Effekte der Relativität

bewegte Uhren gehen langsamer

bewegte Massstäbe sind verkürzt

Geschwindigkeitsaddition anders

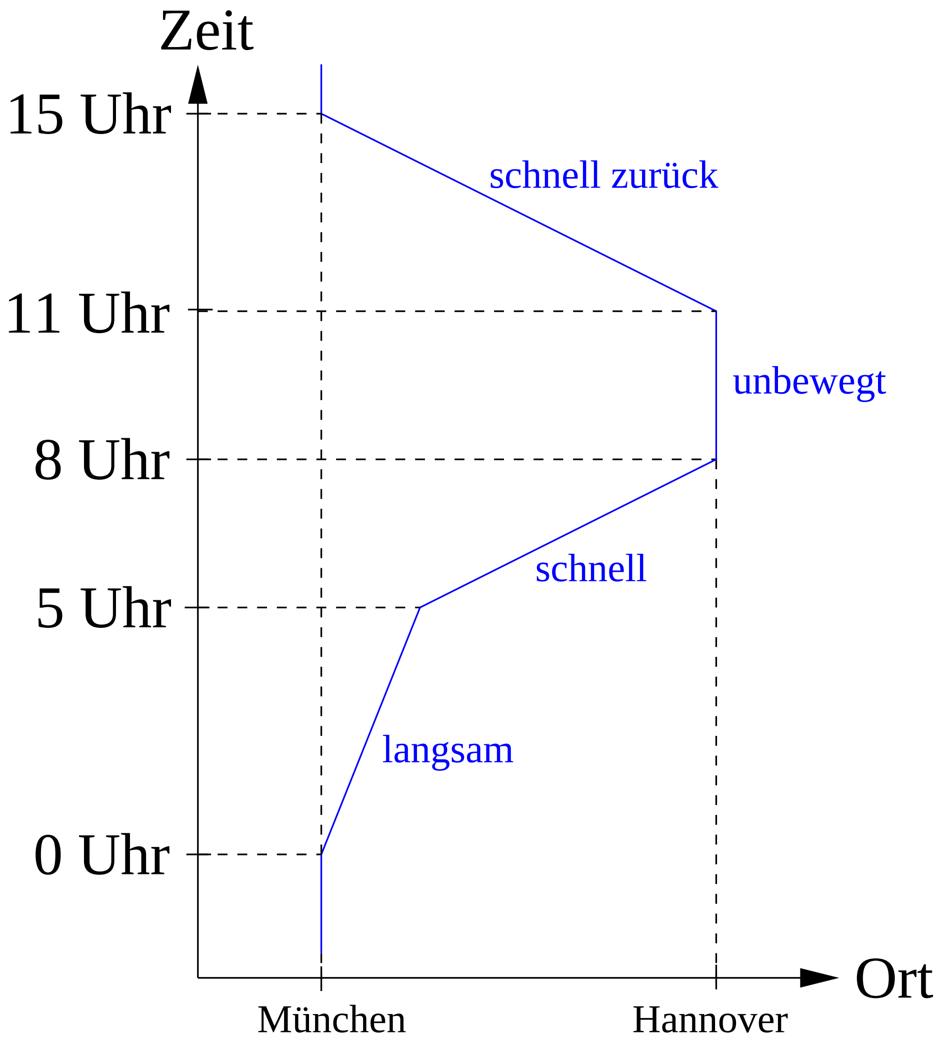
bewegte Beobachter sehen Lichtquellen

− aus anderer Richtung (Aberration)

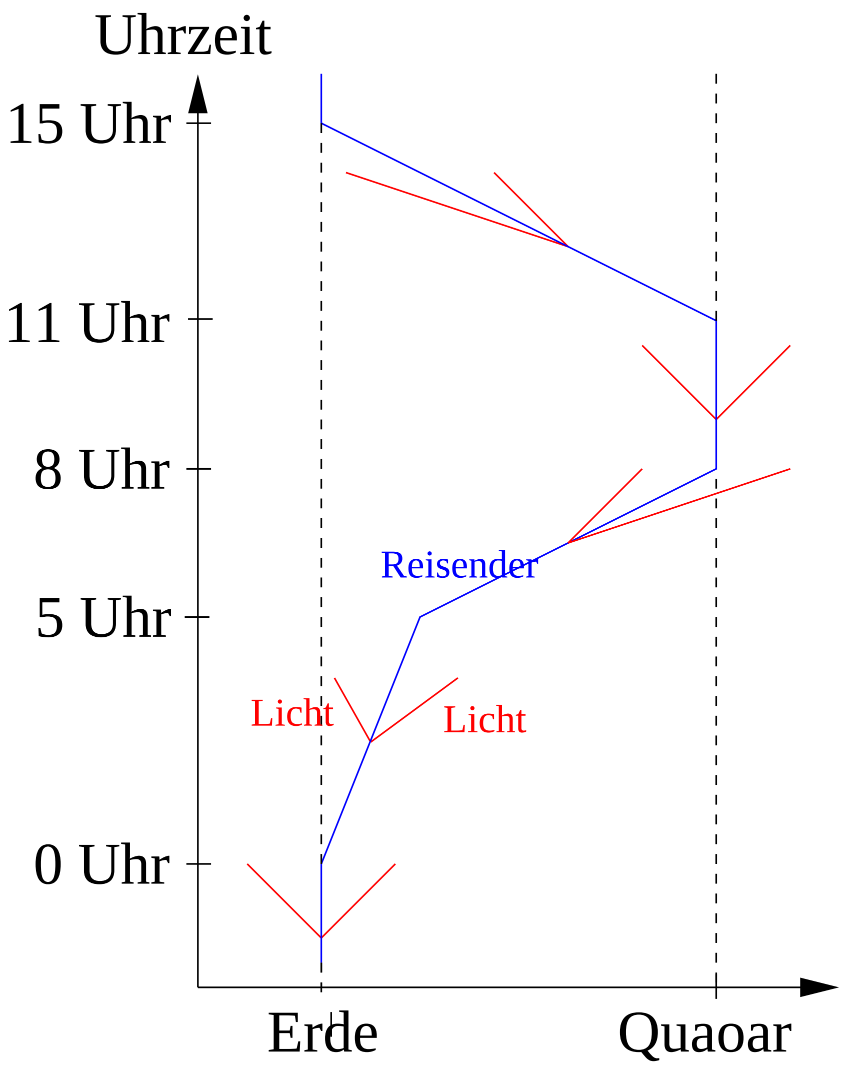
− in anderen Farben (Doppler−Effekt)

− in anderer Intensität

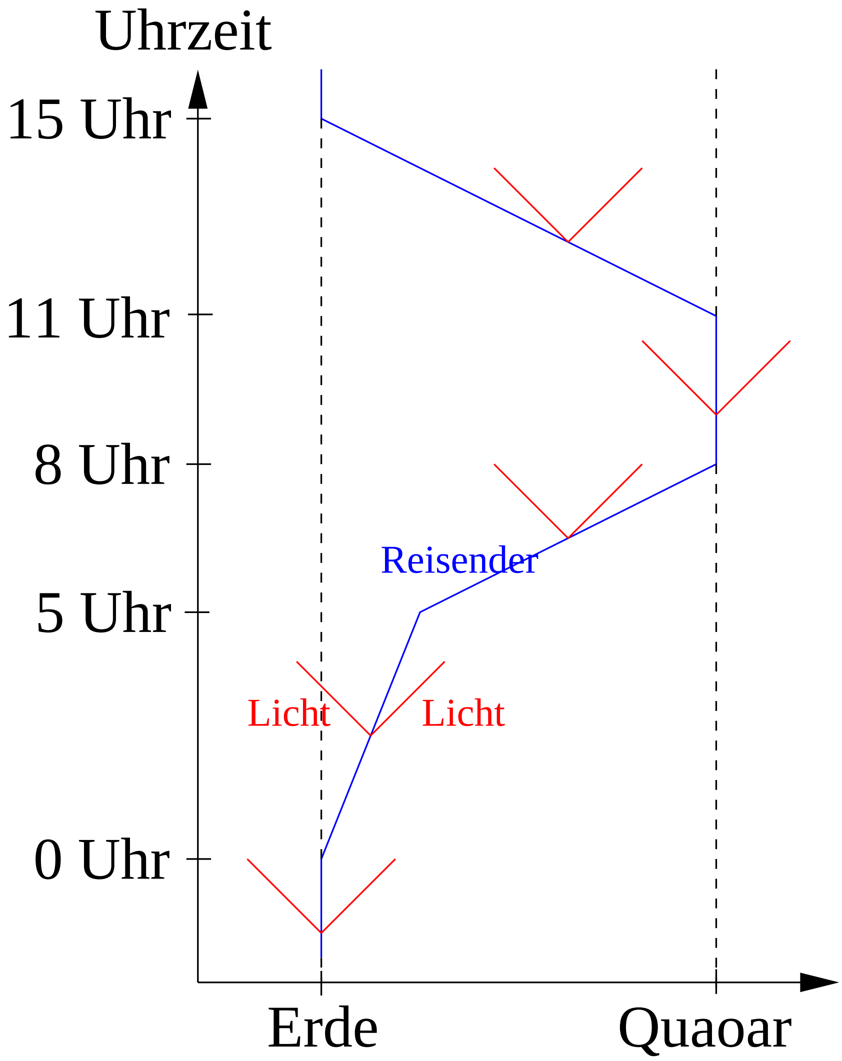
Raum−Zeit−Diagramm



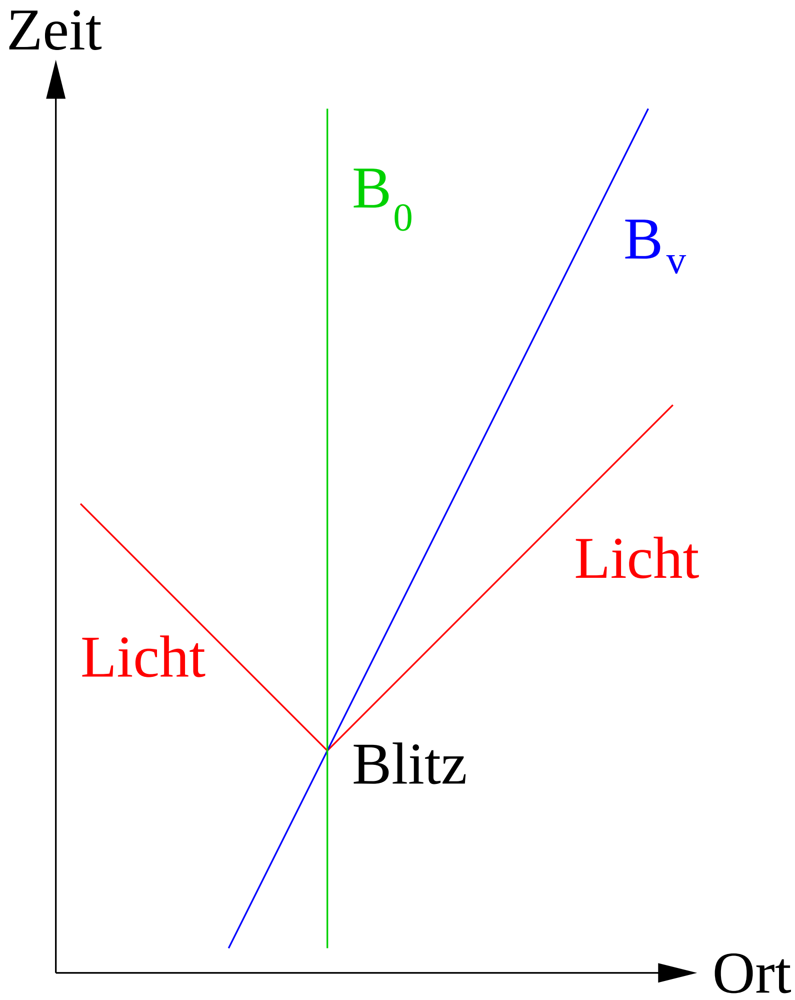
vor Einstein



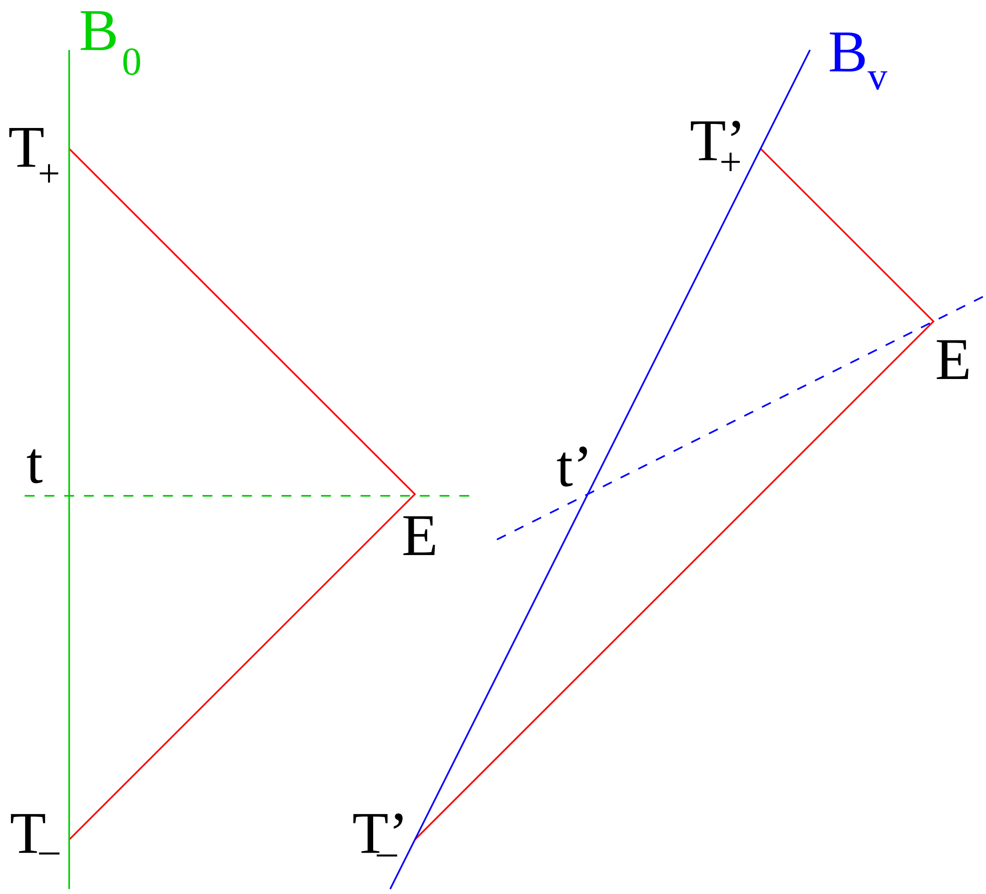
n a c h   E i n s t e i n



Beobachter und Licht



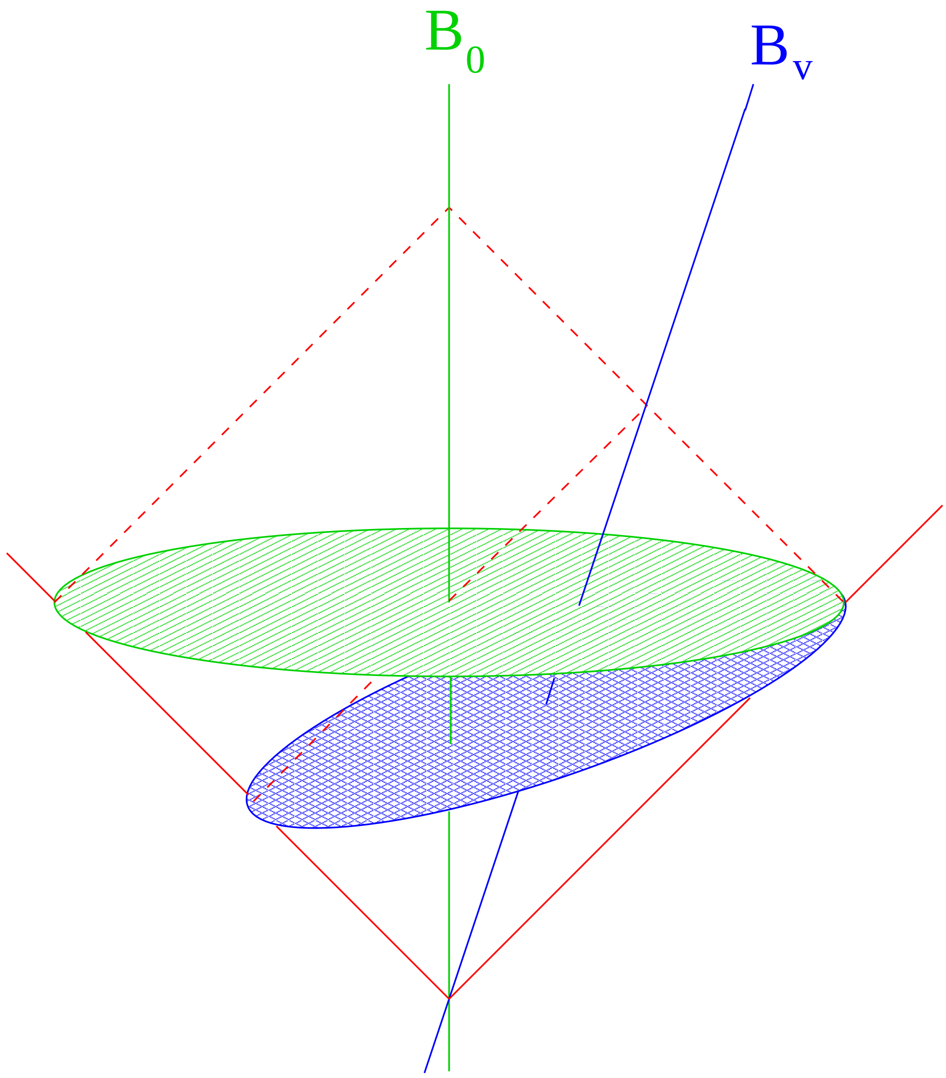
relativ gleichzeitig



$$r = (T_+ - T_-)/2 \qquad T_+ = t + r$$

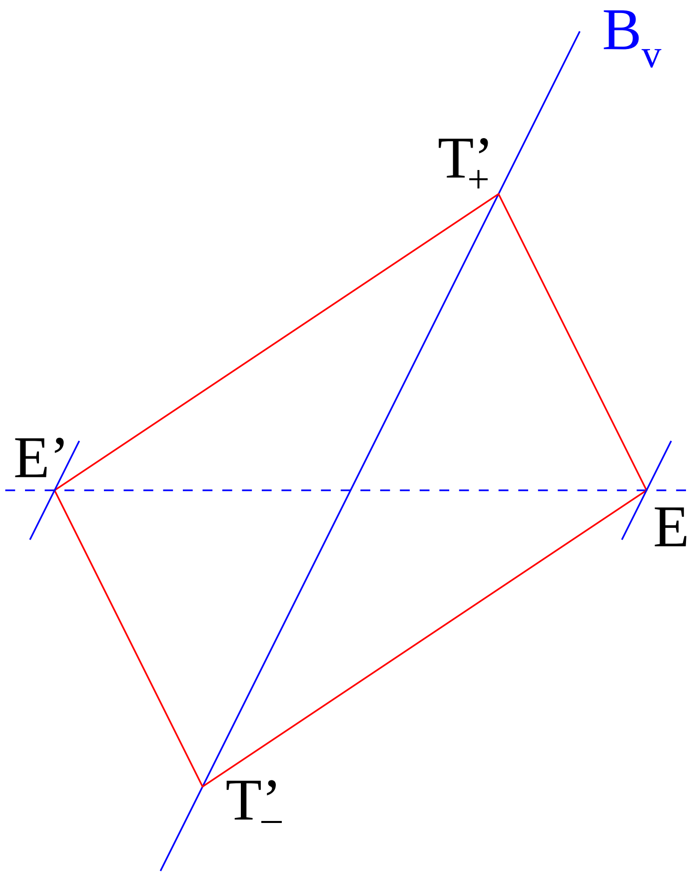
$$t = (T_+ + T_-)/2 \qquad T_- = t - r$$

Zwei Raum−Dimensionen



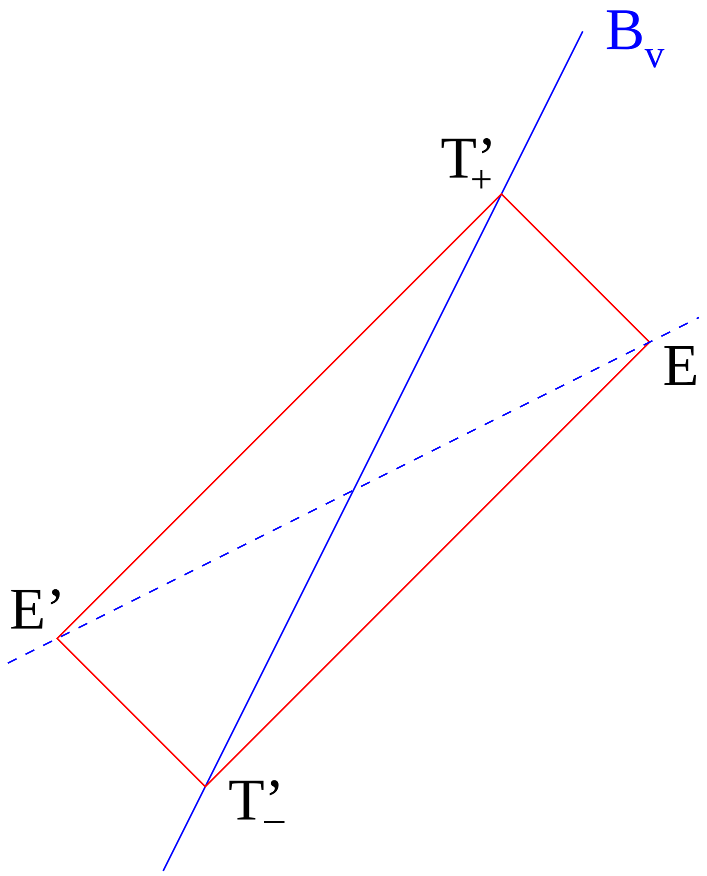
v o r   E i n s t e i n :

Licht−Parallelogramm



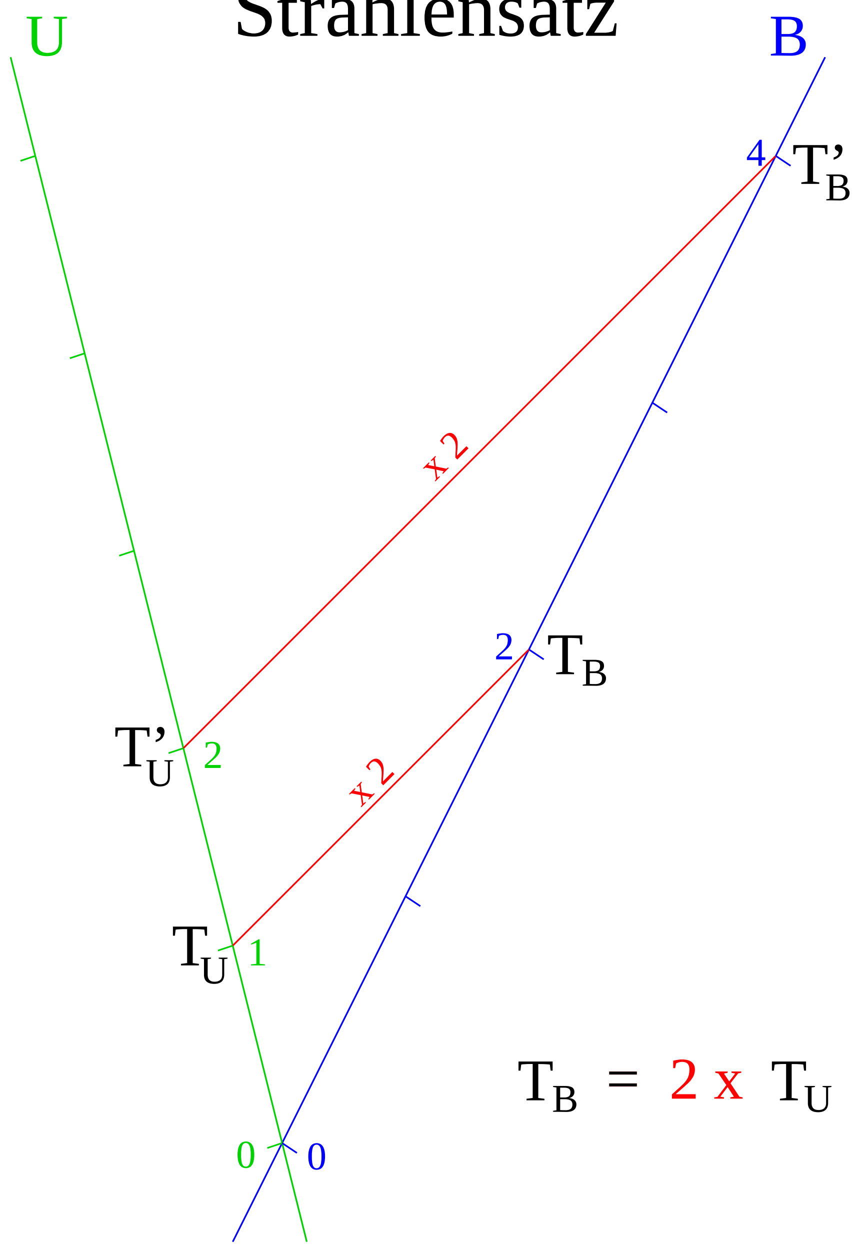
gleich−ortig & gleichzeitig

Licht−Eck

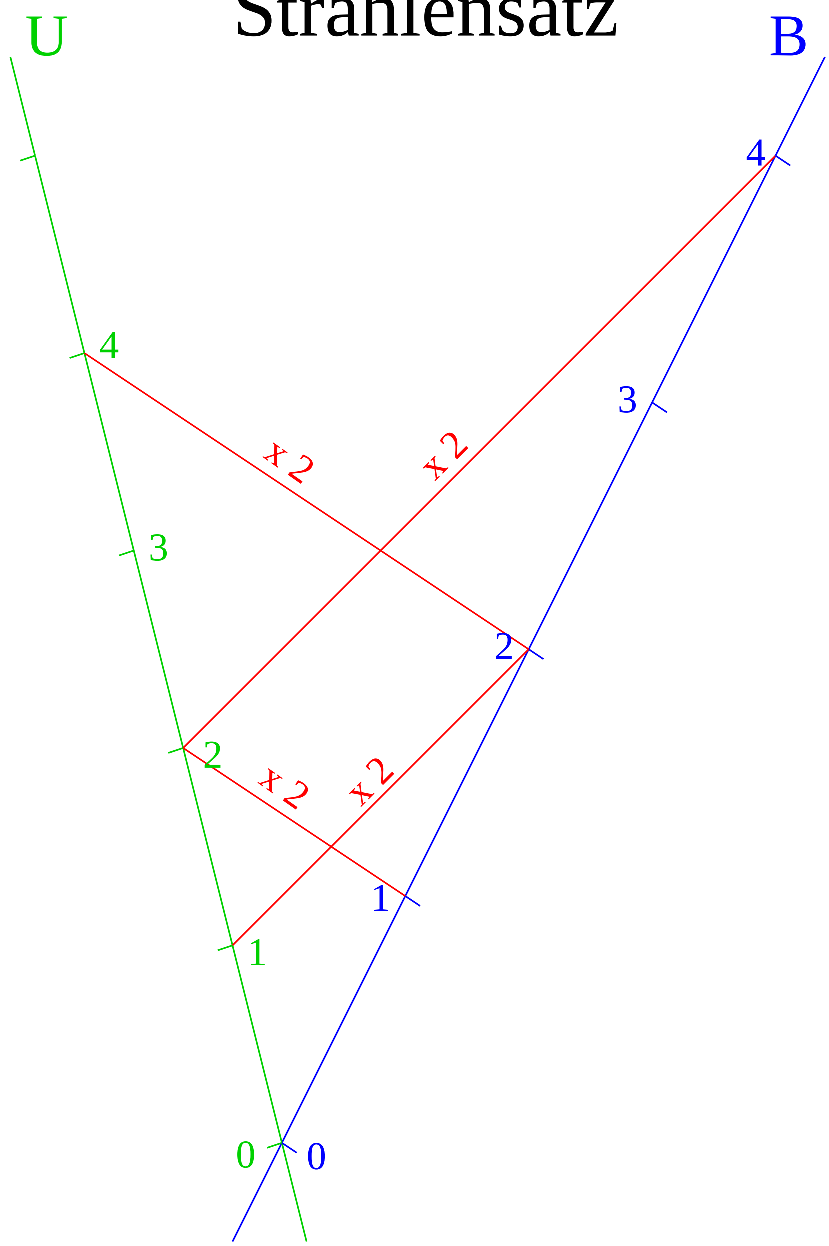


gleich−ortig & gleichzeitig

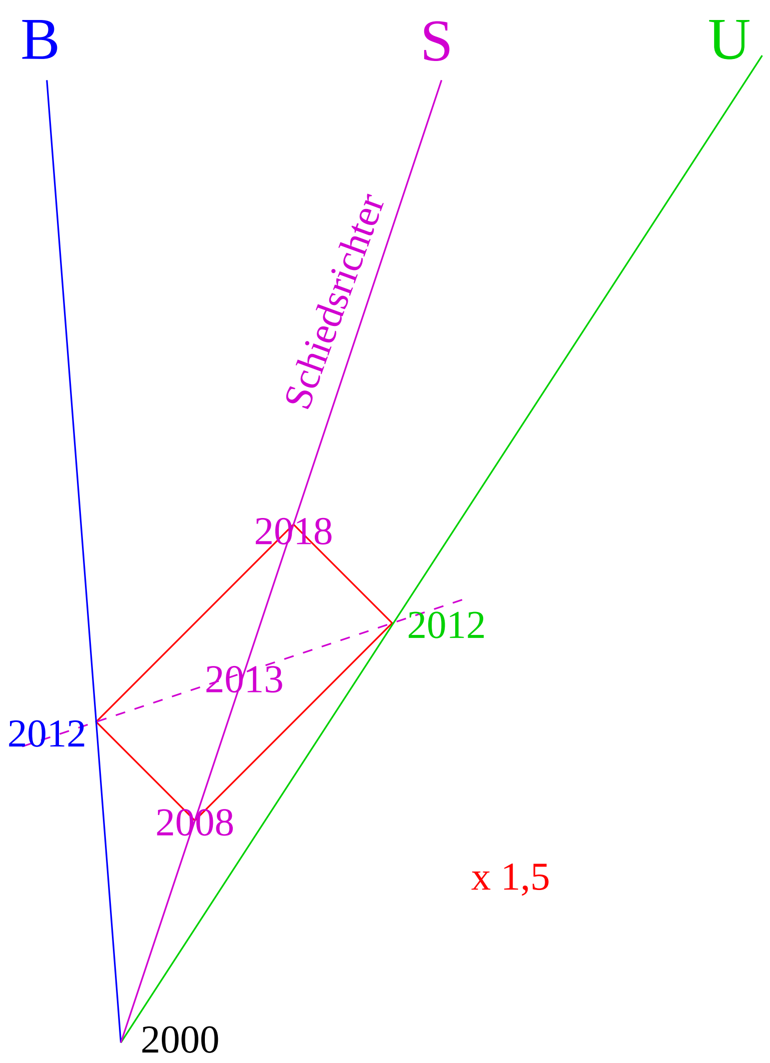
Strahlensatz



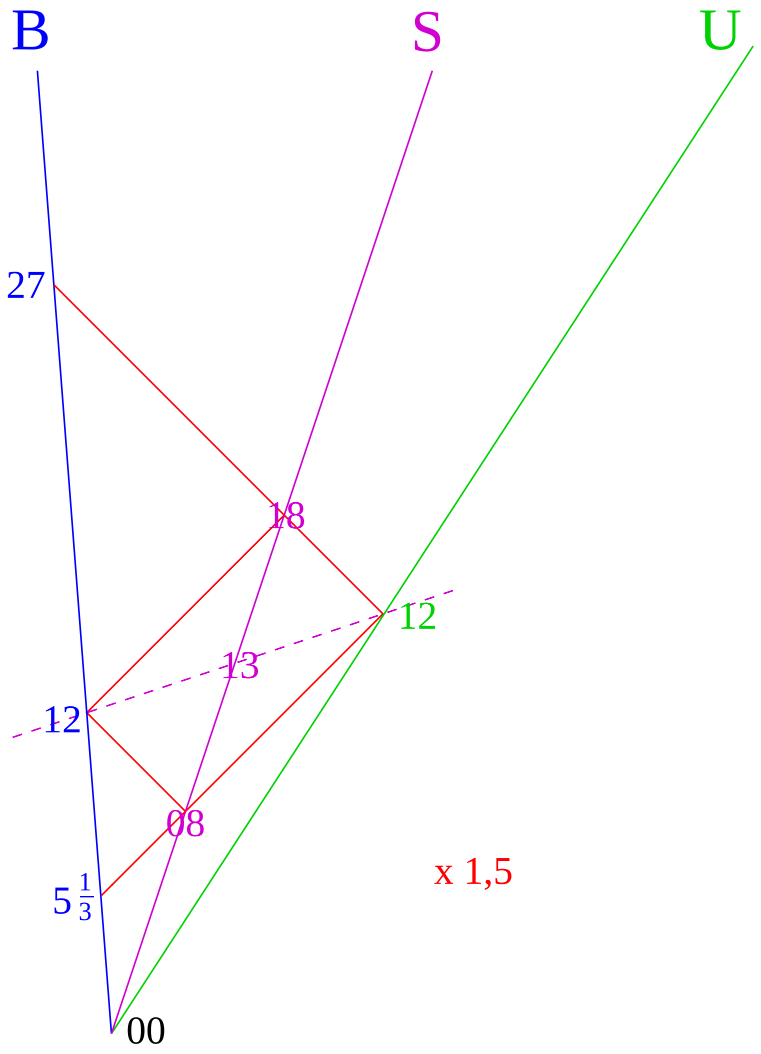
Strahlensatz



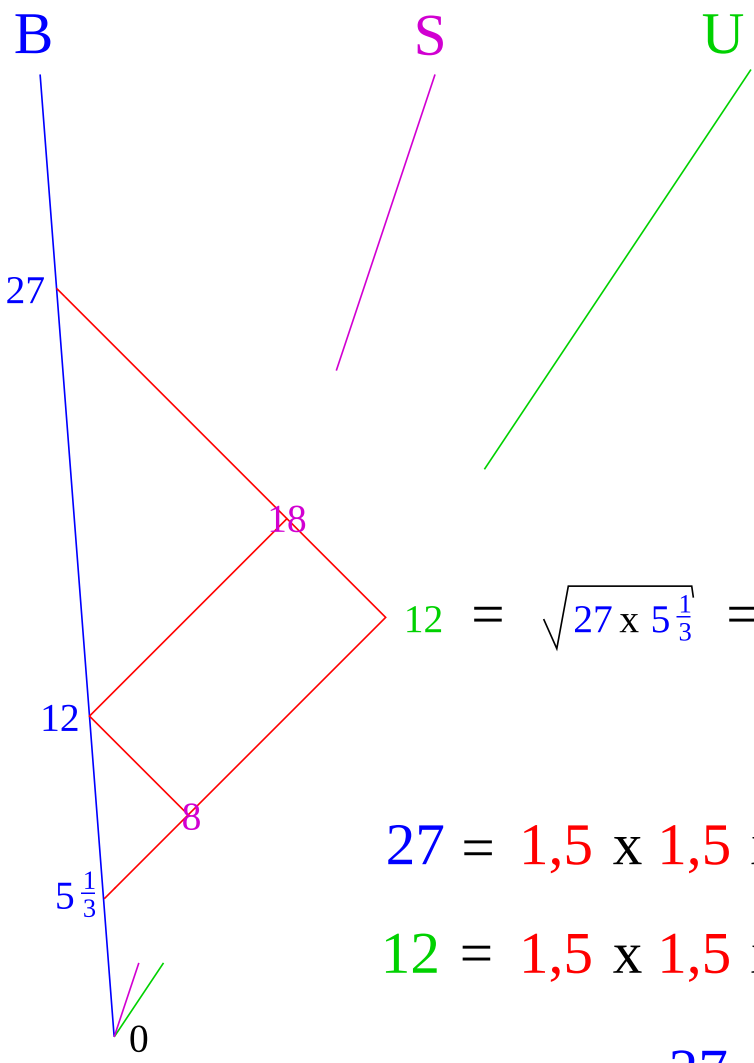
Uhrenvergleich



verlängertes Lichteck



geometrisches Mittel



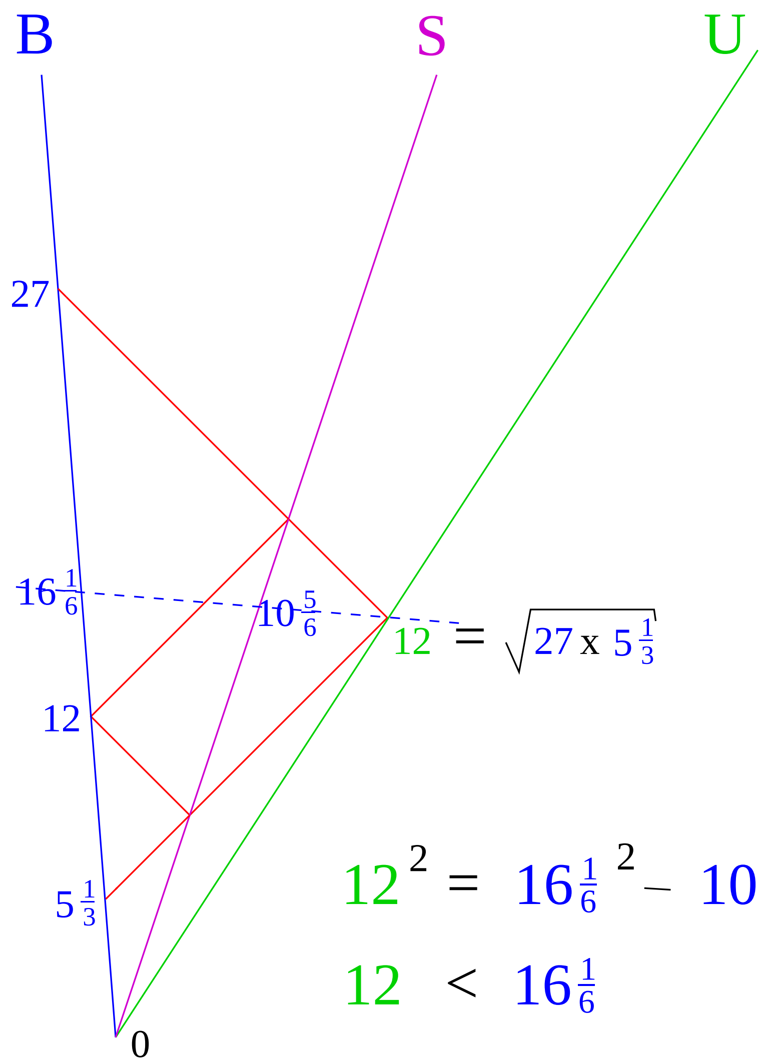
$$12 = \sqrt{27 \times 5\tfrac{1}{3}} = \sqrt{144}$$

$$27 = 1{,}5 \times 1{,}5 \times 12$$

$$12 = 1{,}5 \times 1{,}5 \times 5\tfrac{1}{3}$$

$$\Rightarrow \quad \frac{27}{12} = \frac{12}{5\frac{1}{3}}$$

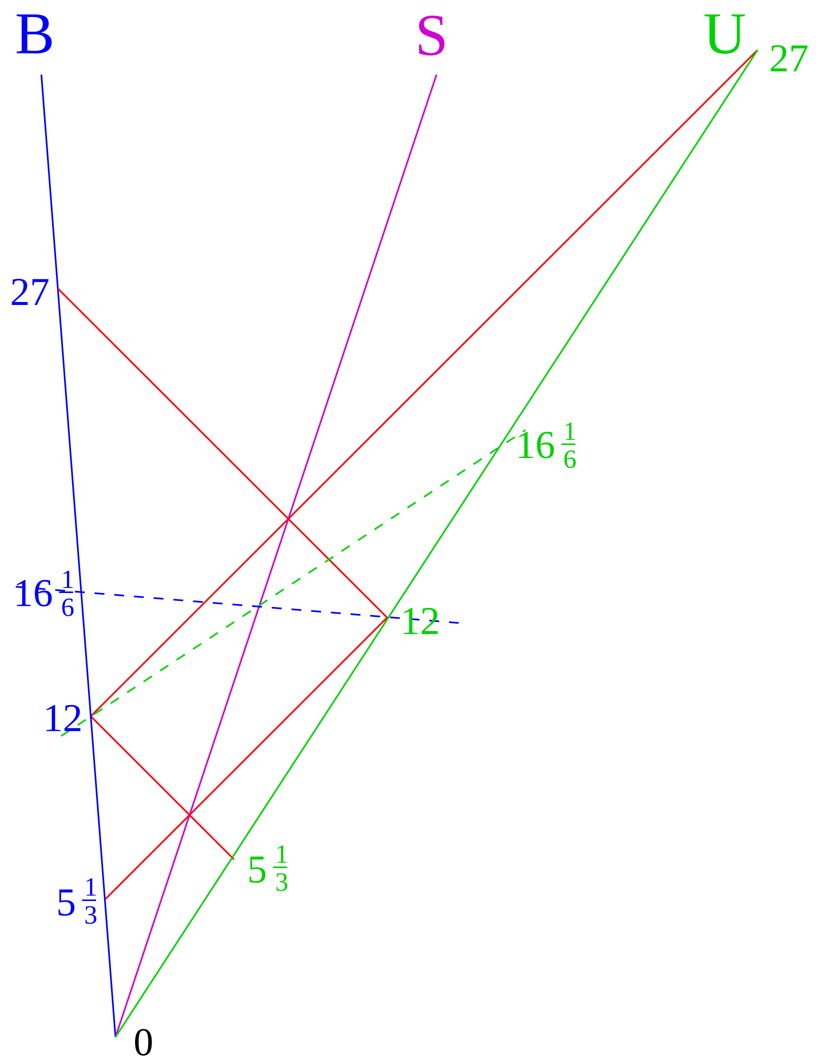
Zeit−Dehnung



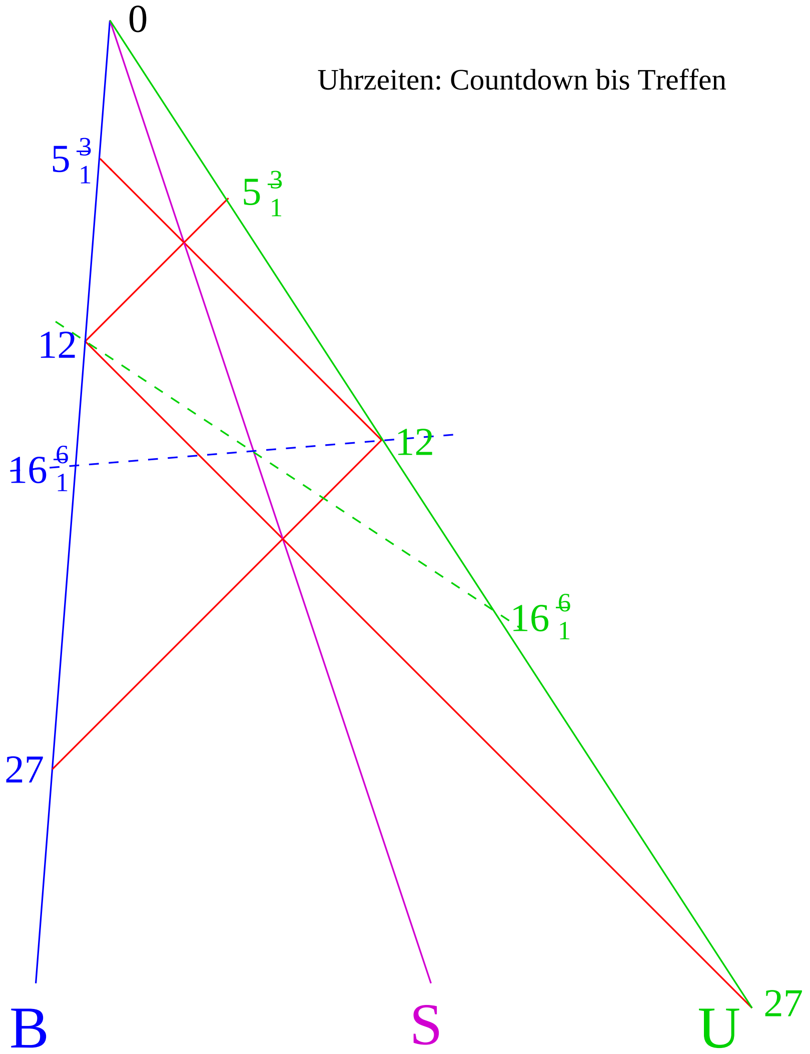
$$12^2 = 16\tfrac{1}{6}^{\,2} - 10\tfrac{5}{6}^{\,2}$$

$$12 < 16\tfrac{1}{6}$$

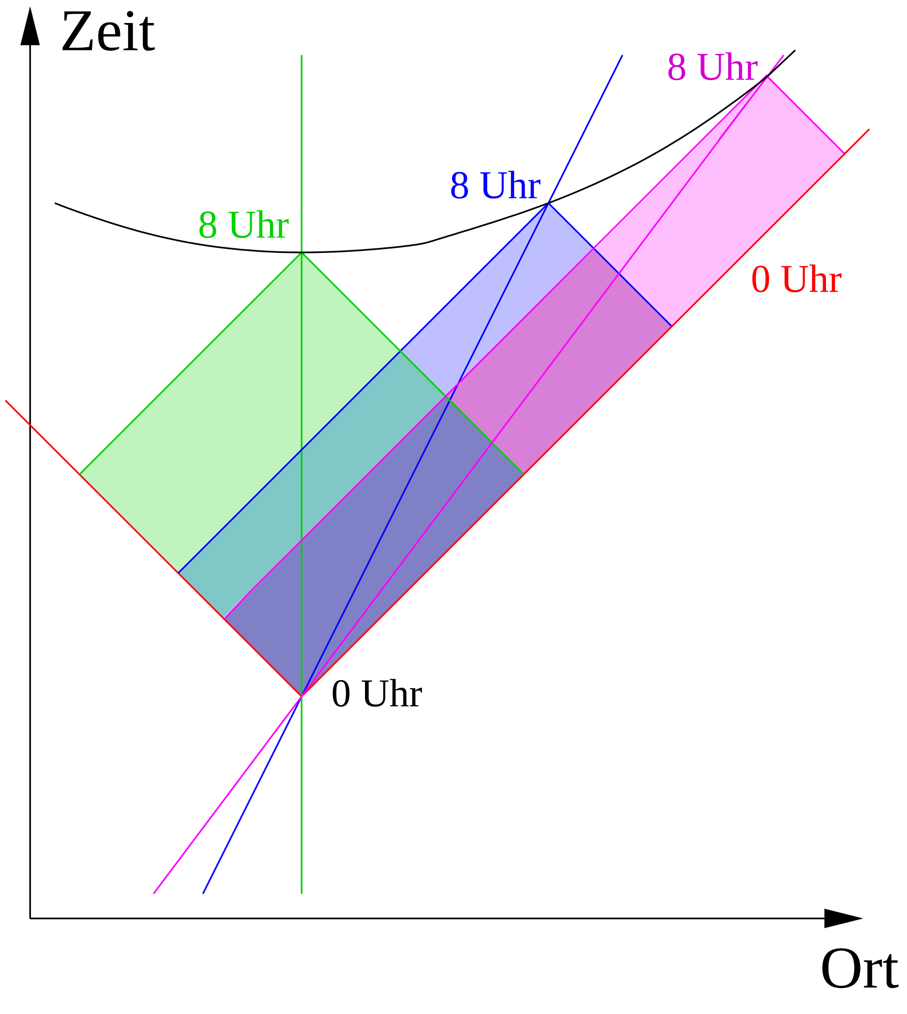
Zeit−Dehnung wechselseitig



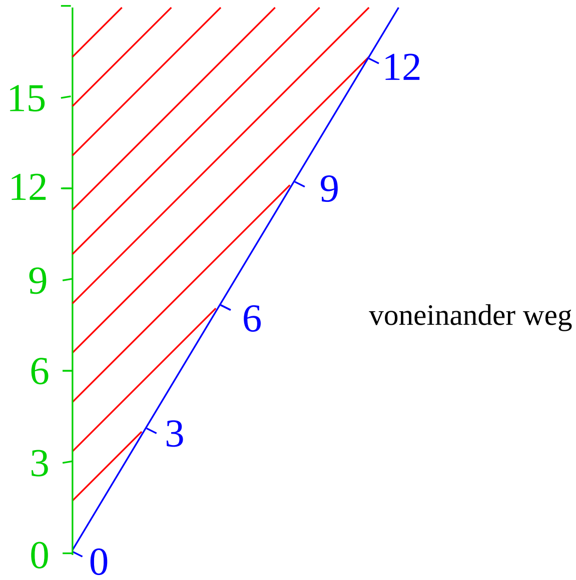
umgekehrte Bewegung

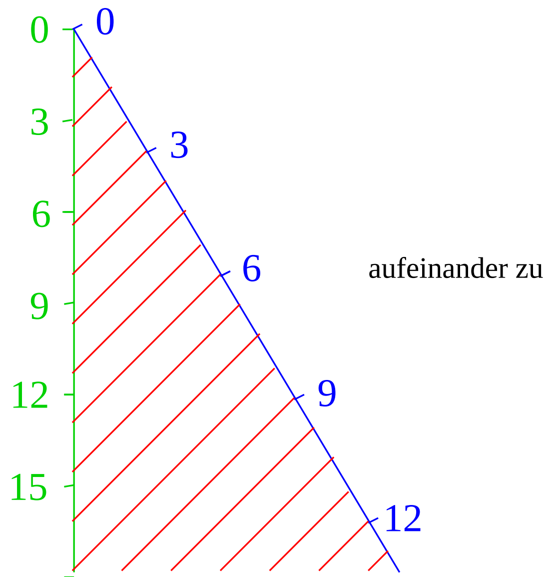


Bewegte Uhren

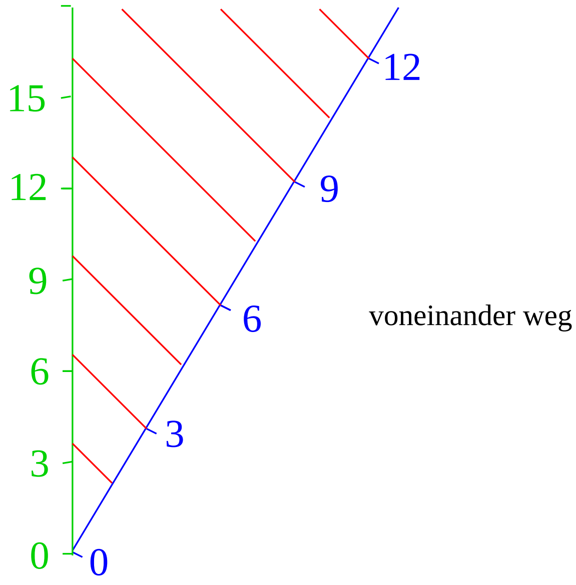


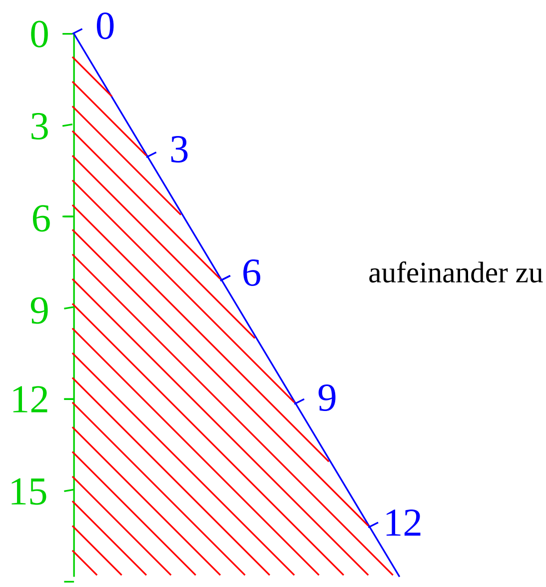
regelmäßige Lichtsignale



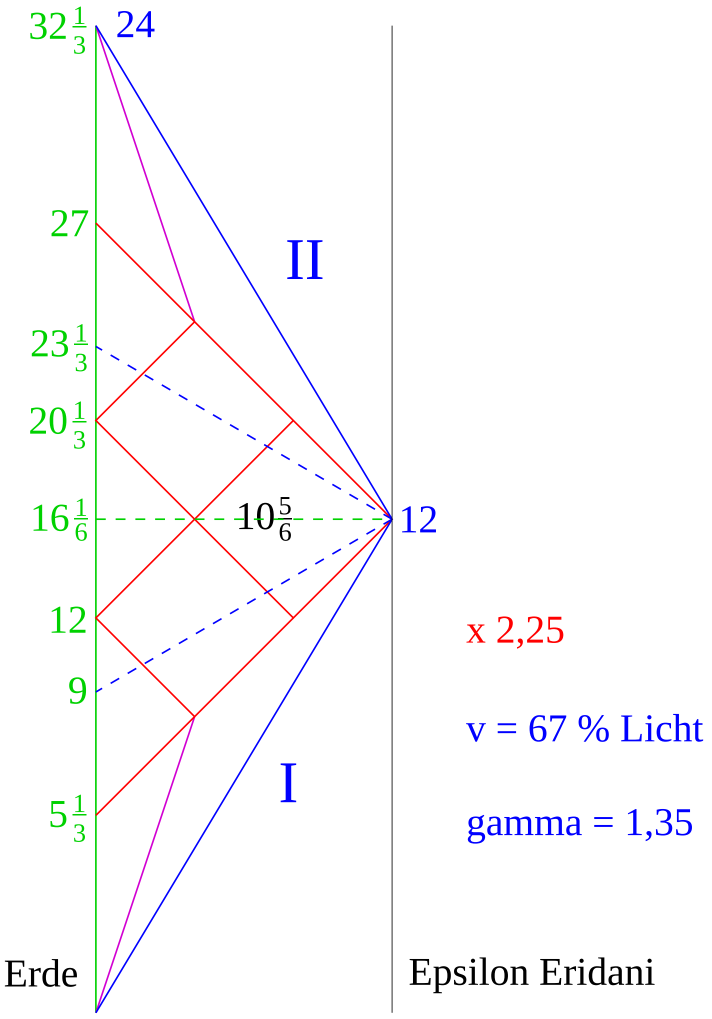


regelmäßige Lichtsignale

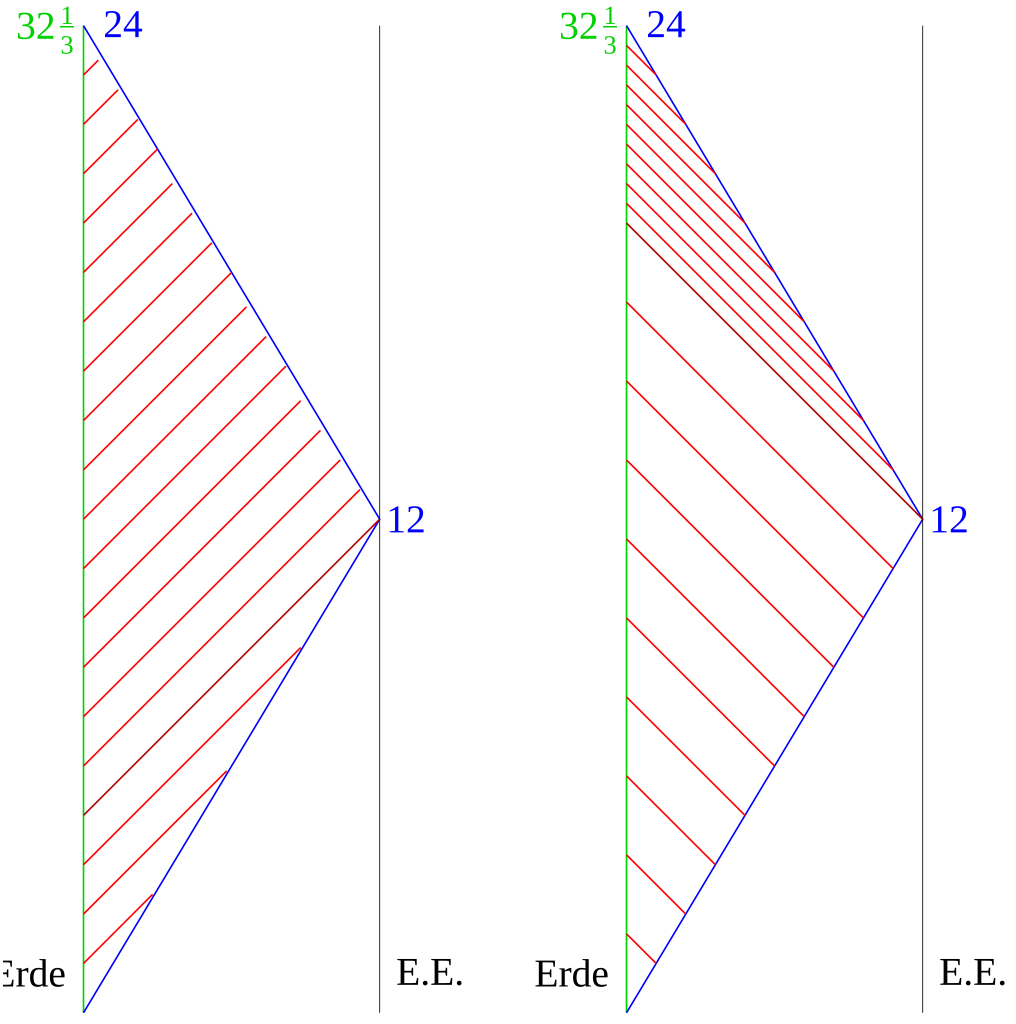




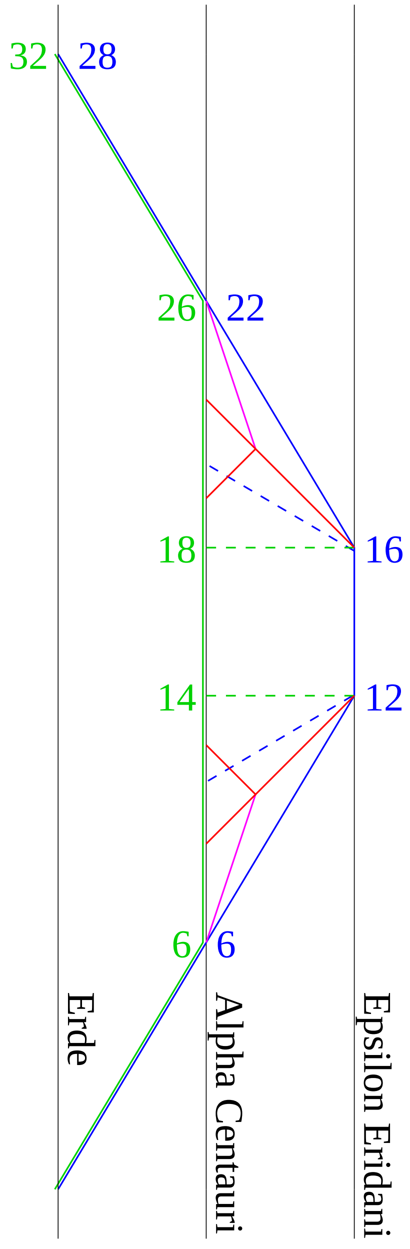
Zwillings−Effekt



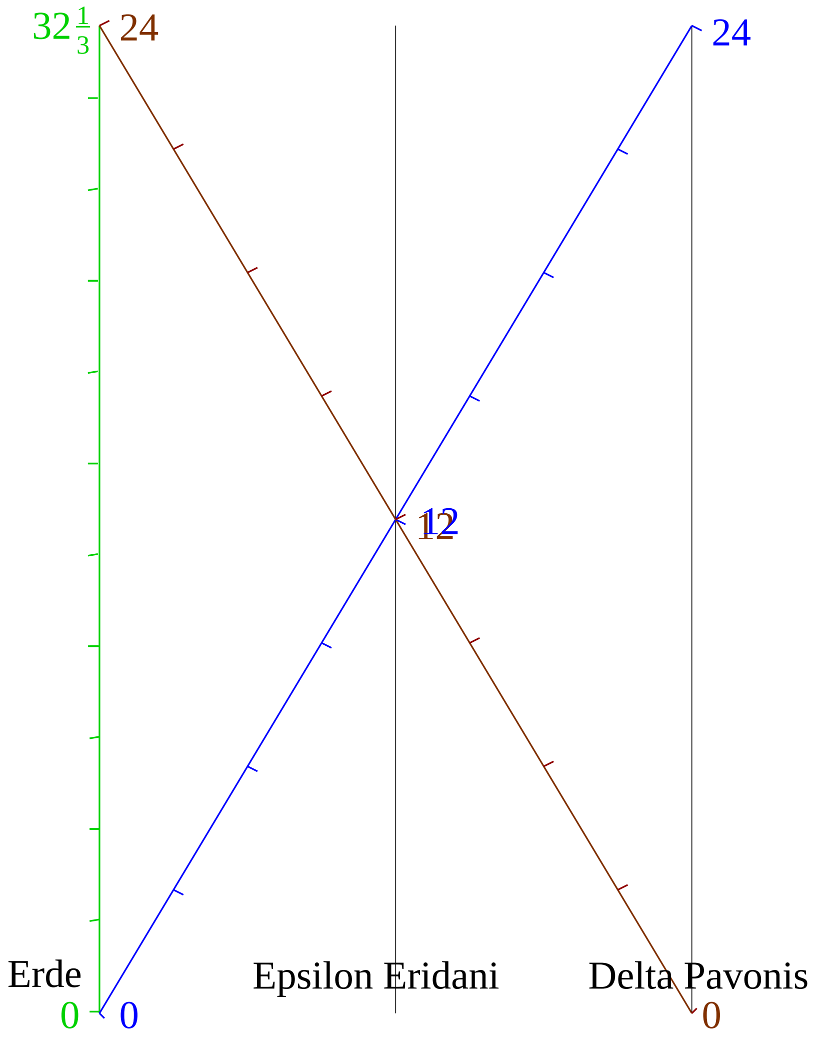
Austausch von Lichtsignalen



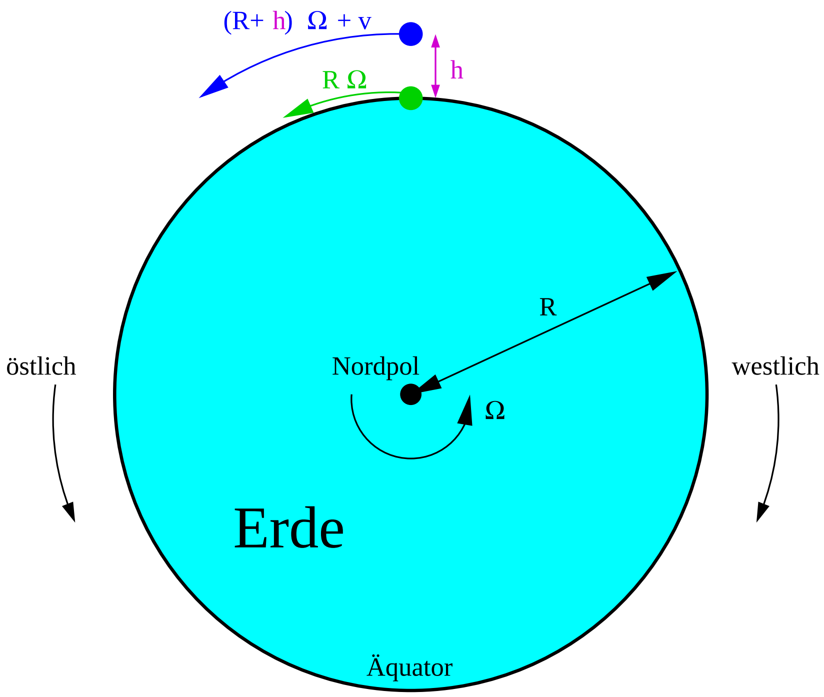
gleiche Beschleunigungen



ohne Beschleunigung



Hafele & Keating 1971



relative Differenz $\delta = \frac{T - T}{T} = \frac{g\,h}{c^2} - \frac{(2\,R\,\Omega + v)\;v}{2\,c^2}$

$c = 300000000$ m/s $\quad g = 9{,}8$ m/s^2 $\quad R\,\Omega = 470$ m/s

$h = 10000$ m $\quad v = \pm 300$ m/s

$\delta_+ = (1 - 2) \times 10^{-12}$

$\delta_- = (1 + 1) \times 10^{-12}$

Flug von 36 Stunden = 130000 Sekunden

Zeitdifferenz: − 130 ns bzw. + 260 ns

mit Caesium−Uhren gemessen

Licht-Koordinaten

$$r = \tfrac{1}{2}(T_+ - T_-) \quad \text{und} \quad t = \tfrac{1}{2}(T_+ + T_-)$$

$$T_\pm = t \pm r \quad \Longrightarrow \quad T_+ T_- = t^2 - r^2 = \tau^2 = (1-v^2)\, t^2 = t^2/\gamma^2$$

Streck-Faktor

$$T_+ = k\,\tau \quad \text{und} \quad \tau = k\, T_- \quad \Longrightarrow \quad k^2 = \frac{T_+}{T_-} = \frac{t+r}{t-r} = \frac{1+v}{1-v}$$

$$k(v) = \sqrt{\frac{1+v}{1-v}} = \frac{1+v}{\sqrt{1-v^2}} \quad \Longleftrightarrow \quad v = \frac{k^2-1}{k^2+1} = \frac{k-\frac{1}{k}}{k+\frac{1}{k}}$$

Rapidität

$$k =: \mathrm{e}^\theta \quad \Longrightarrow \quad v = \frac{\mathrm{e}^\theta - \mathrm{e}^{-\theta}}{\mathrm{e}^\theta + \mathrm{e}^{-\theta}} = \tanh\theta \quad \text{und} \quad \gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\theta$$

Lorentz-Transformation

$$\begin{pmatrix} T'_+ \\ T'_- \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} T_+ \\ T_- \end{pmatrix} = \begin{pmatrix} \mathrm{e}^\theta & 0 \\ 0 & \mathrm{e}^{-\theta} \end{pmatrix} \begin{pmatrix} T_+ \\ T_- \end{pmatrix}$$

$$\begin{pmatrix} t' \\ r' \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix}$$

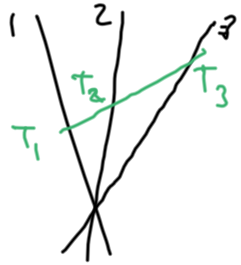
Aberration

$$\tan\frac{\phi'}{2} = k \tan\frac{\phi}{2} = \sqrt{\frac{1+v}{1-v}} \tan\frac{\phi}{2}$$

Relativistische Dynamik von Punktteilchen

$c = 1$

zunächst noch Kinematik. Beispiel: Addition von Geschw.



$$T_3 = k_{32} \cdot T_2 = k_{32} \cdot (k_{21} \cdot T_1) \stackrel{!}{=} k_{31} \cdot T_1$$

$$\rightsquigarrow \quad k_{31} = k_{32} \cdot k_{21} \quad \rightsquigarrow \quad \frac{1+v_{31}}{1-v_{31}} = \frac{1+v_{32}}{1-v_{32}} \cdot \frac{1+v_{21}}{1-v_{21}}$$

oder

$$\rightsquigarrow \quad \theta_{31} = \theta_{32} + \theta_{21} \quad \rightsquigarrow$$

$$v_{31} = \tanh(\theta_{31}) = \tanh(\theta_{32} + \theta_{21}) \stackrel{\text{Add thm}}{=} \frac{\tanh(\theta_{32}) + \tanh(\theta_{21})}{1 + \tanh(\theta_{32}) \tanh(\theta_{21})} = \frac{v_{32} + v_{21}}{1 + v_{32} \cdot v_{21}}$$

Lorentz-Trsf. in 1+3 Dimensionen

x-Boost:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

analog in y- & z- Richtg.

hinzu kommen Drehungen

$$\begin{pmatrix} t' \\ \vec{r}\,' \end{pmatrix} = \begin{pmatrix} 1 & 0 \; 0 \; 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & D \end{pmatrix} \begin{pmatrix} t \\ \vec{r} \end{pmatrix}$$

} erzeugen Lorentzgruppe

$$SO(1,3) \ni \Lambda$$

$\underline{x}' = \Lambda \cdot \underline{x}$ mit $\underline{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ Vierspalte

Index-Schreibweise:

$x^\mu = (x^0, x^{i=1,2,3})$

$x = e_\mu x^\mu$ Vierervektor

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ Lorentz-Trsf.

Invarianz:

in 1+1 Dim.: $T_+' T_-' = T_+ T_- \rightsquigarrow (t+r)(t-r) = t^2 - r^2 = \tau^2$ Lorentz-invariant

mit Drehinv. in 1+3 Dim.: $t^2 - \vec{r}^2 = t^2 - x^2 - y^2 - z^2 = (t, x, y, z) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ inv.

$\eta \rightarrow$ (Matrix)

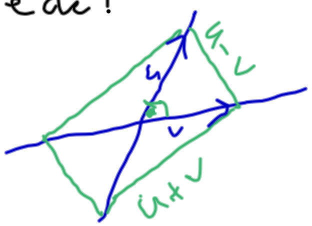
Index-Schreibweise: $u^\mu \eta_{\mu\nu} u^\nu =: u \cdot u$ Bilinearform

Skalarprodukt: $u \cdot v = \frac{1}{4}(u+v)^2 - \frac{1}{4}(u-v)^2 = u^\mu \eta_{\mu\nu} v^\nu$

$= u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3$

Licht: $u \cdot u = 0$

Lichtecke!



$0 = (u+v)^2 = u^2 + 2u \cdot v + v^2$
$0 = (u-v)^2 = u^2 - 2u \cdot v + v^2$
$\Big\}$ $u^2 = -v^2$, $u \cdot v = 0$

$\rightsquigarrow u \perp v$. Kurve $\tau^2 =$ konst. sind Hyperbeln

Index-Stellung relevant, weil

$x^\mu \doteq \begin{pmatrix} t \\ \vec{r} \end{pmatrix} \rightsquigarrow x_\mu = \eta_{\mu\nu} x^\nu \doteq \begin{pmatrix} t \\ -\vec{r} \end{pmatrix}$

kann η verstecken: $x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = x^\mu y_\mu = x_\mu y^\mu$

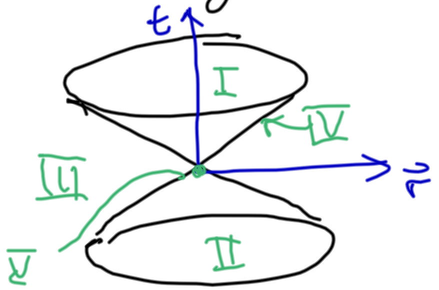
wie transformiert x_μ?

$x'_\mu = \eta_{\mu\nu} x'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} x_\sigma =: \tilde{\Lambda}_\mu{}^\sigma x_\sigma$

mit $\tilde{\Lambda}_\mu{}^\sigma = (\eta \Lambda \eta^{-1})_\mu{}^\sigma = (\Lambda^{-1T})_\mu{}^\sigma$ $\qquad \eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu^\rho$

weil $\Lambda^T \eta \Lambda = \eta \Leftrightarrow \Lambda^{T-1} = \eta \Lambda \eta^{-1}$ def. $SO(1,3)$

Lichtkegel



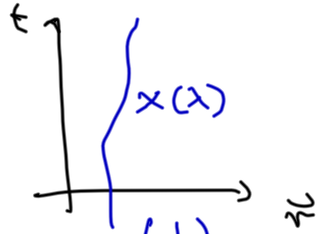
im Minkowskiraum $\mathbb{R}^{1,3}$

lässt Lorentz-Trsf. 5 Bereiche invariant

I zeitartige Zukunft } $x^2 > 0$
II — " — Vergangenheit }
III raumartig — $x^2 < 0$
IV lichtartig — $x^2 = 0$
V Null — $x = 0$

· Weltlinie eines Punktteilchens

$\lambda \longmapsto x^\mu(\lambda) \doteq \begin{pmatrix} x^0(\lambda) \\ x^1(\lambda) \\ x^2(\lambda) \\ x^3(\lambda) \end{pmatrix}$



Newton: $\lambda = t = x^0 \leadsto x^\mu \doteq \begin{pmatrix} t \\ \vec{r}(t) \end{pmatrix} \leadsto \dot{x}^\mu \doteq \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$

Eigenzeit: $\lambda = \tau$ mit $\tau_{12} = \int_1^2 ds = \int_1^2 dt \sqrt{\dot{x}\cdot\dot{x}} = \int_1^2 dt \underbrace{\sqrt{1-\vec{v}^2(t)}}_{\gamma^{-1}}$

↳ Lorentz-Skalar!

$\gamma = \frac{1}{\sqrt{1-v^2}} = dt/d\tau \leadsto d\tau = \sqrt{1-v^2}\, dt$

· Geschwindigkeit & Beschleunigung

teile durch $d\tau$ (Skalar) statt durch dt (Teil von dx^μ)

4-Geschw.: $u = \frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} \doteq \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \leadsto u^2 = \gamma^2(1-\vec{v}^2) = 1$

4-Beschl.: $b = \frac{du}{d\tau} = \frac{d^2x}{d\tau^2} = \frac{dt}{d\tau}\frac{du}{dt} \doteq \gamma^2 \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix} + \gamma^4\, \vec{v}\cdot\vec{a} \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$

$\leadsto 0 = \frac{d}{d\tau} u^2 = 2u\cdot b \leadsto u^2 > 0,\ b^2 < 0,\ b \perp u$

momentanes Ruhesystem: $\gamma = 1, \vec{v} = 0 \leadsto u = \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix},\ b = \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$

kleine Geschw.: $\gamma = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots$

· Dynamik ersetze alle 3-Vektoren durch 4-Vektoren

4-Impuls: $p \doteq \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} \doteq m u \doteq m \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix} = m \begin{pmatrix} 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 + \dots \\ \vec{v} + \frac{1}{2} v^2 \vec{v} + \dots \end{pmatrix}$

4-Kraft: $F \doteq \begin{pmatrix} F^0 \\ \vec{F} \end{pmatrix} \doteq m b = m \begin{pmatrix} \gamma^4 \vec{v} \cdot \vec{a} \\ \gamma^2 \vec{a} + \gamma^4 (\vec{v} \cdot \vec{a}) \vec{v} \end{pmatrix}$

$= m \begin{pmatrix} \vec{v} \cdot \vec{a} + 2 v^2 \, \vec{v} \cdot \vec{a} + \dots \\ \vec{a} + v^2 \vec{a} + (\vec{v} \cdot \vec{a}) \vec{v} + \dots \end{pmatrix}$

räumliche Komponenten: Newton + relativist. Korrekturen

zeitliche Komponenten sind Zugaben:

$E = c p^0 = \gamma m c^2 = \underline{m c^2} + \frac{1}{2} m v^2 + \frac{3}{8} m \, v^4/c^2 + \dots$ Energie

$\hookrightarrow$ Ruheenergie

$P = c F^0 = \vec{v} \cdot \vec{F} + \dots$ Leistung $\Big\}$ $P = \frac{dE}{d\tau}$

Bewegungsgl.: $F^\mu = \frac{dp^\mu}{d\tau} = m \frac{d^2 x^\mu}{d\tau^2} = m b^\mu$

Energie-Impuls-Beziehung: $m^2 c^2 = p^2 = (E/c)^2 - \vec{p}^{\,2} \rightsquigarrow E = c \sqrt{m^2 c^2 + \vec{p}^{\,2}}$

masselose Teilchen $m = 0$ sind möglich! Massenschale