momentum-current density}]^{mn} \end{aligned} \tag{1c}$$

- We further say (following Laue) that T represents a *complete system* (or T *is complete* for short), if

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1d}$$

Energy-Momentum Tensor: Equivariance

- T depends on sets of fields which we separate into $F \in \mathcal{F}$ and $\tilde{F} \in \tilde{\mathcal{F}}$. Here F collectively denotes the fields the energy-momentum distribution of which is represented by T, and $\tilde{F}$ denotes all other fields on which T depends as well ("background fields"), like external currents, metric, etc. We sometimes write

$$T = T[F, \tilde{F}] \tag{2}$$

- The dependence of T on the fields $(F, \tilde{F})$ is complete in the sense that, for $p \in M$, $T(p)$ is determined by $F(p)$ and $\tilde{F}(p)$. Hence, given that the sets $\mathcal{F}$ and $\tilde{\mathcal{F}}$ of fields are endowed with representations of $D : \mathrm{Diff}(M) \to \mathrm{Aut}(\mathcal{F})$ and $\tilde{D} : \mathrm{Diff}(M) \to \mathrm{Aut}(\tilde{\mathcal{F}})$ of $\mathrm{Diff}(M)$, then

$$T\big[D(\Phi)F, \tilde{D}(\Phi)\tilde{F}\big] = \Phi_*\Big(T[F, \tilde{F}]\Big) \tag{3}$$

  This condition is sometimes called *naturality*, *equivariance*, *covariance*, or simply a *physical principle* (V. Fock 1960). It will turn out important later on.

Energy-Momentum Tensor: Integration

- Can we construct global energy-momentum from integrating its local distribution? If so, how do you characterise the value-space of these global quantities? Does global energy-momentum form a "four vector" of sorts?
- Give proper mathematical meaning to expressions like

$$P^{\mu} := \int_{\Sigma} d^3x T^{\mu 0} \tag{4}$$

- Note that in SR, a boost-transformation $x^{\mu} \mapsto \hat{x}^{\mu}$ with $\beta = v/c$ results in

$$\hat{T}^{00} = \gamma^2 \left[T^{00} + 2\beta T^{0\|} + \beta^2 T^{\|\|}\right] \tag{5a}$$

$$\hat{T}^{0\|} = \gamma^2 \left[(1+\beta^2)T^{0\|} + \beta(T^{00} + T^{\|\|})\right] \tag{5b}$$

$$\hat{T}^{\|\|} = \gamma^2 \left[T^{\|\|} + 2\beta T^{0\|} + \beta^2 T^{00}\right] \tag{5c}$$

$$\hat{T}^{0\perp} = \gamma\left[T^{0\perp} + \beta T^{\|\perp}\right] \tag{5d}$$

$$\hat{T}^{\|\perp} = \gamma\left[T^{\|\perp} + \beta T^{0\perp}\right] \tag{5e}$$

$$\hat{T}^{\perp\perp} = T^{\perp\perp} \tag{5f}$$

- A standard text-book statement is, that "4-vector transformation property" of (4) depends on vanishing of integrals of certain components of T ($\rightarrow$ Laue's theorem). **But that makes no unambiguous mathematical sense!**

Energy-Momentum Tensor: Alternative representation

- Energy-Momentum distribution is measured by a 3-form valued 1-form, that is, by an element in $\mathcal{S}\big(T^*M \otimes \bigwedge^3 T^*M\big)$. Suppressing the dependence on $F, \tilde{F}$ for the moment, we have:

$$\begin{aligned}\mathcal{T} = [T^{\flat\flat}]\star = T_{\mu\nu}\theta^{\mu} \otimes \star\theta^{\mu} \\ = \tfrac{1}{3!}\, T_{\mu\nu}\ \varepsilon^{\nu}{}_{\alpha\beta\gamma}\ \theta^{\mu} \otimes \big(\theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^{\gamma}\big)\end{aligned} \tag{6}$$

- It defines a $C^{\infty}(M)$-linear map

$$\mathcal{T} : \mathcal{S}TM \to \mathcal{S}\textstyle\bigwedge^3 T^*M\,, \quad X \mapsto i_x\mathcal{T} =: \mathcal{T}(X) \tag{7}$$

so that for any compact hypersurface $\Sigma \subset M$ and not necessarily complete T the following pairing makes sense (re-introducing $F, \tilde{F}$)

$$\mathfrak{M}\,[\Sigma, F, \tilde{F}](X) := \int_{\Sigma} \mathcal{T}[F, \tilde{F}](X) \tag{8}$$

- If T is complete and, in addition, X is Killing, then integrand is closed:

$$\nabla_{\mu}T^{\mu\nu} = 0 \quad L_X g = 0 \ \Rightarrow\ d\big(\mathcal{T}[F, \tilde{F}](X)\big) = 0 \tag{9}$$

Isometries

- Let G be a Lie group that acts through isometries on $(M; g)$. Hence there is a homomorphisms

$$\Phi : G \longrightarrow \mathrm{Diff}(M)\,, \quad g \mapsto \Phi_g \in \mathrm{Diff}(M) \tag{10}$$

such that

$$\Phi_e = \mathrm{id}_M \quad \text{and} \quad \Phi_g \circ \Phi_h = \Phi_{gh} \tag{11}$$

- This induces an anti-homomorphisms (see (13a)) of Lie algebras, given by $V : \mathrm{Lie}(G) \longrightarrow \mathcal{S}TM$, $\xi \mapsto V_\xi$, where

$$V_\xi(p) := \frac{d}{ds}\Big|_{s=0} \Phi_{\exp(s\xi)}(p) \tag{12}$$

- V_ξ is called the *fundamental vector field* associated to $\xi \in \mathrm{Lie}(G)$. The linear map $\xi \mapsto V_\xi$ satisfies:

$$\big[V_\xi, V_\eta\big] = -V_{[\xi,\eta]} \tag{13a}$$

$$(\Phi_g)_* V_\xi = V_{\mathrm{Ad}_g(\xi)} \tag{13b}$$

- From (9) get for complete T:

$$d\big(\mathcal{T}[F,\tilde{F}](V_\xi)\big) = 0\,, \quad \forall \xi \in \mathrm{Lie}(G) \tag{14}$$

Momentum map

- Given a 3-dimensional submanifold $\Sigma \subset M$ we have the following three maps, each of which is linear:

$$V \;:\; \mathrm{Lie}(G) \;\to\; \mathcal{S}TM\,, \qquad \xi \mapsto V_\xi \tag{15a}$$

$$\mathcal{T} \;:\; \mathcal{S}TM \;\to\; \mathcal{S}\bigwedge^3 TM\,, \qquad X \mapsto \mathcal{T}(X) \tag{15b}$$

$$\int_\Sigma \;:\; \mathcal{S}\bigwedge^3 T^*M \;\to\; \mathbb{R}\,, \qquad F \mapsto \int_\Sigma F \tag{15c}$$

- Hence, given EMT (not necessarily complete) and hypersurface Σ, the composition of these maps result in a linear map $\mathfrak{M}_\Sigma : \mathrm{Lie}(G) \to \mathbb{R}$, i.e. an element in $\mathrm{Lie}^*(G)$, the dual of the Lie algebra. It is called the *momentum map*:

$$\mathfrak{M}[\Sigma, F, \tilde{F}](\xi) := \int_\Sigma \mathcal{T}[F, \tilde{F}](V_\xi) \tag{16}$$

- If T is complete closedness (14) implies that dependence on Σ is only though its homology class, modulo boundary components in the complement of T's support: Suppose $K \subset M$ has $\partial K = \Sigma \cup \Sigma' \cup Z$ with either $Z = \emptyset$ or $T|_Z \equiv 0$, then, for all $\xi \in \mathrm{Lie}(G)$, have

$$\mathfrak{M}[\Sigma, F, \tilde{F}](\xi) + \mathfrak{M}[\Sigma', F, \tilde{F}](\xi) = 0 \tag{17}$$

- This implies independence of $\mathfrak{M}[\Sigma, F, \tilde{F}]$ on Σ within the class of Cauchy hypersurfaces, and hence *existence* and *uniqueness* of global G-charges.

The habitat of global conserved quantities

- In view of the homomorphism $\phi : G \longrightarrow \mathrm{Diff}(M)$ (left action of G on M by isometries), the condition of equivariance (3) becomes

$$T\left[D(\Phi_g)F, \tilde{D}(\Phi_g)\tilde{F}\right] = (\Phi_g)_*\left(T[F, \tilde{F}]\right) \tag{18}$$

- **Proposition:** For not necessarily complete T have

$$\mathfrak{M}\left[\Phi_g(\Sigma)\,,\, D(\Phi_g)F\,,\, \tilde{D}(\Phi_g)\tilde{F}\right] = Ad_g^*\left(\mathfrak{M}\,[\Sigma, F, \tilde{F}]\right) \tag{19}$$

- **Corollary:** If T is complete and $\Phi_g(\Sigma) \sim \Sigma$ (homologous modulo $supp(T)$) and $g \in Stab_G(\tilde{F}) \subseteq G$, then

$$\mathfrak{M}\left[\Sigma\,,\, D(\Phi_g)F\,,\, \tilde{F}\right] = Ad_g^*\left(\mathfrak{M}\,[\Sigma\,,\, F\,,\, \tilde{F}]\right) \tag{20}$$

  Momentum, as function of the relevant fields alone, lives in $Lie^*(G)$ and transforms under co-adjoint representation of $Stab_G(\tilde{F})$. In particular, if $\tilde{F}$ contains the metric only, then $Stab_G(\tilde{F}) = G$ and $\mathfrak{M}\,[F] \in Lie$ transforms via Ad^* under all of G.
- **Claim:** The concept of "local charges" always refers to a global construction with some splitting $T = \sum_i T_i$. In particular: No G, no momenta!

Co-adjoint representation of the Poincaré group

- Let (V, η) be a real, 4-dimensional vector space with Minkowski metric η. We use η to identify V with V^*. In this way we also identify $\mathrm{Lie}(\mathrm{Poin})$ and $\mathrm{Lie}^*(\mathrm{Poin})$ with the same vactor space:

$$\mathrm{Lie}(\mathrm{Poin}) \cong \mathrm{Lie}^*(\mathrm{Poin}) \cong V \oplus (V \wedge V) \tag{21}$$

- As $\mathrm{Poin} \cong V \rtimes \mathrm{Lor}$, we have

$$(a, A)(b, B) = (a + Ab\,,\, AB) \tag{22}$$

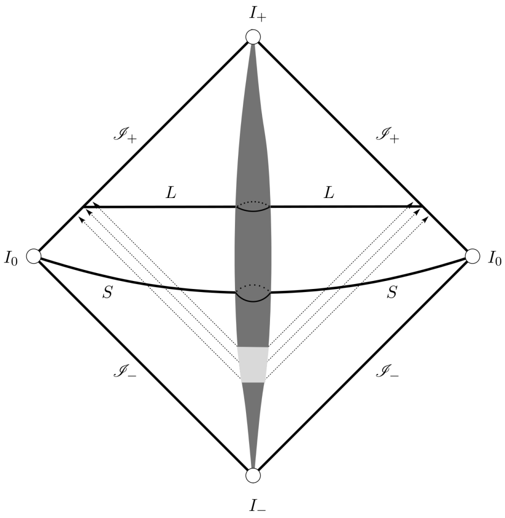
Let $s \mapsto \big(b(s), B(a)\big)$ be a curve in Poin through identity at $s = 0$. Then, with $d/ds|_{s=0}\big(b(s), B(s)\big) = (m, M) \in V \oplus (V \wedge V)$, have

$$\begin{aligned} \mathrm{Ad}_{(a,A)}(m, M) :=& \frac{d}{ds}\bigg|_{s=0} \big(a, A\big)\big(b(s), B(s)\big)\big(a, A\big)^{-1} \\ =& \Big(Am - [(A \otimes A)M]a\,,\, (A \otimes A)M\Big) \end{aligned} \tag{23}$$

- The co-adjoint representation is the transposed-inverse of that:

$$\begin{aligned} \mathrm{Ad}^*_{(a,A)}(m, M) :=& \big(\mathrm{Ad}_{(a,A)^{-1}}\big)^\top (m, M) \\ =& \Big(Am\,,\, (A \otimes A)M - a \wedge Am\Big) \end{aligned} \tag{24}$$

Existence of global conserved charges



Laue's Theorem

7. ***Zur Dynamik der Relativitätstheorie;***
von M. Laue.

Die Dynamik des Massenpunktes hat A. Einstein[1]) schon in seiner ersten grundlegenden Arbeit über das Relativitätsprinzip, kurz darauf auch M. Planck[2]) behandelt. Das wesentlichste Ergebnis ihrer Untersuchung waren die bekannten, seither am Elektron verschiedentlich experimentell bestätigten Formeln für die Abhängigkeit der longitudinalen und der transversalen Masse von der Geschwindigkeit. Als Ausgangspunkt diente die Annahme, daß im Grenzfall unendlich kleiner Geschwindigkeit die Newtonsche Dynamik bestehen bleibt. Später hat Planck[3]) die Theorie nach der thermodynamischen Seite hin erweitert, und hat dabei die mechanische Trägheit völlig auf Energie (und Druck) zurückgeführt. Dabei legte er das Prinzip der kleinsten Wirkung der Betrachtung zugrunde, mußte aber nebenbei noch eine Annahme über die Transformation der Kräfte einführen.

Dennoch gibt es in der Dynamik noch ungelöste Probleme. Z. B. fragt P. Ehrenfest[4]), ob die Dynamik des Massenpunktes auch dann noch für ein Elektron gilt, wenn man diesem nicht — wie üblich — radiale Symmetrie sondern etwa elliptische Gestalt zuschreibt. Einstein[5]) bejaht dies, weil im Grenzfall unendlich kleiner Geschwindigkeit unter allen Umständen die Newtonsche Mechanik gelten müsse. Diese Annahme ist aber in dieser Allgemeinheit sicherlich nicht zutreffend, wie wir später sehen werden. Auch M. Born[6]) glaubt dem Elektron Kugelsymmetrie zuschreiben zu müssen,

1) A. Einstein, Ann. d. Phys. **17.** p. 891. 1905.
2) M. Planck, Verh. d. Deutsch. Physik. Ges. **4.** p. 136. 1906.
3) M. Planck, Berliner Ber. 1907. p. 542; Ann. d. Phys. **26.** p. 1. 1908.
4) P. Ehrenfest, Ann. d. Phys. **23.** p. 204. 1907.
5) A. Einstein, Ann. d. Phys. **23.** p. 206. 1907.
6) M. Born, Ann. d. Phys. **30.** p. 1. 1909.

Ann. d. Physik, 340 (1911), 524

- Laue's theorem explains many of the apparent paradoxial features in special-relativistic dynamics, like: factor 4/3 in momentum of charged structures; momenta not $\|$ velocities; energy-momentum integrals do not form four-vector; stressed systems need torque in oder to be set into translatory motion (Trouton-Noble experiment).
- A recent applications is to coupling of gravity to quantum systems (gravitational decoherence): How does internal energy of a systems of charged particles (large molecule) couple to Newtonian potential? Is is via $H_0 = T_{int} + U$ (Pikovski et al. 2015) or via $H_{eff} = 3T_{int} + 2U$ (Rudnicki 2017). The answer is: It's the same!

Laue's Theorem: Generalised traditional formulation

- **Theorem:** Let $T^{\alpha\beta}$ be the contravariant components of a symmetric tensor in Minkowski space with respect to some inertial frame K. Let T be conserved and stationary:

$$\partial_\mu T^{\mu\nu} = 0\,, \quad \text{(Laue: “complete system”)} \tag{25a}$$
$$\partial_t T^{\mu\nu} = 0\,, \quad \text{(Laue: “stationary system”)} \tag{25b}$$

Let further T have either compact spacelike support or spatial fall-off at least $O(1/r^{3+\varepsilon})$. Then

$$\int_{t=const.} T^{m\nu}\, d^3x = 0 \tag{26}$$

- **Proof:**

$$\partial_a (T^{a\nu} x^m) = (\partial_a T^{a\nu}) x^m + T^{m\nu} \tag{27}$$

The first term on the r.h.s. vanishes due to (25). Upon spatial integration the l.h.s. vanishes on account of Gauß' theorem and fall-off consitions. Hence result follows.

$\mathbb{Q}$ How should the statement and the proof of Laue's theorem be phrased, so as to make proper differential-geometric sense?

Laue's Theorem: Proper differential-geometric formulation

- **Theorem:** Let (M, g) be a spacetime, $T \in S(T^*M \vee T^*M)$ a symmetric and conserved energy-momentum tensor, $V, U \in \mathcal{S}TM$ Killing fields, i.e. $L_V g = L_U g = 0$, such that T and V are invariant under the flow of U:

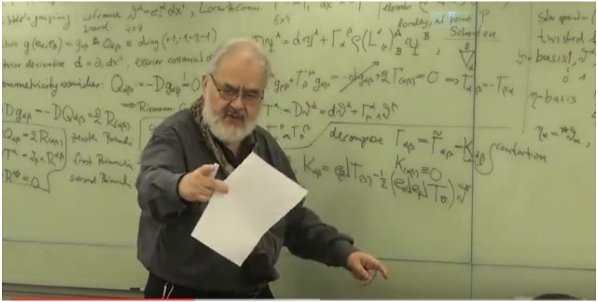
$$L_U T = 0 \tag{28a}$$
$$L_U V = [U, V] = 0 \,. \tag{28b}$$

  Then, for any smooth function $\varphi \in C^\infty(M)$ and any 3-dimensional submanifold $\Omega \subset M$ such that $\varphi T|_{\partial\Omega} \equiv 0$ we have

$$0 = \int_\Omega d\varphi \wedge i_U \mathcal{T}(V) \equiv \int_\Omega \left(U(\varphi) \star (i_V T)^\flat - (i_V T)(\varphi) \star U^\flat \right) \tag{29}$$

- **Proof:** Equations (28) together with $L_U g = 0$ and $d\mathcal{T}(V) = 0$ (here $L_U g = 0$ is used) imply $0 = L_U \mathcal{T}(V) = d i_U \mathcal{T}(V) = 0$; hence $d\varphi \wedge i_u \mathcal{T}(V) = d(\varphi i_U \mathcal{T}(V))$, which proves first equality. The second equality is also immediate from $i_U(d\varphi \wedge \mathcal{T}(V)) = U(\varphi)\mathcal{T}(V) - d\varphi \wedge i_U \mathcal{T}(V)$ and the definition of $\star$: $i_U(d\varphi \wedge \star(i_V T)^\flat) = i_U \varepsilon \, \langle d\varphi, (i_V T)^\flat \rangle = i_v T(\varphi) \star U^\flat$.
- **Special case:** The standard formulation in Minkowski space is recovered by setting $U = \partial/\partial t$, $V = \partial/\partial x^\mu$, $\varphi = x^m$ and $\Omega = \{x^\mu \mid t = \text{const}\}$.

⋆ HAPPY BIRTHDAY FRIEDRICH ⋆



und … weiter so!

Laue's theorem for conserved currents

- Laue's theorem is actually a result for conserved currents J, here applied to $J := i_V T$. To see that, let $J = J^\mu\, \partial/\partial x^\mu \in \mathcal{S}TM$ be a conserved current,

$$\nabla_\mu J^\mu = 0 \tag{30}$$

- This is equivalent to $\mathcal{J} := \star J^\flat \in \mathcal{S}\bigwedge^3 TM$ being closed

$$d\mathcal{J} = 0 \tag{31}$$

- **Theorem:** Let $\mathcal{J} \in \mathcal{S}\bigwedge^3 TM$ be closed and $U \in \mathcal{S}TM$ a symmetry of it:

$$L_U \mathcal{J} = 0 \tag{32}$$

  Then, for any $\varphi \in C^\infty(M)$ and any 3-dimensional submanifold $\Omega \subset M$ such that $\varphi J|_{\partial\Omega} \equiv 0$ we have

$$\int_\Omega d\varphi \wedge i_U \mathcal{J} \equiv \int_\Omega \left(U(\varphi) \star J^\flat - J(\varphi) \star U^\flat\right) \tag{33}$$

- **Proof:** Due to (31), condition (32) is equivalent to $d(i_U \mathcal{J}) = 0$ so that integrand id $d\varphi \wedge i_U \mathcal{J} = d(\varphi i_U \mathcal{J})$ and integral vanishes if $\varphi J|_{\partial\Omega} = 0$. The second equality follows from $i_U(d\varphi \wedge \mathcal{J}) = U(\varphi)\mathcal{J} - d\varphi \wedge i_U \mathcal{J}$ and the definition of $\star$ applied to lhs: $i_U(d\varphi \wedge \star J^\flat) = i_U \epsilon \langle d\varphi, J^\flat \rangle = J(\varphi) \star U^\flat$.