Exercise sheet 4: Covariant derivatives and geodesics

Please prepare your solutions, ready to present in the class on **18.05.2022** at **16:00**.

1. Let the line-element
$$\mathrm{d}s^2 = -\mathrm{e}^{2x}\mathrm{d}t^2 + \mathrm{e}^{2y}\mathrm{d}x^2 + \mathrm{e}^{2t}\mathrm{d}y^2 \tag{1}$$
define a metric g for a $1+2$ dimensional space M.
   (a) For a curve $x^\mu(s)$ in (M, g), compute the Euler–Lagrange equations for the lagrangian
   $$\mathcal{L}(x^\mu, \partial_s x^\mu) = \partial_s x^\mu \partial_s x_\mu, \quad x^\mu = x^\mu(s).$$
   (b) By comparing your answer to part (a) with the geodesic equations, determine the Christoffel symbols for the metric (1). Check your answer against the formula for the Christoffel symbols involving the metric.

2. Using the metric on $\mathbb{R}^2$ described by the line-element
$$\mathrm{d}s^2 = (x + \sin^2 y + 1)^2\, \mathrm{d}x^2 + \mathrm{e}^{2x}\, \mathrm{d}x\mathrm{d}y + (2 + \cos x)^2\, \mathrm{d}y^2,$$
determine the proper length of the curves γ_1 and γ_2, where
   (a) γ_1 goes between $(0, 0)$ and $(2, 0)$, and remains on the x-axis;
   (b) γ_2 goes from $(0, 0)$ to $(1, 1)$ along $x = y$.

   You may leave you answers in terms of an integral. Are γ_1 or γ_2 geodesics?

3. Consider a curve $x^\alpha(\lambda)$ in a spacetime (M, g). For vectors v^α defined along this curve, define the *Fermi–Walker derivative* as
$$\frac{D^F}{\mathrm{d}\lambda} v^\alpha := \frac{D}{\mathrm{d}\lambda} v^\alpha + 2u^{[\beta} a^{\alpha]} v_\beta,$$
where we have denoted $\frac{D}{\mathrm{d}\lambda}$ the directional derivative along the curve, $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$ the velocity (tangent vector), and $a^\alpha = \frac{D}{\mathrm{d}\lambda} u^\alpha$ the (covariant) acceleration. A vector v^α is said to be *Fermi–Walker transported* along x^α if its Fermi–Walker derivative vanishes, i.e. $\frac{D^F}{\mathrm{d}\lambda} v^\alpha = 0$.
   (a) Consider a timelike curve $x^\alpha(\tau)$, parameterised by its proper time (i.e. $\lambda = \tau$). Show that the velocity $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ is itself Fermi–Walker transported along x^α.
   (b) Show that if two vectors v^α and w^α are Fermi–Walker transported along a curve, then their inner product $v^\alpha w_\alpha$ remains constant.
   (c) For which class of curves is Fermi–Walker transport equivalent to parallel transport? Justify your answer.