Aspects of Classical Physics - Tutorial 2

Olaf Lechtenfeld, Gabriel Picanço

05 Nov 2021

1 Baker-Campbell-Hausdorff up to third order

The Baker-Campbell-Hausdorff formula is given by

$$\log(\mathrm{e}^X \mathrm{e}^Y) := Z = X + \int_0^1 \mathrm{d}t\, g(\mathrm{e}^{\mathrm{ad}\, X} \mathrm{e}^{t\, \mathrm{ad}\, Y})(Y), \tag{1}$$

with $g(z) = \frac{\log z}{1 - z^{-1}}$. One can expand this function in z:

$$g(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{6}(z-1)^2 + ... \tag{2}$$

We will use this expansion to find an approximation of the BCH formula.

a) Prove that $(\mathrm{e}^{ad\, X} \mathrm{e}^{t\, ad\, Y} - I)^n$ only contains terms of order n or higher in ad X (or ad Y).

b) Expand $g(\mathrm{e}^{\mathrm{ad}\, X} \mathrm{e}^{t\, \mathrm{ad}\, Y})$ up to 2nd order in ad X or ad Y.

c) Integrate the approximation of $g(\mathrm{e}^X \mathrm{e}^{tY})(Y)$ and show that

$$\log(\mathrm{e}^X \mathrm{e}^Y) \approx (X + Y) + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + ..., \tag{3}$$

that is, the Baker-Campbell-Hausdorff formula can be written entirely as the sum of commutators.

d) Extra item: Expand $\log(\mathrm{e}^X \mathrm{e}^Y)$ up to 3rd order in X and Y in order to get the same approximation.

2 A counterexample

The following matrices are given:

$$X = \pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

Let us show that there is no matrix $Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^Z$.

a) Compute $\mathrm{e}^X, \mathrm{e}^Y$ and $\mathrm{e}^X \mathrm{e}^Y$.

b) Prove that a matrix Z must have zero trace to satisfy what we want. Also, determine Z^2 and e^Z.

c) Prove that such a matrix Z does not exist.

3 The Heisenberg Group

The Lie algebra of the Heisenberg group is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{5}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. The Lie algebra is related to the canonical commutation relations. Indeed, if we define

$$q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{p}{i\hbar} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

then $[q, p] = i\hbar z$ and $[p, z] = [q, z] = 0$.

The *center* $Z(\mathfrak{g})$ of a Lie algebra is the set of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Find the center of the Lie algebra of the Heisenberg group and show that $[X, Y] \in Z(\mathfrak{g})$ for all $X, Y \in \mathfrak{g}$. Use these results to show with an explicit calculation that

$$\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]}, \tag{7}$$

for all $X, Y \in \mathfrak{g}$.

4 Ad and ad

Let matrices X, Y be given. Recall that the adjoint mapping $\mathrm{ad}\, X$ is defined by $(\mathrm{ad}\, X)(Y) = [X, Y]$.

a) Using induction, prove that

$$(\mathrm{ad}\, X)^n(Y) = \sum_{k=0}^{n} \binom{n}{k} X^k \, Y \, (-X)^{n-k}. \tag{8}$$

b) Prove that

$$\mathrm{e}^{\mathrm{ad}\, X}(Y) = \mathrm{Ad}(\mathrm{e}^X)(Y) := \mathrm{e}^X \, Y \, e^{-X}. \tag{9}$$

The last equation is the definition of $\mathrm{Ad}(X)(Y)$.