Aspects of Classical Physics - Tutorial 3

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Dilatation in field theory

Scale invariance plays a very important role in many physical systems, for example in Statistical Mechanics or Conformal Field Theory. Let us study some of its features on scalar fields on d-dimensional Minkowski spacetime under the light of Noether's theorem.

A scalar field ϕ is said to have weight w with respect to dilatations if $x' = \lambda\, x$ induces a transformation on ϕ such that $\phi'(x') = \lambda^w \phi(x)$.

a) Express $\phi'(x)$ in terms of ϕ, x, λ and w. Use this to compute $\delta\phi(x) := \phi'(x) - \phi(x)$.

b) Take the Lagrangian $\mathcal{L}_m = \frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{m^2}{2}\phi^2$. How is $S[\phi']$ related to $S[\phi]$? Under which conditions on m, d and w is $\delta\phi$ a symmetry of the action?

c) In the following we set $m = 0$. Consider an infinitesimal dilatation $\lambda = 1+\epsilon$ with $\epsilon \ll 1$, expand $\delta\phi$ to first order in ϵ and compute $\delta\mathcal{L}_0$. Rederive the condition of b) for the infinitesimal $\delta\phi$ to be a symmetry of the (massless) Lagrangian. Hint: $K^\mu = -\epsilon\, x^\mu \mathcal{L}_0$.

d) Write the Noether current j^μ associated with this symmetry for a general Lagrangian $\mathcal{L}$, then specialize to $\mathcal{L}_0$. Writing $j^\mu =: -\epsilon\, J^\mu$ defines the "dilatation current". For $\mathcal{L} = \mathcal{L}_0$ check the condition under which J^μ becomes divergence-free, i.e. $\partial_\mu J^\mu = 0$. Hint: You must use the equation of motion $\Box\phi = 0$.

e) Write explicitly the energy-momentum tensor $T^\mu_{\ \nu}$ for $\mathcal{L}_0$ and compute its trace $T^\mu_{\ \mu}$. Now express $J^\mu = x^\rho\, T^\mu_{\ \rho} + R^\mu$ and determine the "remainder" R^μ.

f) We want to employ the ambiguity in the definition of the Noether currents to remove the remainder R. For this we should "improve" both currents T and J to $\widehat{T}$ and $\widehat{J}$ in such a way that they obey just $\widehat{J}^\mu = x^\rho\, \widehat{T}^\mu_{\ \rho}$. Show that this requires $\widehat{T}$ to be traceless. Hint: The energy-momentum conservation $\partial_\mu \widehat{T}^\mu_{\ \rho} = 0$ must be assumed.

g) We firstly improve $\widehat{T}^\mu_{\ \rho} = T^\mu_{\ \rho} + a(\partial^\mu\partial_\rho - \delta^\mu_{\ \rho}\Box)\phi^2$. Check that $\widehat{T}$ is still divergence-free. Which choice for a renders $\widehat{T}$ traceless? Hint: Use $\Box\phi^2 = 2\phi\Box\phi + 2\partial\phi\cdot\partial\phi$ and $\Box\phi = 0$.

h) Secondly, compute the improved current $\widehat{J}^\mu =: J^\mu + S^\mu$ by combining the above relations. Show that $\partial_\mu S^\mu = 0$ so that the improvement does not ruin the conservation of the dilatation current. On general grounds, you should find that $S^\mu = \partial_\rho f^{\rho\mu}$ with $f^{\rho\mu} = -f^{\mu\rho}$. Do you?