Quantum Field Theory: Exercise Session 2

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Problem 1: Spinor Representation of the Lorentz Algebra

The Clifford algebra is given by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}. \tag{1}$$

The Weyl representation of this algebra is given by the following choice of gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \text{where} \quad \sigma^\mu = (\mathbb{1}, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i), \tag{2}$$

with σ^i being the Pauli matrices satisfying $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ and $(\sigma^i)^\dagger = \sigma^i$.

Dirac realized that given any representation of the Clifford algebra, one can immediately write down a spinor representation of the Lorentz algebra as

$$S^{\mu\nu} = \frac{\mathrm{i}}{4}\,[\gamma^\mu, \gamma^\nu]\,. \tag{3}$$

The spinor representation of the Lorentz group is then obtained via exponentiation,

$$\Lambda_{(D)} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\epsilon_{\mu\nu}S^{\mu\nu}}, \tag{4}$$

where $\epsilon_{\mu\nu}$ are the parameters for infinitesimal Lorentz transformations.

A four-component field ψ that transforms accordingly as

$$\psi(x) \to \Lambda_{(D)}\psi(\Lambda_{(V)}^{-1}x) \tag{5}$$

under Lorentz transformations is called a Dirac spinor.

(a) Compute the Lorentz generators $S^{\mu\nu}$ using the Weyl representation of the gamma matrices.

(b) Is this representation irreducible? What does this imply for the Dirac spinor?

(c) Is this representation Hermitian? Is the corresponding representation of the Lorentz group $\Lambda_{(D)}$ unitary?

(d) Write down the explicit matrix representing a rotation through an angle θ about the z-axis, that is taking $\epsilon_{12} = -\epsilon_{21} = \theta$ and all other components zero. What is the result for $\theta = 2\pi$?

Problem 2: Weyl spinors, helicity and parity

We can define a "fifth" gamma matrix

$$\gamma^5 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{6}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor with $\epsilon^{0123} = 1$. Using the anticommutation relations (1), one can easily show that $\{\gamma^5, \gamma^\mu\} = 0$.

Left- and right-handed spinors are defined by the following conditions:

$$\gamma^5\psi_L = -\psi_L, \qquad \gamma^5\psi_R = \psi_R. \tag{7}$$

(a) Show that an arbitrary Dirac spinor can be split into a left-handed and right-handed part as

$$\psi = \psi_L + \psi_R, \qquad \psi_{L,R} = P_{L,R}\psi, \qquad P_{L,R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5). \tag{8}$$

Hint: first show that $(\gamma^5)^2 = \mathbb{1}$.

(b) Compute γ^5, $P_{L,R}$ and ψ in the Weyl representation, and show how $\psi_{L,R}$ transform under Lorentz transformations.

(c) Write down the Dirac Lagrangian in terms of the Weyl spinors $\psi_{L,R}$.

(d) Show that in the massless case $\psi_{L,R}$ are eigenstates of the helicity operator

$$h = \frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|}. \tag{9}$$

What are the eigenvalues?

Hint: use the Dirac equation in the Weyl representation.

(e) In the Standard Model of Particle Physics, the fields mediating the weak interaction, the $W^\pm, Z$ bosons, couple only to the left-handed components of spin 1/2 fields. E.g. the vector Z boson, has a coupling of the form

$$Z_\mu\bar{\psi}_L\gamma^\mu\psi_L = Z_\mu\bar{\psi}\gamma^\mu\frac{1}{2}(\mathbb{1} - \gamma^5)\psi \tag{10}$$

in the Lagrangian. Does this interaction respect parity, $(t, \vec{x}) \to (t, -\vec{x})$?

Hint: Show that $(P\psi)(t, \vec{x}) = \gamma^0\psi(t, -\vec{x})$, i.e. parity interchanges left- and right-handed spinors.