Exercise sheet 6: Curvature tensors, and some general relativity

Please prepare your solutions, ready to present in the class on **01.06.2022** at **16:00**.

1. Let $\mathbb{H}^3$ denote the upper half-space model of hyperbolic space, that is the set of all $(x, y, z) \in \mathbb{R}^3$ with $z > 0$, and metric with components $g_{\mu\nu} = \frac{1}{z^2}\delta_{\mu\nu}$. For $0 < R \leq 1$, consider the subspace
$$M_R = \{(x, y, z) \in \mathbb{H}^3 \, : \, x^2 + y^2 + (z-1)^2 = R^2\}.$$
   (a) Parameterise M_R using spherical coordinates (φ, θ), and then compute the metric on M_R in these coordinates as that induced from the metric on $\mathbb{H}^3$.
   (b) Compute the Riemann curvature tensor for M_R. What is the interpretation of the case $R = 1$?

2. Let (M, g) be an n-dimensional $(n \geq 3)$ *Einstein* manifold, that is, where its Ricci curvature is proportional to the metric, i.e. $R_{ab} = \kappa g_{ab}$.
   (a) Determine the factor κ in terms of the dimension n and the scalar curvature R.
   (b) Show that $\kappa \in \mathbb{R}$ must be constant (*Hint:* use a Bianchi identity).

3. A good approximation to the metric outside the surface of Earth is given in spherical polar coordinates (t, r, φ, θ) by the line-element
$$\mathrm{d}s^2 = -(1 + 2\Phi)\,\mathrm{d}t^2 + (1 - 2\Phi)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2),$$
where $\Phi = -\frac{GM}{r}$ is the familiar Newtonian gravitational potential. This coordinate system is fixed to Earth's surface, i.e. a point on the surface has coordinate (t, r, φ, θ) with $r = R_1$ and φ, θ constant.
   (a) Imagine a clock on the surface of Earth at distance R_1 from the centre of Earth, and another clock on top of a tall building at distance R_2 from the centre of Earth. Calculate the time elapsed by each clock as a function of the coordinate time t. Which clock runs faster?
   (b) Solve the geodesic equations in the case of a circular orbit around the equator of Earth $(\theta = \frac{\pi}{2})$. Determine $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ for such a geodesic.
   (c) Determine the proper time elapsed by a satellite completing one orbit at radius R_1 (that is, only just skimming Earth's surface). By using the physical values
$$R_1 = 6.37 \times 10^6 \text{ m}, \quad M = 5.97 \times 10^{24} \text{ kg}, \quad G = 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$
   and restoring the speed of light as $c = 3.00 \times 10^8 \text{ ms}^{-1}$, determine your answer in seconds. How does this number compare to that of the clock from part (a)?