Quantum Field Theory: Exercise Session 3 27 November 2012

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Problem 1: Lorentz-covariant quantization of photon field in the light-cone basis and the physical space of states

After fixing to the Lorentz gauge the photon field has the form

$$A_\mu(x) = \int \tilde{\mathrm{d}}k \sum_\lambda \epsilon_\mu^{(\lambda)}(\vec{k}) \left[a_\lambda(\vec{k}) \mathrm{e}^{-\mathrm{i}k\cdot x} + a_\lambda^\dagger(\vec{k}) \mathrm{e}^{\mathrm{i}k\cdot x} \right]. \tag{1}$$

A convenient basis for the four polarization vectors $\epsilon_\mu^{(\lambda)}(\vec{k})$ corresponding to time-like, longitudinal and transverse polarizations is the light-cone basis with

$$\epsilon_\mu^{(\lambda)}(\vec{k}) = \left\{ \frac{1}{\sqrt{2}} \frac{k_\mu}{|\vec{k}|}, \frac{1}{\sqrt{2}} \frac{\bar{k}_\mu}{|\vec{k}|}, \begin{pmatrix} 0 \\ \vec{e_1} \end{pmatrix}, \begin{pmatrix} 0 \\ \vec{e_2} \end{pmatrix} \right\}, \tag{2}$$

where λ takes values $k, \bar{k}, 1, 2$ and

$$k^\mu \equiv (|\vec{k}|, \vec{k}), \quad \bar{k}^\mu \equiv (|\vec{k}|, -\vec{k}), \quad \text{and } \vec{e_i} \cdot \vec{e_j} = \delta_{ij}, \quad \vec{e_i} \cdot \vec{k} = 0 \text{ for } i, j = 1, 2. \tag{3}$$

(a) Compute the commutators ($\lambda, \sigma = k, \bar{k}, 1, 2$)

$$\left[a_\lambda(\vec{k}), a_\sigma^\dagger(\vec{k'}) \right] \tag{4}$$

using

$$a_\mu(\vec{k}) = \sum_\lambda \epsilon_\mu^{(\lambda)}(\vec{k}) a_\lambda(\vec{k}) \tag{5}$$

and

$$\left[a_\mu(\vec{k}), a_\nu^\dagger(\vec{k'}) \right] = -\eta_{\mu\nu} \tilde{\delta}(\vec{k} - \vec{k'}). \tag{6}$$

Hint: Use the scalar product

$$g^{\lambda\sigma} \equiv \epsilon_\mu^{(\lambda)}(\vec{k}) \eta^{\mu\nu} \epsilon_\nu^{(\sigma)}(\vec{k}). \tag{7}$$

(b) The one-particle Fock space $H^{(1)}$ is the vector space spanned by the states $a_\lambda^\dagger(\vec{k})|0\rangle$. Determine explicitly the subspace $H^{(1)}_{\text{inv}} \subset H^{(1)}$ of states $|\psi\rangle$ which satisfy the Gupta-Bleuler condition

$$\partial_\mu A^{\mu(+)}|\psi\rangle = 0, \quad \text{where } A_\mu^{(+)} \equiv \int \tilde{\mathrm{d}}k \, a_\mu(\vec{k}) \mathrm{e}^{-\mathrm{i}k\cdot x}. \tag{8}$$

(c) Compute the scalar product of the state $a^\dagger_{\bar{k}}|0\rangle \in H^{(1)}_{\text{inv}}$ with an arbitrary vector in $H^{(1)}_{\text{inv}}$. How would you define a physically reasonable state space $H^{(1)}_{\text{phys}}$?

Problem 2: Lorentz-covariant quantization of the photon field with a general gauge-fixing term

An action which gives rise to Maxwell's equations in the Lorentz gauge is

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \alpha \left(\partial_\mu A^\mu \right)^2 \right] = \frac{1}{2} \int d^4x \, A_\mu \left[\Box \eta^{\mu\nu} + (\alpha - 1) \partial^\mu \partial^\nu \right] A_\nu. \quad (9)$$

In Problem 1 we used what is called the Feynman gauge with $\alpha = 1$.

(a) Determine the propagator in momentum space. What happens when $\alpha = 0$?

(b) Write down the equations of motion for $A_\mu(x)$ and show that $A_\mu(x)$ satisfies

$$\Box \partial^\mu A_\mu = 0 \quad \text{and} \quad \Box\Box A_\mu = 0 \quad (10)$$

(c) Solve the equations (10) subject to the constraint $A_\mu = A^\dagger_\mu$ and so that the equations of motion hold.

Hint: With the ansatz $A_\mu(x) = A_\mu(t) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$, the general solution to

$$\left(\frac{d^2}{dt^2} + p^2 \right)^2 A_\mu(t) = 0 \quad (11)$$

is $A_\mu(t) = (a^\dagger_\mu + t\, b^\dagger_\mu) \mathrm{e}^{\mathrm{i}pt} + (a_\mu + t\, b_\mu) \mathrm{e}^{-\mathrm{i}pt}$, for some parameters $a^\dagger_\mu, a_\mu, b^\dagger_\mu, b_\mu$.

(d) Given that $[\mathrm{i}P_\mu, A_\nu] = \partial_\mu A_\nu$, compute the commutators $\left[P_0, a^\dagger_\mu(\vec{k}) \right]$ and $\left[P_0, a^\dagger_\lambda(\vec{k}) \right]$ (where $\lambda = k, \bar{k}, 1, 2$). Can the hermitian operator $P^0 = H$ be diagonalized on the one-particle Fock space?

(e) The commutation relations for creation and annihilation operators read $(i, j = 1, 2)$

$$\left[a_i(\vec{k}), a^\dagger_j(\vec{k}') \right] = \delta_{ij} \tilde{\delta}(\vec{k} - \vec{k}'), \quad (12)$$

$$\left[a_k(\vec{k}), a^\dagger_{\bar{k}}(\vec{k}') \right] = -\frac{\alpha + 1}{2\alpha} \tilde{\delta}(\vec{k} - \vec{k}'). \quad (13)$$

Determine the physical state space in the same manner as in Problems 1 (b) and (c).