Lösungen Blatt 5

Clifford - Algebren & Spinor - Darstellungen

1)
$$[\Sigma^{ab},\Gamma^c] = \frac{i}{4}\left[[\Gamma^a,\Gamma^b],\Gamma^c\right]$$
$$= \frac{i}{4}\left([\Gamma^a\Gamma^b,\Gamma^c] - [\Gamma^b\Gamma^a,\Gamma^c]\right)$$
$$= \frac{i}{4}\left(\Gamma^a\{\Gamma^b,\Gamma^c\} - \{\Gamma^a,\Gamma^c\}\Gamma^b - \Gamma^b\{\Gamma^a,\Gamma^c\} + \{\Gamma^b,\Gamma^c\}\Gamma^a\right)$$
$$= \frac{i}{4}\,2\left(\Gamma^a\eta^{bc} - \eta^{ac}\Gamma^b - \eta^{ac}\Gamma^b + \eta^{bc}\Gamma^a\right)$$
$$= i\left(\Gamma^a\eta^{bc} - \Gamma^b\eta^{ac}\right)$$

2)
$$[\Sigma^{ab},\Sigma^{cd}] = \frac{i}{4}\left[\Sigma^{ab},[\Gamma^c,\Gamma^d]\right]$$
$$= -\frac{i}{4}\left([\Gamma^d,[\Sigma^{ab},\Gamma^c]] + [\Gamma^c,[\Gamma^d,\Sigma^{ab}]]\right) \quad \text{(Jacobi)}$$
$$= -\frac{i}{4}\left([\Gamma^d, i(\Gamma^a\eta^{bc} - \Gamma^b\eta^{ac})] - [\Gamma^c,(i\,\Gamma^a\eta^{bd} - \Gamma^b\eta^{ad})]\right)$$
$$= \frac{1}{4}\left(\eta^{bc}[\Gamma^d,\Gamma^a] - \eta^{ac}[\Gamma^d,\Gamma^b] - \eta^{bd}[\Gamma^c,\Gamma^a] + \eta^{ad}[\Gamma^c,\Gamma^b]\right)$$
$$= -i\left(-\eta^{bc}\Sigma^{ad} + \eta^{ac}\Sigma^{bd} + \eta^{bd}\Sigma^{ac} - \eta^{ad}\Sigma^{bc}\right)$$

3)
$$\Lambda(\Theta) = \exp(i\Theta\,\Sigma^{12})$$
$$= \mathbb{1} + i\Theta\,\Sigma^{12} + \frac{(i\Theta\,\Sigma^{12})^2}{2} + \frac{1}{3!}(i\Theta\,\Sigma^{12})^3 + \frac{1}{4!}(i\Theta\,\Sigma^{12})^4 + \dots$$

$$(\Sigma^{12})^2 = -\frac{1}{16}[\Gamma^1,\Gamma^2]^2$$
$$= -\frac{1}{16}\big(\underbrace{\Gamma^1\Gamma^2 - \Gamma^2\Gamma^1}_{(1)\Rightarrow\ =2\Gamma^1\Gamma^2}\big)^2$$
$$= -\frac{1}{4}\,\Gamma^1\Gamma^2\Gamma^1\Gamma^2$$
$$= \frac{1}{4}\,\underbrace{\Gamma^1\Gamma^1}_{1}\underbrace{\Gamma^2\Gamma^2}_{1}$$
$$= \frac{1}{4}\,\mathbb{1}$$

$$\Rightarrow \Lambda(\Theta) = \mathbb{1} + i\Theta\Sigma^{12} - \frac{\Theta^2}{2}\frac{1}{4}\mathbb{1} - \frac{i\Theta^3}{3!}\Sigma^{12}\cdot\frac{1}{4}\mathbb{1} + \frac{\Theta^4}{4!}\left(\frac{1}{4}\mathbb{1}\right)^2 + \dots$$

$$= \mathbb{1}\left(1 - \frac{1}{2}\left(\frac{\Theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\Theta}{2}\right)^4 - \dots\right) + 2i\frac{\Theta}{2}\Sigma^{12} - \frac{2i}{3!}\left(\frac{\Theta}{2}\right)^3\Sigma^{12} + \dots$$

$$= \mathbb{1}\cdot\cos\frac{\Theta}{2} + i\Sigma^{12}\, 2\sin\frac{\Theta}{2}$$

$$\boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}}$$

$$\rightarrow \quad f(\Theta) = \cos\frac{\Theta}{2}$$
$$g(\Theta) = 2i\sin\frac{\Theta}{2}$$

4) $$\Lambda(\Theta = 2\pi) = \mathbb{1}\cdot\cos\pi + 2i\sin\pi\,\Sigma^{12}$$
$$= -\mathbb{1}$$

Majorana-Bedingung

1) Schur's Lemma (→ Skript Brödel):

$A: V \to V$ lineare Selbstabb., die einen Eigenvektor besitzt
$\mathcal{D}$ irreduzible Darstellung auf VRV

Wenn $\mathcal{D} A = A \mathcal{D}$ gilt, dann hat A die Form

$$A = \lambda \mathbb{1}$$

V endl.dim, kompl. ⇒ A hat einen Eigenvektor.

→ Wir müssen zeigen:

$$B^* B \Gamma^a = \Gamma^a B^* B$$

gl. (7): $\Gamma^{a*} = \eta B \Gamma^a B^{-1}$

$$\Rightarrow \Gamma^{a**} = \eta B^* \Gamma^{a*} (B^{-1})^* = \eta^2 B^* B \Gamma^a B^{-1} (B^{-1})^*$$
$$= \Gamma^a$$

$$\Rightarrow \Gamma^a B^* B = B^* B \Gamma^a$$

Das ist die gesuchte Bedingung. Da B auf einem endl.dim., kompl. VR (dem Darstellungsraum der Γ^a) wirkt, existiert ein Eigenvektor und wir können Schurs Lemma anwenden

$$\rightarrow \quad B^* B = \varepsilon \mathbb{1}$$

2) $B^* B = \varepsilon \mathbb{1} \iff BB^* = \varepsilon^* \mathbb{1}$

$|\det B| = 1 \Rightarrow |\det(B^*)| = |(\det B)^*| = 1$

$\det(B^* B) = \det(BB^*) = \varepsilon^d = \varepsilon^{*d} = (\varepsilon^d)^*$

$\Rightarrow \varepsilon^d \in \mathbb{R}$

$|\det B^* B| = |\det B^*| |\det B| = |\varepsilon^d| = |\varepsilon^{*d}| = 1$

$\Rightarrow \quad \varepsilon = \pm 1$

3) Es gilt: $\Gamma = (-i)^{\frac{d+1}{2}} \Gamma_0 \Gamma_1 \cdots \Gamma_{d-2}$

$$\Gamma_a^* = \eta B \Gamma_a B^{-1}$$

$$\Rightarrow \quad \Gamma^* = \left((-i)^{\frac{d+1}{2}}\right)^* \Gamma_0^* \Gamma_1^* \cdots \Gamma_{d-2}^*$$

$$= i^{\frac{d+1}{2}} \eta^{d-1} B \Gamma_0 B^{-1} B \Gamma_1 B^{-1} \cdots B \Gamma_{d-2} B^{-1}$$

$$= i^{\frac{d+1}{2}} \eta^{d-1} B \Gamma_0 \Gamma_1 \cdots \Gamma_{d-2} B^{-1}$$

(d ungerade) $= i^{\frac{d+1}{2}} B \Gamma_0 \Gamma_1 \cdots \Gamma_{d-2} B^{-1}$

$$\overset{!}{=} \eta (-i)^{\frac{d+1}{2}} B \Gamma_0 \Gamma_1 \cdots \Gamma_{d-2} B^{-1}$$

$$\Rightarrow \quad \cancel{i^{\frac{d+1}{2}}} \overset{!}{=} \eta (-i)^{\frac{d+1}{2}} = \eta (-1)^{\frac{d+1}{2}} \cancel{(i)^{\frac{d+1}{2}}}$$

$$\Leftrightarrow \quad \eta = (-1)^{\frac{d+1}{2}}$$

Diracmatrizen in 4 Dimensionen

1) $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$

$$(\Gamma^0)^2 = -\mathbb{1} = \eta^{00}$$

$$(\Gamma^1)^2 = \mathbb{1} = \eta^{11}$$

2) $\Gamma = \underbrace{(-i)^{\frac{d+1}{2}}}_{(-i)^2 = -1} \Gamma^0 \Gamma^1 = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3) $\left.\begin{array}{l} \gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}\right\} d = 2$

$d = 4$: $\Gamma^0 = \gamma^0 \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & 0 \end{pmatrix}$$

$$\Gamma^1 = \gamma^1 \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix}$$

$$\Gamma^2 = \mathbb{1}_{2\times 2} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

$$\Gamma^3 = \mathbb{1}_{2\times 2} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 0 & -i \\ i & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \end{pmatrix}$$