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Electromagnetic Knots in Minkowski and de Sitter Space

Master Thesis

submitted by

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Hannover, 27.09.2019, Melan Baktiar

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Contents

1. Introduction

The electromagnetic interaction is one of the four fundamental forces of physics. It describes phenomena such as light, electricity, magnetism and optics. It also plays a key role building the structures of atoms and molecules.
Although electromagnetism is well-understood today, there are still some aspects of it, which need a closer consideration.
One of them is the idea of knotted electromagnetic field configurations. An electromagnetic knot is a special electromagnetic field with the property, that any pair of its magnetic lines or any pair of its electric lines is a link corresponding to a certain linking number [1,2,3]. At first, this concept was presented in 1989 by A. Rañada [1] and has been developed since then by a number of papers applicated to other areas in physics [4,5,6].

Knotted field configurations have interesting topological properties to study. Topology itself has always played an important role in theoretical physics, in particular in electromagnetism, for example in the theorems of Gauss and Stokes [4]. Topological arguments were also a key factor in the proposal for magnetic monopoles by Dirac 1931, which implied an explanation for the quantization of the electric charge [4,7]. But also in concepts such as Skyrme model [8] and Yang-Mills theory [9] topological considerations are very important.

Not only by the construction of Rañada, which is based on the Hopf map in the case of $S^3 \to S^2$ with the description of Maxwell equations in differential forms, there are more methods to generate mathematically knotted field configurations. To enumerate the possibilites:
Another way is to use methods from Twistor theory where elementary twistor functions correspond to electromagnetic torus knots.
A different approach is the so called Bateman construction generating null solutions from complex Euler potentials.
A fourth method is relied on special conformal transformations to get new knotted fields from existing ones. More details about these methods are given in [4].

A new approach for knotted electromagnetic fields was given in [10] by O. Lechtenfeld and G. Zhilin. This work was inspired by [11,12], where solutions of Yang-Mills equations are constructed on 4-dimensional de Sitter space dS_4 motivated by string theory. The new method is based on two key points [10,13]:

- The simplicity of analytically solving Maxwell equations on a temporal cylinder over a 3-sphere
- The conformal equivalence of a cylinder patch to the 4-dimensional Minkowski space

The goal of this thesis is to deal with some characteristics of the constructed solutions presented in [10]. More precisely, we will consider the decay behaviour of these solutions and check when we do have so called *null fields*, a special class of electromagnetic fields. Furthermore, we present a detailed consideration of the energy and the helicity of these new solutions.

1.1. Outline

This thesis is organized as follows:

Section 2: We give a short description of de Sitter space, especially the 4-dimensional de Sitter space and show their conformal equivalence to the Lorentzian cylinder $I \times S^3$, where I is a finite interval.

Section 3: We handle the correspondence of solving Yang-Mills equations on Minkowski and de Sitter Space and introduce all the technical tools we need for the progress of this thesis.

Section 4: The new solutions to Maxwell equations are presented which are electromagnetic knots.

Section 5: We discuss the decay behaviour of the corresponding electric and magnetic fields of the solutions in Minkowski space. We will disuss them for 2 cases:
The first one is for large position r with fixed time $t = t_0$ and the second one is for large $r = \pm t$.

Section 6: We check and discuss under which conditions we can generate null fields from the solutions presented in section 4.

Section 7: We derive formulas for computing the energy and the helicity. We also present the calculation of a particular example for these quantities. Furthermore, we discuss the relation between energy and helicity in our context.

Section 8: We summarize our results and give a perspective for possible future research projects.

The sections 2, 3 and 4 are based on [10], but with more further details. Some of the main results in section 7 also appeared already in [10], but the discussion we do goes far beyond the one which was done there. Throughout this thesis we use Einsteins sum convention and we consider all physical quantities with natural units by setting $c = \hbar = 1$.

2. Description and conformal equivalence of the 4-dimensional de Sitter space dS_4 to a finite cylinder $I \times S^3$

In this section, we give a short description of the 4-dimensional de Sitter space dS_4 and show their conformal equivalence to the Lorentzian cylinder $I \times S^3$ with $I := (0, \pi)$. The results of this section were already mentioned in [10,11,12], but for the continuation and the results of this thesis it is indispensable to execute them.

We start with some basics about de Sitter space based on [14,15].
Actually, de Sitter spacetime is a solution of the vacuum Einstein equations with a positive cosmological constant [14,15]. It is interesting to consider de Sitter spacetime since it seems to be relevant for early and late phases of development of the universe [14].
In General Relativity, it is usual to define manifolds and properties like curvature intrinsically and not by embedding them in a higher dimensional spacetime [15]. But in our context, the following is an useful and important fact.

The 4-dimensional de Sitter space dS_4 can be embedded in the 5-dimensional Minkowski space $\mathbb{R}^{1,4}$ [14]. The metric in the embedding Minkowski space is

$$ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^3 + dZ_4^2 \; . \tag{1}$$

The de Sitter space is a hypersurface described by the equation

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = l^2 \; , \tag{2}$$

where l is a non-zero constant with dimensions of length. This equation describes a hyperbola.

Constant Z_0 slices are 3-spheres with varying radius [10,11,12]. We use the parametrization

$$Z_0 = -l \cot \tau \qquad Z_A = \frac{l}{\sin \tau} \omega_A \qquad A = 1, \ldots, 4 \qquad \tau \in I := (0, \pi) \tag{3}$$

with the standard embedding of the *unit* 3-sphere S^3 in $\mathbb{R}^4$ given by

$$\omega_1 = \sin\chi \sin\theta \sin\phi \qquad \omega_2 = \sin\chi \sin\theta \cos\phi \qquad \omega_3 = \sin\chi \cos\theta \qquad \omega_4 = \cos\chi$$

with $\chi, \theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\omega_A \omega_A = 1$.

The metric of dS_4 in such coordinates becomes

$$ds^2 = \frac{l^2}{\sin^2 \tau} \left(-d\tau^2 + d\Omega_3^2 \right) \tag{4}$$

with the metric of the unit 3-sphere

$$d\Omega_3^2 = d\chi^2 + \sin^2 (d\theta^2 + \sin^2 \theta d\phi^2) \; . \tag{5}$$

Hence, the 4-dimensional de Sitter space is conformally equivalent to a finite Lorentzian cylinder over a 3-sphere.

But the $Z_0 + Z_4 < 0$ half of dS_4 is also conformally equivalent to the future half of Minkowski space with the parametrization

$$Z_0 = \frac{t^2 - r^2 - l^2}{2t} \qquad Z_1 = l\frac{x}{t} \qquad Z_2 = l\frac{y}{t} \qquad Z_3 = l\frac{z}{t} \qquad Z_4 = \frac{r^2 - t^2 - l^2}{2t} \tag{6}$$

with

$$x, y, z \in \mathbb{R} \quad t \in \mathbb{R}_+ \quad r^2 = x^2 + y^2 + z^2 \ . \tag{7}$$

The metric of dS_4 becomes

$$ds^2 = \frac{l^2}{t^2}(-dt^2 + dx^2 + dy^2 + dz^2) \ , \tag{8}$$

such that we can cover with these coordinates the future half of Minkowski space $\mathbb{R}^{1,3}_+$. This parametrization can be extended to the whole of Minkowski space by gluing a second copy of dS_4 to provide for the $t < 0$ half corresponding to $Z_0 + Z_4 > 0$.

By comparing (3) and (6), we get

$$-\cot\tau = \frac{t^2 - r^2 - l^2}{2lt} \qquad \omega_1 = \gamma\frac{x}{l} \qquad \omega_2 = \gamma\frac{y}{l} \qquad \omega_3 = \gamma\frac{z}{l}\omega_4 \qquad = \gamma\frac{r^2 - t^2 - l^2}{2l^2} \tag{9}$$

with the factor

$$\gamma = \frac{2l^2}{\sqrt{4l^2t^2 + (r^2 - t^2 + l^2)^2}} \ . \tag{10}$$

We derive this for the ω_1 and for the ω_4 case, since the calculation for ω_2 and ω_3 are analogous to the one for ω_1.

We have

$$\tau = \operatorname{arccot}\left(\frac{r^2 - t^2 + l^2}{2lt}\right)$$

and we know from (3) that

$$Z_1 = \frac{l}{\sin\tau}\omega_1 \ ,$$

hence

$$\frac{l}{\sin\tau}\omega_1 = l\frac{x}{t} \quad \Leftrightarrow \quad \omega_1 = \sin\tau\frac{x}{t} = \sin\left(\operatorname{arccot}\left[\frac{r^2 - t^2 + l^2}{2lt}\right]\right)\frac{x}{t} \ .$$

Using the relation [16]

$$\sin(\operatorname{arccot} a) = \frac{1}{\sqrt{1+a^2}} \ , \tag{11}$$

we have

$$\omega_1 = \frac{1}{\sqrt{1+\frac{(r^2-t^2+l^2)^2}{4l^2t^2}}}\frac{x}{t} = \frac{1}{\frac{1}{2lt}\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}}\frac{x}{t} = \frac{2l}{\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}}x$$
$$= \frac{2l^2}{\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}}\frac{x}{l} = \gamma\frac{x}{l} \ .$$

For the ω_4 case we have

$$\frac{l}{\sin\tau}\omega_4 = \frac{r^2-t^2-l^2}{2t} \quad \Leftrightarrow \quad \omega_4 = \sin\tau\frac{r^2-t^2-l^2}{2lt} \ .$$

Again, using (11), we have

$$\omega_4 = \sin\left(\operatorname{arccot}\left[\frac{r^2-t^2+l^2}{2lt}\right]\right)\frac{r^2-t^2-l^2}{2lt} = \frac{r^2-t^2-l^2}{\sqrt{1+\frac{(r^2-t^2+l^2)^2}{4l^2t^2}}}\frac{1}{2lt} = \frac{r^2-t^2-l^2}{\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}}$$
$$= \frac{2l^2(r^2-t^2-l^2)}{\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}}\frac{1}{2l^2} = \gamma\frac{r^2-t^2-l^2}{2l^2} \ .$$

Since $t = -\infty, 0, \infty$ corresponds to $\tau = -\pi, 0, \pi$, the cylinder is doubled to $2I \times S^3$ and the parameter τ is interpretable as a sphere-time and the full Minkowski space is covered by the cylinder patch ω_4.

We can get a very useful relation by employing Eulers formula

$$\exp(i\tau) = i\sin\tau + \cos\tau \tag{12}$$

and [16]

$$\cos(\operatorname{arccot} a) = \frac{a}{\sqrt{1+a^2}} \quad . \tag{13}$$

Then, we have with (11) and (13)

$$\exp(i\tau) = i\sin\left(\operatorname{arccot}\frac{r^2-t^2+l^2}{2lt}\right) + \cos\left(\operatorname{arccot}\frac{r^2-t^2+l^2}{2lt}\right) = \frac{i+\frac{r^2-t^2+l^2}{2lt}}{\sqrt{1+\frac{(r^2-t^2+l^2)^2}{4l^2t^2}}}$$
$$= \frac{i+\frac{r^2-t^2+l^2}{2lt}}{\frac{1}{2lt}\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}} = \frac{2lti+r^2-t^2+l^2}{\sqrt{4l^2t^2+r^2-t^2+l^2}} \ ,$$

such that

$$\exp(i\tau) = \frac{(l+it)^2+r^2}{\sqrt{4l^2t^2+(r^2-t^2+l^2)^2}} \ . \tag{14}$$

Comparing (4) and (8) we have an explicit conformal equivalence between the full Minkowski space $\mathbb{R}^{1,3}$ and a patch of a finite S^3 cylinder with $2I = (-\pi,\pi) \ni \tau$.

3. About the correspondence

In this section, we describe the correspondence of handling Yang-Mills theory on Minkowski space and on the cylinder $2I \times S^3$. This section is mainly based on [10], but many of the results and ideas are also presented in [11,12]. More general information about Yang-Mills theory, their applications in physics and their mathematical background are given for example in [17,18].

Yang-Mills theory is conformally invariant in four spacetime dimensions [11,12]. Thus, we can solve its equations of motions on the cylinder $2I \times S^3$ instead of doing it on Minkowski space $\mathbb{R}^{1,3}$ directly. It is beneficial to do so because S^3 enables manifestly a $SO(4)$ covariant formalism.

Since S^3 is the group manifold of $SU(2)$ we can make the geometric ansatz for the potential [10]

$$A = \sum_{a=1}^{3} X_a(\tau, \omega) e^a \tag{15}$$

on $2I \times S^3$ with the temporal gauge $A_\tau = 0$.

X_a gives three functions of τ and ω valued in some Lie algebra and the 1-forms e^a represent the three left-invariant ones on S^3. We can translate Yang-Mills solutions on $2I \times S^3$ to solutions on $\mathbb{R}^{1,3}$ simply via a change of coordinates.

To become explicit, we need Minkowski-coordinate expressions for the 1-forms $e^0 = d\tau$ and e^a satisfying the Maurer-Cartan structure equation [12]

$$de^a + \epsilon^a_{\ bc} e^b \wedge e^c = 0 \quad \text{and} \quad e^a e^a = d\Omega_3^2 \ . \tag{16}$$

These 1-forms can be constructed for an embedding $\{\omega^a\}$ via [12]

$$e^a = -\eta^a_{BC} \omega_B d\omega_C \ , \tag{17}$$

where η^a_{BC} is denoting the self-dual 't Hooft symbol taking the values

$$\eta^a_{ij} = \begin{cases} \epsilon^a_{\ ij} & i,j = 1,2,3 \\ +\delta^a_i & j = 4 \\ -\delta^a_j & i = 4 \\ 0 & i = j = 4 \end{cases} \ . \tag{18}$$

We have for the explicit expressions for the 1-forms (for details due to the derivation see appendix A)

$$\begin{aligned} e^0 &= \frac{\gamma^2}{l^3} \left(\frac{1}{2}(t^2 + r^2 + l^2) dt - t x^k dx^k \right) \\ e^a &= \frac{\gamma^2}{l^3} \left(t x^a dt - \left(\frac{1}{2}(t^2 - r^2 + l^2) \delta^a_k + x^a x^k + l \epsilon^a_{\ jk} x^j \right) dx^k \right) \end{aligned} \tag{19}$$

with the standard notation

$$(x^i) = (x, y, z) \qquad (x^\mu) = (x^0, x^i) = (t, x, y, z) \ . \tag{20}$$

The simplest Yang-Mills solutions are available when we impose $SO(4)$ symmetry such that the functions X_a become independent of ω, so $X_a(\tau, \omega) = X_a(\tau)$ [10,13]. Then, the Yang-Mills equations become ordinary matrix differential equations [12]

$$\begin{aligned} &\frac{d^2}{d\tau^2} X_a = -4X_a + 3\epsilon_{abc}\,[X_b, X_c] - [X_b, [X_a, X_b]] \\ &[X_a, \frac{d}{d\tau} X_a] = 0 \ . \end{aligned} \tag{21}$$

But in this thesis, we are interested in abelian solutions, i.e. electromagnetic field configurations, so we have to consider the unitary group $U(1)$. For the abelian case the commutator terms vanish. Since in the $U(1)$ case the matrix structure is irrelevant, we take $X_a(\tau, \omega)$ simply to be real-valued functions and focus on the Maxwell equations. In the $SO(4)$ invariant case, $X_a = X_a(\tau)$ are found to obey the oscillator equation

$$\frac{d^2}{d\tau^2} X_a(\tau) = -4X_a(\tau) \quad \Rightarrow \quad X_a(\tau) = c_a \cos(2(\tau - \tau_a)) \tag{22}$$

such that we have six integration constants in the general solution [10]. Since the X_a functions are oscillating with a frequency of two, we can use the simple expression (14) for $e^{2i\tau}$ to translate the dependence on τ into a rational expression in t and r. From

$$A = X_a(\tau) e^a = A_\mu dx^\mu \tag{23}$$

one gets the components A_μ of the potential and from [10]

$$F = dA = \frac{d}{d\tau} X_a(\tau) e^0 \wedge e^a - \epsilon^a_{\ bc} X_a(\tau) e^b \wedge e^c = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{24}$$

one can extract the electric and magnetic field components via [7,10]

$$E_i = F_{i0} \qquad B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} \ , \tag{25}$$

where F denotes the field strength.

With the solution

$$X_1(\tau) = -\frac{1}{8}\sin 2\tau \quad X_2(\tau) = -\frac{1}{8}\sin 2\tau \quad X_3(\tau) = 0 \tag{26}$$

one gets the famous Hopf-Rañada knot[1]

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \frac{l^2}{((t - il)^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t - il - z)^2 \\ i(x - iy)^2 + i(t - il - z)^2 \\ -2(x - iy)(t - il - z) \end{pmatrix} \tag{27}$$

[1]Although in [10] it was mentioned that it needs a few lines of computation to get the Hopf-Rañada knot, we did not find an easy way to get it. We did the derivation with some pages of computation using some symmetry arguments to simplify a few steps. We used facts that appear in later stages of this thesis. The details are attached in appendix B.

being the easiest, non-trivial and analytically computed electromagnetic knotted field [1,2,4,5,6,13,19]. The term $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ is known as the Riemann-Silberstein vector [4,20,21]. As mentioned in the introduction, and in particular in this case of the Hopf-Rañada knot, a knotted field configuration is characterized by the property that any pair of its field lines is linked.

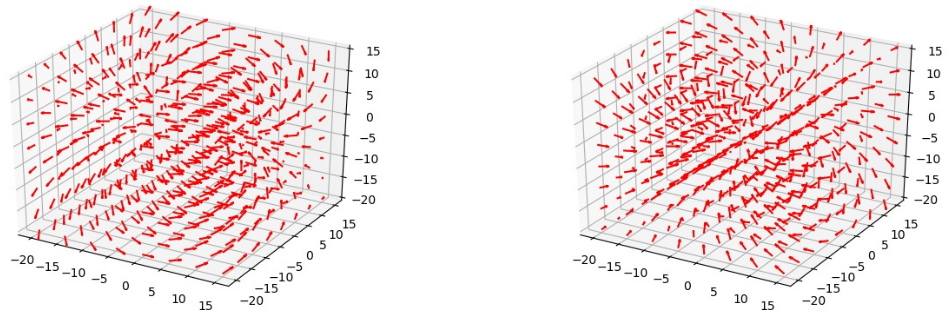

Figure 1: The Hopf-Ranada knot for $t = 0$. The left picture shows the real part (electric field) and the right picture shows the imaginary part (magnetic field) of the Hopf-Rañada knot

4. Construction of electromagnetic knots

In this section, we want to extend our previous solutions to $SO(4)$ non-symmetric ones, such that the solutions X_a will depend on the S^3 coordinates, e.g. $X_a = X_a(\tau, \omega)$. These solutions are knotted electromagnetic fields. The ideas and results of this section are based on [10].

To capture the dependence of the S^3 coordinates in an $SO(4)$-covariant way, we introduce S^3 vector fields R_a building a basis dual to the left-invariant 1-forms such that $e^a(R_b) = \delta^a_b$ via

$$R_a = -\eta^a_{BC}\omega_B \frac{\partial}{\partial \omega_C} \ . \tag{28}$$

These right-invariant vector fields form an $su(2)$ representation by the commutation relation

$$[R_a, R_b] = 2\epsilon_{abc}R_c \ . \tag{29}$$

They realize the infinitesimal right multiplications on $SU(2)$ so that for an arbitrary function ϕ on S^3 we may write

$$d\phi = e^a R_a \phi \ . \tag{30}$$

Another triplet of left-invariant vector fields is given by

$$L_a = -\tilde{\eta}^a_{BC}\omega_B \frac{\partial}{\partial \omega_C} \tag{31}$$

with $\tilde{\eta}^a_{BC}$ denoting the anti-self dual 't Hooft symbol.

They obey the same algebra as the right-invariant ones, so

$$[L_a, L_b] = 2\epsilon_{abc}L_c \ . \tag{32}$$

Since right and left multiplications commute, we have

$$[R_a, L_b] = 0 \ . \tag{33}$$

The space of functions on S^3 can be decomposed into irreducible representations of the $su(2)_L \oplus su(2)_R$ algebra generated by L_a and R_a. We label these representations uniquely by a non-negative number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ such that $2j = 0, 1, 2, \ldots$.

Defining hermitian "angular momentum" operators by

$$I_a := \frac{i}{2}L_a \qquad J_a := \frac{i}{2}R_a \ , \tag{34}$$

then a particular basis of hypersperical harmonics

$$Y_{j;m,n}(\omega) \qquad m,n=-j,-j+1,\ldots,+j-1,+j \qquad 2j=0,1,\ldots \tag{35}$$

is specified by the relations

$$\begin{aligned} I_3 Y_{j;m,n} &= m Y_{j;m,n} \\ J_3 Y_{j;m,n} &= n Y_{j;m,n} \\ I^2 Y_{j;m,n} &= J^2 Y_{j;m,n} = j(j+1) Y_{j;m,n} \end{aligned} \tag{36}$$

with $I^2 = I_a I_a$ and $J^2 = J_a J_a$ beeing the Casimirs of the $su(2)$ subalgebras.

For an explicit expression for $Y_{j;m,n}$ it is helpful to introduce two complex coordinates

$$\alpha = \omega_1 + i\omega_2 \qquad \beta = \omega_3 + i\omega_4 \tag{37}$$

parametrizing S^3 via $\overline{\alpha}\alpha + \overline{\beta}\beta = 1$. Moreover, we define the notation

$$I_\pm = \frac{I_1 \pm iI_2}{\sqrt{2}} \qquad J_\pm = \frac{J_1 \pm iJ_2}{\sqrt{2}} \qquad X_\pm = \frac{X_1 \pm iX_2}{\sqrt{2}}\ . \tag{38}$$

Applying the chain rule for the angular momentum operators for this coordinate transformation gives

$$\begin{aligned} I_+ &= \frac{\overline{\beta}\partial_{\overline{\alpha}} - \alpha\partial_\beta}{\sqrt{2}} & J_+ &= \frac{\beta\partial_{\overline{\alpha}} - \alpha\partial_{\overline{\beta}}}{\sqrt{2}} \\ I_3 &= \frac{\alpha\partial_\alpha + \overline{\beta}\partial_{\overline{\beta}} - \overline{\alpha}\partial_{\overline{\alpha}} - \beta\partial_\beta}{2} & J_3 &= \frac{\alpha\partial_\alpha + \beta\partial_\beta - \overline{\alpha}\partial_{\overline{\alpha}} - \overline{\beta}\partial_{\overline{\beta}}}{2} \\ I_- &= \frac{\overline{\alpha}\partial_{\overline{\beta}} - \beta\partial_\alpha}{\sqrt{2}} & J_- &= \frac{\overline{\alpha}\partial_\beta - \overline{\beta}\partial_\alpha}{\sqrt{2}}\ . \end{aligned} \tag{39}$$

The normalized hyperspherical harmonics are given by

$$Y_{j;m,n} = \sqrt{\frac{2j+1}{2\pi^2}} \sqrt{\frac{2^{j-m}(j+m)!2^{j-n}(j+n)!}{(2j)!(j-m)!(2j)!(j-n)!}} (I_-)^{j-m}(J_-)^{j-n}\alpha^{2j}\ . \tag{40}$$

They satisfy the orthonormal property

$$\int_{S^3} (Y_{j_1;m_1,n_1})^* Y_{j_2;m_2,n_3} d^3\Omega_3 = \delta_{j_1 j_2}\delta_{m_1 m_2}\delta_{n_1 n_2} \tag{41}$$

and its complex conjugate is

$$(Y_{j;m,n})^* = (-1)^{m+n} Y_{j;-m,-n}\ . \tag{42}$$

We will use these properties for later stages of this thesis.

For now, we will work in the Coulomb gauge on $2I \times S^3$, which for a general gauge field

$$A = X_0(\tau,\omega)d\tau + X_a(\tau,\omega)e^a \tag{43}$$

with $\tau \in (-\pi, +\pi)$ and $\omega \in S^3$ means that

$$X_0(\tau,\omega) = 0 \qquad \text{and} \qquad J_a X_a(\tau,\omega) = 0\ . \tag{44}$$

The Maxwell equation in vacuum [7]

$$dF_D = d(*F) = d(*dA) = 0 \tag{45}$$

then takes a very simple form, where $*$ denotes the Hodge-dual operator and the subscript D stands for dual [2]. With $A = X_a e^a$, we get for the exterior derivative

$$dA = R_a X_b e^{ab} - \epsilon_{abc} X_a e^{bc} + \frac{d}{d\tau} X_a e^{0a}\ ,$$

where $e^{ab} \equiv e^a \wedge e^b$. So we get

$$\begin{aligned} *\, dA &= -\epsilon_{abc} R_a X_b e^{0c} + \epsilon_{abc}\epsilon_{bcd} X_a e^{0d} + \frac{1}{2}\frac{d}{d\tau} X_a \epsilon_{abc} e^{bc} \\ &= -\epsilon_{abc} R_a X_b e^{0c} + 2X_a e^{0d} + \frac{1}{2}\frac{d}{d\tau} X_a \epsilon_{abc} e^{bc}\ . \end{aligned}$$

Then, the Maxwell equation (45) becomes

$$\begin{aligned} d * dA = {}& \epsilon_{abc} R_d R_a e^{0dc} - \epsilon_{abc}\epsilon_{0de} R_a X_b e^{0dc} + 2R_a X_b e^{0ba} + 2X_a \epsilon_{abc} e^{0dc} \\ & + \frac{1}{2}\frac{d^2}{d\tau^2} X_a \epsilon_{abc} e^{0dc} + \frac{1}{2} R_d \frac{d}{d\tau} X_a \epsilon_{abc} e^{dbc} \\ = {}& 0\ . \end{aligned}$$

Applying Gauss law

$$\frac{1}{2} R_a \frac{d}{d\tau} X_a = \frac{d}{d\tau} R_a X_a = 0\ , \tag{46}$$

we get

$$\begin{aligned} 0 &= \epsilon_{adc}\epsilon_{ebc} R_d R_e X_b - \epsilon_{ade}\epsilon_{fbc}\epsilon_{cde} R_f X_b + 2\epsilon_{abc} R_c X_b + 2\epsilon_{abc}\epsilon_{dbc} X_d + \frac{1}{2}\frac{d^2}{d\tau^2} X_d \epsilon_{dbc}\epsilon_{abc} \\ &= R_e R_a X_b - R^2 X_a - 2\epsilon_{afb} R_f X_b - 2\epsilon_{abc} R_b X_c + 4X_a + \frac{d^2}{d\tau^2} X_a \\ &= \frac{d^2}{d\tau^2} X_a + 4X_a + 2\epsilon_{abc} R_c X_b + R_a R_b X_b - R^2 X_a - 4\epsilon_{abc} R_b X_c \\ \Leftrightarrow -\frac{d^2}{d\tau^2} X_a &= 4X_a + 2\epsilon_{abc} R_c X_b + R_a R_b X_b - R^2 X_a - 4\epsilon_{abc} R_b X_c \\ \Leftrightarrow -\frac{1}{4}\frac{d^2}{d\tau^2} X_a &= -\frac{1}{4} R^2 X_a + X_a - \frac{1}{2}\epsilon_{abc} R_b X_c + \frac{1}{4} R_a R_b X_b \\ &= (J^2 + 1) X_a + i\epsilon_{abc} J_b X_c - J_a J_b X_b\ . \end{aligned}$$

[2]The dual field strength is given by its Hodge-dual [7].

With the gauge $J_a X_a = 0$ we get the for the Maxwell equation

$$-\frac{1}{4}\frac{d^2}{d\tau^2}X_a = (J^2+1)X_a + i\epsilon_{abc}J_b X_c \tag{47}$$

which is $su(2)_L$ invariant and $su(2)_r$ covariant.

We get for each value

$$\begin{aligned}
-\frac{1}{4}\partial_\tau^2 X_1 &= (J^2+1)X_1 + i(J_2X_3 - J_3X_2)\\
-\frac{1}{4}\partial_\tau^2 X_2 &= (J^2+1)X_2 + i(J_3X_1 - J_1X_3)\\
-\frac{1}{4}\partial_\tau^2 X_3 &= (J^2+1)X_3 + i(J_1X_2 - J_2X_1)\ .
\end{aligned} \tag{48}$$

Expressing these equations in the $\pm$ basis, they take the form (see appendix C for details)

$$\begin{aligned}
-\frac{1}{4}\partial_\tau^2 X_+ &= (J^2+1-J_3)X_+ + J_+X_3\\
-\frac{1}{4}\partial_\tau^2 X_3 &= (J^2+1)X_3 - J_+X_- + J_-X_+\\
-\frac{1}{4}\partial_\tau^2 X_- &= (J^2+J_3+1)X_- - J_-X_3
\end{aligned} \tag{49}$$

with the gauge condition as

$$0 = J_aX_a = J_+X_- + J_-X_+ + J_3X_3\ . \tag{50}$$

We can solve this system of partial differential equations by the separation ansatz of variables

$$X_a(\tau,\omega) = \sum_{j,m,n} X_a^{j;m,n}(\tau)Y_{j;m,n}(\alpha,\beta) \tag{51}$$

using the eigenvalue equations for the hyperspherical harmonics and the exponential ansatz

$$X_a(\tau) = c_a^{j;n}e^{i\Omega_a^{j;n}}\ . \tag{52}$$

The system becomes linear with some constans $c_a^{j;n}$ and frequencies $\Omega_a^{j;n}$ with

$$\Omega_a^{j;n} = \pm 2(j+1) \quad \text{or} \quad \Omega_a^{j;n} = \pm 2j\ . \tag{53}$$

We call the corresponding solutions *type 1* and *type 2* as mentioned in [1], where we extract the coefficients $c_a^{j,n}$ (up to an irrelevant overall factor). The basic solutions read

for *type 1* with
$j \geq 0: \quad m = -j, \ldots, +j \quad n = -j-1, \ldots, +j+1, \quad \Omega^j = \pm 2(j+1)$

$$\begin{aligned} X_+ &= \sqrt{\frac{(j-n)(j-n+1)}{2}} e^{\pm 2(j+1)i\tau} Y_{j;m,n+1} \\ X_3 &= \sqrt{(j+1)^2 - n^2} e^{\pm 2(j+1)i\tau} Y_{j;m,n} \\ X_- &= -\sqrt{\frac{(j+n)(j+n+1)}{2}} e^{\pm 2(j+1)i\tau} Y_{j;m,n-1} \end{aligned} \tag{54}$$

and for *type 2* with
$j \geq 1: \quad m = -j, \ldots, +j \quad n = -j+1, \ldots, +j-1, \quad \Omega^j = \pm 2j$

$$\begin{aligned} X_+ &= -\sqrt{\frac{(j+n)(j+n+1)}{2}} e^{\pm 2ji\tau} Y_{j;m,n+1} \\ X_3 &= \sqrt{j^2 - n^2} e^{\pm 2ji\tau} Y_{j;m,n} \\ X_- &= \sqrt{\frac{(j-n)(j-n+1)}{2}} e^{\pm 2ji\tau} Y_{j;m,n-1} \ . \end{aligned} \tag{55}$$

Let us summarize some properties of these two classes of solutions as in [10]. For a given frequency Ω, the solutions occur with the two values $j = \frac{1}{2}|\Omega|$ and $j = \frac{1}{2}|\Omega| - 1$ (except for $\Omega = 1$). Constant solutions ($\Omega = 0$) are not allowed. *Type 1* solutions with spin j correspond to solutions of *type 2* with spin $j+1$ via a parity transformation $m \leftrightarrow n$ also exchaning left- and right $su(2)$ algebras. Furthermore, electromagnetic duality is realized in a simple way for the solutions above. For *type 1* (*type 2*) at fixed j, shifting $|\Omega^j|\tau$ by $\frac{\pi}{2}$ ($-\frac{\pi}{2}$) produces the dual solution with $\mathbf{B}_D = \mathbf{E}$ and $\mathbf{E}_D = -\mathbf{B}$ [10].

4.1. Sphere-frame fields

Since we will deal with the solutions (54) and (55) a lot on $2I \times S^3$, we also introduce "sphere-frame" fields via [10]

$$F = \mathcal{E}_a e^a \wedge e^0 + \frac{1}{2}\mathcal{B}_a \epsilon^a_{\ bc} e^b \wedge e^c \ . \tag{56}$$

We need to know how to get the sphere-frame electric field $\mathcal{E}$ and the sphere-frame magnetic field $\mathcal{B}$. To get the sphere-frame electric and magnetic field components from the solutions (54) and (55), we do the following calculation :

$$\begin{aligned} F &= \mathcal{E}_a e^{a0} + \frac{1}{2}\epsilon_{abc}\mathcal{B}_a e^{bc} \\ &= dA = d(X_a e^a) \\ &= (dX_a) \wedge e^a + X_a de^a \\ &= R_a X_b e^{ab} - \epsilon_{abc} X_a e^{bc} + \frac{d}{d\tau} X_a e^{0a} \\ &= -\frac{d}{d\tau} X_a e^{a0} + \frac{1}{2}(R_a X_b - R_b X_a - 2\epsilon_{abc} X_c) e^{ab} \ . \end{aligned} \tag{57}$$

Hence

$$\mathcal{E}_a = -\dot{X}_a \tag{58}$$

$$\mathcal{B}_a = \epsilon_{abc} R_b X_c - 2X_a = -2i\epsilon_{abc} J_b X_c - 2X_a \ , \tag{59}$$

where a dot over a variable stands for its derivative in respect to τ. We notice that it is quite easy to get the sphere-frame electric field, but the formula for the sphere-frame magnetic field looks a bit more complicated.

But for $\mathcal{B}_a$ we have much more practial formulas for any fixed j solution. We can write the Maxwell equation (47) since $J^2 X_a = j(j+1)X_a$ equivalently as

$$\begin{aligned} -\frac{1}{4}\ddot{X}_a &= (j(j+1)+1)X_a + i\epsilon_{abc} J_b X_c \\ &= (j^2+j+1)X_a + i\epsilon_{abc} J_b X_c \ . \end{aligned} \tag{60}$$

For *type 1* solutions we have

$$\ddot{X}_a = -4(j+1)^2 X_a \tag{61}$$

and for *type 2* solutions we have

$$\ddot{X}_a = -4j^2 X_a \ . \tag{62}$$

Since that, we can write the Maxwell equation for *type 1* solutions as

$$(j+1)^2 X_a = (j^2+j+1)X_a + i\epsilon_{abc} J_b X_c \tag{63}$$

which is equivalent to

$$jX_a = i\epsilon_{abc}J_bX_c \ . \tag{64}$$

So for *type 1* solutions we can write $\mathcal{B}_a$ with (59) as

$$\begin{aligned} \mathcal{B}_a &= -2i\epsilon_{abc}J_bX_c - 2X_a \\ &= -2jX_a - 2X_a \\ &= -2(j+1)X_a \end{aligned} \tag{65}$$

For *type 2* solutions the Maxwell equation becomes

$$\begin{aligned} j^2X_a &= (j^2 + j + 1)X_a + i\epsilon_{abc}J_bX_c \\ \Leftrightarrow -(j+1)X_a &= i\epsilon_{abc}J_bX_c \ . \end{aligned} \tag{66}$$

Thus for *type 2* solutions the sphere-frame magnetic field components become

$$\begin{aligned} \mathcal{B}_a &= 2(j+1)X_a - 2X_a \\ &= 2jX_a \ . \end{aligned} \tag{67}$$

We remind that these formulas for the sphere-frame magnetic field hold only for some fixed j quantum number. We will use these relations for later stages of this thesis.

5. Decay behaviour of the solutions

In this section, we want to check the decay behaviour of the correspondig electric and magnetic fields of the solutions mentioned in the previous section. In the first subsection, we consider this for large r and fixed time $t = t_0$. In the second subsection, we set $t = r$ and let that getting large. What we would like to have in both cases is that they tend to 0 for $r \to \infty$, since we want to have finity energy in our systems. Since the electric and magnetic field components appear as a sum in the Riemann-Silberstein vector, we will consider that one.

5.1. Large r and fixed time $t = t_0$

First of all, let us have a closer look at the coordinates, all the prefactors in the relevant formulas and all the conditions we have.

We start with the prefactor γ of the ω_A coordinates for constant $t = t_0$:

$$\begin{aligned} \gamma &= \frac{2l^2}{\sqrt{4l^2t_0^2 + (r^2 - t_0^2 + l^2)^2}} \\ &\propto \frac{1}{\sqrt{r^4}} = \frac{1}{r^2} \ . \end{aligned} \tag{68}$$

From that follows that

$$\omega_{1,2,3} \propto \frac{1}{r} \tag{69}$$

for large r. But since for $t = t_0$

$$\omega_4 = \gamma \frac{r^2 - t_0^2 - l^2}{2l^2} \ , \tag{70}$$

we have

$$\omega_4 \propto 1 \tag{71}$$

for large r.

We note that $\omega_A \in S^3$, so ω_A with $A = 1, \ldots, 4$ are unit S^3 coordinates and we have the condition

$$\begin{aligned} \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 1 \\ \Leftrightarrow \omega_1^2 + \omega_2^2 + \omega_3^2 = 1 - \omega_4^2 \ . \end{aligned} \tag{72}$$

Furthermore, we have the coordinates $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ which are complex linear combinations of the ω_A expressions. The condition (72) then becomes

$$\alpha\bar{\alpha} + \beta\bar{\beta} = 1 \ . \tag{73}$$

These $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ can be interpreted as angles of the unit 3-sphere, since ω_A are the corresponding coordinates.

Now let us consider the solutions themself. As we can see, they are proportional to $e^{\pm 2(j+1)i\tau} Y_{j;m,n}$ for *type 1* and proportional for $e^{\pm 2ji\tau} Y_{j;m,n}$ for *type 2* solutions. Let us look at the exponential function $e^{2i\tau}$ in Minkowski coordinates for $t = t_0$ and large r:

$$\begin{aligned} e^{2i\tau} &= \frac{((l+it_o)^2 + r^2)^2}{4l^2 t_0^2 + (r^2 - t_0^2 + l^2)^2} \\ &= \frac{((l+it_o)^2 + r^2)^2}{((l+it_o)^2 + r^2)((l-it_o)^2 + r^2)} \\ &= \frac{(l+it_o)^2 + r^2}{(l-it_o)^2 + r^2} \propto -1 \; . \end{aligned} \tag{74}$$

We see that the exponential function makes no contribution for large r in Minkowski space.

We also need to handle the hyperspherical harmonics

$$Y_{j;m,n} \propto (I_-)^{j-m} (J_-)^{j-n} \alpha^{2j} \; . \tag{75}$$

Since I_- and J_- are operators, we have to check carefully how it behaves for each term appearing right of them. Let us see what result we get acting only one time on α^{2j} and then make some inductively statements.

We start with J_-:

$$\begin{aligned} (J_-)\alpha^{2j} &\propto (\bar{\alpha}\partial_\beta - \bar{\beta}\partial_\alpha)\alpha^{2j} \\ &= -2j\bar{\beta}\alpha^{2j-1} \\ &\propto -\bar{\beta}\alpha^{2j-1} \; , \end{aligned} \tag{76}$$

so we can deduce inductively

$$\begin{aligned} (J_-)^{j-n}\alpha^{2j} &\propto (-1)^{j-n}(\bar{\beta})^{j-n}\alpha^{2j-(j-n)} \\ &= (-1)^{j-n}(\bar{\beta})^{j-n}\alpha^{j+n} \\ &\propto \bar{\beta}^{j-n}\alpha^{j+n} \; . \end{aligned} \tag{77}$$

Now let us check what happens if I_- acts additionally:

$$\begin{aligned} (I_-)(J_-)^{j-n}\alpha^{2j} &\propto (\bar{\alpha}\partial_{\bar{\beta}} - \beta\partial_\alpha)\bar{\beta}^{j-n}\alpha^{j+n} \\ &= (j-n)\bar{\alpha}\bar{\beta}^{j-n-1}\alpha^{j+n} - (j+n)\beta\bar{\beta}^{j-n}\alpha^{j+n-1} \\ &\propto \bar{\alpha}\bar{\beta}^{j-n-1}\alpha^{j+n} - \beta\bar{\beta}^{j-n}\alpha^{j+n-1} \; . \end{aligned} \tag{78}$$

If we let I_- acting again on the right side, we get

$$\begin{aligned} (I_-)^2(J_-)^{j-n}\alpha^{2j} &\propto (\bar{\alpha}\partial_{\bar{\beta}} - \beta\partial_\alpha)\{\bar{\alpha}\bar{\beta}^{j-n-1}\alpha^{j+n} - \beta\bar{\beta}^{j-n}\alpha^{j+n-1}\} \\ &\propto \bar{\alpha}^2\bar{\beta}^{j-n-2}\alpha^{j+n} - \bar{\alpha}\beta\bar{\beta}^{j-n-1}\alpha^{j+n-1} \\ &\quad - \beta\bar{\alpha}\bar{\beta}^{j-n-1}\alpha^{j+n-1} + \beta^2\bar{\beta}^{j-n}\alpha^{j+n-2} \; . \end{aligned} \tag{79}$$

$(I_-)^3(J_-)^{j-n}\alpha^{2j}$ would give 8 terms, $(I_-)^4(J_-)^{j-n}\alpha^{2j}$ would give 16 terms and so on. But from the structure of the results above we can deduce that the sum of the exponents will always be $2j$. So the hyperspherical harmonics are polynomials of degree $2j$ in α, β and their complex conjugates.

But since $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ are angles in the unit 3-sphere and they are complex linear combinations of ω_A, the hyperspherical harmonics tend for large r to a constant value and thus the solutions X_a, too.

We note that considering large r for a constant time $t = t_0$ is the same as approaching the point $\omega_4 = 1$ on the unit 3-sphere. So with the condition (72) the solutions X_a do not play any role for the asymptotic behaviour for large r and constant time in Minkowski space.

The last thing to consider is electromagnetic field strength tensor [7,10]

$$\begin{aligned} F &= \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \\ &= E_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ijk}B_i dx^j \wedge dx^k \\ &= \frac{d}{d\tau}X_a e^0 \wedge e^a - \epsilon^a_{\ bc} X_a e^a \wedge e^b \, . \end{aligned} \tag{80}$$

The 1-forms e^0, e^a remain the only objects the decay behaviour of the electric and magnetic field can be encoded in. Obviously, the behaviour of $e^0 \wedge e^a$ corresponds to the behaviour of the electric field $\mathbf{E}$ and the behaviour of $e^a \wedge e^b$ corresponds to the one of the magnetic field $\mathbf{B}$.

Since we have already seen that $\gamma \propto \frac{1}{r^2}$, we deduce that $e^0, e^a \propto \frac{1}{r^2}$, because we have a γ^2 as a prefactor, but a r^2 term also appears inside the brackets of the 1-forms as it can be seen in (19). Putting everything together, we have

$$e^0 \wedge e^a \propto \frac{1}{r^4} \qquad \text{and} \quad e^a \wedge e^b \propto \frac{1}{r^4} \tag{81}$$

and thus

$$\mathbf{E} \propto \frac{1}{r^4} \qquad \mathbf{B} \propto \frac{1}{r^4} \, , \tag{82}$$

so obviously

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} \propto \frac{1}{r^4} \, . \tag{83}$$

So all Riemann-Silberstein vectors which are computable from the solutions (54) and (55) behave for large r and fixed time $t = t_0$ as $\frac{1}{r^4}$ in Minkowski space.

5.2. Large $r = \pm t$

We now want to check how the electric and magnetic fields from the solutions X_a behave for large $r = \pm t$ in Minkowski space. The behaviour for the electric and magnetic fields for $r = \pm t$ describes them on the light cone.

In the following, we will deal with $r = t$. The $r = -t$ is analogues since only squares of them are appearing in the relevant formulas.

As in the previous section, we start again by considering each relavant term and condition.

For $r = t$ the prefactor γ becomes

$$\begin{aligned} \gamma &= \frac{2l^2}{\sqrt{4l^2r^2 + l^4}} \\ &= \frac{2l}{\sqrt{4r^2 + l^2}} \\ &\propto \frac{1}{r} \ . \end{aligned} \tag{84}$$

This means for the embedded 3-sphere coordinates

$$\omega_{1,2,3} \propto 1 \tag{85}$$

and

$$\begin{aligned} \omega_4 &= -\gamma \frac{l^2}{2l^2} \\ &= -\frac{\gamma}{2} \\ &\propto -\frac{1}{r} \ . \end{aligned} \tag{86}$$

The condition (72) remains.

On the $r = t$ slice the exponential function becomes

$$\begin{aligned} e^{2i\tau} &= \frac{(l+it)^2 + r^2}{(l-it)^2 + r^2} \\ &= \frac{l^2 + 2ilr}{l^2 - 2ilr} \\ &\propto -1 \ . \end{aligned} \tag{87}$$

We see that the exponential function makes no contribution for large $r = t$ in Minkowski space.

Since the consideration of the hyperspherical harmonics does not change, the solutions tend to some constant value again.

Thus, as in the previous subsection, we have to consider the 1-forms e^0 and e^a again, because they are the only objects left where the electric and magnetic fields can be encoded in.

Let us set each coordinate as

$$x^a = r\theta^a \tag{88}$$

to see things better, while θ^a is a dimensionless angle. With that we get for the 1-forms

$$\begin{aligned} e^0 &= \frac{\gamma^2}{l^3}\left(\left(r^2 + \frac{1}{2}l^2\right)dt - r^2\theta^k dx^k\right) \\ e^a &= \frac{\gamma^2}{l^3}\left(r^2\theta^a dt - \left(\frac{1}{2}l^2\delta^a_k + r^2\theta^a\theta^k + l\epsilon^a_{\ jk} r\theta^j\right)dx^k\right) \ . \end{aligned} \tag{89}$$

We see that the highest order term is r^2 written in the brackets. With $\gamma^2 \propto \frac{1}{r^2}$ the potential

$$A = X_a e^a \tag{90}$$

has no decay behaviour.

Now let us consider the electromagnetic field strength

$$F = dA = \frac{d}{d\tau}X_a e^0 \wedge e^a - \epsilon^a_{\ bc} X_a e^b \wedge e^c \tag{91}$$

and have a closer look a the wedge-product of the 1-forms. We have the Maurer-Cartan structure equation

$$\begin{aligned} 0 &= de^a + \epsilon^a_{\ bc} e^b \wedge e^c \\ \Leftrightarrow de^a &= -\epsilon^a_{\ bc} e^b \wedge e^c \ . \end{aligned} \tag{92}$$

In other words, the wedge product of our 1-forms is proportional to de^a and thus we can only consider the term de^a. Since e^0 and e^a have the same structure, we just consider proportionalities of de.

We have

$$e \propto \gamma^2 r^2 \ , \tag{93}$$

thus

$$\begin{aligned} de &\propto d(\gamma^2)r^2 + \gamma^2 d(r^2) \\ &\propto d\left(\frac{1}{r^2}\right)r^2 + \frac{1}{r^2}r \\ &\propto \frac{1}{r^3}r^2 + \frac{1}{r} \\ &\propto \frac{1}{r} \ . \end{aligned} \tag{94}$$

So the field strength behaves for large $r = t$ (same holds for $r = -t$) as

$$F \propto \frac{1}{r} \tag{95}$$

and thus the corresponding electric and magnetic fields, too

$$\mathbf{E} \propto \frac{1}{r} \qquad \mathbf{B} \propto \frac{1}{r} \; . \tag{96}$$

Hence, we have for the Riemann-Silberstein vector

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} \propto \frac{1}{r} \; . \tag{97}$$

So all the Riemann-Silberstein vectors which are computable from the solutions (54) and (55) behave for large $r = \pm t$ as $\frac{1}{r}$ in Minkowski space. We note again that $r = \pm t$ is the light cone, hence we described the decay behaviour of the electric and magnetic field on the light cone.

6. Null field construction

If we want to solve problems in electromagnetism, it is useful to know Lorentz invariant quantities. Two of them can be constructed via the field strength F [22,23]. In terms of differential forms with the basis $\{dx^\mu \wedge dx^\nu\} = \{dx^0 \wedge dx^i, dx^i \wedge dx^k\}$ we can write it as [7]

$$\begin{aligned} F &= E_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ijk} B_i dx^j \wedge dx^k \\ &= E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt \\ &\quad + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \ . \end{aligned} \tag{98}$$

The dual field strength is given by [7]

$$\begin{aligned} F_D = *F &= -B_i dx^i \wedge dx^0 + \frac{1}{2}\epsilon_{ijk} E_i dx^j \wedge dx^k \\ &= -(B_x dx \wedge dt + B_y dy \wedge dt + B_z dz \wedge dt) \\ &\quad + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \ . \end{aligned} \tag{99}$$

Computing their wedge products, we arrive at

$$F \wedge F = -2\mathbf{E} \cdot \mathbf{B} \ dt \wedge dx \wedge dy \wedge dz \tag{100}$$

$$F \wedge F_D = (\mathbf{B}^2 - \mathbf{E}^2) dt \wedge dx \wedge dy \wedge dz \ . \tag{101}$$

Indeed, the terms

$$\mathbf{E} \cdot \mathbf{B} \tag{102}$$

$$\mathbf{B}^2 - \mathbf{E}^2 \tag{103}$$

are fundamental Lorentz invariants [22], e.g. every other Lorentz invariant of the electromagnetic field can be expressed in terms of these two.

The quantity (103) implies that if for example $|\mathbf{E}| < |\mathbf{B}|$ holds in one reference frame, $|\mathbf{E}| < |\mathbf{B}|$ holds in every other reference frame. Obviously, the same statement is also true for $|\mathbf{E}| > |\mathbf{B}|$. In case for same magnitudes in one reference frame, they are equal in every other reference frame, too [23].

The quantity (102) gives the same information but about the angle between $\mathbf{E}$ and $\mathbf{B}$. For example, if they are orthogonal in one reference frame, they are orthogonal in all reference frames [23].

So, with these quantities it is possible to classify electromagnetic fields. We have a so called *null field* if it satisfies

$$0 = \mathbf{B}^2 - \mathbf{E}^2 \quad \Leftrightarrow \quad \mathbf{E}^2 = \mathbf{B}^2 \tag{104}$$

$$0 = \mathbf{E} \cdot \mathbf{B} \ . \tag{105}$$

Equivalently, these null field conditions can be written as

$$0 = F \wedge F \tag{106}$$
$$0 = F \wedge F_D \ . \tag{107}$$

At the same time, the field strength is also given by [10]

$$F = \mathcal{E}_a e^a \wedge e^0 + \frac{1}{2}\mathcal{B}_a \epsilon^a_{\ bc} e^b \wedge e^c \ . \tag{108}$$

Luckily, for 2-forms such as the field strength F it does not matter which metric one uses whenever we have conformal equivalence, in our case the Minkowski or the $I \times S^3$ metric. The reason is, that the conformal factor drops out from the calculation, in particular for the dual 2-form (for a detailed general description of this see appendix D).

Hence, the Lorentz invariants in Minkowski space $\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E} \cdot \mathbf{B}$ are proportional to the same expressions in sphere-frame coordinates, such that we can write the null field conditions as

$$\mathcal{E}^2 = \mathcal{B}^2 \tag{109}$$
$$\mathcal{E} \cdot \mathcal{B} = 0 \ . \tag{110}$$

In this section, we will discuss the null field construction of the solutions (54) and (55). At first, we will pick the solutions (54) and (55) and check for which fixed quantum numbers we do have null fields. After this, we will introduce a different formalism to handle and discuss the null field construction to gain more interpretation about the results.

6.1. Null fields for fixed quantum numbers

To see for which quantum numbers we have null fields, we will compute from our solutions (54) and (55) the sphere-frame fields $\mathcal{E}$ and $\mathcal{B}$ and then insert them in the null field conditions (109), (110). We focus on *type 1* solutions. After finishing this computation, we are also able to make a statement about *type 2* solutions by their parity relation.

First of all, we note that the solutions (54) and (55) are complex valued functions. They are convenient to deal with, but physically meaningful solutions should be real. So what we want to have are null fields for real solutions. The relation

$$X_a = \operatorname{Re} X_a + i \operatorname{Im} X_a \tag{111}$$

gives

$$X_a + X_a^* = 2 \operatorname{Re} X_a \tag{112}$$

$$\frac{1}{i}\left(X_a - X_a^*\right) = 2 \operatorname{Im} X_a \ . \tag{113}$$

To see for which quantum numbers we do have null fields, we have to compute $\mathcal{E}$ and $\mathcal{B}$ and their squares for the real solutions, i.e. for $\operatorname{Re} X_a$ and $\operatorname{Im} X_a$.

Computation for $\operatorname{Re} X_a$

To check when we do have null fields for $\operatorname{Re} X_a$, we write the solutions (54) as

$$\begin{aligned} X_+ &= \sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\ X_3 &= \sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\ X_- &= -\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \ . \end{aligned} \tag{114}$$

Actually we do the computation for $2 \operatorname{Re} X_a$, but the prefactor 2 has no influence on the results we get.

Staying in $+3-$ basis, the sphere-frame electric and magnetic field components are given with (58) and (65) by

$$\begin{aligned} \mathcal{E}_+ &= -2i(j+1)\sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\ \mathcal{E}_3 &= -2i(j+1)\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\ \mathcal{E}_- &= 2i(j+1)\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \end{aligned} \tag{115}$$

and

$$
\begin{aligned}
\mathcal{B}_+ &= -2(j+1)\sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
\mathcal{B}_3 &= -2(j+1)\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\
\mathcal{B}_- &= 2(j+1)\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \ .
\end{aligned}
\tag{116}
$$

We start with the consideration of condition (109). We have for the squared sphere-frame electric field

$$
\begin{aligned}
\mathcal{E}^2 &= 2\mathcal{E}_+\mathcal{E}_- + \mathcal{E}_3^2 \\
&= 4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad - 4\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right)^2 \\
&= 4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\{Y_{j;m,n+1}Y_{j;m,n-1}e^{4i(j+1)\tau} \\
&\quad - Y_{j;m,n+1}(Y_{j;m,n-1})^* - (Y_{j;m,n+1})^*Y_{j;m,n-1} + (Y_{j;m,n+1})^*(Y_{j;m,n-1})^* e^{-4i(j+1)\tau}\} \\
&\quad - 4\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right)^2
\end{aligned}
\tag{117}
$$

and

$$
\begin{aligned}
\mathcal{B}^2 &= 2\mathcal{B}_+\mathcal{B}_- + \mathcal{B}_3^2 \\
&= -4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad + 4\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right)^2 \\
&= -4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\{Y_{j;m,n+1}Y_{j;m,n-1}e^{4i(j+1)\tau} \\
&\quad + Y_{j;m,n+1}(Y_{j;m,n-1})^* + (Y_{j;m,n+1})^*Y_{j;m,n-1} + (Y_{j;m,n+1})^*(Y_{j;m,n-1})^* e^{-4i(j+1)\tau}\} \\
&\quad + 4\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right)^2 \ .
\end{aligned}
\tag{118}
$$

We see that the null field condition (109) is only satisfied when $\mathcal{E}^2 = \mathcal{B}^2 = 0$. This is fulfilled if

$$
\begin{aligned}
0 &= (j+1)^2 - n^2 \\
\Leftrightarrow n &= \begin{cases} j+1 \\ -j-1 \end{cases} \ .
\end{aligned}
$$

Now we turn to the condition (110). In +3− basis the scalar product is given by

$$\begin{aligned}
\mathcal{E}\cdot\mathcal{B} &= \mathcal{E}_+\mathcal{B}_- + \mathcal{E}_-\mathcal{B}_+ + \mathcal{E}_3\mathcal{B}_3 \\
&= -2i(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad - 2i(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad + 4i(j+1)^2\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) .
\end{aligned} \tag{119}$$

We notice that this expression vanishes when

$$\begin{aligned}
0 &= (j+1)^2 - n^2 \\
\Leftrightarrow n &= \begin{cases} j+1 \\ -j-1 \end{cases} .
\end{aligned}$$

We see that we get for both conditions the same result.

Computation for $\operatorname{Im} X_a$

Now we handle the case for $\operatorname{Im} X_a$. Actually we will do the computation again for $2\operatorname{Im} X_a$, but as mentioned earlier, the prefactor 2 has no influence on the results we get. For the imaginary part the *type 1* solutions are written as

$$\begin{aligned}
X_+ &= \frac{1}{i}\sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
X_3 &= \frac{1}{i}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\
X_- &= -\frac{1}{i}\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) .
\end{aligned} \tag{120}$$

Then, the sphere-frame fields are given by

$$\begin{aligned}
\mathcal{E}_+ &= -2(j+1)\sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
\mathcal{E}_3 &= -2(j+1)\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\
\mathcal{E}_- &= 2(j+1)\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right)
\end{aligned} \tag{121}$$

and

$$
\begin{aligned}
\mathcal{B}_{+} &= \frac{-2(j+1)}{i}\sqrt{\frac{(j-n)(j-n+1)}{2}}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^{*}e^{-2i(j+1)\tau}\right)\\
\mathcal{B}_{3} &= \frac{-2(j+1)}{i}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^{*}e^{-2i(j+1)\tau}\right)\\
\mathcal{B}_{-} &= \frac{2(j+1)}{i}\sqrt{\frac{(j+n)(j+n+1)}{2}}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^{*}e^{-2i(j+1)\tau}\right) .
\end{aligned}
\tag{122}
$$

We start again with the condition (109). We have for the squared fields

$$
\begin{aligned}
\mathcal{E}^2 &= 2\mathcal{E}_{+}\mathcal{E}_{-} + \mathcal{E}_3^2\\
&= -4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^{*}e^{-2i(j+1)\tau}\right)\\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^{*}e^{-2i(j+1)\tau}\right)\\
&\quad + 4(j+1)^2\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^{*}e^{-2i(j+1)\tau}\right)^2\\
&= -4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\{Y_{j;m,n+1}Y_{j;m,n-1}e^{4i(j+1)\tau}\\
&\quad + Y_{j;m,n+1}(Y_{j;m,n-1})^{*} + (Y_{j;m,n+1})^{*}Y_{j;m,n-1} + (Y_{j;m,n+1})^{*}(Y_{j;m,n-1})^{*}e^{-4i(j+1)\tau}\}\\
&\quad + 4(j+1)^2\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^{*}e^{-2i(j+1)\tau}\right)^2
\end{aligned}
\tag{123}
$$

and

$$
\begin{aligned}
\mathcal{B}^2 &= 2\mathcal{B}_{+}\mathcal{B}_{-} + \mathcal{B}_3^2\\
&= 4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^{*}e^{-2i(j+1)\tau}\right)\\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^{*}e^{-2i(j+1)\tau}\right)\\
&\quad - 4(j+1)^2\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^{*}e^{-2i(j+1)\tau}\right)^2\\
&= 4(j+1)^2\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\{Y_{j;m,n+1}Y_{j;m,n-1}e^{4i(j+1)\tau}\\
&\quad - Y_{j;m,n+1}(Y_{j;m,n-1})^{*} - (Y_{j;m,n+1})^{*}Y_{j;m,n-1} + (Y_{j;m,n+1})^{*}(Y_{j;m,n-1})^{*}e^{-4i(j+1)\tau}\}\\
&\quad - 4(j+1)^2\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^{*}e^{-2i(j+1)\tau}\right)^2 .
\end{aligned}
\tag{124}
$$

As in the previous case we see that these squares are the same if they are zero and this is fulfilled if

$$
\begin{aligned}
0 &= (j+1)^2 - n^2\\
\Leftrightarrow n &= \begin{cases} j+1 \\ -j-1 \end{cases} .
\end{aligned}
$$

For the scalar product we have

$$
\begin{aligned}
\mathcal{E}\cdot\mathcal{B} &= \mathcal{E}_+\mathcal{B}_- + \mathcal{E}_-\mathcal{B}_+ + \mathcal{E}_3\mathcal{B}_3 \\
&= -\frac{2(j+1)^2}{i}\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n+1}e^{2i(j+1)\tau} + (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n-1}e^{2i(j+1)\tau} - (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad - \frac{2(j+1)^2}{i}\sqrt{j^2-n^2}\sqrt{(j+1)^2-n^2}\left(Y_{j;m,n-1}e^{2i(j+1)\tau} + (Y_{j;m,n-1})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n+1}e^{2i(j+1)\tau} - (Y_{j;m,n+1})^* e^{-2i(j+1)\tau}\right) \\
&\quad + \frac{4(j+1)^2}{i}\left((j+1)^2-n^2\right)\left(Y_{j;m,n}e^{2i(j+1)\tau} + (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right) \\
&\quad \left(Y_{j;m,n}e^{2i(j+1)\tau} - (Y_{j;m,n})^* e^{-2i(j+1)\tau}\right)\ .
\end{aligned} \tag{125}
$$

This scalar product is zero if

$$
\begin{aligned}
0 &= (j+1)^2 - n^2 \\
\Leftrightarrow n &= \begin{cases} j+1 \\ -j-1 \end{cases} \quad .
\end{aligned}
$$

We notice that we get for both cases $\operatorname{Re} X_a$ and $\operatorname{Im} X_a$ the same condition for the quantum number n. Hence, for *type 1* solutions the fixed quantum numbers

$$
(j;m,j+1) \quad \text{and} \quad (j;m,-j-1) \tag{126}
$$

give null field solutions.

Statement about type 2 solutions

The previous computations only dealt with *type 1* solutions. But *type 1* and *type 2* solutions are related to each other. Since that, we can deduce for which fixed quantum numbers we do have null fields for *type 2* solutions. The transition from *type 1* to *type 2* in terms of the quantum numbers is given by [10]

$$
\begin{aligned}
j &\longrightarrow j+1 \\
(m,n) &\longrightarrow (n,m)\ .
\end{aligned} \tag{127}
$$

Hence, for null fields from *type 1* solutions we get

$$
\begin{aligned}
(j;m,j+1) &\longrightarrow (j+1;j+1,m) \\
(j;m,-j-1) &\longrightarrow (j+1;-j-1,m)\ ,
\end{aligned} \tag{128}
$$

such that for *type 2* solutions the fixed quantum numbers

$$
(j;j,n) \quad \text{and} \quad (j;-j,n) \tag{129}
$$

give null fields.

6.2. A different approach

Let us have a closer look at the ansatz

$$X_a(\tau,\omega) = \sum_{j,m,n} X_a^{(j;m,n)}(\tau) Y_{j;m,n}(\alpha,\beta) \ . \tag{130}$$

As mentioned earlier, we can deal with complex solutions, but physically relevant are only real solutions. Thus, we can parametrize the solutions (of *type 1* on which we will focus now) for some fixed j as

$$X_a(\tau,\omega) = X_a^+(\tau,\omega) + X_a^-(\tau,\omega) \tag{131}$$

where the upper index + stands for positive frequency parts and the upper index − for negative ones. We write them as

$$\begin{aligned} X_a^+ &= \sum_{m,n} c_{m,n} Z_a^{m,n} e^{2i(j+1)\tau} \\ X_a^- &= \sum_{m,n} c_{m,n}^* \left(Z_a^{m,n}\right)^* e^{-2i(j+1)\tau} \end{aligned} \tag{132}$$

with $Z_a^{m,n} = X_a^{m,n}(\tau = 0)$, some complex numbers $c_{m,n}$ where $m = -j, \ldots, j$ and $n = -j-1, \ldots, j+1$. Note that X_a^+ and X_a^- are related to each other via complex conjugation, i.e.

$$(X_a^+)^* = X_a^- \ . \tag{133}$$

Putting all together, with this parametrization the solutions can be written as [3]

$$X_a(\tau,\omega) = \sum_{m,n} c_{m,n} Z_a^{m,n}(\alpha,\beta) e^{2i(j+1)\tau} + c_{m,n}^* \left(Z_a^{m,n}\right)^* (\alpha,\beta) e^{-2i(j+1)\tau} \ . \tag{134}$$

We note that $Z_a^{m,n}$ is a complex polynom of degree $2j$ in $\alpha, \bar{\alpha}, \beta, \bar{\beta}$.

Inserting this parametrization into the potential for a fixed j gives

$$A = \sum_a \sum_{m,n} \left(c_{m,n} Z_a^{m,n} e^{2i(j+1)\tau} + c_{m,n}^* \left(Z_a^{m,n}\right)^* e^{-2i(j+1)\tau} \right) e^a \ . \tag{135}$$

Setting

$$c_a := \sum_{m,n} c_{m,n} Z_a^{m,n} \qquad c_a^* := \sum_{mn} c_{m,n}^* \left(Z_a^{m,n}\right)^* \ , \tag{136}$$

[3] A similar expression can be written for *type 2* solutions by swapping m and n and writing j instead of $j+1$.

we can write the solutions in a shorter expression as [4]

$$X_a = c_a e^{2i(j+1)\tau} + c_a^* e^{-2i(j+1)\tau} \tag{137}$$

with complex c_a. The dependence of the angles α and β are in c_a now, such that they can be interpreted as functions

$$c_a : S^3 \to \mathbb{C}^3 \cong \mathbb{R}^6 \; . \tag{138}$$

With this parametrization, the sphere-frame electric and magnetic field with (58) and (65) is given by

$$\mathcal{E}_a = -2i(j+1)(c_a e^{2i(j+1)\tau} - c_a^* e^{-2i(j+1)\tau}) \tag{139}$$

$$\mathcal{B}_a = -2(j+1)(c_a e^{2i(j+1)\tau} + c_a^* e^{-2i(j+1)\tau}) \; . \tag{140}$$

Computing the Riemann-Silberstein vector, we notice that the negative frequency parts of the the sphere-frame electric and magnetic field cancel each other:

$$\mathcal{E}_a + i\mathcal{B}_a = -4i(j+1)c_a e^{2i(j+1)\tau} \tag{141}$$

For null fields, the squared Riemann-Silberstein vector has to vanish [6], so

$$(\mathcal{E}_a + i\mathcal{B}_a)^2 = \mathcal{E}_a^2 - \mathcal{B}_a^2 + 2i\mathcal{E}_a\mathcal{B}_a = 0 \; . \tag{142}$$

Here we have

$$(\mathcal{E}_a + i\mathcal{B}_a)^2 = -16(j+1)^2 c_a c_a e^{4i(j+1)\tau} \; . \tag{143}$$

Hence, to get a null field we have the condition

$$\boxed{c_a c_a = 0} \; . \tag{144}$$

Explicitly written, the condition is

$$c_1^2 + c_2^2 + c_3^2 = 0 \Leftrightarrow c_3 = \sqrt{-c_1^2 - c_2^2} \tag{145}$$

$$\Leftrightarrow 2c_+c_- + c_3^2 = 0 \; . \tag{146}$$

Equation (145) is satisfied for any $(c_1, c_2) \in \mathbb{C}^2 \setminus \{0\}$ [5].

Alternatively, we could deal with the complex conjugate of the Riemann-Silberstein vector, where the postive frequency parts then would cancel

$$\mathcal{E}_a - i\mathcal{B}_a = -4i(j+1)c_a^* e^{-2i(j+1)\tau} \; . \tag{147}$$

[4] We could also consider $X_a = \frac{1}{i}\left(c_a e^{2i(j+1)\tau} - c_a^* e^{-2i(j+1)\tau}\right)$, but this is the same as (137) inserting for c_a the expression $\frac{1}{i}c_a$, which is allowed since c_a is complex. The appearing results would be the same.

[5] The origin 0 is trivial in this discussion.

The null field condition for the complex conjugate Riemann-Silberstein vector is

$$c_a^* c_a^* = 0 \,. \tag{148}$$

We note that we notice the $(m, j+1)$ part has only a $X_-^{\pm}$ component and the $(m, -j-1)$ part has only a $X_+^{\mp}$ component all in agreement with the previous subsection.

Since c_a is complex, we can write it as

$$c_a = x_a + iy_a \quad \Leftrightarrow \quad \mathbf{c} = \mathbf{x} + i\mathbf{y} \tag{149}$$

with $x_a, y_a \in \mathbb{R}$.

The null field condition becomes

$$\mathbf{c}^2 = 0 = \mathbf{x}^2 - \mathbf{y}^2 + 2i\mathbf{x} \cdot \mathbf{y} \tag{150}$$

which is equivalent to

$$\mathbf{x}^2 = \mathbf{y}^2 \tag{151}$$

$$\mathbf{x} \cdot \mathbf{y} = 0 \,. \tag{152}$$

With 6 real parameters x_a, y_a, we have with these conditions then $6 - 2 = 4$ free real parameters. Condition (151) can be geometrically interpreted as a hyperbola.

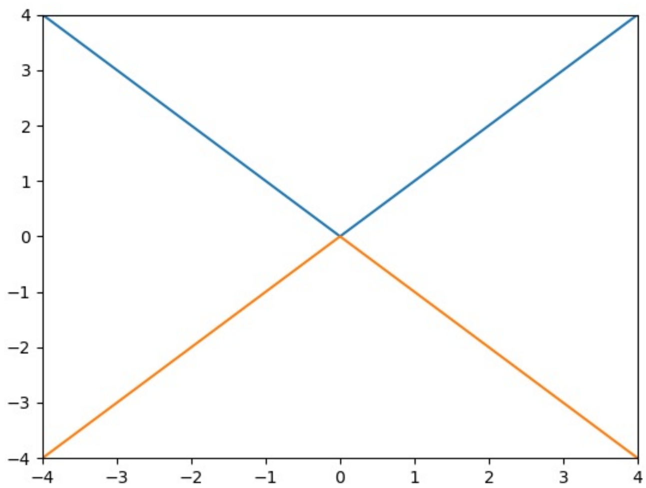

Figure 2: Condition (151) as a hyperbola in $\mathbb{C} \cong \mathbb{R}^2$

6.3. Further consideration

In the previous subsections we checked and discussed under which conditions we can generate null fields for our solutions for fixed quantum numbers $(j; m, n)$. But usually, for a fixed j we have a big parameter space of possible solutions since we could consider all solutions for the values which m and n are taking. Namely, for *type 1* we have $2(2j+1)(2j+3)$ and for *type 2* we have $2(2j+1)(2j-1)$ solutions [10].

Let us discuss this with the easiest case for *type 1* solutions with $j = 0$. The solution for $j = 0$ is given by

$$\begin{aligned} X_a^{(j=0)} &= c_{0,-1} Z_a^{0,-1} e^{2i\tau} + c_{0,-1}^* \left(Z_a^{0,-1}\right)^* e^{-2i\tau} \\ &\quad + c_{0,0} Z_a^{0,0} e^{2i\tau} + c_{0,0}^* \left(Z_a^{0,0}\right)^* e^{-2i\tau} \\ &\quad + \left(c_{0,1} Z_a^{0,0} e^{2i\tau} + c_{0,1}^* Z_a^{0,0}\right)^* e^{-2i\tau} \\ &= c_a e^{2i\tau} + c_a^* e^{-2i\tau} , \end{aligned} \tag{153}$$

where we have

$$\begin{aligned} c_a &= c_{0,-1} Z_a^{0,-1} + c_{0,0} Z_a^{0,0} + c_{0,1} Z_a^{0,1} \\ c_a^* &= c_{0,-1}^* \left(Z_a^{0,-1}\right)^* + c_{0,0}^* \left(Z_a^{0,0}\right)^* + c_{0,1}^* \left(Z_a^{0,1}\right)^* \ . \end{aligned} \tag{154}$$

As we have condition (144) to get null fields, it is sufficient to consider c_a. Comparing (154) with (144) and (54), the first term corresponds to c_+, the second one to c_3 and the last one to c_-. For $j = 0$ we have no dependence of the angles (α, β) on S^3 and the null field condition appears as in (146).

But things become much more complicated if we consider a non-zero value for j. Just to deal with an example, we will look at $j = \frac{1}{2}$ for the solutions of *type 1*. There we have $m = -\frac{1}{2}, \frac{1}{2}$ and $n = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$.

Thus, the solution is given by

$$
\begin{aligned}
X_a^{(j=\frac{1}{2})} &= c_{-\frac{1}{2},-\frac{3}{2}} Z_a^{-\frac{1}{2},-\frac{3}{2}} e^{3i\tau} + c^*_{-\frac{1}{2},-\frac{3}{2}} \left(Z_a^{-\frac{1}{2},-\frac{3}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{-\frac{1}{2},-\frac{1}{2}} Z_a^{-\frac{1}{2},-\frac{1}{2}} e^{3i\tau} + c^*_{-\frac{1}{2},-\frac{1}{2}} \left(Z_a^{-\frac{1}{2},-\frac{1}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{-\frac{1}{2},\frac{1}{2}} Z_a^{-\frac{1}{2},\frac{1}{2}} e^{3i\tau} + c^*_{-\frac{1}{2},\frac{1}{2}} \left(Z_a^{-\frac{1}{2},\frac{1}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{-\frac{1}{2},\frac{3}{2}} Z_a^{-\frac{1}{2},\frac{3}{2}} e^{3i\tau} + c^*_{-\frac{1}{2},\frac{3}{2}} \left(Z_a^{-\frac{1}{2},\frac{3}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{\frac{1}{2},-\frac{3}{2}} Z_a^{\frac{1}{2},-\frac{3}{2}} e^{3i\tau} + c^*_{\frac{1}{2},-\frac{3}{2}} \left(Z_a^{\frac{1}{2},-\frac{3}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{\frac{1}{2},-\frac{1}{2}} Z_a^{\frac{1}{2},-\frac{1}{2}} e^{3i\tau} + c^*_{\frac{1}{2},-\frac{1}{2}} \left(Z_a^{\frac{1}{2},-\frac{1}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{\frac{1}{2},\frac{1}{2}} Z_a^{\frac{1}{2},\frac{1}{2}} e^{3i\tau} + c^*_{\frac{1}{2},\frac{1}{2}} \left(Z_a^{\frac{1}{2},\frac{1}{2}} \right)^* e^{-3i\tau} \\
&\quad + c_{\frac{1}{2},\frac{3}{2}} Z_a^{\frac{1}{2},\frac{3}{2}} e^{3i\tau} + c^*_{\frac{1}{2},\frac{3}{2}} \left(Z_a^{\frac{1}{2},\frac{3}{2}} \right)^* e^{-3i\tau} \\
&= c_a e^{3i\tau} + c_a^* e^{-3i\tau}
\end{aligned}
\tag{155}
$$

with

$$
\begin{aligned}
c_a &= c_{-\frac{1}{2},-\frac{3}{2}} Z_a^{-\frac{1}{2},-\frac{3}{2}} + c_{-\frac{1}{2},-\frac{1}{2}} Z_a^{-\frac{1}{2},-\frac{1}{2}} + c_{-\frac{1}{2},\frac{1}{2}} Z_a^{-\frac{1}{2},\frac{1}{2}} + c_{-\frac{1}{2},\frac{3}{2}} Z_a^{-\frac{1}{2},\frac{3}{2}} \\
&\quad + c_{\frac{1}{2},-\frac{3}{2}} Z_a^{\frac{1}{2},-\frac{3}{2}} + c_{\frac{1}{2},-\frac{1}{2}} Z_a^{\frac{1}{2},-\frac{1}{2}} + c_{\frac{1}{2},\frac{1}{2}} Z_a^{\frac{1}{2},\frac{1}{2}} + c_{\frac{1}{2},\frac{3}{2}} Z_a^{\frac{1}{2},\frac{3}{2}} \\
c_a^* &= c^*_{-\frac{1}{2},-\frac{3}{2}} \left(Z_a^{-\frac{1}{2},-\frac{3}{2}} \right)^* + c^*_{-\frac{1}{2},-\frac{1}{2}} \left(Z_a^{-\frac{1}{2},-\frac{1}{2}} \right)^* + c^*_{-\frac{1}{2},\frac{1}{2}} \left(Z_a^{-\frac{1}{2},\frac{1}{2}} \right)^* + c^*_{-\frac{1}{2},\frac{3}{2}} \left(Z_a^{-\frac{1}{2},\frac{3}{2}} \right)^* \\
&\quad + c^*_{\frac{1}{2},-\frac{3}{2}} \left(Z_a^{\frac{1}{2},-\frac{3}{2}} \right)^* + c^*_{\frac{1}{2},-\frac{1}{2}} \left(Z_a^{\frac{1}{2},-\frac{1}{2}} \right)^* + c^*_{\frac{1}{2},\frac{1}{2}} \left(Z_a^{\frac{1}{2},\frac{1}{2}} \right)^* + c^*_{\frac{1}{2},\frac{3}{2}} \left(Z_a^{\frac{1}{2},\frac{3}{2}} \right)^* .
\end{aligned}
\tag{156}
$$

We have for $j = \frac{1}{2}$ 8 possible solutions. But even in this complicated case condition (144) remains to get null fields. Condition (144) is then a complex equation for 8 terms $c_{m,n} Z_a^{m,n}$, where $Z_a^{m,n}$ is a linear function in α, β and their complex conjugates for $j = \frac{1}{2}$.

Having a closer look at $Z_a^{m,n}$ for the particular $j = \frac{1}{2}$ case, we recognize that c_a gets for any fixed $a = +, 3, -$ only contributions from half of the (m, n) values.

(m,n)	Z_+	Z_3	Z_-
$\left(-\frac{1}{2},-\frac{3}{2}\right)$	$\sqrt{3}Y_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$	0	0
$\left(-\frac{1}{2},-\frac{1}{2}\right)$	$Y_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$	$\sqrt{2}Y_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$	0
$\left(-\frac{1}{2},\frac{1}{2}\right)$	0	$\sqrt{2}Y_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$	$-Y_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$
$\left(-\frac{1}{2},\frac{3}{2}\right)$	0	0	$-\sqrt{3}Y_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}$
$\left(\frac{1}{2},-\frac{3}{2}\right)$	$\sqrt{3}Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}$	0	0
$\left(\frac{1}{2},-\frac{1}{2}\right)$	$Y_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	$\sqrt{2}Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}$	0
$\left(\frac{1}{2},\frac{1}{2}\right)$	0	$\sqrt{2}Y_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$	$-Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}$
$\left(-\frac{1}{2},\frac{3}{2}\right)$	0	0	$-\sqrt{3}Y_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$

Table 1: Results for $Z_a^{m,n}$ for $j=\frac{1}{2}$ in +3- basis

With Table 1 the condition (146) becomes for $j=\frac{1}{2}$

$$\begin{aligned}
0 &= 2c_+c_- + c_3^2 \\
&= 2\left(\sqrt{3}c_{-\frac{1}{2},-\frac{3}{3}}Y_{\frac{1}{2};-\frac{1}{2},-\frac{1}{2}} + c_{-\frac{1}{2},-\frac{1}{2}}Y_{\frac{1}{2};-\frac{1}{2},\frac{1}{2}} + \sqrt{3}c_{\frac{1}{2},-\frac{3}{2}}Y_{\frac{1}{2};\frac{1}{2},-\frac{1}{2}} + c_{\frac{1}{2},-\frac{1}{2}}Y_{\frac{1}{2};\frac{1}{2},\frac{1}{2}}\right) \\
&\quad \left(-c_{-\frac{1}{2},\frac{1}{2}}Y_{\frac{1}{2};-\frac{1}{2},-\frac{1}{2}} - \sqrt{3}c_{-\frac{1}{2},\frac{3}{2}}Y_{\frac{1}{2};-\frac{1}{2},\frac{1}{2}} - c_{\frac{1}{2},\frac{1}{2}}Y_{\frac{1}{2};\frac{1}{2},-\frac{1}{2}} - \sqrt{3}c_{-\frac{1}{2},\frac{3}{2}}Y_{\frac{1}{2};\frac{1}{2},\frac{1}{2}}\right) \\
&\quad + \left(\sqrt{2}c_{-\frac{1}{2},-\frac{1}{2}}Y_{\frac{1}{2};-\frac{1}{2},-\frac{1}{2}} + \sqrt{2}c_{-\frac{1}{2},\frac{1}{2}}Y_{\frac{1}{2};-\frac{1}{2},\frac{1}{2}} + \sqrt{2}c_{\frac{1}{2},-\frac{1}{2}}Y_{\frac{1}{2};\frac{1}{2},-\frac{1}{2}} + \sqrt{2}c_{\frac{1}{2},\frac{1}{2}}Y_{\frac{1}{2};\frac{1}{2},\frac{1}{2}}\right)^2 .
\end{aligned} \tag{157}$$

To get null fields for linear combinations of $j=\frac{1}{2}$ solutions, one just has to check for which $c_{m,n}$ equation (157) is satisfied.

That all was about the discussion for $j=\frac{1}{2}$. But the higher the value of j, the more terms we get: $j=1$ has 15 solutions, so 15 terms are appearing for c_a, $j=\frac{3}{2}$ has 24 terms, $j=2$ has 35 terms and so on for *type 1* solutions. The question whether which possible linear combinations of solutions are null fields is answered by condition (144), too. One just has to check for which $c_{m,n}$ this is satisfied. Doing this by hand can bring some lengthy expressions, but the condition is quite simple.

7. Energy and helicity

Energy and helicity are two important quantities to characterize knotted field configurations. Especially, helicity is a very important one since it gives a chance to quantify the topology of the knotted field [4]. It gives a measure of the mean value of the linking number of theses knotted field lines [4,21,24]. In this section, we derive useful formulas for the helicity and the energy to compute them in context of our formalism. We will also demonstrate the computation for a particular example. Finally, we will make a statement about the relation between energy and helicity.

7.1. Useful expressions for energy and helicity

In this section, we derive formulas for the helicity and the energy which are useful for our formalism. We will use them in the next subsection by demonstrating the effectiveness of the constructed solutions (54) and (55) in the computation of energy and helicity. The main results of this section were also mentioned in [10], but we give further details which were not demonstrated and described there.

We start with some statements about helicity. The magnetic helicity is defined as [4,24]

$$h_m = \frac{1}{2}\int_{\mathbb{R}^3} d^3x\ \mathbf{A}\cdot\mathbf{B} \tag{158}$$

recalling that the vector potential $\mathbf{A}$ is defined by

$$\mathbf{B} = \nabla\times\mathbf{A}\ . \tag{159}$$

To have a description of the magnetic helicity also with the field strength F, we can rewrite it as

$$h_m = \frac{1}{2}\int_{\mathbb{R}^3} A\wedge F\ , \tag{160}$$

since

$$A\wedge F = \frac{1}{2}A_iF_{jk}dx^{ijk} + A_iE_jdx^{ij0} + \frac{1}{2}A_0F_{jk}dx^{0jk}\ ,$$

so

$$\frac{1}{2}\int_{\mathbb{R}^3} A\wedge F = \frac{1}{2}\int_{\mathbb{R}^3} A_i(\epsilon_{jkl}B_l)\epsilon_{ijk}\ dx^{123} = \frac{1}{2}\int_{\mathbb{R}^3} A\wedge F\ .$$

Since the electric field in vacuum has zero divergence, as the magnetic field always has this property, with the concept of duality one can also define the electric helicity [4]

$$h_{el} = \frac{1}{2}\int_{\mathbb{R}^3} d^3x\ \mathbf{C}\cdot\mathbf{E}\ , \tag{161}$$

where $\mathbf{C}$ is defined by

$$\mathbf{C} = \nabla \times \mathbf{E} \,. \tag{162}$$

With the same argumentation as above, we have for the electric helicity also

$$h_{el} = \frac{1}{2} \int_{\mathbb{R}^3} A_D \wedge F_D \,. \tag{163}$$

With [4]

$$\begin{aligned} \frac{dh_m}{dt} &= -\frac{1}{2} \int_{\mathbb{R}^3} d^3x \, \mathbf{E} \cdot \mathbf{B} \\ \frac{dh_{el}}{dt} &= \frac{1}{2} \int_{\mathbb{R}^3} d^3x \, \mathbf{E} \cdot \mathbf{B} \,, \end{aligned} \tag{164}$$

the sum of it, which we call helicity

$$h = h_m + h_{el} = \frac{1}{2} \int_{\mathbb{R}^3} A \wedge F + A_D \wedge F_D \,, \tag{165}$$

is a conserved quantity.

The field energy [3,4,10]

$$E = \frac{1}{2} \int_{\mathbb{R}^3} d^3x \, (\mathbf{E}^2 + \mathbf{B}^2) \tag{166}$$

is also a conserved quantity. We introduced already sphere-frame electric and magnetic fields via [10]

$$F = \mathcal{E}_a e^a \wedge e^0 + \frac{1}{2} \mathcal{B}_a \epsilon^a_{\ bc} e^b \wedge e^c \,. \tag{167}$$

With these new fields we want to find integral expressions for the energy on the $t = \tau = 0$ slice. We use the $t = \tau = 0$ slice to simplify our calculations in the next steps.

First of all, we have the usual expression [7]

$$F = E_i dx^i \wedge dx^0 + \frac{1}{2} B_i \epsilon_{ijk} dx^j \wedge dx^k \,. \tag{168}$$

A comparison of coefficients gives that

$$E_i = \mathcal{E}_a e^a_i e^0_0 \qquad B_i = \frac{1}{2} \epsilon_{abc} \epsilon_{ijk} e^b_j e^c_k \mathcal{B}_a \,. \tag{169}$$

Since we use $t = \tau = 0$, the prefactor γ is then

$$\gamma = \frac{2l^2}{r^2 + l^2} \,. \tag{170}$$

For the 1-forms we get for $t = \tau = 0$

$$e^0 = \frac{4l}{(r^2+l^2)^2}\frac{1}{2}(r^2+l^2)dt = \frac{2l}{r^2+l^2}dt \tag{171}$$

$$e^a = \frac{4l}{(r^2+l^2)^2}\left(\frac{1}{2}(l^2-r^2)\delta^a_k + x^a x^k + l\epsilon^a_{\ jk}x^j\right)dx^k \ . \tag{172}$$

Explicitly, we have

$$\begin{aligned} e^0_0 &= \frac{2l}{r^2+l^2} & e^0_i &= 0 \\ e^a_0 &= 0 & e^a_k &= \frac{4l}{(r^2+l^2)^2}\left(\frac{1}{2}(l^2-r^2)\delta^a_k + x^a x^k + l\epsilon^a_{\ jk}x^j\right) \end{aligned} \tag{173}$$

with $a, i, k = 1, 2, 3$.

From equation (169) we see that these coefficients give the transition from the sphere-frame fields to the fields in Minkowski space. Here, they are given for $t = \tau = 0$.

For the squares of the electric and magnetic field we have

$$\mathbf{E}^2 = E_i E_i = (e^0_0)^2 e^a_i e^b_i \mathcal{E}_a \mathcal{E}_b \tag{174}$$

and

$$\begin{aligned} \mathbf{B}^2 = B_i B_i &= \frac{1}{4}\epsilon_{abc}\epsilon_{def}\mathcal{B}_a\mathcal{B}_d\epsilon_{ijk}\epsilon_{ilm}e^b_j e^c_k e^e_l e^f_m \\ &= \frac{1}{4}\epsilon_{abc}\epsilon_{def}\mathcal{B}_a\mathcal{B}_d(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})e^b_j e^c_k e^e_l e^f_m \\ &= \frac{1}{4}\epsilon_{abc}\epsilon_{def}\mathcal{B}_a\mathcal{B}_d(e^b_j e^e_j e^c_k e^f_k - e^b_j e^f_j e^c_k e^e_k) \\ &= \frac{1}{2}\epsilon_{abc}\epsilon_{def}\mathcal{B}_a\mathcal{B}_d e^b_j e^e_j e^c_k e^f_k \ . \end{aligned} \tag{175}$$

We focus our consideration for $\mathbf{E}^2$. The following computation is completely analogous for $\mathbf{B}^2$.

So now, what we want to do is to calculate the term $e^a_i e^b_i$. We could calculate it by brute force, but there is an easier way to get things done. The term $e^a_i e^b_i$ is symmetric in a and b and x^1, x^2 and x^3 should enter symmetrically, so we have a kind of rotational symmetry. This all means we can write the ansatz

$$e^a_i e^b_i = c_1(r^2)\delta_{ab} + c_2(r^2)x^a x^b \ . \tag{176}$$

Now the task is to find the coefficients c_1 and c_2 which depend on r^2.

To do so, we set $x^1 = x^2 = 0$. We then have

$$e^1_i e^1_i = c_1\left((x^3)^2\right) \qquad e^3_i e^3_i = c_1\left((x^3)^2\right) + (x^3)^2 c_2\left((x^3)^2\right) \ . \tag{177}$$

Inserting the values for a and k we have

$$\begin{aligned} e_1^1 &= \frac{4l}{(r^2+l^2)^2}\left[\frac{1}{2}\left(l^2-(x^3)^2\right)\right] \\ e_2^1 &= \frac{-4l^2}{l^2+(x^3)^2}x^3 \\ e_3^1 &= 0 \end{aligned} \tag{178}$$

and

$$\begin{aligned} e_1^3 &= 0 \\ e_2^3 &= 0 \\ e_3^3 &= \frac{4l}{(r^2+l^2)^2}\left[\frac{1}{2}(l^2-(x^3)^2)+(x^3)^2\right] \\ &= \frac{4l}{(r^2+l^2)^2}\left[\frac{1}{2}(l^2+(x^3)^2)\right] \\ &= \frac{2l}{l^2+(x^3)^2} \ . \end{aligned} \tag{179}$$

With these results we get

$$\begin{aligned} e_i^1 e_i^1 &= (e_1^1)^2 + (e_2^1)^2 \\ &= \frac{4l^2(l^2-(x^3)^2)+16l^4(x^3)^2}{(l^2+(x^3)^2)^4} \\ &= \frac{4l^2}{(l^2+(x^3)^2)^4}\left[(l^2-(x^3)^2)^2+4l^2(x^3)^2\right] \\ &= \frac{4l^2}{(l^2+(x^3)^2)^4}\left(l^2+(x^3)^2\right) \\ &= \frac{4l^2}{(l^2+(x^3)^2)^2} \ , \end{aligned}$$

so we have

$$c_1 = \frac{4l^2}{(l^2+(x^3)^2)} \ , \tag{180}$$

Now let us compute c_2. We have

$$e_i^3 e_i^3 = (e_3^3)^2 = \frac{4l^2}{(l^2+(x^3)^2)^2} \ . \tag{181}$$

From that we can deduce that $c_2 = 0$, so

$$e_i^a e_i^b = \frac{4l^2}{(l^2+r^2)^2} \ .$$

From all that we see that

$$\mathbf{E}^2 = \frac{16l^4}{(l^2+r^2)^4}\mathcal{E}^2 \ . \tag{182}$$

The same holds for the squared magnetic field

$$\mathbf{B}^2 = \frac{16l^4}{(l^2+r^2)^4}\mathcal{B}^2 \ . \tag{183}$$

The term $e^1 \wedge e^2 \wedge e^3$ is the volume form of S^3, so we need to compute it. The result is

$$\begin{aligned} e^1 \wedge e^2 \wedge e^3 &= e^1_i e^2_j e^3_k \, dx^i \wedge dx^j \wedge dx^k \\ &= \epsilon_{ijk} e^1_i e^2_j e^3_k \, dx^1 \wedge dx^2 \wedge dx^3 \\ &= \det(e^a_i) \, dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{184}$$

with $dx^1 \wedge dx^2 \wedge dx^3 \equiv dx^{123}$ being the volume form on $\mathbb{R}^3$ and $\det(e^a_i)$ denoting the determinant of e^a_i.

The next step is to find $\det(e^a_i)$. With $SO(3)$ symmetry the term $\det(e^a_i)$ can only depend on $r = \sqrt{x^2+y^2+z^2}$ with $x^1 = x$, $x^2 = y$, $x^3 = z$. To get $\det(e^a_i)$, let us set $x^1 = x^2 = 0$ and $x^3 = r$. Then we have (still on the $t = \tau = 0$ slice)

$$\begin{aligned} e^1 &= \frac{\gamma^2}{l^3}\left(\frac{1}{2}(r^2-l^2)dx^1 + lr dx^2\right) \\ e^2 &= \frac{\gamma^2}{l^3}\left(\frac{1}{2}(r^2-l^2)dx^2 - lr dx^1\right) \\ e^3 &= \frac{\gamma^2}{l^3}\left(-\frac{1}{2}(r^2+l^2)dx^3\right) \ . \end{aligned}$$

For e^a_i we have then

$$e^a_i = \frac{\gamma^2}{2l^3}\begin{pmatrix} l^2-r^2 & -2lr & 0 \\ 2lr & l^2-r^2 & 0 \\ 0 & 0 & -(l^2+r^2) \end{pmatrix} . \tag{185}$$

With that result, it is easy to calculate the determinant:

$$\begin{aligned} \det(e^a_i) &= \frac{\gamma^6}{8l^9}\left(-\left(l^2+r^2\right)\cdot\left(\left(l^2-r^2\right)^2 + 4l^2r^2\right)\right) \\ &= -\frac{\gamma^6}{8l^9}\left(\left(l^2+r^2\right)\cdot\left(l^2+r^2\right)^2\right) \\ &= -\frac{\gamma^6}{8l^9}\left(l^2+r^2\right)^3 \end{aligned}$$

Inserting $\gamma = \frac{2l^2}{r^2+l^2}$ gives

$$\det(e^a_i) = -\frac{8l^3}{(l^2+r^2)^3} \ . \tag{186}$$

So we get

$$\mathbf{E}^2 d^3x = \frac{2l}{l^2+r^2}\mathcal{E}^2 d^3\Omega_3$$
$$\mathbf{B}^2 d^3x = \frac{2l}{l^2+r^2}\mathcal{B}^2 d^3\Omega_3 \ .$$

The volume form on S^3 is given by

$$d^3\Omega_3 = |\det(e_i^a)| d^3x = \frac{8l^3}{(r^2+l^2)^3} d^3x \ . \qquad (187)$$

Since $t=0$, we have

$$\omega_4 = \frac{2l^2}{l^2+r^2}\frac{r^2-l^2}{2l^2} = \frac{r^2-l^2}{l^2+r^2} \ ,$$

so

$$1-\omega_4 = \frac{2l^2}{r^2+l^2}$$

and we arrive at the integrals

$$\int_{\mathbb{R}^3} \mathbf{E}^2 d^3x = \frac{1}{l}\int_{S^3} d^3\Omega_3 (1-\omega_4)\mathcal{E}_a\mathcal{E}_a \qquad \int_{\mathbb{R}^3} \mathbf{B}^2 d^3x = \frac{1}{l}\int_{S^3} d^3\Omega_3 (1-\omega_4)\mathcal{B}_a\mathcal{B}_a \ , \qquad (188)$$

such that

$$E = \frac{1}{2l}\int_{S^3} d^3\Omega_3 (1-\omega_4)[\mathcal{E}_a\mathcal{E}_a + \mathcal{B}_a\mathcal{B}_a] \ . \qquad (189)$$

To sum up, to calculate the energy, firstly we have to compute the sphere-frame electric and magnetic field components and then, secondly, plug in these fields in the integral (189).

For the helicity, we have to compute the potential and the field strength (and their dual ones) which is quite straight forward.

7.2. A non-trivial example for computing energy and helicity

In the following, we will demonstrate the effectiveness of the presented method by computing the helicity and the energy on the $t = \tau = 0$ slice. We consider the *type 1* $(j; m, n) = (1; 0, 0)$ solution and handle the special case where we put the τ dependence into a cos function. The technical steps from the previous subsection will be applicated now.

We choose the orientation of $\mathbb{R}^3$ so that if $\omega = \omega_{123}\; dx^{123}$ is a 3-form, then

$$\int_{\mathbb{R}^3} \omega = \int_{\mathbb{R}^3} \omega_{123}\; d^3x \; . \tag{190}$$

Let us assume that in (e^a) basis the 3-form has components $\omega = \tilde{\omega}_{123}\; e^{123}$.
Since $dx^{123} \propto -e^{123}$, we have

$$\int_{\mathbb{R}^3} \omega = -\int_{S^3} d^3\Omega_3\; \tilde{\omega}_{123} \; , \tag{191}$$

so integrating over $\mathbb{R}^3$ is basically the same as integrating over S^3 up to a sign.

For $(1; 0, 0)$ we have for the solutions with the previous statement about the τ dependence

$$\begin{aligned} X_+ &= \cos 4\tau\; Y_{1;0,1} \\ X_3 &= 2\cos 4\tau\; Y_{1;0,0} \\ X_- &= -\cos 4\tau\; Y_{1;0,-1} \; . \end{aligned} \tag{192}$$

In 1,2,3 basis the solutions are written as

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}} \cos 4\tau\, (Y_{1;0,1} - Y_{1;0,-1}) \\ X_2 &= -\frac{i}{\sqrt{2}} \cos 4\tau\, (Y_{1;0,1} + Y_{1;0,-1}) \\ X_3 &= 2\cos 4\tau\; Y_{1;0,0} \; . \end{aligned} \tag{193}$$

The very first step is to extract the sphere-frame electric and magnetic field components. For this example, we see that the differentiation to τ (needed for the sphere-frame electric field due to (58)) gives a sin and since $\tau = 0$, we get zero components for the electric field. For the magnetic field components we use (65). Thus the sphere-frame field components are given by

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E}_2 = \mathcal{E}_3 = 0 \\ \mathcal{B}_1 &= -2\sqrt{2}(Y_{1;0,1} - Y_{1;0,-1}) \\ \mathcal{B}_2 &= i2\sqrt{2}(Y_{1;0,1} + Y_{1;0,-1}) \\ \mathcal{B}_3 &= -8Y_{1;0,0} \; . \end{aligned} \tag{194}$$

What we get for our vector potential for $\tau = 0$ is

$$A = \frac{1}{\sqrt{2}}(Y_{1;0,1} - Y_{1;0,-1})e^1 - \frac{i}{\sqrt{2}}(Y_{1;0,1} + Y_{1;0,-1})e^2 + 2Y_{1;0,0}e^3 \ . \tag{195}$$

As mentioned above, the dual solution for a *type 1* solution is given by shift of $\frac{\pi}{2}$ in the frequency [10] such that we have for our example

$$A_D = 0 \ . \tag{196}$$

The field strength in this particular case is

$$\begin{aligned} F &= \mathcal{B}_1 e^{23} + \mathcal{B}_2 e^{31} + \mathcal{B}_3 e^{12} \\ &= -2\sqrt{2}(Y_{1;0,1} - Y_{1;0,-1})e^{23} + i2\sqrt{2}(Y_{1;0,1} + Y_{1;0,-1}) - 8Y_{1;0,0}e^{12} \ . \end{aligned} \tag{197}$$

At first, let us consider the helicity h. Here we get

$$\begin{aligned} h &= \frac{1}{2}\int_{S^3} [2(Y_{1;0,1} - Y_{1;0,-1})^2 - 2(Y_{1;0,1} + Y_{1;0,-1})^2 + 16Y_{1;0,0}^2] \ d^3\Omega_3 \\ &= \frac{1}{2}\int_{S^3} [-8Y_{1;0,1}Y_{1;0,-1} + 16Y_{1;0,0}Y_{1;0,0}] \ d^3\Omega_3 \\ &= \int_{S^3} [-4Y_{1;0,1}Y_{1;0,-1} + 8Y_{1;0,0}Y_{1,0,0}] \ d^3\Omega_3 \ . \end{aligned}$$

We remind that the orthonormal property of $Y_{j;m,n}$ is

$$\int_{S^3} (Y_{j_1;m_1,n_1})^* Y_{j_2;m_2,n_3} d^3\Omega_3 = \delta_{j_1 j_2}\delta_{m_1 m_2}\delta_{n_1 n_2} \tag{198}$$

and also we have

$$(Y_{j;m,n})^* = (-1)^{m+n} Y_{j;-m,-n} \ . \tag{199}$$

This means for the helicity

$$h = 4 + 8 = 12 \ . \tag{200}$$

Now let us consider the energy E. Since we have

$$\begin{aligned} Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} &= -\frac{1}{\pi}\bar{\beta} \\ Y_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} &= -\frac{1}{\pi}\beta \end{aligned}$$

and

$$\omega_4 = i\frac{\bar{\beta} - \beta}{2} \ ,$$

thus

$$\omega_4 = \frac{i\pi}{2}\left(-Y_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} + Y_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}\right) \ . \tag{201}$$

This short computation shows that we can discard the ω_4 term in the energy formula (189), because it will be impossible to balance all the quantum numbers to get non-zero integrals. So we get for the energy

$$\begin{aligned}
E &= \frac{1}{2l}\int_{S^3} [8(Y_{1;0,1} - Y_{1;0,-1})^2 - 8(Y_{1;0,1} + Y_{1;0,-1})^2 + 64Y_{1;0,0}^2] \, d^3\Omega_3 \\
&= \frac{1}{2l}\int_{S^3} [-32Y_{1;0,1}Y_{1;0,-1} + 64Y_{1;0,0}Y_{1;0,0}] \, d^3\Omega_3 \\
&= \frac{1}{l}\int_{S^3} [-16Y_{1;0,1}Y_{1;0,-1} + 32Y_{1;0,0}Y_{1;0,0}] \, d^3\Omega_3
\end{aligned}$$

and again with (198) and (199) our result is

$$E = \frac{1}{l}[16 + 32] = \frac{48}{l} \ . \tag{202}$$

We see that it is relatively easy to compute the energy and the helicity for these knotted fields because of the properties of the hyperspherical harmonics on S^3.

7.3. A relation between energy and helicity

In the previous subsection we computed the energy and the helicity for a particular *type 1* solution with a cos function as its τ dependence. Comparing these two results from the previous subsection, we see that

$$E^{(1;0,0)}l = 4h^{(1;0,0)} \ . \tag{203}$$

This observations leads to the assumption that there is a relation between energy and helicity.

The goal of this subsection is to derive a relation between energy and helicity. With a general relation, these two very important quantities are easy to convert into each other.

Let us consider *type 1* solutions and let us have a closer look at the potential

$$A = X_a e^a \ .$$

We use the notation

$$\begin{aligned} A_a &= X_a \\ (A_D)_a &= (X_D)_a \ . \end{aligned} \tag{204}$$

Applying Eulers Formula (12) to the parametrization (137) mentioned in section 6.2., we have

$$\begin{aligned} A_a = X_a &= c_a e^{2i(j+1)\tau} + c_a^* e^{-2i(j+1)\tau} \\ &= (\operatorname{Re} c_a + i \operatorname{Im} c_a)(\cos 2(j+1)\tau + i \sin 2(j+1)\tau) \\ &\quad + (\operatorname{Re} c_a - i \operatorname{Im} c_a)(\cos 2(j+1)\tau - i \sin 2(j+1)\tau) \\ &= 2 \operatorname{Re} c_a \cos 2(j+1)\tau - 2 \operatorname{Im} c_a \sin 2(j+1)\tau \ . \end{aligned} \tag{205}$$

We remind that c_a, c_a^* are defined by (136).

To get the dual solution, we have to shift the frequency by $\frac{\pi}{2}$ for *type 1* solutions [10]. sin and cos functions become by a shift of $\frac{\pi}{2}$

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \ . \end{aligned} \tag{206}$$

Hence, the dual solution is given by

$$(A_D)_a = (X_D)_a = -2 \operatorname{Re} c_a \sin 2(j+1)\tau - 2 \operatorname{Im} c_a \cos 2(j+1)\tau \ . \tag{207}$$

For $t = \tau = 0$ we get

$$\begin{aligned} A_a = X_a &= 2 \operatorname{Re} c_a \\ (A_D)_a = (X_D)_a &= -2 \operatorname{Im} c_a \ . \end{aligned} \tag{208}$$

Now, we can compute the helicity

$$
\begin{aligned}
h &= \frac{1}{2}\int_{\mathbb{R}^3} A\wedge F + A_D \wedge F_D \\
&= \frac{1}{2}\int_{\mathbb{R}^3} e^{123}\left(A_a\mathcal{B}_a + (A_D)_a\mathcal{E}_a\right)
\end{aligned} \tag{209}
$$

on the $t=\tau=0$ slice.

The sphere-frame electric field (139) for $\tau = 0$ is given by

$$
\begin{aligned}
\mathcal{E}_a &= -2i(j+1)(c_a - c_a^*) \\
&= 4(j+1)\,\mathrm{Im}\, c_a \ ,
\end{aligned} \tag{210}
$$

while for the sphere-frame magnetic field (140) we have for $\tau = 0$

$$
\begin{aligned}
\mathcal{B}_a &= -2(j+1)(c_a + c_a^*) \\
&= -4(j+1)\,\mathrm{Re}\, c_a \ .
\end{aligned} \tag{211}
$$

Thus, we get for the helicity

$$
\begin{aligned}
h &= \frac{8(j+1)^2}{2}\int_{S^3} d^3\Omega_3 \left((\mathrm{Re}\, c_a)^2 + (\mathrm{Im}\, c_a)^2\right) \\
&= 4(j+1)\int_{S^3} d^3\Omega_3 \ c_a c_a^* \ .
\end{aligned} \tag{212}
$$

We remind that we have to take care about the volume form on S^3 for the helicity computation, since it has to be $-e^{123}$ to preserve orientation.

Now we can consider the energy. It becomes with (210) and (211)

$$
\begin{aligned}
E &= \frac{1}{2l}\int_{S^3} d^3\Omega_3 \left(\mathcal{E}_a\mathcal{E}_a + \mathcal{B}_a\mathcal{B}_a\right) \\
&= \frac{16(j+1)^2}{2l}\int_{S^3} d^3\Omega_3 \left((\mathrm{Re}\, c_a)^2 + (\mathrm{Im}\, c_a)^2\right) \\
&= \frac{8(j+1)^2}{l}\int_{S^3} d^3\Omega_3 \ c_a c_a^* \ .
\end{aligned} \tag{213}
$$

We mention again that the ω_4 term is irrelevant, that is why it is not written down here.

Comparing the expressions (212) for h and (213) for E, we can see that they are related with each other via

$$
\begin{aligned}
El &= 2(j+1)h \\
\Leftrightarrow \frac{E}{h} &= \frac{2(j+1)}{l} \ .
\end{aligned} \tag{214}
$$

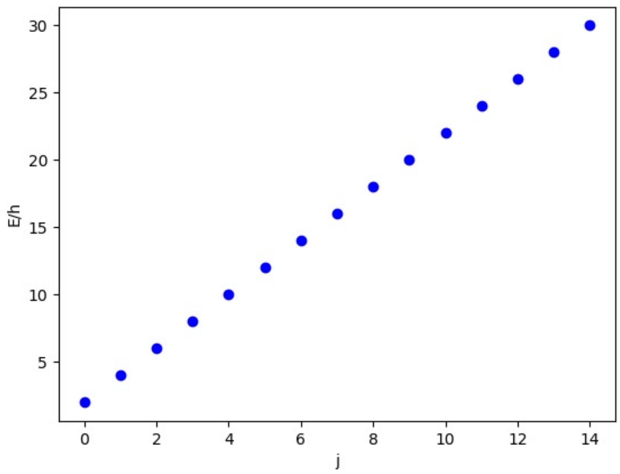

Figure 3: Visualization of the relation between energy and helicity with $l = 1$ for *type 1* solutions

Statement about type 2 solutions

With the same argumentation and since *type 1* solutions and *type 2* solutions are related to each other by a parity transform, for *type 2* solutions the ratio between energy and helicity is given by

$$\frac{E}{h} = -\frac{2j}{l} \ . \tag{215}$$

Since the energy is always positive, we can also call *type 1* solutions to have "positive helicity" and *type 2* solutions to have "negative helicity" . This designation agrees with the fact that *type 1* and *type 2* solutions are related with each other by spatial inversion

$$(x, y, z) \rightarrow (-x, -y, -z) \ .$$

We are lucky to have a simple relation between energy and helicity, which are important quantities to characterize electromagnetic fields, especially the helicity for knotted electromagnetic fields. With the relations (214) and (215) they are easy to convert into each other for *type 1* and *type 2* solutions.

8. Summary and outlook

Knotted electromagnetic fields have been an active area of research since 1989. To the 4 methods presented in [4], a new fifth method is presented in [10]. In this thesis we gave a detailed description of this new method. As mentioned in the introduction, it is based on the simplicity of solving Maxwell equations on a temporal cylinder over a 3-sphere and the conformal equivalence of a part of the latter space to 4-dimensional Minkowski space.

The goal of this thesis was to work on some details of the new presented method in [10]. We computed the decay behaviour of these new solutions in Minkowski space, which is important to check since we are interested in solutions with finite energies. Furthermore, we constructed a condition to generate null fields, a special important class of electromagnetic fields and checked for which fixed quantum numbers of each type of solutions null fields are generated. Finally, we did a detailed discussion of the two important quantities energy and helicity. We derived formulas for them so that they are relatively easy computable with our formalism. We presented the computation of energy and helicity of a particular example and also prooved a general relation between them.

Of course there are many more problems which can be worked on. For possible future research and thesis projects we would like to mention a few of these problems.
It would be interesting to check how rotated and boosted solutions on $2I \times S^3$ transform over to Minkowski space. More generally, one could look at how symmetries transform and check these new solutions with their new properties.
Throughout this thesis and in [10], the Maxwell equations were handled in vacuum, or in other words, we had no sources for electric and magnetic charges and currents. But for practical purposes it would be useful to add sources to the fields and investigate their transformation from the cylinder to Minkowski space.
Besides all that, we only dealt with abelian solutions. But for a general Yang-Mills description we need to handle the non-abelian case. The problem with the non-abelian extension is the coupling of different j components of the solutions X_a in (51). The analysis of now nonlinear ordinary differential equations generalizing the Maxwell equation (47) will be much harder. But it could be a useful tool for numerical studies of Yang-Mills dynamics in Minkowski space.

Although the authors in [10] had the intention to find new Yang-Mills solutions, they found this new method of constructing electromagnetic knots. The research on knotted field configurations is still quite active. As science is, answering an important question or a new discovery always raises new questions. We hope that these questions will be answered in future research projects.

Appendices

A. Derivation of the 1-forms e^0, e^a

Inserting in (17) for $a = 1, 2, 3$ and for $B, C = 1, 2, 3, 4$, we arrive at

$$\begin{aligned} e^1 &= \omega_4 d\omega_1 - \omega_1 d\omega_4 + \omega_3 d\omega_2 - \omega_2 d\omega_3 \\ e^2 &= \omega_1 d\omega_3 - \omega_3 d\omega_1 + \omega_4 d\omega_2 - \omega_2 d\omega_4 \\ e^3 &= \omega_2 d\omega_1 - \omega_1 d\omega_2 + \omega_4 d\omega_3 - \omega_3 d\omega_4 \ . \end{aligned} \tag{216}$$

Calculating the exterior derivative of ω_1, ω_2, ω_3 and ω_4 would lead to some long expressions which are difficult to handle. To get the correct 1-forms, we make the following educated guess for them due to $SO(4)$ symmetry

$$\begin{aligned} e^0 &= c_1 dt + c_2 x^k dx^k \\ e^a &= c_3 x^a dt + c_4 dx^a + c_5 x^a x^k dx^k + c_6 \epsilon_{ajk} x^j dx^k, \end{aligned} \tag{217}$$

where the coefficients $c_1, \ldots, c_6$ only depend on t and r. The first step to get the 1-forms is to get all these coefficients. To do so, we go to the $x = y = 0$, $z = r$ frame to simplify things. We remind that we use the notation $x^1 = x$, $x^2 = y$, $x^3 = z$.

In this frame we have

$$\begin{aligned} \omega_1 &= \gamma \frac{x}{l} = 0 \\ \omega_2 &= \gamma \frac{y}{l} = 0 \\ \omega_3 &= \gamma \frac{z}{l} \\ \omega_4 &= \gamma \frac{z^2 - t^2 - l^2}{2l^2} \ , \end{aligned} \tag{218}$$

so that the exterior derivatives become

$$\begin{aligned} d\omega_1 &= d\gamma \frac{x}{l} + \gamma \frac{dx}{l} = \gamma \frac{dx}{l} \\ d\omega_2 &= d\gamma \frac{y}{l} + \gamma \frac{dy}{l} = \gamma \frac{dy}{l} \\ d\omega_3 &= d\gamma \frac{z}{l} + \gamma \frac{dz}{l} \\ d\omega_4 &= d\gamma \frac{z^2 - t^2 - l^2}{2l^2} + \gamma \frac{z}{l^2} dz - \gamma \frac{t}{l^2} dt \ . \end{aligned} \tag{219}$$

In this frame, the initial expressions (216) for the 1-forms become

$$\begin{aligned} e^1 &= \omega_4 d\omega_1 + \omega_3 d\omega_2 \\ e^2 &= -\omega_3 d\omega_1 + \omega_4 d\omega_2 \\ e^3 &= \omega_4 d\omega_3 - \omega_4 d\omega_3 \end{aligned} \tag{220}$$

and our educated guess (217) takes the form

$$\begin{aligned} e^0 &= c_1 dt + c_2 x^3 dx^3 \\ e^1 &= c_4 dx^1 - c_6 x^3 dx^3 \\ e^2 &= c_4 dx^2 + c_6 x^3 dx^1 \\ e^3 &= c_3 x^3 dt + c_4 dx^3 + c_5 (x^3)^2 dx^3 \ . \end{aligned} \tag{221}$$

We put all the ω expressions (218) and (219) in (220) and with a comparison of coefficients with (221) we get $c_1, \ldots, c_6$. We start with e^1:

$$\begin{aligned} e^1 &= \omega_4 d\omega_1 + \omega_3 d\omega_2 \\ &= \gamma \frac{z^2 - t^2 - l^2}{2l^2} \gamma \frac{dx}{l} + \gamma \frac{z}{l} \gamma \frac{dy}{l} \\ &= \gamma^2 \frac{z^2 - t^2 - l^2}{2l^3} dx + \frac{\gamma^2}{l^2} z dy \ . \end{aligned}$$

From that we can deduce that

$$\begin{aligned} c_4 &= \gamma \frac{z^2 - t^2 - l^2}{2l^3} \\ c_6 &= -\frac{\gamma^2}{l^2} \ . \end{aligned} \tag{222}$$

We get the same results for e^2:

$$\begin{aligned} e^2 &= -\omega_3 d\omega_1 + \omega_4 d\omega_2 \\ &= -\gamma \frac{z}{l} \gamma \frac{dx}{l} + \gamma \frac{z^2 - t^2 - l^2}{2l^2} \gamma \frac{dy}{l} \\ &= -\frac{\gamma^2}{l^2} z dx + \gamma^2 \frac{z^2 - t^2 - l^2}{2l^3} dy \end{aligned}$$

and we see again

$$\begin{aligned} c_4 &= \gamma \frac{z^2 - t^2 - l^2}{2l^3} \\ c_6 &= -\frac{\gamma^2}{l^2} \ . \end{aligned} \tag{223}$$

Now, let us handle e^3:

$$\begin{aligned} e^3 &= \omega_4 d\omega_3 - \omega_3 d\omega_4 \\ &= \gamma \frac{z^2 - t^2 - l^2}{2l^2} \left(d\gamma \frac{z}{l} + \gamma \frac{dz}{l} \right) - \gamma \frac{z}{l} \left(d\gamma \frac{z^2 - t^2 - l^2}{2l^2} + \gamma \frac{z}{l^2} dz - \gamma \frac{t}{l^2} dt \right) \\ &= \gamma z \frac{z^2 - t^2 - l^2}{2l^3} d\gamma + \frac{\gamma^2}{2l^3} (z^2 - t^2 - l^2) dz - \gamma z \frac{z^2 t^2 - l^2}{2l^3} d\gamma - \gamma^2 \frac{z^2}{l^3} dz + \gamma^2 \frac{zt}{l^3} dt \\ &= \frac{\gamma^2}{2l^3} (z^2 - t^2 - l^2) dz - \gamma^2 \frac{z^2}{l^3} dz + \gamma^2 \frac{zt}{l^3} dt \ . \end{aligned}$$

We can deduce that

$$\begin{aligned} c_3 &= \frac{\gamma^2}{l^3} t \\ c_4 &= \frac{\gamma^2}{2l^3}(z^2 - t^2 - l^2) \\ c_5 &= -\frac{\gamma^2}{l^3} \;. \end{aligned} \tag{224}$$

Now we have all the coefficients we need to compute $e^{1,2,3}$. Instead of the term z^2 in c_4, we write r^2 and we go to the usual frame in our educated guess and plug in our coefficients. We start again with e^1:

$$\begin{aligned} e^1 &= c_3 x^1 dt + c_4 dx^1 + c_5 x^1 x^k dx^k + c_6 \epsilon_{1jk} x^j dx^k \\ &= \frac{\gamma^2}{l^3} t x^1 dt + \frac{\gamma^2}{2l^3}(r^2 - t^2 - l^2) dx^1 - \frac{\gamma^2}{l^3}\left((x^1)^2 dx^1 + x^1 x^2 dx^2 + x^1 x^3 dx^3\right) - \frac{\gamma^2}{l^2}(x^2 dx^3 - x^3 dx^2) \\ &= \frac{\gamma^2}{l^3}\left(t x^1 dt + \left(\frac{1}{2}(r^2 - t^2 - l^2) - (x^1)^2\right) dx^1 - (x^1 x^2 - l x^3) dx^2 - (x^1 x^3 + l x^2) dx^3\right) \\ &= \frac{\gamma^2}{l^3}\left(t x^1 dt - \left(\frac{1}{2}(t^2 - r^2 + l^2) + (x^1)^2\right) dx^1 - (x^1 x^2 - l x^3) dx^2 - (x^1 x^3 + l x^2) dx^3\right) \;. \end{aligned}$$

We do the same calculationf for e^2 and e^3:

$$\begin{aligned} e^2 &= c_3 x^2 dt + c_4 dx^2 + c_5 x^2 x^k dx^k + c_6 \epsilon_{2jk} x^j dx^k \\ &= \frac{\gamma^2}{l^3} t x^2 dt + \frac{\gamma^2}{2l^3}(r^2 - t^2 - l^2) dx^2 - \frac{\gamma^2}{l^3}(x^2 x^1 dx^1 + (x^2)^2 dx^2 + x^2 x^3 dx^3) - \frac{\gamma^2}{l^2}(x^3 dx^1 - x^1 dx^3) \\ &= \frac{\gamma^2}{l^3}\left(t x^2 dt + \frac{1}{2}(r^2 - t^2 - l^2) dx^2 - x^2 x^1 dx^1 - (x^2)^2 dx^2 - x^2 x^3 dx^3 - l x^3 dx^1 + l x^1 dx^3\right) \\ &= \frac{\gamma^2}{l^3}\left(t x^2 dt - \left(\frac{1}{2}(t^2 - r^r - l^2) + (x^2)^2\right) dx^2 - (x^2 x^1 + l x^3) dx^1 - (x^2 x^3 - l x^1) dx^3\right) \end{aligned}$$

and

$$\begin{aligned} e^3 &= c_3 x^3 dt + c_4 dx^3 + c_5 x^3 x^k dx^k + c_6 \epsilon_{3jk} x^j dx^k \\ &= \frac{\gamma^2}{l^3} t x^3 dt + \frac{\gamma^2}{2l^3}(r^2 - t^2 - l^2) dx^3 - \frac{\gamma^2}{l^3}(x^3 x^1 dx^1 + x^3 x^2 dx^2 + (x^3)^2 dx^3) - \frac{\gamma^2}{l^2}(x^1 dx^2 - x^2 dx^1) \\ &= \frac{\gamma^2}{l^3}\left(t x^3 dt + \frac{1}{2}(r^2 - t^2 - l^2) dx^3 - x^3 x^1 dx^1 - x^3 x^2 dx^2 - (x^3)^2 dx^3 - l x^1 dx^2 + l x^2 dx^1\right) \\ &= \frac{\gamma^2}{l^3}\left(t x^3 dt - \left(\frac{1}{2}(t^2 - r^2 + l^2) + (x^3)^2\right) dx^3 - (x^3 x^2 + l x^1) dx^2 - (x^3 x^1 - l x^2) dx^1\right) \;. \end{aligned}$$

Summarized our results are

$$
\begin{aligned}
e^1 &= \frac{\gamma^2}{l^3}\left(tx^1dt - \left(\frac{1}{2}(t^2 - r^2 + l^2) + (x^1)^2\right)dx^1 - (x^1x^2 - lx^3)dx^2 - (x^1x^3 + lx^2)dx^3\right)\\
e^2 &= \frac{\gamma^2}{l^3}\left(tx^2dt - \left(\frac{1}{2}(t^2 - r^r - l^2) + (x^2)^2\right)dx^2 - (x^2x^1 + lx^3)dx^1 - (x^2x^3 - lx^1)dx^3\right)\\
e^3 &= \frac{\gamma^2}{l^3}\left(tx^3dt - \left(\frac{1}{2}(t^2 - r^2 + l^2) + (x^3)^2\right)dx^3 - (x^3x^2 + lx^1)dx^2 - (x^3x^1 - lx^2)dx^1\right)\ ,
\end{aligned}
\tag{225}
$$

which can be written in a shorter notation as

$$
e^a = \frac{\gamma^2}{l^3}\left(tx^a dt - \left(\frac{1}{2}(t^2 - r^2 + l^2)\delta^a_k + x^a x^k + l\epsilon^a_{\ jk}x^j\right)dx^k\right)\ . \tag{226}
$$

Now let us consider $e^0 = d\tau$. The ansatz was

$$
e^0 = c_1 dt + c_2 x^k dx^k. \tag{227}
$$

In the $x = y = 0$, $z = r$ frame we have

$$
e^0 = c_1 dt + c_2 x^3 dx^3 \tag{228}
$$

and

$$
-\cot\tau = \frac{t^2 - z^2 - l^2}{2lt} \tag{229}
$$

from (9). We have to compute $d\tau$ with

$$
\tau = \operatorname{arccot}\left(\frac{z^2 - t^2 + l^2}{2lt}\right). \tag{230}
$$

To do so, we use the derivation rule [16]

$$
\frac{d}{dx}\operatorname{arccot}(x) = \frac{-1}{x^2 + 1} \tag{231}
$$

of the arccot function. So we get the following:

$$
\begin{aligned}
e^0 = d\tau &= \frac{2z}{2lt}\frac{(-1)}{\frac{(z^2-t^2+l^2)^2}{4l^2t^2} + 1}dz + \frac{t^2 + z^2 + l^2}{2lt^2}\frac{1}{\frac{(z^2-t^2+l^2)^2}{4l^2t^2} + 1}dt\\
&= -\frac{z4lt}{(z^2 - t^2 + l^2)^2 + 4l^2t^2}dz + \frac{(t^2 + z^2 + l^2)2l}{(z^2 - t^2 + l^2)^2 + 4l^2t^2}dt\\
&= -z\frac{\gamma^2}{l^3}tdz + (t^2 + z^2 + l^2)\frac{\gamma^2}{2l^3}dt\ .
\end{aligned}
$$

We see that the coefficients c_1 and c_2 are

$$
\begin{aligned}
c_1 &= \frac{(t^2+z^2+l^2)\gamma^2}{2l^3} \\
c_2 &= -\frac{\gamma^2}{l^3}t \, .
\end{aligned} \tag{232}
$$

Going back to the usual frame and hence writing instead of z^2 in c_1 the term r^2, we get for e^0

$$
\begin{aligned}
e^0 &= \frac{(t^2+r^2+l^2)\gamma^2}{2l^3}dt - \frac{\gamma^2}{l^3}t(x^1dx^1+x^2dx^2+x^3dx^3) \\
&= \frac{\gamma^2}{l^3}\left(\frac{1}{2}\left(t^2+r^2+l^2\right)dt - t(x^1dx^1+x^2dx^2+x^3dx^3)\right) \, ,
\end{aligned}
$$

which can be written in a shorter notation as

$$
e^0 = \frac{\gamma^2}{l^3}\left(\frac{1}{2}(t^2+r^2+l^2)dt - tx^k dx^k\right) \, . \tag{233}
$$

B. Derivation of the Hopf-Rañada knot

If we consider rotations in ω_1-ω_2-plane, the point $(\omega_1, \omega_2, \omega_3, \omega_4)$ goes to $(\omega_1 \cos\varphi - \omega_2 \sin\varphi, \omega_1 \sin\varphi + \omega_2 \cos\varphi, \omega_3, \omega_4)$.
In Minkowski coordinates this is a rotation in the x-y-plane and we get the transformation

$$(x, y, z) \mapsto (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi, z) \ .$$

The sphere-frame electric and magnetic field components of the solutions (26) with (58) and (59) are given by

$$\begin{aligned} \mathcal{E}_1 &= \frac{1}{4}\cos 2\tau & \mathcal{B}_1 &= -\frac{1}{4}\sin 2\tau \\ \mathcal{E}_2 &= -\frac{1}{4}\sin 2\tau & \mathcal{B}_2 &= -\frac{1}{4}\cos 2\tau \\ \mathcal{E}_3 &= 0 & \mathcal{B}_3 &= 0 \ . \end{aligned} \tag{234}$$

These fields do not depend on the ω_a coordinates, but they still transform under rotations, since they are vectors

$$(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) \mapsto (\mathcal{E}_1\cos\varphi - \mathcal{E}_2\sin\varphi, \mathcal{E}_1\sin\varphi + \mathcal{E}_2\cos\varphi, 0)$$
$$(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \mapsto (\mathcal{B}_1\cos\varphi - \mathcal{B}_2\sin\varphi, \mathcal{B}_1\sin\varphi + \mathcal{B}_2\cos\varphi, 0) \ .$$

So a complex vector $\mathcal{F} = \mathcal{E} + i\mathcal{B}$ in our case transforms into $\mathcal{F}e^{i\varphi}$ under rotations. But this is actually a duality transformation, which interchanges electric and magnetic fields.

So the solution (26) has the special property that it does not change under simultaneous rotation in the x-y-plane and a duality transformation of the same magnitude.

Using this property we can write down an ansatz for $\mathbf{F} = \mathbf{E} + i\mathbf{B}$.

Indeed, if the solution was just axially symmetric, we would pick a basis of symmetric fields, e.g.

$$\mathbf{b}_r = (x, y, 0) \quad \mathbf{b}_l = (-y, x, 0) \quad \mathbf{b}_z = (0, 0, 1)$$

and we would write

$$\mathbf{F} = f_1(\rho, z, t)\mathbf{b}_r + f_2(\rho, z, t)\mathbf{b}_l + f_3(\rho, z, t)\mathbf{b}_z \ ,$$

where the labels r and l stand for *radial* and *longitudinal.*

But the solution should change by $e^{i\varphi}$ after a rotation. So we should pick a basis of fields that transforms accordingly.

One such choice is

$$\mathbf{d}_1 = ((x - iy)^2, i(x - iy)^2, 0) \quad \mathbf{d}_2 = (1, -i, 0) \quad \mathbf{d}_3 = (0, 0, x - iy) \ . \tag{235}$$

So the ansatz is

$$\mathbf{F} = f_1(\rho, z, t)\mathbf{d}_1 + f_2(\rho, z, t)\mathbf{d}_2 + f_3(\rho, z, t)\mathbf{d}_3 \ , \tag{236}$$

while $f_{1,2,3}$ are complex.

We can fix $x = \rho$ and $y = 0$ to simplify things.

We have

$$F = dA = \frac{1}{4}\cos 2\tau e^{10} - \frac{1}{4}\sin 2\tau e^{20} + \frac{1}{4}\sin 2\tau e^{23} + \frac{1}{4}\cos 2\tau e^{31} \ . \tag{237}$$

To extract $\mathbf{E}$ and $\mathbf{B}$, we need to rewrite this in $dx^{\mu\nu}$ basis.

Firstly, let us do this for E_3 and B_3 to find $f_3(\rho, z, t)$. For that we have to pick out the parts with $dz \wedge dt$ and $dx \wedge dy$.

Since we picked $x = \rho$ and $y = 0$, we have

$$\begin{aligned}
e^1 &= \frac{\gamma^2}{l^3}\left(t\rho dt - \frac{1}{2}\left(t^2 + \rho^2 - z^2 + l^2\right)dx + lzdy - \rho zdz\right) \\
e^2 &= \frac{\gamma^2}{l^3}\left(-lzdx - \frac{1}{2}\left(t^2 - r^2 + l^2\right)dy + l\rho dz\right) \\
e^3 &= \frac{\gamma^2}{l^3}\left(tzdt - \frac{1}{2}\left(t^2 - \rho^2 + z^2 + l^2\right)dz - \rho zdx - l\rho dy\right) \\
e^0 &= \frac{\gamma^2}{l^3}\left(\frac{1}{2}\left(t^2 + r^2 + l^2\right)dt - t\rho dx - tzdz\right) \ .
\end{aligned} \tag{238}$$

Then we have

$$\begin{aligned}
e^{10} &= \frac{\gamma^4}{l^6}\left(\frac{1}{2}\rho z\left(t^2 - r^2 - l^2\right)dz \wedge dt + lzt\rho dx \wedge dy + \dots\right) \\
e^{20} &= \frac{\gamma^4}{l^6}\left(\frac{1}{2}l\rho\left(t^2 + r^2 + l^2\right)dz \wedge dt - \frac{1}{2}t\rho\left(t^2 - r^2 + l^2\right)dx \wedge dy + \dots\right) \\
e^{23} &= \frac{\gamma^4}{l^6}\left(l\rho tzdz \wedge dt - \frac{1}{2}\rho z\left(t^2 - r^2 + l^2\right)dx \wedge dy + \dots\right) \\
e^{31} &= \frac{\gamma^4}{l^6}\left(-\frac{1}{2}\rho t\left(t^2 - r^2 + l^2\right)dz \wedge dt - \frac{1}{2}\rho l\left(t^2 + r^2 + l^2\right)dx \wedge dy + \dots\right) \ .
\end{aligned} \tag{239}$$

So what we get for the third component of the electric and magnetic field is

$$\begin{aligned}
E_3 &= \frac{\gamma^4}{4l^6}\{\frac{1}{2}\rho z\left(t^2 - r^2 - l^2\right)\cos 2\tau - \frac{1}{2}l\rho\left(t^2 + r^2 + l^2\right)\sin 2\tau \\
&\quad + l\rho t z \sin 2\tau - \frac{1}{2}\rho t(t^2 - r^2 + l^2)\cos 2\tau\} \\
&= \frac{\gamma^4}{8l^6}\{\rho z(t^2 - r^2 - l^2)\cos 2\tau - l\rho(t^2 + r^2 + l^2)\sin 2\tau \\
&\quad + 2l\rho t z \sin 2\tau - \rho t(t^2 - r^2 + l^2)\cos 2\tau\}
\end{aligned}$$

and

$$\begin{aligned}
B_3 &= \frac{\gamma^4}{4l^6}\{lzt\rho\cos 2\tau + \frac{1}{2}t\rho(t^2 - r^2 + l^2)\sin 2\tau \\
&\quad - \frac{1}{2}\rho z(t^2 - r^2 - l^2)\sin 2\tau - \frac{1}{2}\rho l(t^2 + r^2 + l^2)\cos 2\tau\} \\
&= \frac{\gamma^4}{8l^6}\{2lzt\rho\cos 2\tau + t\rho(t^2 - r^2 + l^2)\sin 2\tau \\
&\quad - \rho z(t^2 - r^2 - l^2)\sin 2\tau - \rho l(t^2 + r^2 + l^2)\cos 2\tau\}\ .
\end{aligned}$$

So

$$\begin{aligned}
F_3 &= E_3 + iB_3 \\
&= \frac{\gamma^4}{8l^6}(\rho z(t^2 - r^2 - l^2)\cos 2\tau - \rho z(t^2 - r^2 - l^2)i\sin 2\tau \\
&\quad + i2lzt\rho\cos 2\tau + 2ltz\rho\sin 2\tau \\
&\quad - i\rho l(t^2 + r^2 + l^2)\cos 2\tau - \rho l(t^2 + r^2 + l^2)\sin 2\tau \\
&\quad - \rho t(t^2 - r^2 + l^2)\cos 2\tau + it\rho(t^2 - r^2 + l^2)\sin 2\tau) \\
&= \frac{\gamma^4}{8l^6}(\rho z(t^2 - r^2 - l^2)(\cos 2\tau - i\sin 2\tau) + 2ilzt\rho(\cos 2\tau - i\sin 2\tau) \\
&\quad - i\rho l(t^2 + r^2 + l^2)(\cos 2\tau - i\sin 2\tau) - \rho t(t^2 - r^2 + l^2)(\cos 2\tau - i\sin 2\tau)) \\
&= \frac{\gamma^4 e^{-2i\tau}}{8l^6}(\rho z(t^2 - r^2 - l^2) + 2ilzt\rho - i\rho l(t^2 + r^2 + l^2) - \rho t(t^2 - r^2 + l^2))\ .
\end{aligned}$$

With

$$e^{-2i\tau} = \frac{((l - it)^2 + r^2)^2}{4l^2t^2 + (r^2 - t^2 + l^2)^2}$$

and

$$\gamma^4 = \frac{16l^8}{(4l^2t^2 + (r^2 - t^2 + l^2)^2)^2}$$

we get

$$\begin{aligned}
F_3 &= \frac{16l^8((l - it)^2 + r^2)^2}{8l^6(4l^2t^2 + (r^2 - t^2 + l^2)^2)^3}\rho\left(z(t^2 - r^2 - l^2) + 2ilzt - il(t^2 + r^2 + l^2) - t(t^2 - r^2 + l^2)\right) \\
&= \frac{2l^2((l - it)^2 + r^2)^2}{(4l^2t^2 + (r^2 - t^2 + l^2)^2)^3}\rho(-(t + il)^2 + r^2)(-z + t - il)\ .
\end{aligned}$$

Using

$$(4l^2t^2 + (r^2 - t^2 + l^2)^2)^3 = ((l + it)^2 + r^2)^3((l - it)^2 + r^2)^3 ,$$

F_3 becomes

$$F_3 = \frac{2l^2((l - it)^2 + r^2)^2}{((l + it)^2 + r^2)^3((l - it)^2 + r^2)^3}\rho(r^2 - (t + il)^2)(t - il - z)$$

and with the simple equation

$$r^2 - (t + il)^2 = r^2 + (l - it)^2$$

we arrive at

$$\begin{aligned} F_3 &= \frac{2l^2((l - it)^2 + r^2)^3}{((l + it)^2 + r^2)^3((l - it)^2 + r^2)^3}\rho(t - il - z) \\ &= \frac{2l^2}{((l + it)^2 + r^2)^3}\rho(t - il - z) \\ &= \frac{-2l^2}{((t - il)^2 - r^2)^3}\rho(t - il - z) . \end{aligned}$$

Comparing the last equation with our ansatz (236), we have for $x = \rho$ and $y = 0$

$$F_3 = \rho f_3 ,$$

so that

$$f_3 = \frac{-2l^2}{((t - il)^2 - r^2)^3}(t - il - z) \tag{240}$$

and finally

$$F_3 = E_3 + iB_3 = \frac{-2l^2}{((t - il)^2 - r^2)^3}(x - iy)(t - il - z) . \tag{241}$$

To get the first and the second component and to simplify some steps of calculation, let us make some statements about duality.

At fixed j making a shift $|\Omega_j|\tau \to |\Omega_j|\tau \pm \frac{\pi}{2}$ (+ for type 1, $-$ for type 2) [10] produces a dual solution

$$\mathbf{E}_D = -\mathbf{B} \qquad \mathbf{B}_D = \mathbf{E} . \tag{242}$$

The coordinate transformation $(\tau, \omega^A) \mapsto (t, x, y, z)$ changes orientation due the choice of sign in (6). In the cylinder frame this duality looks like

$$\mathcal{E}_D = \mathcal{B} \qquad \mathcal{B}_D = -\mathcal{E} . \tag{243}$$

The field strength is given by [7]

$$dA = E_j dx^{j0} + \frac{1}{2}\epsilon_{jkl} B_j dx^{kl} \tag{244}$$

and the dual one is given as

$$(dA)_D = *dA = -B_j dx^{j0} + \frac{1}{2}\epsilon_{jkl} E_j dx^{kl} \ , \tag{245}$$

so

$$dA - i(dA)_D = (E_j + iB_j)dx^{j0} + \dots \ . \tag{246}$$

For the solution in our example we have

$$dA = \frac{1}{4}\left(\cos 2\tau e^{10} - \sin 2\tau e^{20} + \sin 2\tau e^{23} + \cos 2\tau e^{31}\right) \tag{247}$$

$$(dA)_D = \frac{1}{4}\left(\sin 2\tau e^{10} + \cos 2\tau e^{20} - \cos 2\tau e^{23} + \sin 2\tau e^{31}\right) \tag{248}$$

so we get

$$\begin{aligned} dA - i(dA)_D = &\frac{1}{4}(\cos 2\tau e^{10} - i\sin 2\tau e^{10} - \sin 2\tau e^{20} - i\cos 2\tau e^{10} \\ &+ \sin 2\tau e^{23} + i\cos 2\tau e^{23} + \cos 2\tau e^{31} - i\sin 2\tau e^{31}) \\ = &\frac{1}{4}((\cos 2\tau - i\sin 2\tau)e^{10} - i(\cos 2\tau - i\sin 2\tau)e^{20} \\ &+ i(\cos 2\tau - i\sin 2\tau)e^{23} + (\cos 2\tau - i\sin 2\tau)e^{31}) \end{aligned}$$

and finally

$$dA - i(dA)_D = \frac{1}{4}e^{-2i\tau}(e^{10} + e^{31} + ie^{23} - ie^{20}) \ . \tag{249}$$

Now, to extract $F_1 = E_1 + iB_1$ we only need to separate the $dx \wedge dt$ part and for $F_2 = E_2 + iB_2$ the $dy \wedge dt$ part.

What we get for the term of all the wedge products is

$$\begin{aligned} e^{10} + e^{31} + ie^{23} - ie^{20} = &\frac{4l^2(\rho^2 - (t - il - z)^2)}{((t+il)^2 - r^2)((t-il)^2 - r^2)^2} dx \wedge dt \\ &+ \frac{4il^2(\rho^2 + (t - il - z)^2)}{((t+il)^2 - r^2)((t-il)^2 - r^2)^2} dy \wedge dt + \dots \ . \end{aligned}$$

We arrive at this result by some algebra. Let us explain this for the $dx \wedge dt$ component. With $x = \rho$ and $y = 0$ we have

$$
\begin{aligned}
e^{10} &= \frac{\gamma^4}{4l^6}\left(z^4 - t^4 - \rho^4 - l^4 - 2l^2t^2 - 2l^2\rho^2 + 2t^2\rho^2\right) dx \wedge dt \ldots \\
e^{20} &= \frac{\gamma^4}{4l^6}\left(-2lz\left(l^2 + t^2 + z^2 + \rho^2\right)\right) dx \wedge dt \ldots \\
e^{31} &= \frac{\gamma^4}{4l^6}\left(2tz\left(l^2 + t^2 - z^2 - \rho^2\right)\right) dx \wedge dt \ldots \\
e^{23} &= \frac{\gamma^4}{4l^6}\left(-4ltz^2\right) dx \wedge dt\ .
\end{aligned}
$$

We get with these expressions

$$
\begin{aligned}
e^{10} + ie^{23} &= \frac{\gamma^4}{4l^6}\left(z^4 - t^4 - \rho^4 - l^4 - 2l^2t^2 - 2l^2\rho^2 + 2t^2\rho^2 - 4iltz^2\right) dx \wedge dt \ldots \\
&= \frac{\gamma^4}{4l^6}\left(-\rho^4 + 2\rho^2(t^2 - l^2) + z^4 - t^4 - l^4 - 2l^2t^2 - 4iltz^2\right) dx \wedge dt \ldots\ ,
\end{aligned}
$$

where we view the term ρ^4 in the brackets as a polynomial in ρ^2. For the discriminant we have

$$
\begin{aligned}
D &= 4t^4 + 4l^4 - 8t^2l^2 + 4z^4 - 4t^4 - 4l^4 - 8l^2t^2 -^6 iltz^2 \\
&= 4z^4 - 16t^2l^2 - 16iltz^2 \\
&= 4(z^4 - 4t^2l^2 - 4iltz^2) \\
&= 4(z^2 - 2ilt)^2
\end{aligned}
\tag{250}
$$

and the roots are given by

$$
(t^2 - l^2) \pm (z^2 - 2ilt) = \left\{ \begin{array}{c} z^2 + (t - il)^2 \\ -z^2 + (t + il)^2 \end{array} \right\}, \tag{251}
$$

so

$$
e^{10} + ie^{23} = -\frac{\gamma^4}{4l^6}\left(\rho^2 - z^2 - (t - il)^2\right)\left(\rho^2 + z^2 - (t + il)^2\right) dx \wedge dt + \ldots\ . \tag{252}
$$

We also have

$$
\begin{aligned}
e^{31} - ie^{20} &= \frac{\gamma^4}{4l^6}\left(2tz(l^2 + t^2 - z^2 - \rho^2) + 2ilz(l^2 + t^2 + z^2 + \rho^2)\right) dx \wedge dt + \ldots \\
&= \frac{\gamma^4}{4l^6}\left(2tzl^2 + 2t^3z - 2tz\rho^2 + 2il^3z + 2ilzt^2 + 2ilz^3 + 2ilz\rho^2\right) dx \wedge dt + \ldots \\
&= \frac{\gamma^4}{4l^6}\left((t^2 + l^2)(2tz + 2ilz) - (z^2 + \rho^2)(2tz - 2ilz)\right) dx \wedge dt + \ldots \\
&= \frac{\gamma^4}{4l^6}\left(2z(t - il)((t + il)^2 - z^2\rho^2)\right) dx \wedge dt + \ldots\ .
\end{aligned}
\tag{253}
$$

So all together we have

$$e^{10} + e^{31} + ie^{23} - ie^{20} = \frac{4l^2(\rho^2 - (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)^2} dx \wedge dt + \dots \ .$$

Since in this frame

$$r^2 = \rho^2 + z^2 \ ,$$

it follows that

$$e^{10} + e^{31} - ie^{20} + ie^{23} = \frac{\gamma^4}{4l^6} \left((t + il)^2 - r^2\right) \left(\rho^2 - (t - il - z)^2\right) dx \wedge dt + \dots \ .$$

Finally, we get the $dx \wedge dt$ component

$$e^{10} + e^{31} - ie^{20} + ie^{23} = \frac{4l^2(\rho^2 - (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)} dx \wedge dt + \dots \ . \tag{254}$$

With the complete analogous argumentation we get for the $dy \wedge dt$ component

$$e^{10} + e^{31} - ie^{20} + ie^{23} = \frac{4il^2(\rho^2 - (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)} dy \wedge dt + \dots \ , \tag{255}$$

so both components together give as mentioned

$$\begin{aligned} e^{10} + e^{31} + ie^{23} - ie^{20} = &\frac{4l^2(\rho^2 - (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)^2} dx \wedge dt \\ &+ \frac{4il^2(\rho^2 + (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)^2} dy \wedge dt + \dots \ . \end{aligned} \tag{256}$$

And all this means that

$$\begin{aligned} F_1 &= \frac{1}{4} e^{-2i\tau} \frac{4l^2(\rho^2 - (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)^2} \\ &= \frac{l^2}{((t - il)^2 - r^2)^3} (\rho^2 - (t - il - z)^2) \end{aligned} \tag{257}$$

and

$$\begin{aligned} F_2 &= \frac{1}{4} e^{-2i\tau} \frac{4il^2(\rho^2 + (t - il - z)^2)}{((t + il)^2 - r^2)((t - il)^2 - r^2)^2} \\ &= \frac{il^2}{((t - il)^2 - r^2)^3} (\rho^2 + (t - il - z)^2) \ . \end{aligned} \tag{258}$$

Considering the ansatz (236) for $x = \rho$ and $y = 0$, we have

$$F_1 = \rho^2 f_1 + f_2 \tag{259}$$

$$F_2 = i\rho^2 f_1 - if_2 \ , \tag{260}$$

thus

$$f_1 = \frac{l^2}{((t-il)^2 - r^2)^3} \tag{261}$$

$$f_2 = \frac{-l^2(t-il-z)^2}{((t-il)^2 - r^2)^3} \ . \tag{262}$$

Putting this back in our ansatz, we get

$$F_1 = \frac{l^2}{((t-il)^2 - r^2)^3}((x-iy)^2 - (t-il-z)^2) \tag{263}$$

and

$$F_2 = \frac{l^2}{((t-il)^2 - r^2)^3}(i(x-iy)^2 + i(t-il-z)^2) \ . \tag{264}$$

and the Hopf-Rañada knot is given by all its components

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \frac{l^2}{((t-il)^2 - r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-il-z)^2 \\ i(x-iy)^2 + i(t-il-z)^2 \\ -2(x-iy)(t-il-z) \end{pmatrix} .$$

C. Derivation of the Maxwell equations for each component

Let us insert for a each value in (47). We start with $a = 3$:

$$-\frac{1}{4}\partial_\tau^2 X_3 = (J^2+1)X_3 + i(J_1X_2 - J_2X_1) \ .$$

From the definitions of J_+, J_-, X_+ and X_-, we see that

$$J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-) \qquad J_2 = \frac{i}{\sqrt{2}}(J_- - J_+) \qquad X_1 = \frac{1}{\sqrt{2}}(X_+ + X_-) \qquad X_2 = \frac{i}{\sqrt{2}}(X_- - X_+) \ .$$

Let us consider the terms J_1X_2 and J_2X_1, so

$$\begin{aligned} J_1X_2 &= \frac{i}{2}(J_+ + J_-)(X_- - X_+) \\ &= \frac{i}{2}(J_+X_- - J_+X_+ + J_-X_- - J_-X_+) \end{aligned}$$

and

$$\begin{aligned} J_2X_1 &= \frac{i}{2}(J_- - J_+)(X_+ + X_-) \\ &= \frac{i}{2}(J_-X_+ + J_-X_- - J_+X_+ - J_+X_-) \ , \end{aligned}$$

such that

$$J_1X_1 - J_2X_1 = i(J_+X_- - J_-X_+)$$

gives

$$-\frac{1}{4}\partial_\tau^2 X_3 = (J^2+1)X_3 - J_+X_- + J_-X_+ \ .$$

Now let us look at $a = 1, 2$:

$$\begin{aligned} -\frac{1}{4}\partial_\tau^2 X_1 &= (J^2+1)X_1 + i(J_2X_3 - J_3X_2) \\ -\frac{1}{4}\partial_\tau^2 X_2 &= (J^2+1)X_2 + i(J_3X_1 - J_1X_3) \ . \end{aligned}$$

Let us express the Maxwell equations with X_+ and X_-:

$$\begin{aligned} -\frac{1}{4}\partial_\tau^2 X_+ &= \frac{1}{\sqrt{2}}\left(-\frac{1}{4}\partial_\tau^2 X_1 - i\frac{1}{4}\partial_\tau^2 X_2\right) \\ &= \frac{1}{\sqrt{2}}\big((J^2+1)X_1 + i(J_2X_3 - J_3X_2) + i(J^2+1)X_2 - (J_3X_1 - J_1X_3)\big) \\ &= \frac{1}{\sqrt{2}}\big((J^2+1)(X_1 + iX_2) + i(J_2X_3 - iJ_3X_2) - J_3X_1 + J_1X_3\big) \\ &= \frac{1}{\sqrt{2}}\big((J^2+1)(X_1 + iX_2) + (J_1 + iJ_2)X_3 - J_3(X_1 + iX_2)\big) \\ &= (J^2+1)X_+ + J_+X_3 - J_3X_+ \\ &= (J^2+1-J_3)X_+ + J_+X_3 \ , \end{aligned}$$

$$
\begin{aligned}
-\frac{1}{4}\partial_\tau^2 X_- &= \frac{1}{\sqrt{2}}\left(-\frac{1}{4}\partial_\tau^2 X_1 + i\frac{1}{4}\partial_\tau^2 X_2\right)\\
&= \frac{1}{\sqrt{2}}((J^2+1)X_1 + i(J_2X_3 - J_3X_2) - i(J^2+1)X_2 + J_3X_1 - J_1X_3)\\
&= \frac{1}{\sqrt{2}}((J^2+1)(X_1 - iX_2) + iJ_2X_3 - iJ_3X_2 + J_3X_1 - J_1X_3)\\
&= \frac{1}{\sqrt{2}}((J^2+1)(X_1 - iX_2) + J_3(X_1 - iX_2) - (J_1 - iJ_2)X_3)\\
&= (J^2+1)X_- + J_3X_- - J_-X_3\\
&= (J^2 + J_3 + 1)X_- - J_-X_3 \ .
\end{aligned}
$$

Now we look at the gauge condition $0 = J_aX_a$:

$$
\begin{aligned}
0 &= J_aX_a\\
&= J_1X_1 + J_2X_2 + J_3X_3\\
&= \frac{1}{2}(J_+ + J_-)(X_+ + X_-) - \frac{1}{2}(J_- - J_+)(X_- - X_+) + J_3X_3\\
&= \frac{1}{2}(J_+X_+ + J_+X_- + J_-X_+ + J_-X_-) - \frac{1}{2}(J_-X_- - J_-X_+ - J_+X_- + J_+X_+) + J_3X_3\\
&= J_+X_- + J_-X_+ + J_3X_3 \ .
\end{aligned}
$$

D. Discussion of the change of the metric for 2-forms by a conformal tranformation

Suppose we have a 2-form ω. In some local coordinate patch it can be written as

$$\omega = \frac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu \ . \tag{265}$$

The Hodge star duality operation can be introduced if we have a metric g. For that we need the Levi-Civita tensor ϵ, which is defined by its components

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{|\det g_{\alpha\beta}|}\Delta_{\mu\nu\rho\sigma} \ . \tag{266}$$

Δ denotes the Levi-Civita totally antisymmetric *symbol* with the choice $\Delta_{0123} = 1$.

Although we defined ϵ by giving its components explicitly in some coordinate frame, it is indeed a tensor. One can check this by considering the effect which a coordinate transformation has on it. The symbol does not change, but the determinant of the metric does: It produces exactly the transformation properties we expect from a tensor. Hence, it does not matter which coordinates we choose to define the Levi-Civita tensor.

The Hodge dual of ω is given by

$$*\omega_{\mu\nu} = \frac{1}{4}\omega_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}\, dx^\rho \wedge dx^\sigma \ . \tag{267}$$

Since

$$\epsilon^{\mu\nu}{}_{\rho\sigma} = \sqrt{|\det g_{\alpha\beta}|}g^{\mu\mu'}g^{\nu\nu'}\Delta_{\mu'\nu'\rho\sigma} \ , \tag{268}$$

the Hodge dual $*\omega$ refers to the metric and one expects different results for different choices of the metric.

But when the metric ist perturbed by a conformal factor

$$g_{\alpha\beta} \to e^\phi g_{\alpha\beta} \ , \tag{269}$$

something interesting happens for 2-forms [25]. Since we also have

$$\begin{aligned} g^{\alpha\beta} &\to e^{-\phi}g^{\alpha\beta} \\ \det g_{\alpha\beta} &\to e^{4\phi}\det g_{\alpha\beta} \ , \end{aligned} \tag{270}$$

we see that $\epsilon^{\mu\nu}{}_{\rho\sigma}$ remains unchanged. So the Hodge-dual of 2-forms in 4 dimensions is insensitive to conformal transformations.

E. Python Code for plots

The few plots presented in this thesis were made because of illustrative and aesthetical reasons. They were made with Python 3.7.2. The source code of each plot is presented in this appendix.

E.1. Source code for Figure 1

```
from mpl_toolkits.mplot3d import Axes3D

import matplotlib.pyplot as plt
import numpy as np
import math
import cmath

fig = plt.figure()
ax = fig.gca(projection='3d')

# Make the grid
x, y, z = np.meshgrid(np.arange(-20, 20, 5),
np.arange(-20, 20, 5),
np.arange(-20, 20, 5))


# Make the direction data for the arrows
u = (1-(x*x+y*y+z*z)**3)*((x-1j*y)**2 - (-1j-z)**2).real
v = (1-(x*x+y*y+z*z)**3)*(1j*(x-1j*y)**2 + 1j*(-1j-z)**2).real
w = (1-(x*x+y*y+z*z)**3)*(-2*(x-1j*y)*(-1j-z)).real

ax.quiver(x, y, z, u, v, w, length = 2.3, normalize=True, color='red')

plt.show()
```

E.2. Source code for Figure 2

```
import sys
import matplotlib.pyplot as plt
import numpy as np

X = np.linspace(-4, 4, 1000, endpoint=True)
F = np.sqrt(X*X)
plt.plot(X,F)
plt.plot(X,-F)
startx, endx = -4,4
starty, endy = -4,4
plt.axis([startx, endx, starty, endy])
plt.show()
```

E.3. Source code for Figure 3

```
import sys
import matplotlib.pyplot as plt
import numpy as np

plt.ylabel('E/h')
plt.xlabel('j')
x = np.arange(0,15,1)
y = 2*(x+1)
plt.plot(x,y, 'bo')
plt.show()
```

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