Tutorial 5 - Fundamental Interactions

Olaf Lechtenfeld, Gabriel Picanço

Jun 2023

1 Gamma matrices

1.1 Useful properties

Let $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\not{p} = p_\mu\gamma^\mu$. Using those definitions and the algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, without any explicit matrix forms, prove the following statements:

a) $\mathrm{Tr}(\gamma^\mu) = 0$

b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

c) $(\gamma^5)^2 = 1$

d) $\mathrm{Tr}(\gamma^5) = 0$

e) $\mathrm{Tr}(\not{p}\not{q}) = 4p \cdot q$

f) $\mathrm{Tr}(\not{p}_1 ... \not{p}_n) = 0$ if n is odd

g) $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$

h) $\gamma_\mu \not{p}_1 \not{p}_2 \gamma^\mu = 4p_1 \cdot p_2$

It is also true that

$$\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)] \quad (1)$$

and

$$\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma. \quad (2)$$

Feel free to try to show those if you'd like.

1.2 Construction in any dimension

For this part let us work in d dimensions and use the Euclidian signature such that $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ (to obtain these we just need to redefine the gamma matrices that square to -1 with an extra factor of $-i$). Let σ_1, σ_2 and σ_3 be the usual Pauli matrices. We can always use them to construct d $\lfloor d/2 \rfloor \times \lfloor d/2 \rfloor$ gamma matrices that form a representation of the algebra.

a) For $d = 4$, take $\gamma^1 = \sigma_1 \otimes 1$ and $\gamma^2 = \sigma_2 \otimes 1$. Find γ^3 and γ^4.

b) Add one more tensor factor and extend the above idea to guess how to generate the gamma matrices for $d = 6$. Then generalize it for any even dimension, $d = 2n$, and write the expressions for the gamma matrices.

c) Finally, let us construct the representations for any odd d. Find how to add one more matrix γ^{2n+1} in the set $\{\gamma^1, ..., \gamma^{2n}\}$ that represents the algebra for $d = 2n$ such that the set $\{\gamma^1, ..., \gamma^{2n}, \gamma^{2n+1}\}$ represents it for $d = 2n + 1$.

2 Covariant Derivative of a Dirac field

The Lagrangian for the free Dirac field reads

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{3}$$

This Lagrangian has a global symmetry with $\psi \to \mathrm{e}^{-i\lambda}\psi$ (and $\bar{\psi} \to \mathrm{e}^{i\lambda}\bar{\psi}$), for q and $\lambda \in \mathbb{R}$, forming a global $U(1)$ symmetry group.

a) Check if the Lagrangian is invariant by a local $U(1)$ transformation, that is, with $\psi \to \mathrm{e}^{-i\lambda(x)}\psi$.

As you were able to check, if we want to impose a (gauge) local $U(1)$ symmetry we need to make other changes such that the Lagrangian becomes symmetric. The way to do this is to insert a vector field $A_\mu(x)$, defining a covariant derivative operator $D_\mu = \partial_\mu + iqA_\mu(x)$. This is similar to what we do in General Relativity when we use the Christoffel symbols (connection) to define a covariant derivative operator.

b) Find how the connection/vector field $A_\mu(x)$ transforms as a function of $\lambda(x)$ such that $D_\mu\psi \to \mathrm{e}^{-iq\lambda(x)}D_\mu\psi$ (which makes the Lagrangian symmetric).

c) Write the Lagrangian $\mathcal{L}_1$ including in $\mathcal{L}_0$ the kinetic term associated with the vector field $A_\mu(x)$. Can we insert a mass term for $A_\mu(x)$ as well while preserving the gauge symmetry?

d) Use the action associated with $\mathcal{L}_1$ to find the equations of motion for both fields.