Exercise sheet 8: Einstein–Maxwell theory, and spherically-symmetric spacetime

Please prepare your solutions, ready to present in the class on **22.06.2022** at **16:00**.

1. The Lagrange density for an electromagnetic potential A_μ in a spacetime (M, g) is
$$\mathcal{L} = \sqrt{-g}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right),$$
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, and J^μ is the conserved current.
   (a) Derive from $\mathcal{L}$ the energy-momentum tensor for electromagnetism on (M, g), and show that it satisfies the dominant energy condition.
   (b) Derive Maxwell's equations on (M, g).
   (c) Now consider the additional term added to the Lagrange density of the form
$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}F_{\nu\sigma}, \quad \beta > 0.$$
      i. How does this affect Maxwell's and Einstein's equations?
      ii. Is the current J^μ still conserved?

2. A static, spherically-symmetric metric in four-dimensional spacetime takes the form
$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{1}$$
   (a) Verify that the only non-zero components of the Ricci tensor are
$$R_{tt} = \frac{1}{h}\left(\frac{1}{2f}(f''f - f'^2) - \frac{1}{4}\frac{f'h'}{h} + \frac{f'}{r}\right), \quad R_{rr} = -\left(\frac{1}{2f^2}(f''f - f'^2) - \frac{1}{4}\frac{f'h'}{fh} - \frac{1}{r}\frac{h'}{h}\right),$$
$$R_{\theta\theta} = \frac{1}{\sin^2\theta}R_{\phi\phi} = \frac{1}{h}\left(\frac{r}{2}\left(\frac{h'}{h} - \frac{f'}{f}\right) - 1\right) + 1.$$
   (b) Solve the vacuum Einstein's equations with cosmological constant Λ for the metric (1) (*Hint: consider the combination* $\frac{h}{f}R_{tt} + R_{rr}$). You should choose the integration constants by comparing with the Schwarzschild case $\Lambda = 0$.
   (c) Write down the geodesic equations for an affinely parameterised geodesic. Show that these allow for equatorial geodesics (i.e. when $\theta = \frac{\pi}{2}$).
   (d) Recognise there is a timelike and a spacelike Killing vector field for the metric (1). Recall that, given a Killing vector field K^μ, $p_\mu K^\mu$ is conserved along a geodesic $x^\mu(\tau)$ with tangent vector $p^\mu = \dot{x}^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$. Use this to find two conserved quantities for geodesics. Obtain a third conserved quantity from the normalization of p^μ and express it in terms of the first two conserved quantities and $\dot{r}^2$ for the case of a spacelike, timelike, and null equatorial geodesic.
   (e) Show that conservation of the quantities derived in (d) imply the geodesic equations.