Quantum Field Theory: Exercise Session 1 23 April 2012

Lecturer: Olaf Lechtenfeld
Assistant: Susha Parameswaran

Problem 1: Poincaré algebra for the real scalar field

The real Klein-Gordon field $\phi(x)$ is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2\,. \tag{1}$$

We can write $\phi(x)$ and its conjugate momentum $\pi(x)$ in terms of the creation and annihilation operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$

$$\phi(\vec{x}) \;=\; \int \tilde{dk}\left[a_{\vec{k}}\,e^{i\vec{k}.\vec{x}} + a^\dagger_{\vec{k}}\,e^{-i\vec{k}.\vec{x}}\right] \tag{2}$$

$$\pi(\vec{x}) \;=\; -i\int \tilde{dk}\omega_k\left[a_{\vec{k}}\,e^{i\vec{k}.\vec{x}} - a^\dagger_{\vec{k}}\,e^{-i\vec{k}.\vec{x}}\right]\,, \tag{3}$$

where

$$\tilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k}, \qquad \omega_k \equiv \sqrt{\vec{k}^2+m^2}, \qquad \vec{k}.\vec{x} \equiv k^i\delta_{ij}x^j\,. \tag{4}$$

In the lectures, you have seen that the energy and momentum may be written in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ as

$$P^0 \equiv H \equiv \int d^3x\,T^{00} \;=\; \int d^3x\,\frac{1}{2}\left(\pi^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2\right)$$
$$= \int \tilde{dk}\,\omega_k\,a^\dagger_{\vec{k}}\,a_{\vec{k}},\,, \tag{5}$$

and

$$P^i \equiv \int d^3x\,T^{0i} = \int d^3x\,\pi(x)\nabla^i\phi(x) = \int \tilde{dk}\,k^i\,a^\dagger_{\vec{k}}\,a_{\vec{k}}\,, \tag{6}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. The Lorentz generators are given by:

$$M^{\mu\nu} \equiv \int d^3x\,\left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right)\,. \tag{7}$$

(a) Use the above expressions to write down the boost and rotation generators, M_{i0} and M_{ij}, in terms of $\phi(x)$ and $\pi(x)$.

(b) Use the Fourier expansions of $\phi(x)$ and $\pi(x)$ to express the rotation generators M_{ij} in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

Tip:

$$\int d^3x x_i\, e^{i\vec{k}.\vec{x}} = \int d^3x \left(-i\frac{\partial}{\partial k_i}\right) e^{i\vec{k}.\vec{x}} = -i(2\pi)^3 \frac{\partial}{\partial k_i}\delta^3(\vec{k})\,. \tag{8}$$

(c) Compute the commutators $[P^i, \phi(x)]$ and $[M^{ij}, \phi(x)]$ in terms of $\phi(x)$, with the help of the commutator relations for $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(d) *Optional* Repeat exercises (b) and (c) for the boost generators M_{0i}, to find:

$$M_{i0} = -i \int \tilde{dk}\omega_k a^\dagger_{\vec{k}} \partial_{k_i} a_{\vec{k}} \tag{9}$$

and $[M^{i0}, \phi(x)]$. Check that the commutator of M_{i0} with M_{j0} satisfies the Lorentz algebra:

$$\left[M^{\sigma\tau}, M^{\alpha\beta}\right] = i\left(\eta^{\tau\alpha}M^{\sigma\beta} + \eta^{\sigma\beta}M^{\tau\alpha} - \eta^{\sigma\alpha}M^{\tau\beta} - \eta^{\tau\beta}M^{\sigma\alpha}\right)\,. \tag{10}$$

Problem 2: The complex scalar field

The theory of a complex Klein-Gordon scalar field is given by the Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi\,. \tag{11}$$

Since the complex scalar field carries two degrees of freedom, quantizing it gives rise to two independent creation operators. The mode expansion for ϕ is

$$\phi(\vec{x}) = \int \tilde{dk}\left[a_{\vec{k}}\, e^{i\vec{k}.\vec{x}} + b^\dagger_{\vec{k}}\, e^{-i\vec{k}.\vec{x}}\right]\,, \tag{12}$$

where the operators $a_{\vec{k}}$ and $b_{\vec{k}}$ satisfy the commutation relations:

$$\left[a_{\vec{k}}, a^\dagger_{\vec{p}}\right] = \left[b_{\vec{k}}, b^\dagger_{\vec{p}}\right] = \tilde{\delta}(\vec{k} - \vec{p}) \tag{13}$$

with all other commutators vanishing. The creation operators $a^\dagger_{\vec{k}}$ and $b^\dagger_{\vec{k}}$ create two types of particle, both of mass m and spin zero, which are interpreted as particles and anti-particles.

Notice that $\mathcal{L}$ is invariant under the rigid phase transformation $\phi \to e^{i\alpha}\phi$. Noether's theorem gives a conserved charge associated to this symmetry:

$$Q = i\int d^3x \left(\partial_0\phi^\dagger\phi - \phi^\dagger\partial_0\phi\right)\,. \tag{14}$$

(a) Write down the mode expansion for $\phi^\dagger$ and the conjugate momenta, $\pi, \pi^\dagger$.

(b) Express H and Q in terms of the creation and annihilation operators. Show that $[H, Q] = 0$ and give the interpretation of Q. Comment also on the implications that the theory is free.