Exercise sheet 3: Differential geometry and metrics

Please prepare your solutions, ready to present in the class on **11.05.2022** at **16:00**.

1. Consider the following vector fields on $\mathbb{R}^3$:

$$U(x, y, z) = z\partial_x - 2z\partial_y + (2y - x)\partial_z,$$
$$V(x, y, z) = y\partial_x - x\partial_y.$$

   (a) Compute the Lie bracket $[U, V]$.

   (b) By considering the one-form $\omega = x\ \mathrm{d}x + y\ \mathrm{d}y + z\ \mathrm{d}z$, show that U and V, when restricted to $S^2 \subset \mathbb{R}^3$, define vector fields on S^2, i.e. lie in the tangent bundle TS^2.

2. The hyperboloid H^2 consists of all $X = (t, x, y) \in \mathbb{R}^{1,2}$, $t > 0$, with $X^\mu X_\mu = -1$, i.e.

$$-t^2 + x^2 + y^2 = -1.$$

   (a) By considering the Euler–Lagrange equations for the lagrangian

$$\mathcal{L}[X(s)] = \dot{X}^\mu \dot{X}_\mu + \lambda(X^\mu X_\mu + 1), \quad \dot{X}^\mu := \partial_s X^\mu,$$

   show that the geodesics on H^2, embedded in $\mathbb{R}^{1,2}$, are of the form

$$X^\mu(s) = A^\mu e^{\sqrt{\lambda}s} + B^\mu e^{-\sqrt{\lambda}s}, \quad \lambda > 0,$$

   where A^μ and B^μ are null vectors in $\mathbb{R}^{1,2}$ satisfying $A^\mu B_\mu = -\frac{1}{2}$.

   (b) Show that H^2 may be parameterised as $(t, x, y) = (\cosh\theta, \sinh\theta\cos\phi, \sinh\theta\sin\phi)$. Let g be the metric on H^2 determined by restriction of the Minkowski metric on $\mathbb{R}^{1,2}$ to H^2. Using the above parameterisation, determine the components $g_{\mu\nu}$ of the metric in the coordinates $(r, \phi) \in [0, \infty) \times [0, 2\pi)$, where $r = \sqrt{x^2 + y^2}$.

   (c) Consider new coordinates $(\rho, \varphi) \in [0, 1) \times [0, 2\pi)$ where $r = \frac{2\rho}{1-\rho^2}$ and $\phi = \varphi$.

      i. Calculate the components of the metric g on H^2 in the new coordinates (ρ, φ).

      ii. For some choice of coordinates x^α on H^2, define $D_{\alpha\beta\gamma} = \partial_\alpha g_{\beta\gamma}$. Calculate $D_{\alpha\beta\gamma}$ for the coordinates (r, ϕ) and (ρ, φ). Use your answer to explain why $D_{\alpha\beta\gamma}$ does not define a tensor.

   (d) Using your answer to part (a), determine general formulae for the geodesics on H^2 in the new coordinates as curves $(r(s), \phi(s))$ and $(\rho(s), \varphi(s))$. Plot examples of these in the (r, ϕ) and (ρ, φ) regions.

3. Consider the metric described by the line-element

$$\mathrm{d}s^2 = \left(y^2 - 4x^2 + 2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + 4xy\left(\mathrm{d}x^2 - \mathrm{d}y^2\right) + (8x^2 - 2y^2)\mathrm{d}x\mathrm{d}y$$

   Is the metric well-defined for all $(x, y) \in \mathbb{R}^2$? Determine, in terms of $(x, y) \in \mathbb{R}^2$ where it is well-defined, the signature of the metric.