Quantum Field Theory: Exercise Session 4 11 December 2015

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Perturbation theory for interacting ϕ^4 scalar field theory

Consider a scalar field theory with quartic self-interaction, described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

In quantum field theory, all n-point correlation functions can be encoded in a single object called the generating functional, $Z[J]$, as

$$G_n(x_1, x_2, \ldots, x_n) = (-\mathrm{i})^n \left[\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\right]_{J=0}. \tag{2}$$

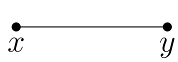
For the ϕ^4 theory, the generating functional is given by

$$Z[J] = \frac{\exp\left(\frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4z\left(-\mathrm{i}\frac{\delta}{\delta J(z)}\right)^4\right)Z_0[J]}{\left\{\exp\left(\frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4z\left(-\mathrm{i}\frac{\delta}{\delta J(z)}\right)^4\right)Z_0[J]\right\}_{J=0}}, \tag{3}$$

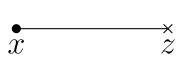
where Z_0 is the free generating functional

$$Z_0[J] = Z_0[0]\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)D_\mathrm{F}(x-y)J(y)\right). \tag{4}$$

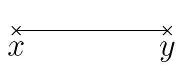
Assuming small interaction coupling, $\lambda \ll 1$, we can use perturbation theory. The Feynman rules for the ϕ^4 theory read:



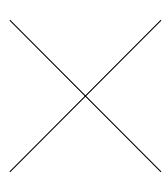
$= D_\mathrm{F}(x-y)$



$= \int \mathrm{d}^4z\, D_\mathrm{F}(x-z)\,\mathrm{i}J(z)$



$= \int \mathrm{d}^4x \int \mathrm{d}^4y\,\mathrm{i}J(x)D_\mathrm{F}(x-y)\,\mathrm{i}J(y)$



$= \left(\frac{-\mathrm{i}\lambda}{4!}\right)\int \mathrm{d}^4z$

1 (a) Apply the functional derivative $-\mathrm{i}\frac{\delta}{\delta J(z)}$ to $Z_0[J]$, and draw the corresponding diagram. What does $-\mathrm{i}\frac{\delta}{\delta J(z)}$ do to a diagram?

(b) By expanding $Z[J]$ to first order in λ, and applying $-\mathrm{i}\frac{\delta}{\delta J(z)}$ four times, show that to $\mathcal{O}(\lambda)$

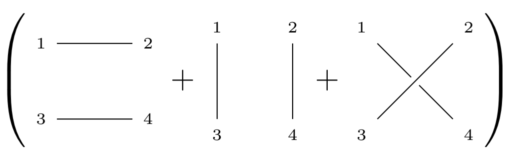
$$Z[J] = \frac{Z_0[J]}{\left[1 - \frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x\,(3(D_\mathrm{F}(0))^2)\right] Z_0[0]} \times \left[1 - \frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x \Bigg(3(D_\mathrm{F}(0))^2 + 6D_\mathrm{F}(0)\left(\int \mathrm{d}^4y D_\mathrm{F}(x-y)J(y)\right)^2 + \left(\int \mathrm{d}^4y D_\mathrm{F}(x-y)J(y)\right)^4\Bigg)\right]. \tag{5}$$

You may choose whether to work explicitly or to use the Feyman diagrams.

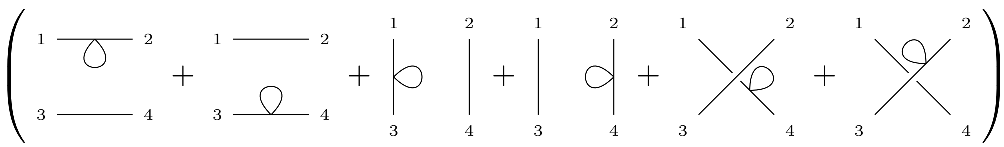
(c) Show that the vacuum diagrams, which diverge, cancel thanks to the normalization.

2 In terms of Feynman diagrams, the four-point function reads

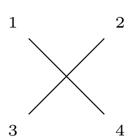
$$G_4(x_1, x_2, x_3, x_4) =$$



$$-\frac{i\lambda}{2}$$



$$-i\lambda$$



$$+\mathcal{O}\left(\lambda^2\right). \tag{6}$$

What diagrams appear in $G_4(x_1, x_2, x_3, x_4)$ at $\mathcal{O}(\lambda^2)$? Take the symmetry factors into account!

3 Only the connected Feynman diagrams in a correlation function contribute to the non-trivial (off-diagonal) part of the S-matrix. Show to $\mathcal{O}(\lambda)$ that the functional $W[J] = -\mathrm{i}\ln Z[J]$ generates only the connected diagrams of $G_4(x_1, x_2, x_3, x_4)$.