Tutorial 3 - Fundamental Interactions

Olaf Lechtenfeld, Gabriel Picanço

May 2021

1 Yukawa Potential

Both in the lecture and in the book (equation (5.16)) you saw that we can obtain the non-relativistic limit for the Yukawa potential from the Fourier transform of a scattering amplitude, as seen in the following equation

$$V_{\text{Yukawa}} = -g^2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^2 + m^2}. \tag{1}$$

Let us compute this Fourier transform.

a) Partially integrate the above expression to get it only in terms of m, $r = |\vec{x}|$ and $q = |\vec{q}|$.

b) You should have obtained an odd function of q in the integrand. Using this fact, extend the domain of integration and then close the contour of integration in the complex plane to get the correct result (be careful to pick up the correct pole). If you don't know residue theorem, use https://www.wolframalpha.com/ instead to compute the integral that you obtained in item a) and reach the intended result.

2 Explicit Lorentz invariance

When computing physical relevant quantities it is usual to get integrals of the form

$$I = \int \frac{d^3\vec{p}}{2E_{\vec{p}}} f(E, \vec{p}, t, \vec{x}), \tag{2}$$

where $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} =: p^0$. Even if $f(p, x)$ is Lorentz invariant (with $p = (p^0, \vec{p})$, $x = (x^0, \vec{x})$, and $x^0 = ct$) it is not obvious that the integral itself is invariant because of the integration measure and the energy in the integrand. Show that

$$\frac{d^3\vec{p}}{2E_{\vec{p}}} \propto d^4p\, \delta(p^2 - m^2)\, \Theta(p^0) \tag{3}$$

and argue that this expression is manifestly (proper-)Lorentz invariant.

3 Extra: Intro to Part III of the book

a) The Levi Civita symbol is not a tensor (it is in fact a tensor density) but it is left invariant by any volume-preserving linear transformation (that is, linear transformations with determinant equals 1), be it real or complex, in any dimension. Show explicitly that this is true for 2-dimensional volume-preserving complex linear transformations. That is, for $M \in SL(2, \mathbb{C})$, show that $M_{ai} M_{bj} \epsilon_{ij} = \epsilon_{ab}$. Conclude that it is also true for $SL(2, \mathbb{R})$, $SU(2)$ and $SO(2)$ transformations.

b) Rotations are in general not commutative (equivalently, the product of rotation matrices is in general not commutative). But, using the so-called exponential map, we can construct all of them using a linear vector space of matrices (the algebra of the rotation group), which is easier to deal with. In matrix groups the exponential map is the usual exponential function. Show explicitly that $\theta_i T_i$ (no sum), where the θ_i's are constants (angles) and the T_i's are the three matrices such that $[T_i]_{jk} = -\epsilon_{ijk}$, generate, via the exponential map, the rotations of $SO(3)$ around the three cartesian axis x, y and z with the respective angles.

Remarks: The three T matrices form a basis of the $\mathfrak{so}(3)$ algebra.

It is worth remembering that, as these matrices doesn't commute, there is a difference between taking the exponential of a linear combination of algebra elements, $\exp(\theta_i T_i)$ (sum intended), or composing rotations generated individually by each of the algebra elements with the respective angles, $\exp(\theta_1 T_1) \cdot \exp(\theta_2 T_2) \cdot \exp(\theta_3 T_3)$. Look for Baker-Campbell-Hausdorff (on the Wikipedia, for example) for details if you want to remember this.