Quantum Field Theory: Exercise Session 4 12 June 2012

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Perturbation theory for interacting ϕ^4 scalar field theory

Consider a scalar field theory with quartic self-interaction, described by:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{1}$$

In quantum field theory, all n-point correlation functions can be encoded in a single object called the generating functional, $Z[J]$, as:

$$G_n(x_1, x_2, \ldots, x_n) = i^{-n}\left[\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\right]_{J=0} . \tag{2}$$

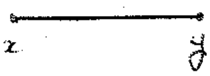
For the ϕ^4 theory, the generating functional is given by:

$$Z[J] = \frac{\exp\left[\left(-i\frac{\lambda}{4!}\right)\int d^4z\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right]Z_0[J]}{\left\{\exp\left[\left(-i\frac{\lambda}{4!}\right)\int d^4z\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right]Z_0[J]\right\}_{J=0}} , \tag{3}$$

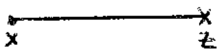
where Z_0 is the free generating functional:

$$Z_0[J] = Z_0[0]\exp[-\frac{1}{2}\int d^4x d^4y J(x)D_F(x-y)J(y)] . \tag{4}$$

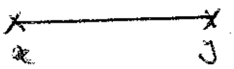
Assuming small interaction coupling, $\lambda \ll 1$, we can use perturbation theory. The Feynman rules for the ϕ^4 theory read:



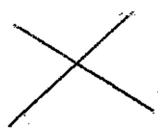
$= D_F(x-y)$



$= \int d^4z\, D_F(x-z)\, iJ(z)$



$= \int d^4x \int d^4y\ iJ(x)\, D_F(x-y)\, iJ(y)$



$= \left(\frac{-i\lambda}{4!}\right)\int d^4z$

1 (a) Apply the functional derivative $\frac{1}{i}\frac{\delta}{\delta J(z)}$ to $Z_0[J]$, and draw the corresponding diagram. What does $\frac{1}{i}\frac{\delta}{\delta J(z)}$ do to a diagram?

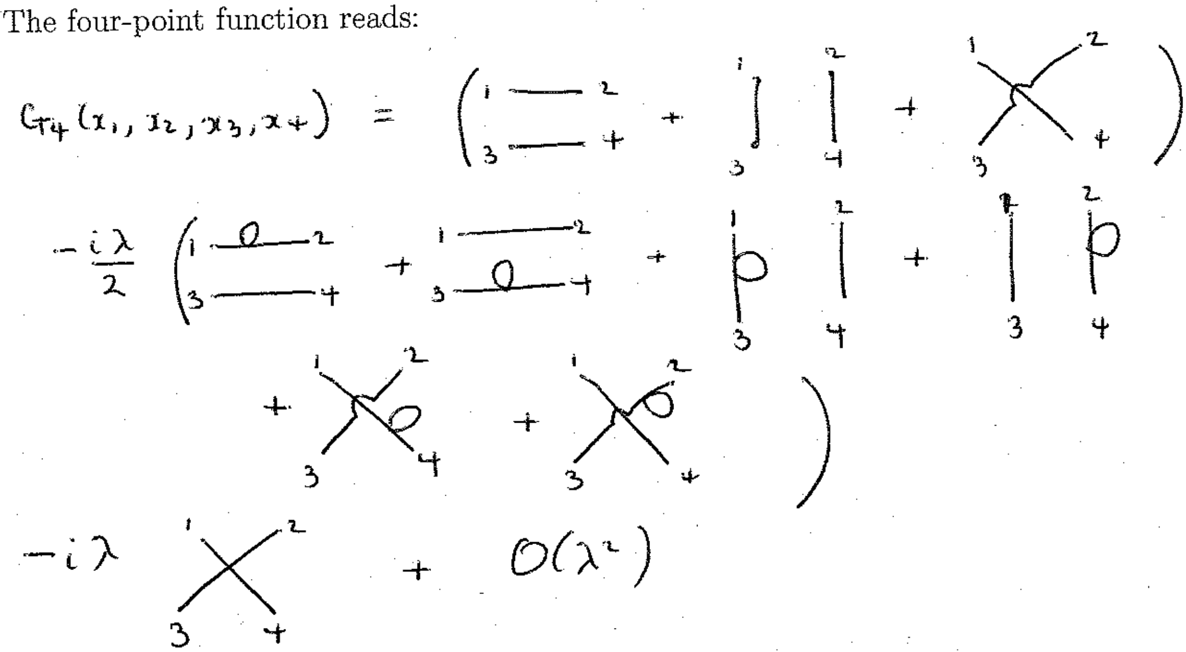
(b) By expanding $Z[J]$ to first order in λ, and applying $\frac{\delta}{i\delta J(z)}$ four times, show that to $\mathcal{O}(\lambda)$:

$$Z[J] = \frac{\left[1 - i\frac{\lambda}{4!}\int d^4x \left(3(D_F(0))^2 + 6D_F(0)\left(\int d^4y D_F(x-y)J(y)\right)^2 + \left(\int d^4y D_F(x-y)J(y)\right)^4\right)\right] Z_0}{\left[1 - i\frac{\lambda}{4!}\int d^4x\, (3(D_F(0))^2)\right] Z_0[0]} \tag{5}$$

You may choose whether to work explicitly or to use the Feyman diagrams.

(c) Show that the vacuum diagrams, which diverge, cancel thanks to the normalization.

2 The four-point function reads:



What diagrams appear in $G_4(x_1, x_2, x_3, x_4)$ at $\mathcal{O}(\lambda^2)$? Take the symmetry factors into account!

3 Only the connected Feynman diagrams in a correlation function contribute to the non-trivial (off-diagonal) part of the S matrix. Show to $\mathcal{O}(\lambda)$ that the functional $W[J] = -i \ln Z[J]$ generates only the connected diagrams of $G_4(x_1, x_2, x_3, x_4)$.