11. Präsenzübung, **Statistische Physik**

zu bearbeiten am Donnerstag, 5.1.2012

Aufgabe P25 *Bethe-Peierls approximation for the Ising model*

An improvement to the mean-field theory for an Ising spin system defined by the Hamiltonian $H_{\text{exact}} = -J\sum_{ij}\sigma_i\sigma_j$ can be obtained as follows: the interaction of a spin σ_0 with its z neighbors is treated exactly. The remaining interactions are taken into account by a mean field h', which acts only on the z neighbors. If the original Hamiltonian is translation invariant, it does not matter which spin we choose as σ_0. The simplified Hamiltonian is then given by

$$H = -h'\sum_{j=1}^{z}\sigma_j - J\sum_{j=1}^{z}\sigma_0\sigma_j.$$

The mean field h' is determined self-consistently from the condition $\langle\sigma_0\rangle = \langle\sigma_j\rangle$.

a. Show that the partition function $Z(h',\tau)$ for H has the form $Z = Z_+ + Z_-$ where

$$Z_\pm = \left[2\cosh\left(\frac{h' \pm J}{\tau}\right)\right]^z.$$

b. Express the average values $\langle\sigma_0\rangle$ and $\langle\sigma_j\rangle$ as functions of h'.

c. Show that $h' = 0$ is a solution of the self-consistency equation $\langle\sigma_0\rangle = \langle\sigma_j\rangle$.

d. Show that the self-consistency equation is equivalent to

$$\frac{h'}{\tau} = \frac{z-1}{2}\ln\frac{\cosh[h'/\tau + J/\tau]}{\cosh[h'/\tau - J/\tau]}.$$

This has a nonzero solution provided that the temperature is smaller than a transition temperature τ_c. Determine approximations to τ_c and h' by expanding the right hand side in powers of h'/τ and truncating to order $(h'/\tau)^3$.