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Diploma Thesis

Yang-Mills configurations on nearly Kähler and G_2-manifolds

by

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Erklärung

Hiermit versichere ich, dass ich die Diplomarbeit selbständig und lediglich unter Benutzung der angegebenen Quellen und Hilfsmittel verfasst habe.

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Chapter 1

Introduction

It was noticed a long time ago that the physical structures of the visible world we are living in can elegantly be described by using a mathematical framework. Especially the Einstein's vision of space-time as a dynamical object of Riemannian geometry demonstrates how the mathematical understanding of physical processes helps in logical admission of complicated lows which governs all observed physical phenomena. In generally, it is very exiting to see how just theoretical initially apparently useless mathematical models can once be applied in physics.

If we look at the existing physical theories, we see that while the combination of the "Standard Model", a specific four-dimensional quantum field theory based on Yang-Mills theory, and general relativity makes it difficult to describe some phenomena (for example physics at the energy over the Plank scale $M_{pl} \sim 10^{19}$ GeV, "coupling unification" and the "hierarchy problem"), string, superstring and more general M-theory are mathematical models which at present are the best candidates for a unification of quantum theory of all matter and interactions with gravity. Although there is still missing a theory with a geometrical interpretation, which one actually would expect from a new concept, they produces various remarkable results at the frontier between mathematics and physics.

Historically, it was one of the achievements of the last century attained in accelerator experiments which allowed to unify the three fundamental forces of nature, namely electromagnetism, the weak interaction, and the strong interaction in one gauge theory SU(3)×SU(2)×U(1), showing that quaks and W-bosons as well as photons form representations of the gauge groups SU(3) and SU(2)×U(1), respectively. The Standard Model based on this knowledge has therefore a special meaning in the constructions of new theories. One takes it for granted that additional structures of string theory, M theory or whatever candidate fundamental theory must lie beyond the Standard Model. In string theory this prerequisite is described by configurations of type $M_4 \times X^d$ based on a choice of inner structure of the space-time M_4 given by a compact manifold X^d. The latter furnishes the space-time with d extra dimensions as well as remarkable features. In physics this mechanism became the name compactification or Kaluza-Klein theory. It is clear that independently of what kind of interior structure would be preferred, this additional setup is preserved by the gauge symmetry, corresponding to the isometry group of global transformations of the manifold, which must be contained in the SU(3)×SU(2)×U(1)-symmetry in some sense in order to lead to the Standard Model. The best auxiliary by the description and handling of this topic is of course the language of differential geometry. In any case one of the most important aspects of the theory is the type of the internal compact manifold, which has to be of a very small size and is required to yield dimensional reduction of the full theory to the well-known theory on four dimensional space-time.

As to string theory, born in the 70's and became a large extent after 1984, today the research in string theoretical field focuses on the recently most interesting areas such as compactification on special holonomy manifolds and approaches to mirror symmetry. However, it is still a central problem to derive the particular effective field theory, the Standard Model, from string theory. But nevertheless string theory successfully and above all naturally solves some of the problems of the Standard Model such as the ultraviolet divergences and the inclusion of gravity in the theory, which is impossible in the pure quantum field theory of the Standard Model as well in its generalizations GUT's due to the renormalization problem. In particular, combination of string theory with supersymmetry and its way of thinking of the elementary particles as of the extended objects (strings) smoothes singularities of the theory and appends additional vibration excitations interpreted as spin-2 particles - gravitons.

In general, supersymmetry is a necessary ingredient of a fundamental theory. Since all particles in nature are distributed in two groups - fermions and bosons, the theory must contain both of these types. When we include fermions in string theory, supersymmetry which relates each bosonic particle a fermionic superpartner arises automatically so that fermions and bosons are grouped pairwise together into supermultiplets via supersymmetry. One of the conditions to the theory of superstrings to be consistent and especially free from anomalies is to fix the dimension number $D = 4 + d = 10$. Out of it results the above-mentioned necessary compactification, i.e., curling up of the extra 6 dimensions into a small compact submanifold of the physical space which can not be detected in the real life (the string scale is about 10^{-33} cm). In terms of weak coupling perturbation theory there exist only five different consistent superstring theories known as Type I SO(32), SO(32) Heterotic, E8×E8 Heterotic, Type IIA and Type IIB. The first three of them have only one ($\mathcal{N} = 1$) supersymmetry in ten dimensions, while the last two have $\mathcal{N} = 2$.

As it turns out this fact narrows the variety of possible ways of comactifications of the 10-dimensional superstring theory on a 6-dimensional compact manifold in a very strong manner, because the most trivial compactifications such as curling up of the 6 extra dimension to circles would preserve too much supersymmetries. In order to impose only one supersymmetry in 4 dimensions, which one predicts at the energy scales above 1 TeV, we can compactify on a special kind of 6-manifold, namely on Calabi-Yau manifolds with SU(3)-holonomy. However, the multiplicity of different Calabi-Yau manifolds suggests the possibility of arising of too many various 4-dimensional theories. Furthermore, Calabi-Yau compactifications cause many massless moduli fields in the resulting four-dimensional effective theories (in general, massless moduli fields, as intrinsic components of effective theories below the Planck scale, exist in all known superstring theories which parameterize continuous vacuum-state degeneracies [1]). One possibility to solve this so-called moduli-stabilization problem is to allow for non-trivial p-form fluxes on X^d. String vacua with p-form fields along the extra dimensions, called flux compactifications, relax the restriction of the internal space to be Calabi-Yau demanding only presence of a SU(3)-structure on the internal manifold.Besides that, supersymmetry-preserving compactifications on space-times of the form $M_{10-d} \times X^d$ with further reduction to M^{10-d} requires validity of the first-order BPS-type (Bogomol'nyi-Prasad-Sommerfeld) gauge equations on X^d (see [2] and also references therein), in particular, generalization of the Yang-Mills anti-self-duality equations in four to higher-dimensional manifolds with a special holonomy.

Furthermore, it is natural in six dimensions, $d = 6$, that there appears a 2-form on X^6 which we can use to define a three-form $\mathcal{H}$ considered as torsion. So we may search for an appropriate compact manifold X^d having a non-vanishing torsion field. Some of the rear candidates for the internal space X^6 with torsionful inner geometry in the context of heterotic string theories are the nearly Kähler manifolds which especially carry a natural almost not integrable complex structure and are contained in a class of coset spaces referred to as non-symmetric coset spaces.

In this diploma thesis we are going to derive non-perturbative instanton solutions to the equations of motion of Lie G-valued Yang-Mills fields on the space $\mathbb{R} \times G/H$, where G/H is a compact nearly Kähler six-dimensional homogeneous space and the manifold $R \times G/H$ carries a G_2-structure. Initially instantons were found as topologically non-trivial solutions of the duality equations of the Euclidean Yang-Mills theory with finite action [3], i.e., as soliton solutions of Yang-Mills equations localized in space and time. The first such solutions were discovered in the case of four-dimensional Euclidean space compactified to the four-dimensional sphere and earned the names pseudoparticle and instanton. In four Euclidean dimensions instantons are non-perturbative BPS configurations solving the first-oder anti-self-duality equation and forming a subset of solutions to the full Yang-Mills equations. In higher dimensions they can also be found as solutions of BPS configurations known as generalized anti-self-duality equations or Σ-anti-self-duality already mentioned above. One of the aspects associated to the instantons is that they can be understood as tunneling particles characterized by trajectories connecting degenerated minima of the potential energy or in other words of the Yang-Mills vacuum.

Therefore, in this thesis we are going to investigate the Yang-Mills theory on the only four known examples of compact nearly Kähler 6-manifolds SU(3)/U(1)×U(1), Sp(2)/Sp(1)×U(1), G_2/SU(3)= S^6 and SU(2)^3/SU(2) = $S^3 \times S^3$, which are in particular non-symmetric reductive and homogeneous. Further we consider the corresponding equations of motion on spaces of the form $\mathbb{R} \times G/H$, where G/H stands for one of the coset spaces just listed, and endow it with a G_2-structure. One of the future perspectives could be in the extension of the found solutions on this simplified model to solutions on the Minkowski spaces of the form $\mathbb{R}^4 \times G/H$.

Firstly, we introduce the general ansatz of a Yang-Mills theory with gauge group G over the base space $\mathbb{R} \times G/H$. For all four cases these ansätze are G-equivariant which performs the dimensional reduction and cancels a part of the additional coset variables. The gauge potential of the theory is given by a connection on an associated principal bundle parameterized by complex functions ϕ_i, $i = 1, 2, 3$. In this case we are able to write down the ansatz in general form without any knowledge about the explicit type of the coset. From the G-equivariant ansatz for the gauge potential we derive the corresponding field strength and the corresponding Yang-Mills equations. Then we distinguish between the four cases of different coset spaces and analyze specifying the Yang-Mills equations at first on $\mathbb{R}$×SU(3)/U(1)×U(1). Our ansatz reduces the Yang-Mills equations to coupled Newtons equations of type $\ddot{\phi}_i = f(\phi_1, \phi_2, \phi_3)$ describing a particle in complex three-dimensional space under influence of a cubic force f. We also show the symmetry characteristic to the equations. Then we go over to the same consideration of the case $\mathbb{R}$×Sp(2)/Sp(1)×U(1) seeking for common relations between the two cases. Furthermore, we give for each case the explicit form of the action functional, the found equation of motions can be derived from, and determine the quartic potential V. Then we construct for specific values of torsion some BPS-type solutions coming either from differential gradient or Hamiltonian flow equations and yielding instanton configurations of finite action. By a duality transformation, which relates solutions for different values of torsion, infinite-action solutions to the Yang-Mills equations are presented as well. At last we give some non-BPS solutions on G_2/SU(3) such as instanton-anti-instanton solutions (sphalerons) and dyonic solutions (bounces). They can be lifted up to the solutions on SU(3)/U(1)×U(1) and Sp(2)/Sp(1)×U(1).

The structure of thesis has the following order. The content is divided into two main chapters. Chapter 2 is a purely mathematical part of the thesis and consists of a detailed introduction to the basic notions of algebra, especially, it efforts an outlook about the major structure groups and corresponding invariant tensor fields, algebraic structure such as Clifford algebras, Lie algebras and related Lie groups. It also has a look at the theory of homogeneous spaces, specifically of nearly Kähler coset spaces having a SU(3) structure. To understand the latter we are going to introduce differential structures on manifolds, e.g., almost complex structure, metric, connection, torsion etc. Chapter 3 is dedicated to the dealing with manifolds specified to be nearly Kähler. Before consideration of Yang-Mills theories on this spaces we need to understand additional structures given by fibre bundle, principal bundle and the generalized notions from Chapter 2, i.e., connections and curvature on such bundles. Further, we introduce the definition of the Yang-Mills theory on coset spaces and give the notion of a G_2-structure on 7-dimensional manifolds specializing in the case of cylinder spaces over 6-dimensional nearly Kähler coset spaces. The rest of Chapter 3 is dedicated to the explicit derivation of the Yang-Mills equations on nearly Kähler spaces as well as to the finding of their solutions according to the conditions described above.

Chapter 2

Mathematical preliminaries

During the complete chapter we want to provide the concept of basic mathematical structures needed as a background knowledge for understanding of the Yang-Mills theory. We start with an introduction to fundamental notions on vector spaces and subsequently go over to algebraic, group theoretical and geometrical definitions. After that we will use the theoretical approach established in the first part of the chapter to deepen in the topic specifying and concretizing the subject in the theory of nearly Kähler coset spaces.

2.1 Metric structure on vector spaces

In this section we introduce the vector space as a fundamental object and make it clear that any structure on it corresponds to some group of transformations. In this way every structure can be interpreted as an invariant of a certain group preserving it.

2.1.1 Vector space $\mathbb{R}^n$

Let $\mathbb{R}$ denote the field of real numbers. The system of all possible ordered n-tuples $x = (x^1, x^2, ..., x^n)$, where $x^i \in \mathbb{R}$ for each $i = 1, ..., n$, defines an n-dimensional Euclidean space $\mathbb{R}^n$. Each point $x \in \mathbb{R}^n$ is determined by its corresponding n-tuple (Cartesian coordinates) in a unique way, i.e., two points $x, y \in \mathbb{R}^n$ are equal if and only if their coordinates coincide: $x^i = y^i$. We introduce on $\mathbb{R}^n$ the following vector operations:

$$x + y = (x^1 + y^1, ..., x^n + y^n), \tag{2.1}$$
$$c \cdot x = (c \cdot x^1, ..., c \cdot x^n) \tag{2.2}$$

for two given points $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. With the sum and the multiplication on scalars the Euclidean space acquires the structure of n-dimensional vector space. The simplest example is the line of real numbers which can be described by only one coordinate x^1, i.e., its dimension is 1. The plane $\mathbb{R}^2$ can be furnished with two coordinates x^1, x^2 and the ordinary space $\mathbb{R}^3$ with three; they are 2- and 3-dimensional spaces, respectively.

2.1.2 Coordinates on $\mathbb{R}^n$

In order to describe an arbitrary point x of $\mathbb{R}^n$ one can always write it in terms of its coordinates. The most natural way to do this is to represent it as linear combination of the standard basis vectors $\{e_i\}$, $i = 1, ..., n$:

$$x = (x^1, ..., x^n) = \sum_{i=1}^{n} x^i e_i, \tag{2.3}$$

where $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, ..., 0, 1)$. In particular, this construction gives rise to the identification of the vector with its coordinates, i.e., with the coefficients of e_i in the decomposition relating to standard basis. This choice of basis vectors is quite standard. In general case, one uses an arbitrary n-vector tuple as a basis up to a point that they are linear independent or, in other words, the basis allows us to construct every vector of $\mathbb{R}^n$.

2.1.3 Linear transformation of $\mathbb{R}^n$

As it was mentioned above, there are many possible ways to associate a basis to $\mathbb{R}^n$. Suppose two different sets of basis vectors are given: $\{e_i\}$ and $\{e'_j\}$. We may write every vector e'_i in terms of the other basis vectors $\{e_i\}$:

$$e'_i = \sum_{i=1}^{n} a^j_i e_j, \tag{2.4}$$

$\forall i = 1, ..., n$ and some real valued constants $a^j_i \in \mathbb{R}^n$. The matrix $A = (a^j_i)$ corresponds to a transformation of the coordinate system, but can be considered as a transformation of the vector space $\mathbb{R}^n$ itself. This transformation is obviously linear and maps $\mathbb{R}^n$ on $\mathbb{R}^n$ bijectively, i.e., it maps an n-basis to another n-basis. On the other hand, considering such transformations as transformations of $\mathbb{R}^n$, one associates smoothly every region of $\mathbb{R}^n$ in a one-to-one and onto manner to another region with corresponding coordinates. In this sense all possible transformations form an automorphism group of $\mathbb{R}^n$. One can easily show that the corresponding transformation matrix A is always invertible, since bijective, and thus non-singular, $\det A \neq 0$. The matrix product leaves the non-singularity unaffected. Therefore, equipped with the matrix multiplication, the $n \times n$-matrices A build a group. We call it the general linear group $\mathrm{GL}(n, \mathbb{R})$ [4, 5, 6].

2.1.4 Euclidean metric on $\mathbb{R}^n$

The Euclidean metric is the function $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that assigns to any two points $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ on Euclidean space the distance between them. This is given by

$$d(x, y) :=\mid x - y \mid= \sqrt{(x^1 - y^1)^2 + ... + (x^n - y^n)^2}. \tag{2.5}$$

The Euclidean metric on $\mathbb{R}^n$ is the standard metric on this space, i.e., it is a positive definite, symmetric bilinear function, which obeys the triangle inequality. Shortly we can write these constraints as

1. $d(x, x) = 0$; $d(x, y) > 0$, $(x \neq y)$;

2. $d(x, y) = d(y, x)$;

3. $d(x, y) \leqslant d(x, z) + d(z, y)$, for all $z \in \mathbb{R}^n$.

It is straightforward to show that the Euclidean metric is symmetric, non-negative and $d(x, y) = 0$ if and only if $x = y$. The proof of the triangle inequality makes use of the Cauchy-Schwarz inequality. Relative to the standard basis the Euclidean metric on $\mathbb{R}^n$ is given by the matrix

$$(g_{ij}) := (\delta_{ij}) = \mathrm{diag}(1, \ldots, 1) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & 1 \end{pmatrix} \tag{2.6}$$

The Euclidean metric tensor g assigns to every pair of points $x, y \in \mathbb{R}^n$ a scalar product given by non-degenerate bilinear symmetric map:

$$g(x, y) = x^1 y^1 + ... + x^n y^n. \tag{2.7}$$

So that we may write $d(x, y) = \sqrt{g(x - y, x - y)}$. The squared infinitesimal distance between two infinitely closed points x and $x + dx$ then is

$$\mathrm{d}s^2 := \mathrm{d}^2(x, x + dx) = g_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{2.8}$$

where Einstein summation convention has been used.

2.1.5 Metric of signature (p, q) on $\mathbb{R}^n$

Let us now omit the condition of positive definiteness in the definition of metric. Thus, we say that $\mathbb{R}^n$ is endowed with a metric of signature (p, q) if on the space $\mathbb{R}^n$ with respect to an arbitrary chosen basis a smooth bilinear non-degenerate symmetric function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined, whose representation matrix

(g_{ij}) has p positive and q negative eigenvalues ($p + q = n$). Since the matrix (g_{ij}) is symmetric and non-degenerate it can be diagonalized in a way that the metric can be written as: $g(x, y) = \sum_{i=1}^{p} x^i y^i - \sum_{j=1}^{q} x^j y^j$. In this case we call the metric g either pseudo-Euclidean or simply metric of type (p, q). Riemannian metrics are examples of metrics with a positive definite signature $(n, 0)$. Whereas a Lorentzian metric is one with signature $(1, n-1)$ or $(n-1, 1)$.

2.1.6 Groups $\mathrm{O}(p, q)$ and $\mathrm{SO}(p, q)$

Let us now consider linear coordinate transformations of the space $\mathbb{R}^n$, which fix the origin, but also leave every non-degenerate, symmetric bilinear form g of signature (p, q) invariant ($p + q = n$). These transformations form a group, the pseudo-orthogonal group O(p,q), which in case, if $p = 0$ or $q = 0$, contains rotations and reflections of the Euclidean space.

Since the orthogonal group preserves any metric on $\mathbb{R}^n$, it in particular preserves angles[1] between the basis vectors and consequently each matrix $A \in \mathrm{O}(p, q)$ maps an orthonormal basis to an orthogonal basis. This fact gives the matrices $A \in \mathrm{O}(p, q)$ the special property: $A^{-1} = \mathrm{Id}_{p,q} A^T \mathrm{Id}_{p,q}$, where $\mathrm{Id}_{p,q}$ is the identity matrix of signature (p, q). To see this we consider $\mathrm{O}(p, q)$ as a group of transformations preserving the pseudo-Euclidean scalar product $< x, y >= g_{ij} x^i y^i = x^1 y^1 + ... + x^p y^q - ... - x^n y^n$ (relative to an appropriate basis), i.e., it has to satisfy:

$$
\begin{aligned}
< Ax, Ay > &= < x, y >, \quad \forall \quad x, y \in \mathbb{R}^n \quad \text{or} && (2.9) \\
A^T g A &= g = (g_{ij}). && (2.10)
\end{aligned}
$$

Relative to an appropriate basis, g is a unit diagonal matrix $\mathrm{Id}_{p,q}$ of signature (p, q). Hence, $A^{-1} = \mathrm{Id}_{p,q} A^T \mathrm{Id}_{p,q}$. In the simplest case, if $p = 0$ or $q = 0$, this implies that $A^{-1} = A^T$. Also we can conclude that the dimension of $\mathrm{O}(p, q)$ is $n(n-1)/2$. Moreover, one sees that elements of the coset space $\mathrm{GL}(p+q, \mathbb{R})/\mathrm{O}(p, q)$ describe all possible metrics on $\mathbb{R}^{p,q}$ by the mapping $[A] \longmapsto A^T g A$, where $[A]$ is the equivalence class of $A \in \mathrm{GL}(p+q, \mathbb{R})$ and g the standard unit diagonal metric on $\mathbb{R}^{p,q}$ of signature (p, q).

The special orthogonal group, $\mathrm{SO}(p, q)$, is the subgroup of $\mathrm{O}(p, q)$ consisting of all elements with the determinant 1.

2.1.7 Translations and group $\mathrm{IO}(p, q)$

Extending the notion of linear coordinate transformation S, we look at transformations of $\mathbb{R}^n$ preserving the metric which additionally includes transitions of

[1]The angle between two vectors $x, y \in \mathbb{R}^n$ is defined via $\cos\alpha(x, y) = \frac{<x,y>}{\sqrt{<x,x>}\sqrt{<y,y>}}$.

the origin. Then in general we can write the coordinate transformations in the following form

$$x'^i = x^j S^i_j + a^i, \quad S^i_j \in \mathrm{O}(n) \quad a^i \in \mathbb{R}. \tag{2.11}$$

Together with the group of orthogonal transformations $\mathrm{O}(p,q)$ transitions define the indefinite orthogonal group $\mathrm{IO}(p,q)$. These transformations form the largest group preserving the metric. Its elements are called isometries of $\mathbb{R}^n$. A special meaning in mathematical physics has the group $\mathrm{IO}(1, n-1)$ which one calls the Poincare group. It is the group of isometries of Minkowski spacetime $\mathbb{R}^{1,n-1}$.

2.1.8 Conformal group on $\mathbb{R}^{p,q}$

We have already note that the orthogonal group preserves angles. Now we turn to the largest group of transformations which has this property. It is called the conformal group of $\mathbb{R}^{p,q}$. One can show that it can be defined as transformations leaving the metric invariant up to a scalar factor: $g(x) \longmapsto \Omega^2 g(x)$. It is the minimal group that contains the Poincare group as well as the inversion $x \longmapsto \frac{x}{x^2}$. It is isomorphic to the group $\mathrm{SO}(p+1, q+1)$, of which the rotation group $\mathrm{SO}(p,q)$ forms a subgroup. The conformal group consists of 4 sorts of transformations. In coordinate representation they have the following forms.
Translations: $x^i \longmapsto x'^i = x^i + a^i$, $a^i \in \mathbb{R}$.
Rotations/boosts: $x^i \longmapsto x'^i = x^j \Lambda^i_j$, where Λ is an element of the Lorentz group.
Dilatation: $x^i \longmapsto x'^i = \lambda x^i$, where $\lambda \in \mathbb{R}$.
Special conformal transformations: $x^i \longrightarrow x'^i = \frac{x^i - b^i x^2}{1 - b\cdot x + b^2 x^2}$, where $b \in \mathbb{R}^{p,q}$.

2.1.9 $\mathbb{R}^{p,q}$ as a topological space

In order to go over later to the notion of a manifold we introduce the definition of a topological space.

Definition 1. *: A topological space is a pair $(X, \Im)$ consisting of the set of points $x \in X$ and some family of subsets of X, $\{\mathbb{U}_i\}_{i\in I} = \Im$, satisfying the following conditions:*

1. *The empty set and X are in $\Im$.*
2. *The union of any collection of sets in $\Im$ is also in $\Im$.*
3. *The intersection of any finite collection of sets in $\Im$ is also in $\Im$.*

The elements of $\Im$ are called the open sets of space X and their complements in X are called the closed sets. The collection of open sets of $\Im$ is called a topology on X. Some family $\{\mathbb{B}_j\}_{j\in J} = \mathfrak{J}$ is called a basis of the topology $(X, \Im)$ if every set of $\Im$ is a union of some sets of $\mathfrak{J}$. In general, a given set X may have many different topologies. But there is a standard way to associate a topology to the Euclidean spaces $\mathbb{R}^n$. For the Euclidean space equipped with the standard metric d the open sets are defined as a set of open balls with radius $R > 0$: $B(x, R) = \{y \in \mathbb{R}^n | d(x, y) < R\}$. In particular, this means that a set is open if there exists an open ball of non-zero radius around every point in the set.

2.2 Symplectic structures on $\mathbb{R}^{2n}$

Continuing our discussion about basic transformation groups, we turn to a little more complicated form, namely the symplectic form, which in particular will later play not less meaningful role than the metric.

2.2.1 Symplectic form ω on the space $\mathbb{R}^{2n}$

A symplectic form on the vector space $\mathbb{R}^n$ is a map $\omega : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is bilinear, skew-symmetric and non-degenerate, i.e, ω satisfies:

- $\omega(\lambda x_1 + \mu x_2, y) = \lambda\omega(x_1, y) + \mu\omega(x_2, y)$, $\forall x_1, x_2, y \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$;
- $\omega(x, y) = -\omega(y, x)$, $\forall x, y \in \mathbb{R}^n$;
- $\omega(x, y) = 0$, $\forall x$ implies that $y = 0$.

In a fixed basis, ω can be represented by a matrix. One can show that non-degenerate symplectic form on $\mathbb{R}^n$ exists for $n < \infty$ only if n is even, since every skew-symmetric matrix of odd size has determinant zero. Typically ω is chosen to be the block matrix

$$\omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}, \tag{2.12}$$

where Id_n is the $n \times n$ identity matrix. Any symplectic form on $\mathbb{R}^{2n}$ looks like that after choosing a suitable basis. In this basis the form ω can be written as

$$\omega(x, y) = \sum_{i=0}^{n} x^i y^{i+n} - x^{n+i} y^i, \tag{2.13}$$

where $x, y \in \mathbb{R}^{2n}$ [7, 5, 8, 9, 7, 10].

For further applications we note that locally any symplectic manifold resembles this simple one. For the proof of this proposition, which is known as Darboux's theorem, see e.g. [6].

2.2.2 Transformation group $\mathrm{Sp}(2n, \mathbb{R})$ preserving ω

Suppose $\mathbb{R}^{2n}$ carries a symplectic form ω. A linear map $f : \mathbb{R}^n \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is called a symplectic map if it preserves the symplectic form, that is, if

$$\omega(f(x), f(y)) = \omega(x, y), \qquad \forall x, y \in \mathbb{R}^{2n}. \tag{2.14}$$

The set of all such linear transformations f forms the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, which is in particular a (matrix) Lie group.
Any symplectic form ω on $\mathbb{R}^{2n}$ looks like the standard symplectic form after choosing a suitable basis. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^{2n}$. Then we have

$$(\omega(e_i, e_j)) = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} =: J \tag{2.15}$$

and the matrix J satisfies $J^T = -J$, $J^2 = -\mathrm{Id}_{2n}$ in $\mathbb{R}^{2n}$. Using J in terms of the standard inner product on $\mathbb{R}^{2n}$ we have $\omega(x, y) =< x, Jy >$.

Thus, A is a matrix of a symplectic transformation iff $< x, Jy >=< Ax, JAy >=< x, A^T JAy >$. Hence, $A \in \mathrm{Sp}(2n)$ if and only if $A^T JA = J$. Therefore, we can conclude that the coset space $\mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$ gives all symplectic forms on $\mathbb{R}^{2n}$ by the correspondence $[A] \longmapsto A^T \omega A$, where $[A]$ is the equivalence class of $A \in \mathrm{GL}(2n, \mathbb{R})$ and ω is the standard symplectic from on $\mathbb{R}^{2n}$ [5, 9, 7].

2.2.3 Degenerate and non-degenerate symplectic forms on $\mathbb{R}^{2n}$

In the definition of symplectic form ω it is possible to relax the condition of non-degeneracy assuming that there exists some $x \in \mathbb{R}^{2n}$, $x \neq 0$, such that $\omega(x, y)$ vanishes for every $y \in \mathbb{R}^{2n}$. In fact, this means that the function from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ given by $f : x \longmapsto \omega(x, y)$ is not an isomorphism anymore or, in another words, the kernel of the map is non-trivial. Then the determinant of the associated matrix of ω is zero. In this situation the symplectic form is called degenerate.

Returning to the previous definition, we note that the above map f for a non-degenerate symplectic form defines an isomorphism. The determinant of the associated matrix is then non-zero and the associated matrix is not singular. In particular, in any way both statements for degenerate and for non-degenerate symplectic forms are independent of the chosen basis [5].

2.3 Complex structure on $\mathbb{R}^{2n}$

Until now we considered the simplest case of vector spaces, i.e., one built on a field of real numbers, but now consider more general notion of complex vector spaces. We will see that they are highly related.

2.3.1 Complex coordinates on $\mathbb{R}^{2n}$

Many geometrical problems are most conveniently formulated and solved by using complex numbers. Also the real vector space $\mathbb{R}^{2n}$ can be identified with a complex vector space. The most simple way to make it clear is to postulate on $\mathbb{R}^{2n}$ complex coordinates. We do that as follows. At first we might write any point on $z \in \mathbb{R}^{2n}$ in the unique form

$$z = x^i e_i + y^{i+n} e_{i+n}, \tag{2.16}$$

where $i = 1, ..., n$, $x^i, y^{i+n} \in \mathbb{R}$ and $\{e_m\}$, $m = 1, ..., 2n$, is a basis of $\mathbb{R}^{2n}$. Then we associate to z the complex valued coordinates $z^i = x^i + iy^{i+n}$. Therefore, each point on $\mathbb{R}^{2n}$ can be interpreted as a point of an n-dimensional vector space over the complex numbers, which we denote by $\mathbb{C}^n$ (a vector space is determined up to isomorphism by its dimension n). Obviously, the space $\mathbb{C}^n$ then is isomorphic to $\mathbb{R}^{2n}$ [6, 5].

2.3.2 A complex structure J on $\mathbb{R}^{2n}$

Let us define on $\mathbb{R}^m$ a real linear transformation $J : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ from $\mathbb{R}^m$ to itself such that

$$J^2 \quad = \quad -\mathrm{Id}_m, \tag{2.17}$$

where Id_m is the identity mapping of $\mathbb{R}^m$. Then J is called a complex structure on $\mathbb{R}^m$. Also we can equip $\mathbb{R}^m$ with the structure of a complex vector space in the following manner:

$$(\alpha + i\beta)z := \alpha z + \beta J z, \qquad \alpha, \beta \in \mathbb{R}. \tag{2.18}$$

Thus, scalar multiplication on $\mathbb{R}^m$ by complex numbers is provided and it is easy to convince yourself that $\mathbb{R}^m$ becomes a complex vector space. We call this process a complexification of $\mathbb{R}^m$. Suppose that the matrix of the complex structure J with respect to the basis $\{e_i\}, i = 0, ..., m$ is $J = (a^j_i)$, where a^j_i are real numbers. It is obvious that the eigenvalues of J are $\pm i$, and they must appear in pairs. Therefore, the dimension of $\mathbb{R}^m$ must be even, i.e., m = 2n [11].

All this is indeed very similar to the previous identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ which we have performed by introduction of complex coordinates on $\mathbb{R}^{2n}$. We see now that it can also be done by using complex structure tensor J. To make this fact more clear we consider the converse direction by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \{x_1, ..., x_n, y_1, ..., y_n\}, x_j, y_j \in \mathbb{R}$. Scalar multiplication by i in $\mathbb{C}^n$ induces a mapping

$$J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \text{ given by } J(x_1, ..., x_n, y_1, ..., y_n) = (-y_1, ..., -y_n, x_1, ..., x_n),$$

and, moreover, $J^2 = -\mathrm{Id}_{2n}$. The complex structure J defined in this way is called the standard complex structure on $\mathbb{R}^{2n}$ [12, 11, 6, 13, 5].

2.3.3 Subgroup $\mathrm{GL}(n, \mathbb{C})$ of $\mathrm{GL}(2n, \mathbb{R})$ as a group preserving the complex structure J

As it was mentioned above, we can always consider the real space $\mathbb{R}^{2n}$ equipped with a complex structure as the complex vector space $\mathbb{C}^n$. The non-singular linear transformations of the vector space $\mathbb{C}^n$ map $\mathbb{C}^n$ to $\mathbb{C}^n$ and consequently preserve the complex structure. Moreover, they form a group, called the general linear group of degree n over $\mathbb{C}$, denoted by $\mathrm{GL}(n,\mathbb{C})$. This group is isomorphic to the group of $n \times n$ invertible matrices with complex entries. Upon realization of $\mathbb{C}^n$, each of these matrices corresponds to a linear transformation of the real vector space $\mathbb{R}^{2n} \cong \mathbb{C}^n$. That is why the group $\mathrm{GL}(n,\mathbb{C})$ can be regarded as a subgroup of $\mathrm{GL}(2n,\mathbb{R})$. The coset space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ determines all complex structures on $\mathbb{R}^{2n}$ by the mapping $[A] \longmapsto A^{-1}JA$, where $[A]$ is the equivalence class of $A \in \mathrm{GL}(2n,\mathbb{R})$ and J the standard complex structure on $\mathbb{R}^{2n}$ [6, 11, 5].

2.3.4 Complex structure J on the space $\mathbb{R}^{2p,2q}$ and Hermitian metric g of signature (p,q)

Suppose we have the real vector space $\mathbb{R}^{2n}$, equipped with a complex structure J. Complexification of $\mathbb{R}^{2n}$ makes it possible to regard this space as the complex vector space $\mathbb{C}^n$. In the same way, as before for a real space, we can define on $\mathbb{C}^n$ a symmetric sesquilinear form $<\cdot,\cdot>_{\mathbb{C}}: \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$, which satisfy

$$
\begin{aligned}
< ax + y, z >_{\mathbb{C}} &= a < x, z >_{\mathbb{C}} + < y, z >_{\mathbb{C}}, && (2.19)\\
< x, ay + z >_{\mathbb{C}} &= < x, y >_{\mathbb{C}} + \bar{a} < x, z >_{\mathbb{C}}, && (2.20)\\
< x, y >_{\mathbb{C}} &= \overline{< y, x >}_{\mathbb{C}}, && (2.21)
\end{aligned}
$$

where we denote by $\bar{x}$ the complex conjugate of x. We call the scalar product $<\cdot,\cdot>_{\mathbb{C}}$ the complex scalar product on $\mathbb{C}^n$.

In general, we can write the complex scalar product in the form expressed by the corresponding metric h of signature (p,q) (of course, $p+q=n$):

$$< x, y >_{\mathbb{C}} = h_{ij} x^i \bar{y}^j, \qquad \forall x, y \in \mathbb{C}^n. \tag{2.22}$$

For an appropriate basis of $\mathbb{C}^{p,q}$ it can be transformed to

$$< x, y >_{\mathbb{C}} = h_{ij} x^i \bar{y}^j = \sum_{i=1}^{p} x^i \bar{y}^i - \sum_{j=p+1}^{n} x^j \bar{y}^j, \qquad \forall x, y \in \mathbb{C}^n, \tag{2.23}$$

Here we see the connection with the metric h. The scalar product $<\cdot,\cdot>_{\mathbb{C}}$ is called Hermitian scalar product or Hermitian form. In general, the matrix (h_{ij}) can take complex values. However, there is always a basis with respect to which the metric h is diagonal and real. Since the pseudo-Euclidean metric h has the signature (p,q), it is more convenient to write $\mathbb{R}^{2p,2q}$ for the space $\mathbb{R}^{2n}$ and $\mathbb{C}^{p,q}$ for $\mathbb{C}^n$.

If we now consider the Hermitian form h on the complex vector space $\mathbb{C}^n$, understood as a complexification of $\mathbb{R}^{2n}$ by J, we will find out that the 'realized' form of h, i.e. its corresponding form on $\mathbb{R}^{2n}$, can be given by a positive definite real bilinear form g on $\mathbb{R}^{2n}$, i.e., by

$$h(\xi_1, \xi_2) = g(\xi_1, \xi_2) + ig(J\xi_1, \xi_2), \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^{2n}. \tag{2.24}$$

A special feature of the pseudo-Euclidean metric g is that it satisfies the compatibility condition with the complex structure J in the sense that

$$g(J\xi_1, J\xi_2) = g(\xi_1, \xi_2), \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^{2n}. \tag{2.25}$$

In this fashion one says that g is J invariant and also calls it Hermitian metric or J-metric on $\mathbb{R}^{2n}$. We note that (pseudo-)Hermitian metrics h on $\mathbb{C}^{p,q}$ and J-invariant (pseudo-)Euclidean metrics g stand in one to one correspondence to each other [6, 5, 8].

2.3.5 The group $\mathrm{U}(p, q)$ preserving J and g

Let $\mathbb{R}^{2p,2q}$ be equipped with a complex structure J and a pseudo-Hermitian metric g of type (p, q). As the unitary group $\mathrm{U}(p, q)$ we define a group of linear transformations on $\mathbb{R}^{2p,2q} \cong \mathbb{C}^{p,q}$, which preserve the complex scalar product defined in previous section. In other words, if

$$< Ax, Ay >_{\mathbb{C}} = < x, y >_{\mathbb{C}}, \tag{2.26}$$

then $A \in \mathrm{U}(p, q)$. In fact, $\mathrm{U}(p, q)$ consists of those complex $n \times n$ matrices A which are elements of $\mathrm{GL}(n, \mathbb{C})$ and satisfy

$$\bar{A}^T g A = g = (g_{ij}), \tag{2.27}$$

because

$$< Ax, Ay >_{\mathbb{C}} = (Ax)^T g \overline{(Ay)} = x^T A^T g \bar{A} \bar{y} = x^T g \bar{y}. \tag{2.28}$$

The last equation implies

$$\bar{A}^T \mathrm{Id}_{p,q} A = \mathrm{Id}_{p,q}, \tag{2.29}$$

if we put $g_{ij} = (\mathrm{Id}_{p,q})_{ij}$. Consequently we can identify $\mathrm{U}(p, q)$ with a group consisting of the matrices A, which obey $A^\dagger := \bar{A}^T = \mathrm{Id}_{p,q} A^{-1} \mathrm{Id}_{p,q}$.

We now characterize the (pseudo-)unitary group $\mathrm{U}(p, q)$ as intersection of the groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SO}(2p, 2q)$. Obviously, $\mathrm{U}(p, q)$ as a set of non-degenerate transformations of the complex space is a subset of $\mathrm{GL}(n, \mathbb{C})$. Further we introduce on $\mathbb{R}^{2p,2q}$ the usual pseudo-Euclidean scalar product $< \cdot, \cdot >_{\mathbb{R}}$, defined for a pair of vectors $\xi_1 = (x_1^1, ..., x_1^1, y_1^1, ..., y_1^n)$ and $\xi_2 = (x_2^1, ..., x_2^1, y_2^1, ..., y_2^n)$ as

$$< \xi_1, \xi_2 >_{\mathbb{R}} \quad = \quad \sum_{i=1}^{p} (x_1^i x_2^i + y_1^i y_2^i) - \sum_{j=p+1}^{n} (x_1^j x_2^j + y_1^j y_2^j). \tag{2.30}$$

Then the complex scalar product is related to this one in the following way:

$$\mathrm{Re} < x, y >_{\mathbb{C}} \quad = \quad < x, y >_{\mathbb{R}}, \tag{2.31}$$

where Re stands for a real part of a complex number. The explicit computation confirms this equation,

$$\begin{aligned} \mathrm{Re} < \xi_1, \xi_2 >_{\mathbb{C}} &= \sum_{i=1}^{p} (x_1^i + iy_1^i)(x_2^i - iy_2^i) - \sum_{j=p+1}^{n} (x_1^j + iy_1^j(x_2^j - iy_2^j)) \\ &= \sum_{i=1}^{p} (x_1^i x_2^i + y_1^i y_2^i) - \sum_{j=p+1}^{n} (x_1^j x_2^j + y_1^j y_2^j). \end{aligned} \tag{2.32}$$

Since $< \xi, \xi >_{\mathbb{C}}$ is real, it follows

$$< \xi, \xi >_{\mathbb{C}} = < \xi, \xi >_{\mathbb{R}} . \tag{2.33}$$

We conclude that a linear transformation $A \in \mathrm{GL}(n, \mathbb{C})$ preserves the complex scalar product $< \xi, \xi >_{\mathbb{C}}$ if and only if A preserves $< \xi, \xi >_{\mathbb{R}}$. We infer that indeed A has to be an element of $\mathrm{SO}(2p, 2q)$, because $\mathrm{SO}(2p, 2q)$ is exactly the group leaving the form $< \xi, \xi >_{\mathbb{R}}$ invariant (for the fact that the determinant of realized form of A is 1 look [6]). Hence, it follows that

$$\mathrm{U}(p, q) = \mathrm{GL}(2n, \mathbb{C}) \cap \mathrm{SO}(2p, 2q). \tag{2.34}$$

2.3.6 Euclidean case as a subcase

In the case when in the above considerations the metric g is simply the Hermitian metric, i.e., g is of type $(n, 0)$, the unitary group $\mathrm{U}(p, q)$ with $p = n, q = 0$ turns out to be the intersection of the general linear n-dimensional group on $\mathbb{C}$ and the usual special orthogonal group of $2n$ dimension:

$$\mathrm{U}(n) \quad = \quad \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{SO}(2n). \tag{2.35}$$

2.3.7 Symplectic structure ω on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2p,2q}$

Suppose we have endowed the space $\mathbb{R}^{2n}$ with a complex structure J and a Hermitian metric g. Since g is J-invariant, one can define a symplectic form ω on $\mathbb{R}^{2n}$ by

$$\omega(\xi_1, \xi_2) = g(J\xi_1, \xi_2), \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^{2n}. \tag{2.36}$$

The form ω is in fact antisymmetric, since $\forall x, y \in \mathbb{R}^{2n}$

$$\omega(x, y) = g(Jx, y) = g(J^2x, Jy) = -g(x, Jy) = -g(Jy, x) = -\omega(y, x). \tag{2.37}$$

Furthermore, ω is non-degenerate, because g is non-singular. Moreover, we observe that ω is invariant under the action of J:

$$\omega(Jx, Jy) = g(J^2x, Jy) = g(J^3x, J^2y) = g(Jx, y) = \omega(x, y). \tag{2.38}$$

The two-form ω is called the Kähler form. In components we can write it as $\omega_{ij}x^iy^j = g_{kj}J_i^kx^iy^j$ or $\omega_{ij} = J_i^kg_{kj}$. That implies that in matrix form: $\omega = Jg$ [14].

2.3.8 $\mathrm{U}(p, q)$ as a subgroup of $\mathrm{Sp}(2n, \mathbb{R})$

We have seen that the unitary group is the group which preserves the standard complex structure J as well as any pseudo-Hermitian metric g of signature (p, q). Since the Kähler form ω is constructed from g and J, it follows automatically that the symplectic group, defined as a group preserving a symplectic form ω, contains linear transformation $A \in \mathrm{U}(p, q)$. Hence, we conclude that

$$\mathrm{U}(p, q) \subset \mathrm{Sp}(2n, \mathbb{R}). \tag{2.39}$$

2.4 Algebraic structures on vector spaces

Now let us take a look at vector spaces from the algebraic side. We will confirm ourself that group theoretical properties stand in an non-separable conjunction with algebraic structures.

2.4.1 General notion of algebra

Before proceeding further we introduce a general notion of algebra. To do that we need a definition of objects such as field and vector space. In mathematical sense a field K is a set equipped with two binary operations, $+ : K \times K \longrightarrow K$ and $\cdot : K \times K \longrightarrow K$, called addition and multiplication, respectively. The result of applying them to elements $a, b \in K$ is denoted by $a + b$ and ab. The set K additionally must satisfy the following requirements known as field axioms.
Addition:

- Commutativity: $a + b = b + a$;
- Associativity: $a + (b + c) = (a + b) + c$;
- Existence of negative: for any $a \in K$ there is an element $-a$ in K, which obeys $a + (-a) = 0$ (written also as $a - a = 0$);
- Existence of zero: there exists an element $0 \in K$ such that $a + 0 = a$, $\forall a \in K$;

Multiplication:

- Commutativity: $ab = ba$;

- Associativity: $a(bc) = (ab)c$;
- Existence of unity: $\forall a \in K$ there exists an element $1 \in K$ with the property $1a = a1 = a$;
- Existence of inverse: for any $a \in K$ there exists an element a^{-1} such that $aa^{-1} = a^{-1}a = 1$ (it can be shown that this element is unique);

Addition and multiplication:

- Distributivity: $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$.

Moreover, we exclude the trivial case when K contains only zero element.

Now we go over to the notion of a vector space. Under a vector space over a field K we understand a set V together with two binary operations $V \times V \longrightarrow V$ and $K \times V \longrightarrow V$, called vector addition and scalar multiplication. The elements of V are called vectors. The sum of two vectors is denoted by $v + w$, the product of a scalar $a \in K$ and a vector v is denoted $a \cdot v$ or simply av.

To qualify the space as a vector space, addition and multiplication have to fulfill a number of axioms described below. Let $u, v, w \in V$ and $a, b \in K$. First of all, for a vector space the addition field axioms must hold (see above). Further, a vector space is required to satisfy some axioms for the multiplication on scalars:

- $a(bv) = (ab)v$;
- $(a + b)v = av + bv$;
- $a(v + w) = av + aw$;
- $1v = v$.

Now we have all ingredients to give the definition of an algebra. Suppose K is a field and V is a vector space over K equipped with an additional binary operation from $V \times V$ to V, denoted here by $\cdot$ and called multiplication on V. Then $A = (V, \cdot)$ is an algebra over K or a K-algebra if the following identities hold for any three elements v, w, and u in A, and all elements a and b in K:

- Left distributivity: $(x + y)z = xz + yz$;
- Right distributivity: $x(y + z) = xy + xz$;
- Compatibility with multiplication on scalars: $(ax)(by) = (ab)(xy)$.

These three axioms are equivalent to the statement that the binary operation is bilinear. We note that multiplication of elements of an algebra is not necessarily associative. We discuss in the following two possible kinds of algebra: associative and non-associative one [5, 15].

2.4.2 Associative and non-associative algebras

An associative algebra A is a K-algebra, for which the associativity law holds:

$$v(wu) = (vw)u = vwu, \qquad \forall v, w, u \in A. \tag{2.40}$$

If A contains an identity element, i.e., an element 1 such that $1v = v1 = v$ for all x in A, then we call A an associative algebra with a unit. In this case all elements a of the field K identified with $1a$ are in A. As examples of associative algebras we mention the algebra of $n \times n$ matrices, the algebra of polynomials with real coefficients and Clifford algebras.

Under a non-associative algebra we understand an algebra over a field K, for which the associativity law does not hold. In this case we distinguish the right multiplication $R_a : A \times A \longrightarrow A, v \longmapsto va$ and the left multiplication $L_a : A \times A \longrightarrow A, v \longmapsto av$. We call an algebra unital or unitary if it has a unit or identity element I with $Iv = v = vI$ for all v in the algebra. Well known examples of non-associative algebras are Lie algebras, hyperbolic quaternion algebra (the mathematical concept was first suggested by A. Macfarlane), Poisson algebras and many others [5].

2.4.3 Algebra of quaternions $\mathbb{H}$

A special kind of associative algebra is the quaternion algebra. At first we introduce the notion of quaternions. The set of quaternions $\mathbb{H}$ is a four dimensional vector space over real numbers consisting of all linear combinations

$$q \in \mathbb{H}, \qquad q = a + bi + cj + dk, \tag{2.41}$$

where a, b, c, d are real numbers and $1, i, j, k$ are linear independent vectors. Multiplication among the vectors i, j, k is established as

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1. \tag{2.42}$$

Thus, the vectors i, j, k have the property of imaginary unities. We can extend the definition of multiplication to all elements of $\mathbb{H}$ if use the distributive laws and assume that the unity 1 commutes with all quaternions (then for example $-1 = ijk$). Therefore, with this multiplication and with the usual linear addition $\mathbb{H}$ becomes an associative non-commutative algebra over the field of real numbers [6, 5].

2.4.4 The group $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ as a group preserving the quaternion structure

Note that the imaginary units of the quaternion algebra admit a representation in the algebra of 2×2-matrices, where the real unit vector is the diagonal unit matrix Id_2. If we construct out of two matrices

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \tag{2.43}$$

with arbitrary components a, b, c, d, e, f and vanishing trace, two imaginary units

$$j = \tfrac{A}{\sqrt{\det A}} \Longrightarrow j^2 = -\mathrm{Id}_2, \quad k = \tfrac{B}{\sqrt{\det B}} \Longrightarrow k^2 = -\mathrm{Id}_2,$$

then the product of them

$$jk = \frac{AB}{\sqrt{\det AB}} = \frac{1}{\sqrt{\det AB}} \begin{pmatrix} ad + bf & ae - bd \\ cd - af & ec + ad \end{pmatrix} \tag{2.44}$$

is again an imaginary unit under the condition that the product matrix is traceless: $2ad + bf + ec = 0$. If we denote $jk = l$, then it is straightforward to prove that the matrices $I = \mathrm{Id}_2, j, k, l$ obey the multiplication rules of the quaternion algebra. An example of such imaginary triple j, k, l is well known Pauli matrices (with factor complex factor $-i$) appearing in the quantum mechanics:

$$j = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad l = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.45}$$

This correspondence leads to the realization of (2.42) in terms of 2×2 matrices. We then say that Pauli matrices are a 2×2 complex realization of abstract quaternion algebra. We note that there are other possibilities to represent imaginary units of the quaternion algebra. We mention only that the quaternion basis unit vectors can also be written using real 4×4-matrices.

Suppose, we choose an appropriate representation of the quaternion algebra. Then one can rise the question which transformation of the units does not change the quaternion multiplication rules. To give an answer it is convenient to change the notation for quaternion algebra and to rewrite the rules in the compact form:

$$1q_i = q_i 1 = q_i, \qquad q_i q_j = -\delta_{ij} + \epsilon_{ijk} q_k, \tag{2.46}$$

where $q_{1,2,3} = j, k, l$ and ϵ_{ijk} is the antisymmetric Levi-Civita tensor in three dimensions.

So let U be an arbitrary transformation. Let U also be non-zero element of some set G with unity Id (identity transformation), so that there exists U^{-1} with

$$UU^{-1} = U^{-1}U = \mathrm{Id}.$$

Since U^{-1} is also an element of G, the last relation assumes that a multiplication in G is provided and thus requires G to be a group. Then we can impose the conditions

$$q_{k'} = Uq_k U^{-1}, \qquad 1' = U1U^{-1} = \mathrm{Id}1 = 1, \tag{2.47}$$

leaving the multiplication rules (2.46) invariant

$$\begin{aligned} 1' q_{k'} = 1 U q_k U^{-1} &= U q_k 1 U^{-1} = U q_k U^{-1} = q_{k'}, & (2.48)\\ q_{i'} q_{j'} &= U q_i U^{-1} U q_j U^{-1} = U q_i q_j U^{-1} & (2.49)\\ &= -U \delta_{ij} U^{-1} + \epsilon_{ijk} U q_k U^{-1} = -\delta_{ij} + \epsilon_{ijk} q_{k'}. & (2.50) \end{aligned}$$

It should be noted that any transformation U acts on the real unit as the identity transformation, so that in fact only imaginary units are involved. That is easy to understand, because in any matrix representation there exists a unique real unit - the unit diagonal matrix - which is preserved by action of an invertible matrix U.

For this reason the transformation U is actually a transformation of the three vector units and does not concern the scalar part of a quaternion in any way. The number of free parameters of the transformation can be easily counted if we use the 2×2 matrix representation of the algebra. In this case the transformation U is an unimodal 2×2 matrix, $\det U = 1$, because the transformations (2.48) do not depend on the determinant of U. Therefore only three components of U are independent. In general case when the parameters are complex, the total number of parameters is 6 and the group G is the group $\mathrm{SL}(2, \mathbb{C})$. In the case when these parameters are real, the number of free components is 3 and the group contracts to the group SU(2).

In particular, since $\det U = 1$, the transformation U in the 2×2-matrix representation can be expressed as an unit quaternion, that is, if the Euclidean norm $|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2$ for $q = a + bj + ck + dl$ is 1. Then any linear invertible transformation of the space $\mathbb{H}$ preserving the quaternion structure can be written in the form

$$\alpha_p : q \longmapsto pqp^{-1},$$

where $p \in \mathbb{H}$ with $|p|^2 = 1$ and $p^{-1} = \bar{p}/|p|^2$. An alternative method to regard transformation of quaternions is to write them as vectors and consider their transformations given by 3×3-matrices O:

$$q_{k'} = O_{k'k} q_k. \tag{2.51}$$

We remark only that the transformation group then is the group $\mathrm{SO}(3, \mathbb{R})$ (we refer the reader to [16, 5]).

Another way to treat the quaternion transformations is expressing quaternions q as

$$q = a + bj + ck + dl = (a + jb) + (c + jd)k = x + yk, \tag{2.52}$$

where $x = a + jb, y = c + jd$ are complex numbers. Then q can be considered as a vector in $\mathbb{C}^2$. Moreover, we can define (analogous to the Hermitian form) on pairs of vectors q_1, q_2 in $\mathbb{H}$ a form

$$< q_1, q_2 >_{\mathbb{H}} = q_1 \bar{q}_2 = (x_1 + y_1 k)\overline{(x_2 + y_2 k)} = (x_1 \bar{x}_2 + y_1 \bar{y}_2) + (y_1 x_2 - x_1 y_2)k. \quad (2.53)$$

We denote by Sp(1) the group of transformation of $\mathbb{H}$ which preserves the form (2.53). It follows that Sp(1) consists of transformations leaving the Hermitian form $x_1\bar{x}_2 + y_1\bar{y}_2$ invariant and also preserving the skew-symmetric form $y_1x_2 - x_1y_2$. Hence, Sp(1) contains those unitary transformations of $\mathbb{C}^2$ (i.e., elements of U(2)) which preserve the above skew-symmetric form. So we have shown that in fact Sp(1) is a subgroup of U(2). Moreover, we prove that Sp(1) is isomorphic to SU(2) in the following way. We consider the form $y_1x_2 - x_1y_2$ as the area of a complex parallelogram in $\mathbb{C}$ spanned by the side vectors (x_1, y_1) and (x_2, y_2). We conclude that Sp(1) is a group of unitary matrices with determinant 1, since they do not change the orientation of the parallelogram [6].

2.4.5 Vector space $\mathbb{H}^n$ and the group $\mathrm{Sp}(n)$

We have seen that the space $\mathbb{H}$ can be expressed in terms of the complex space $\mathbb{C}^2$. In a simple way we can expand our construction on a multidimensional quaternion space $\mathbb{H}^n$. For that purpose we write again $q \in \mathbb{H}$ as

$$q = a + bi + cj + dk = (a + ib) + (c + id)j = x^1 e_1 + y^1 (je_1), \quad (2.54)$$

so that we can now identify the space $\mathbb{H}^n$ with the complex space $\mathbb{C}^{2n}$ equipped with the basis $e_1, ..., e_n, je_1, ..., je_n$ and with complex coordinates $x^1, ..., x^n, y^1, ..., y^n$. Then for each $q \in \mathbb{H}^n$ we can write $q = q^k e_k = (x^k e_k + y^k je_k)$.

We denote by $\mathrm{GL}(n, \mathbb{H})$ the group of linear invertible transformation on $\mathbb{H}^n$ and consider again the group $\mathrm{Sp}(n) \subset \mathrm{GL}(n, \mathbb{H})$ preserving the form $< \cdot, \cdot >_{\mathbb{H}}$, which in n dimensions has the form

$$\begin{aligned} < q_1, q_2 >_{\mathbb{H}} &= \sum_{i=1}^{n} q_1^i \bar{q}_2^i = \sum_{i=1}^{n} (x_1^i + y_1^i j)\overline{(x_2^i + y_2^i j)} \\ &= \sum_{i=1}^{n} (x_1^i \bar{x}_2^i + y_1^i \bar{y}_2^i) + \sum_{i=1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j, \end{aligned} \quad (2.55)$$

Again as before it follows that the group $\mathrm{Sp}(n)$ consists of transformations which leave the Hermitian form on $\mathbb{C}^{2n}$ and the skew-symmetric form $\sum_{i=1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j$ invariant. This implies that $\mathrm{Sp}(n)$ is a subgroup of $\mathrm{U}(2n)$ preserving this skew-symmetric form [6].

2.4.6 Metric of signature (p, q) with $p + q = n$ on $\mathbb{H}^n$ and the group Sp(p, q)

Analogously to the preceding steps we can extend the concept defined in the previous section to an arbitrary metric g with signature (p, q) (with $p + q = n$) on

$\mathbb{H}^n$,

$$< q_1, q_2 >_{\mathbb{H}} = g_{ij} q_1^i \bar{q}_2^j, \tag{2.56}$$

which for an appropriate basis vectors takes the form

$$\begin{aligned} < q_1, q_2 >_{\mathbb{H}} &= \sum_{i=1}^{p} q_1^i \bar{q}_2^i - \sum_{i=p+1}^{n} q_1^i \bar{q}_2^i \qquad (2.57)\\ &= \sum_{i=1}^{p} (x_1^i \bar{x}_2^i + y_1^i \bar{y}_2^i) - \sum_{i=p+1}^{n} (x_1^i \bar{x}_2^i + y_1^i \bar{y}_2^i) \\ &+ \sum_{i=1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j - \sum_{i=p+1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j. \qquad (2.58) \end{aligned}$$

We see that it is quite obvious to identify the group Sp(p,q) preserving the above form of signature (p, q) as the subgroup of U(p,q), which leaves the metric g in variant and preserves the antisymmetric form $\sum_{i=1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j - \sum_{i=p+1}^{n} (y_1^i x_2^i - x_1^i y_2^i) j$ of type (p, q).

2.4.7 Algebra $\mathbb{O}$ of octonions

Graves's octonions form the last one after Hamilton's quaternions in a sequence of four increasingly complicated normed division algebras (i.e., algebras with well defined division operation and norm obeying $||xy|| = ||x|| \cdot ||y||$): the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

The octonions $\mathbb{O}$ forming a vector space can be formally expressed as 8-tuples (or as a octave) of real numbers. Every octonion can be written as a linear combination of unit octonions $\{1, e_1, ..., e_7\}$:

$$\mathbb{O} = \{a^0 + \sum_{i=1}^{7} a^i e_i : a^0, ..., a^7 \in \mathbb{R}\}.$$

Defined in this way octonions constitute an algebra over the reals generated by units $1, e_1, ..., e_7$ required to satisfy usual addition and certain multiplication laws. Addition of octonions is accomplished by adding corresponding coefficients, as it is for quaternions and complex numbers. Multiplication properties of octonions one can find in the Table 1.1. Analyzing commutativity and associativity we observe that the multiplication table is not commutative and is not associative.

Furthermore, we note that the table does not seem to have any pattern. However, we can generate an equivalent table by swapping rows and columns as follows: we swap e_7 and e_2 then e_4 and e_6 then e_6 and e_7. The result is presented in Table 1.2, where we use another basis i, j, k, l, li, lj, lk. In this table we can see

$e_i \cdot e_j$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 2.1: Multiplication table for octonions [5]

$a \cdot b$	1	i	j	k	l	li	lj	lk
1	1	i	j	k	l	il	jl	kl
i	i	-1	k	$-j$	li	$-l$	$-lk$	lj
j	j	$-k$	-1	i	lj	lk	$-l$	$-li$
k	k	j	$-i$	-1	lk	$-lj$	li	$-l$
l	l	$-li$	$-lj$	$-lk$	-1	i	j	k
li	il	l	$-lk$	lj	$-i$	-1	$-k$	j
lj	jl	lk	l	$-li$	$-j$	k	-1	$-i$
lk	kl	$-lj$	li	l	$-k$	$-j$	i	-1

Table 2.2: Multiplication table for octonions [5]

that octonions can be thought as pairs of quaternions, i.e., they can in analogy with quaternions be written as

$$\begin{aligned} Y &= a^0 + a^1 i + a^2 j + a^3 k + a^4 l + a^5 li + a^6 lj + a^7 lk && (2.59)\\ &= (a^0 + a^1 i + a^2 j + a^3 k) + l(a^4 + a^5 i + a^6 j + a^7 k) \\ &= q_1 + l q_2, && (2.60) \end{aligned}$$

where $q_1, q_2 \in \mathbb{H}$. In fact, it can be done via Cayley-Dickson construction which defines an octonion as a system of two quaternions, $q = (a, b)$, with addition defined pairwise and multiplication defined as

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}), \tag{2.61}$$

where $(a, b), (c, d)$ are pairs of quaternions and $\bar{a}$ denotes the conjugate of the quaternion a. This definition is equivalent to the one given above when the eight unit octonions are identified with the pairs

$$(1, 0), (i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k). \tag{2.62}$$

For more information we refer the reader to [17, 18, 5, 19].

2.4.8 Properties of the octonion structure constants

As it was shown above multiplication table for octonions determines relations between the single unit octonions entirely. Nevertheless, there is another way to express the multiplication rules between them. This is because octonions algebra has the same structure as the algebra of quaternions (compare (2.46)) and can be written in a similar short form:

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k, \quad e_i e_0 = e_0 e_i = e_i, \quad e_0 e_0 = 1, \tag{2.63}$$

where i, j, k run from 1 to 7 and f_{ijk} are the structure constants. Using the multiplication table it can be proved that the structure constants are completely antisymmetric in their indices. Also it can be shown that only for the combinations of indices [20]:

$$(123), (145), (176), (246), (257), (347), (365), \tag{2.64}$$

the structure constants do not vanish and are equal to one.

2.4.9 Group G_2 and $\mathrm{Spin}(7)$ related with the octonion algebra

In many cases it can be useful to consider linear transformations of the vector space of octonions, so we want to find inner symmetries of the space $\mathbb{O}$ and analyze transformation behavior of its elements. The group of invertible linear transformations A from $\mathbb{O}$ to $\mathbb{O}$, satisfying $A(xy) = A(x)A(y)$, is called the automorphism group of $\mathbb{O}$. It is easy to show that automorphisms of $\mathbb{O}$ in fact form a group, denoted by $\mathrm{Aut}(\mathbb{O})$ and also denoted by G_2. One can show that G_2 is a simply connected compact simple real Lie group of dimension 14. It is a subgroup of the group Spin(7). Moreover, this group is the smallest exceptional Lie group [17, 5, 21]. Furthermore, G_2 can be understood as a simple, 14-dimensional stabilizer of the three-form

$$\omega := \mathrm{d}x^{127} + \mathrm{d}x^{347} + \mathrm{d}x^{567} + \mathrm{d}x^{135} - \mathrm{d}x^{146} - \mathrm{d}x^{236} - \mathrm{d}x^{245} \in \Lambda^3(\mathbb{R}^7). \tag{2.65}$$

Here $\mathrm{d}x^{123} = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Under the action of G_2 familiar vector spaces split

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{g}^{\perp}, \tag{2.66}$$

$$\Lambda^2(\mathbb{R}^7) = \Lambda^2_{14} \oplus \Lambda^2_7, \tag{2.67}$$

$$\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}. \tag{2.68}$$

We will consider them later.

Furthermore, it can be proven that the group G_2 is the intersection of any two subgroups Spin(7) (of dimension 21) in the group Spin(8) (dim 28). The groups $\mathrm{Spin}(n)$ will be introduced in section 2.4.16 [17, 21].

2.4.10 Grassmann algebra: definition and main properties

Here, we wish to construct a space that is universal with respect to alternating multilinear maps. To do this, it is necessary at first to introduce the notion of tensor product of two vector spaces V, W over a field K.

To construct $V \otimes W$, one begins with the set of ordered pairs in the Cartesian product $V \times W = \{(v, w) | v \in V, w \in W\}$. Taking as a basis elements of the form $(v, w) \in V \times W$, we define the free vector space of $V \times W$:

$$F(V \times W) = \{\sum_{i=1}^{n} \alpha_i e_{(v_i \times w_i)} | n \in \mathbb{N},\ \alpha_i \in K,\ v_i, w_i \in V \times W\},$$

where the symbol $e_{(v \times w)}$ is used to emphasize that these are taken to be linearly independent for distinct $(v, w) \in V \times W$.

Denoting by R the space generated by three equivalence relations

- $e_{((v_1+v_2)\times w)} \sim e_{(v_1 \times w)} + e_{(v_2 \times w)}$,
- $e_{(v\times(w_1+w_2))} \sim e_{(v \times w_1)} + e_{(v \times w_2)}$,
- $ce_{(v\times w)} \sim e_{(cv \times w)} \sim e_{(v \times cw)}$,

where $v, v_i \in V$, $w, w_i \in W$ and $c \in K$, the tensor product of the two vector spaces V and W is then the quotient space

$$V \otimes W = F(V \times W)/R.$$

We call the space $V \otimes W$ the tensor product of V and W. Since in $V \otimes W$ the space R is set to zero, vectors in $V \otimes W$ obey the following relations:

- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$,
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$,
- $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.

This construction of the tensor product can be extended to the tensor product of more than two vector spaces. It arises in a natural way if we consider for example the case of three vector spaces $V \otimes (W \otimes U)$ and $(V \otimes W) \otimes U$. It turns out that these spaces are equal up to an isomorphism. Then we say the tensor product is associative and write $V \otimes W \otimes U$. For this reason we define for an arbitrary natural number k the k-th tensor power of the vector space V as the k-fold tensor product of V with itself. That is

$$T^k(V) := V \otimes ... \otimes V = V^{k\otimes} \text{ (k-times tensor product).}$$

Elements of $T^k(V)$ are called tensors on V of rank k. By convention $T^0(V)$ is the field K (as a one-dimensional vector space over itself). We then construct $T(V)$ consisting of all possible $T^k(V)$, $k \in \mathbb{N}$. That is the direct sum of $T^k(V)$ (under the direct sum of two vector spaces V, W over a field K we understand the Cartesian product $V \times W$ equipped with the following structure: $(v_1, w_2) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$, $a(v, w) = (av, aw)$, where $v_i \in V$, $w_i \in W$, $a \in K$; we denote it by $V \oplus W$). We can write

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V) = K \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus ....$$

The multiplication in $T(V)$ is determined by the canonical isomorphism

$$T^k(V) \otimes T^l(V) \longrightarrow T^{k+l}(V)$$

given by the tensor product defined in components as

$$(F \otimes G)_{i_1 i_2 ... i_{n+m}} = F_{i_1 ... i_n} G_{i_{n+1} ... i_{n+m}}, \tag{2.69}$$

where F is a tensor of rank n and G is a tensor of rank m.

This multiplication rule implies that the tensor algebra $T(V)$ is a naturally graded algebra, i.e., it can be represented in the form $A = \bigoplus_i A_i$, where i runs through $\mathbb{Z}$, and multiplication satisfies the following condition: $A^i A^j \subset A^{i+j}$. Some A^i subspaces can be empty. The unit of algebra (if it exists) always belongs to A_0 (exterior algebra). Let A be the two-sided ideal[2] generated by all elements of the form $x \otimes x$, $x \in T(V)$. That is the set of all the elements of the form

$$v_1 \otimes ... \otimes v_i \otimes ... \otimes v_i \otimes ... \otimes v_k$$

where $v \in V$, $k = 0, 1, 2, ...$ Then we define the space of all k-vectors, $k \in \mathbb{N}$, called the exterior algebra or Grassmann algebra $\Lambda(V)$ over a vector space V, as the quotient space

$$\Lambda(V) := T(V)/A, \tag{2.70}$$

i.e., the space $T(V)$, where elements of A required to be set to zero. Since in $\Lambda(V)$ we ignore the elements of A, it is convenient to introduce a new product called wedge product or exterior product given for a pair of $v, w \in \Lambda(V)$ by

$$v \wedge w = v \otimes w \mod A. \tag{2.71}$$

It can be easily shown that wedge-product is antisymmetric. In particular, it obeys

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge ... x_{\sigma(k)} = \mathrm{sgn}(\sigma) x_1 \wedge ... \wedge x_k, \tag{2.72}$$

[2] The term two-sided ideal is used in non-commutative rings (algebras) to denote a subset that is both a right ideal and a left ideal. In general, this is a subset I of a ring (or a algebra) $R(\cdot, +)$, for which $(I, +)$ is a subgroup of $(R, +)$ and $xr, ry \in I$, $\forall r \in R$, $\forall x, y \in I$.

where $x_1, x_2, ..., x_k \in V$, σ is an element of the group of permutations of $[1, ..., k]$ and sgn is the signature of the permutation σ.

The space $\Lambda^k(V)$ is the vector subspace of $\Lambda(V)$ spanned by elements of the form

$$x_1 \wedge ... \wedge x_k, \quad x_i \in V, \quad i = 1, ..., k$$

and is called an k-th exterior power of the space V. Elements of $\Lambda^k(V)$ are said to be k-multivectors and can in general be written as

$$A = \sum_{1 \le i_1 < ... < i_k \le n} A^{i_1 ... i_k} e_{i_1} \wedge ... \wedge e_{i_k} = \frac{1}{k!} A^{i_1 ... i_k} e_{i_1} \wedge ... \wedge e_{i_k}. \tag{2.73}$$

By counting the basis elements of $\Lambda^k(V)$ we obtain that the dimension of $\Lambda^k(V)$ is equal to the number of all possible unordered permutations of $[1, ..., k]$, that is the binomial coefficient C_n^k.

Moreover, the space of alternating forms of degree k on V is naturally isomorphic to the dual vector space $(\Lambda^k V)^*$. If V is finite-dimensional, then the latter is naturally isomorphic to $\Lambda^k(V^*)$. Under this identification, the wedge product takes a concrete form. Suppose $\omega : V^k \longrightarrow K$ and $\eta : V^m \longrightarrow K$ are two antisymmetric maps. The wedge product is defined as follows,

$$\omega \wedge \eta = \frac{(k+m)!}{k!m!} \mathrm{Alt}(\omega \otimes \eta), \tag{2.74}$$

where the alternation Alt of a multilinear map is given by

$$\mathrm{Alt}(\omega)(x_1, ..., x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \omega(x_{\sigma(1)}, ..., x_{\sigma(k)}). \tag{2.75}$$

Besides it is clear that any element of the exterior algebra can be written as a sum of k-multivectors. Hence, we obtain:

$$\Lambda(V) = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus ....$$

Thus, the dimension of the space $\Lambda(V)$ is a sum of binomial coefficients, which is equal to 2^n [5, 8, 13, 15, 22].

2.4.11 Clifford algebra of the vector space $\mathbb{R}^{p,q}$ of signature (p, q)

Clifford algebra constitutes an essential tool in the study of quadratic forms and have important applications in a variety of fields including geometry and theoretical physics. Quadratic forms are intimately connected with the theory of rotations in $\mathbb{R}^{p,q}$ generated by a the group Spin(p,q). The group Spin(p,q), called a spinor group, is defined as a certain subgroup of units of an algebra, $Cl_{p,q}$, the

Clifford algebra associated with $\mathbb{R}^{p,q}$. Since the spinor groups are certain well-defined subgroups of units of Clifford algebras, it is necessary to investigate Clifford algebras to get a firm understanding of spinor groups. This section provides the main facts about Clifford algebra $Cl(p,q)$ associated with a non-degenerate symmetric bilinear form of signature (p,q) and the corresponding group Spin(p,q). It also includes a study of the structure of Clifford algebras, which culminates in the 8-periodicity theorem of Elie Cartan and Raoul Bott [23].

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension n, $p+q=n$ together with a bilinear non-degenerate quadratic form $\phi : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \longrightarrow \mathbb{R}$ of signature (p,q), and associated quadratic form Φ, $v^2 = \Phi(v) = \phi(v,v)$, $v \in \mathbb{R}^{p,q}$. A Clifford algebra associated with $\mathbb{R}^{p,q}$ and Φ, denoted by $Cl(\mathbb{R}^{p,q},\Phi) = Cl_{p,q}(\mathbb{R})$, is a real associative algebra satisfying the following conditions [23, 24]:

1. $Cl_{p,q}(\mathbb{R})$ is an associative real algebra with unit element e_0,

2. There exists a map $i_\Phi : \mathbb{R}^{p,q} \longrightarrow Cl_{p,q}(\mathbb{R})$, obeying the fundamental identity:

$$i_\Phi(v)^2 = \Phi(v)\cdot 1, \qquad \forall v \in \mathbb{R}^{p,q}. \tag{2.76}$$

3. For any real algebra A, and any linear map $f : \mathbb{R}^{p,q} \longrightarrow A$, with[3]

$$(f(v))^2 = \Phi(v)\cdot 1, \qquad \forall v \in \mathbb{R}^{p,q}, \tag{2.77}$$

there is a unique algebra homomorphism, $\bar{f} : Cl_{p,q}(\mathbb{R}) \longrightarrow A$, such that

$$f = \bar{f}\cdot i_\Phi \tag{2.78}$$

and the diagram below commutes

$$\begin{array}{ccc} \mathbb{R}^{p,q} & \xrightarrow{i_\Phi} & Cl_{p,q}(\mathbb{R}) \\ f\searrow & & \swarrow \bar{f} \\ & A & . \end{array} \tag{2.79}$$

Since

$$\Phi(u+v) - \Phi(u) - \Phi(v) \quad = \quad 2\phi(u,v) \tag{2.80}$$

and

$$(i(u+v))^2 \quad = \quad (i(u))^2 + (i(v))^2 + i(u)i(v) + i(v)i(u), \tag{2.81}$$

$$(i(v))^2 \quad = \quad \Phi(v)\cdot 1, \tag{2.82}$$

[3] We warn the readers that for example [8] adopt the opposite to our sign convention in defining Clifford algebras, i.e., there is used the condition

$$(f(v))^2 = -\Phi(v)\cdot 1, \quad \forall v \in \mathbb{R}^{p,q}.$$

The most confusing consequence of this is that then $Cl_{p,q}(\mathbb{R})$ in [8] is our $Cl_{q,p}(\mathbb{R})$.

we get

$$i(v)i(w) + i(w)i(v) = 2\phi(v, w), \qquad \forall v, w \in \mathbb{R}^{p,q}. \tag{2.83}$$

As a consequence, if $(u_1, ..., u_n)$ is an orthogonal basis of $\mathbb{R}^{p,q}$ (which means that $u_j u_k = \phi(u_j, u_k) = 0$ for all $j \neq k$), we have [23, 8]:

$$i(u_j)i(u_k) + i(u_k)i(u_j) = 0, \quad \forall j \neq k. \tag{2.84}$$

Moreover, one can show that for each vector space equipped with a quadratic form there exists up to isomorphism an unique ($\mathbb{Z}_2$-graded) Clifford algebra. We describe an approach how it can be constructed.

Let us start with the most general algebra that contains $\mathbb{R}^{p,q}$, namely the tensor algebra $T(\mathbb{R}^{p,q})$, and then enforce the fundamental identity by taking a suitable quotient. Denote by $I(\Phi)$ the two-sided ideal of the full tensor algebra $T(\mathbb{R}^{p,q})$ generated by elements of the form $x \otimes x - \Phi(x) \cdot 1$. Then a Clifford algebra is given by

$$Cl_{p,q}(\mathbb{R}) = T(\mathbb{R}^{p,q})/I(\Phi). \tag{2.85}$$

It is straightforward to show that $Cl_{p,q}(\mathbb{R})$ contains $\mathbb{R}^{p,q}$ and satisfies the above commutation relations (2.84), so that $Cl_{p,q}(\mathbb{R})$ is unique up to an isomorphism. Thus, one speaks of "the" Clifford algebra of $\mathbb{R}^{p,q}$ [8, 5, 23].

2.4.12 Generators of Clifford algebra

For a given basis $e_1, ..., e_n$ of $\mathbb{R}^{p,q}$, the $2^n - 1$ products

$$e_{i_1} e_{i_2} ... e_{i_k}, \quad 1 \leq i_1 < i_2 ... < i_k \leq n, \quad 1 \leq k \leq n,$$

and 1 form the Clifford algebra $Cl_{p,q}(\mathbb{R})$. The empty product ($k = 0$) is defined as the multiplicative identity element. One can show that the dimension of $Cl_{p,q}(\mathbb{R})$ is 2^n unless $e_1 ... e_n$ is a scalar multiple of the identity, otherwise it is $2^n - 1$ [25, 23]. This exceptional case occurs for $p - q - 1$ divisible by 4[26]. Then if $(e_1, ..., e_n)$ is an orthogonal basis of $\mathbb{R}^{p,q}$, $Cl_{p,q}(\mathbb{R})$ is the algebra presented by the generators $(e_1, ..., e_n)$ satisfying the relations

$$e_j^2 = \Phi(e_j) \cdot 1, \quad 1 \leq j \leq n, \quad \text{and} \tag{2.86}$$

$$e_j e_k = -e_k e_j, \quad 1 \leq j, k \leq n, \quad j \neq k. \tag{2.87}$$

Examples:
Since in $Cl_{0,0}(\mathbb{R})$ there is no non-zero vectors it is easy to see that $Cl_{0,0}(\mathbb{R}) \simeq \mathbb{R}$. Let $q = n = 1$, $e_1 = 1$, and assume that $\Phi(x^1 e_1) = -x_1^2$. Then, $Cl_{0,1}(\mathbb{R})$ is spanned by the basis $\{1, e_1\}$. We have $e_1^2 = -1$. Under the bijection $e_1 \longmapsto i$, the Clifford algebra $Cl_{0,1}(\mathbb{R})$, also denoted by $Cl_{0,1}(\mathbb{R})$, is isomorphic to the algebra of complex numbers $\mathbb{C}$.

Now, let $n = 2$, $\{e_1, e_2\}$ be the canonical basis, and assume that $\Phi(x^1e_1 + x^2e_2) = -(x_1^2 + x_2^2)$. Then, $Cl_{0,2}(\mathbb{R})$ is spanned by the basis $\{1, e_1, e_2, e_1e_2\}$. Furthermore, we have $e_2e_1 = -e_1e_2$, $e_1^2 = -1, e_2^2 = -1$, $(e_1e_2)^2 = -1$.[4] Under the bijection

$$e_1 \longmapsto i, \quad e_2 \longmapsto j, \quad e_1e_2 \longmapsto k,$$

we can easily prove that quaternion identities hold, and thus the Clifford algebra $Cl_{0,2}(\mathbb{R})$ is isomorphic to the algebra of quaternions $\mathbb{H}$.

For $\Phi \equiv 0$, it follows that $e_i^2 = 0$, $\forall i = 1, ..., n$, and the Clifford algebra $Cl_n(\mathbb{R})$ is equivalent to the exterior algebra $\Lambda^n(\mathbb{R})$ [5, 23].

2.4.13 The 8-fold periodicity of Clifford algebra

Surprisingly, but the three examples in the last paragraph are enough to provide us with a complete classification of $Cl_{p,q}(\mathbb{R})$. It turns out that the real algebras $Cl_{p,q}(\mathbb{R})$ can always be constructed as tensor products of the basic algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Elie Cartan was the first who studied the real algebras $Cl_{p,q}(\mathbb{R})$ as matrix algebras and discovered the 8-periodicity in 1908. After Cartan, the first proof was independently given by Raoul Bott in the 1960s, but after that many different alternative ways of showing this property have been given [5, 23].

To begin we recall that every non-degenerate quadratic over $\mathbb{R}^n$ has the form $\Phi(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ after a suitable choice of basis. If $p + q = n$, then for simplicity here we write $Cl_{p,q}$ for the Clifford algebra $Cl_{p,q}(\mathbb{R}, \Phi)$ associated to this quadratic form Φ. Of particular importance are the Clifford algebras $Cl_{n,0}$ and $Cl_{0,n}$. We shall now give a compact introduction to these algebras [27].

We use the notation $\mathbb{R}(n)$ (resp. $\mathbb{C}(n)$) for the algebra Mat($\mathbb{R}$) of all $n \times n$ real matrices (resp. the algebra Mat($\mathbb{C}$) of all $n \times n$ complex matrices). As we have mentioned in the last section, it is not hard to show that

$$\begin{aligned} Cl_{0,1} &= \mathbb{C}, \quad Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}, \quad Cl_{0,2} = \mathbb{H}, && (2.88) \\ Cl_{2,0} &= \mathbb{R}(2) \quad \text{and} \quad Cl_{1,1} = \mathbb{R}(2). && (2.89) \end{aligned}$$

Then one can prove the following isomorphisms (the reader can find the proof in Gallier [23]):

$$Cl_{0,n+2} \cong Cl_{n,0} \otimes Cl_{0,2}, \qquad Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0}, \tag{2.90}$$

$$Cl_{p+1,q+1} \cong Cl_{p,q} \otimes Cl_{1,1}, \tag{2.91}$$

[4] Here we extend the quadratic form on $\mathbb{R}^n$ to a quadratic form on $Cl_{p,q}(\mathbb{R})$ by requiring that distinct elements of the form $e_{i_1}...e_{i_k}$ are orthogonal to one another whenever the e_{i_j}'s are orthogonal to each other. Since then generators e_{i_j} anti-commute, we can transform the basis vectors in the form $\Phi(e_{i_1}...e_{i_k})$ to standard order, such that

$$\Phi(e_{i_1}...e_{i_k}) = e_{i_1}...e_{i_k}e_{i_1}...e_{i_k} = (-1)^r e_{i_1}^2...e_{i_k}^2 = (-1)^r \Phi(e_{i_1})...\Phi(e_{i_k}),$$

where $(-1)^r$ is an overall sign corresponding to the number of flips needed to correctly order the basis vectors (i.e., the signature of the ordering permutation).

$p-q \diagdown p+q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		${}^2\mathbb{R}$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					${}^2\mathbb{H}$		$\mathbb{C}(2)$		${}^2\mathbb{R}(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		${}^2\mathbb{H}(2)$		$\mathbb{C}(4)$		${}^2\mathbb{R}(4)$		$\mathbb{C}(4)$		${}^2\mathbb{H}(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	${}^2\mathbb{R}(8)$		$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$		${}^2\mathbb{R}(8)$		$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$

Table 2.3: Clifford Algebras $Cl_{p,q}, p+q < 8$.

for all $n, p, q \geq 0$. Using these isomorphisms and the fact that the following propositions (see [23, 28]) hold,

$$\begin{aligned}
\mathbb{R}(m) \otimes \mathbb{R}(n) &\cong \mathbb{R}(mn) \quad \forall m, n \geq 0, && (2.92)\\
\mathbb{R}(n) \otimes_{\mathbb{R}} K &\cong K(n), \quad \text{where} \quad K = \mathbb{C} \quad \text{or} \quad \mathbb{H},\ n \geq 0, && (2.93)\\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \oplus \mathbb{C}, && (2.94)\\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &\cong \mathbb{C}(2), && (2.95)\\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\cong \mathbb{R}(4), && (2.96)
\end{aligned}$$

one can prove the periodicity theorem. This theorem, called after its discoverers also Cartan/Bott theorem, gives a complete classification of real Clifford algebras. It says for $n \geq 0$ that

$$\begin{aligned}
Cl_{0,n+8} &\cong Cl_{0,n} \otimes Cl_{0,8}, && (2.97)\\
Cl_{n+8,0} &\cong Cl_{n,0} \otimes Cl_{8,0}. && (2.98)\\
Cl_{0,8} &= Cl_{8,0} = \mathbb{R}(16). && (2.99)
\end{aligned}$$

(for the proof see for instance [23, 27]). More general relations for $Cl_{p,q}$ are:

$$\begin{aligned}
Cl_{p,q} &\cong Cl_{p-4,q+4}, \quad \text{where} \quad p \geq 4, && (2.100)\\
Cl_{p+8,q} &\cong \mathrm{Mat}(16, Cl_{p,q}), && (2.101)
\end{aligned}$$

where $\mathrm{Mat}(16, Cl(p,q))$ is the algebra of 16-dimensional matrices with entries in $Cl_{p,q}$ [8, 28, 29]. Based on the knowledge of these relations one can construct Table 2.3 of isomorphisms for every value of (p,q) (this table can be found in [5, 29]).

Using Table 2.3 we can in particular write down the classification of the real Clifford algebras $Cl_{n,0}$ and $Cl_{0,n}$. The Cartan/Bott theorem also tells us how to obtain the result for $n > 8$. The main point is that any real Clifford algebra $Cl_{r,s}$ is isomorphic to one of the matrix algebras $K(2^n) \cong \mathrm{End}(K^{2^n})$ or $K(2^n) \oplus K(2^n)$, where K is either the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$ [28].

Moreover, it turns out that from the classification for the Clifford algebras $Cl_{n,0}$ and $C_{0,n}$ one can actually obtain the classification for Clifford algebras over $\mathbb{C}$. It is much simpler to provide the classification for Clifford algebras over $\mathbb{C}$ due to the fact that every non-degenerate quadratic form over $\mathbb{C}^n$ can be written as $\Phi^{\mathbb{C}}(x) = x_1^2 + ... + x_n^2$. For this reason to each dimension n there corresponds only one Clifford algebra $Cl_n(\mathbb{C}, \Phi^{\mathbb{C}})$ over $\mathbb{C}^n$ in the Clifford algebra. We denote $Cl_n(\mathbb{C}, \Phi^{\mathbb{C}})$ simply by $Cl_n^{\mathbb{C}}$.

As to indefinite Clifford algebras $Cl_{p,q}$ over $\mathbb{R}^{p,q}$, we can complexify $Cl_{p,q}$ by changing the real quadratic form by the form taken over complex numbers. We then obtain $Cl_{p,q} \otimes \mathbb{C} = Cl_{p+q}^{\mathbb{C}}$. Hence, using results on the classification of Clifford algebras over $\mathbb{R}$, we arrive at the classification for Clifford algebras over $\mathbb{C}$. Thus, due to the periodicity theorem, we get for $n \geq 0$:

$$Cl_{n+2}^{\mathbb{C}} \cong Cl_n^{\mathbb{C}} \otimes Cl_2^{\mathbb{C}},\ Cl_1^{\mathbb{C}} \cong \mathbb{C} \otimes \mathbb{C},\ Cl_2^{\mathbb{C}} \cong \mathbb{C}(2). \tag{2.102}$$

The proofs can be found, e.g., in [23, 27].

2.4.14 Matrix representations of Clifford algebra over $\mathbb{C}$ and $\mathbb{R}$

In order to get finished with Clifford algebras, we will need to use representations of the Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$. We start recalling the basics concerning representation theory.
Let V be a vector space over K' ($K' = \mathbb{R}$ or $\mathbb{C}$) and let Φ be a quadratic form on V. Let $K \supset K'$ be a field containing K'. Then a K-representation of $Cl(V, \Phi)$ is a K'-algebra homomorphism

$$\rho : Cl(V, \Phi) \longrightarrow \mathrm{Hom}_K(W)$$

into the algebra of linear transformations of a finite dimensional vector space W over K.

We call the vector space of representations W a $Cl(V, \Phi)$-module[5]. We can also write

$$\phi \cdot w := \rho(\phi)(w), \tag{2.103}$$

where $\phi \in Cl(V, \Phi), w \in W$. The product $\phi \cdot w$ is referred to as Clifford multiplication [27]. We call the map ρ a representation of $Cl(V, \Phi)$ in W. Furthermore, for many purposes it is important to know how the $Cl(V, \Phi)$-module W decomposes, that is why we introduce the following definition. Namely, using the same notation as above, we say that the representation ρ is reducible if the vector space W can be written as a non-trivial direct sum $W = W_1 \oplus W_2$ such that $\phi \cdot W_i \subset W_i$, for $i = 1, 2$ and $\forall \phi \in Cl(V, \Phi)$. In this case we write the representation ρ as $\rho = \rho_1 \oplus \rho_2$, where ρ_i is a restriction of ρ to W_i. A representation is called irreducible if it

[5] According to [5] the abstract theory of Clifford modules was postulated by M.F.Atiyah, R.Bott and A.Shapiro.

is not reducible. One can show that any representation ρ of a Clifford algebra $Cl(V, \Phi)$ can be decomposed into a direct sum $\rho = \rho_1 \oplus ... \oplus \rho_r$ of irreducible representations (the proof can be found in [28, 27, 5]).

We now concentrate on representations of the Clifford algebra $Cl_{p,q}$ on $W = \mathbb{R}^n$ determining an $(p+q)$-dimensional subspace in $\mathrm{End}(\mathbb{R}^n) \cong \mathbb{R}(n)$. Because of the classification in the previous sections, we can find all irreducible representations of Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$.

Initially we will devote ourself to the construction of matrix representations of the Dirac matrices in $Cl_{p,q}(\mathbb{C})$. We begin with the Clifford algebras of type $\mathbb{C}_{p,0}$ where p is even.

Similar to the construction of the Clifford algebra of quaternions $Cl_{0,2}(\mathbb{C}) \cong \mathbb{H}$ we can use the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.104}$$

to find a matrix representation for $Cl_{p,0}(\mathbb{C})$. Namely for $Cl_{2,0}(\mathbb{C})$, we can use $\gamma_1 = \sigma_1$ and $\gamma_2 = \sigma_2$. For $\mathbb{C}_{4,0}$ we set $\gamma_1 = \sigma_1 \otimes \sigma_1$, $\gamma_2 = \sigma_1 \otimes \sigma_2$, $\gamma_3 = \sigma_1 \otimes \sigma_3$ and $\gamma_4 = \sigma_2 \otimes \mathrm{Id}_2$. Here we use the tensor product, also named Kronecker product, of matrices. Via induction this construction can be continued for all $Cl_{p,0}(\mathbb{C})$. If a matrix representation of size $2m \times 2m$ for $\mathbb{C}_{2m,0}$ is provided, we write for the representations of the basis elements simplified $\gamma_k(2m)$, where $k = 1, ..., 2m$. Then a matrix representation for the Dirac matrices of $Cl_{2m+2,0}(\mathbb{C})$ can be expressed as:

$$\gamma_k(2m+2) = \sigma_1 \otimes \gamma_k(2m), \quad \text{for} \quad k = 1, ..., 2m, \tag{2.105}$$

$$\gamma_{2m+1}(2m+2) = i^m \sigma_1 \otimes J(2m), \qquad \gamma_{2m+2}(2m+2) = \sigma_2 \otimes \mathrm{Id}_{2m}, \tag{2.106}$$

where $J(2m) = \gamma_1(2m)...\gamma_{2m}(2m)$ [25].Note that the factor i^m, which appears in the second equation, guarantees that $\gamma_{2m+1}^2(2m+2) = +\mathrm{Id}_{2m+2}$.

One can easily convince himself that the system of matrices $\{\gamma_k(2m+2)\}$ is orthonormal and obeys $\gamma_k^2(2m+2) = +\mathrm{Id}_{2m+2}$ $\forall k = 1, ..., 2m+2$. That is why it generates the Clifford algebra $Cl_{2m+2,0}(\mathbb{C})$.

After all that results it turns out that a construction of the Dirac matrices for Clifford algebras of the type $Cl_{p,q}(\mathbb{C})$, where $p + q = 2m$, becomes much easier. All what one should do is to take the Dirac matrices constructed for $Cl_{2m,0}(\mathbb{C})$ and simply leave the first p matrices unmodified multiplying the rest q of them by i. Obviously, for $p + q = 2m$ the elements of $Cl_{p,q}(\mathbb{C})$ form a vector space of dimension 2^{2m}. Simultaneously, over $\mathbb{C}$ the set of complex $2m \times 2m$-matrices also form a vector space of dimension 2^{2m}. Hence, we see that for $p + q = 2m$, the Clifford algebra $Cl_{p,q}(\mathbb{C})$ is isomorphic to $\mathbb{C}(2m)$, the algebra of complex $2m \times 2m$-matrices. Consequently, all complex universal Clifford algebras for non-degenerate pseudo-Euclidean spaces of a given dimension are equivalent.

In the case if the dimension of the pseudo-Euclidean space is odd, it is not much harder to show that $Cl_{p,q}(\mathbb{C})$ is isomorphic to ${}^2\mathbb{C}(2m) = \mathbb{C}(2m) \oplus \mathbb{C}(2m)$. The corresponding Dirac matrices can be constructed iteratively by

$$\gamma_k(2m+1) = \sigma_3 \otimes \gamma_k(2m), \quad \text{for} \quad k = 1, ..., 2m, \tag{2.107}$$
$$\gamma_{2m+1}(2m+1) = (-i)^m \sigma_3 \otimes J(2m), \tag{2.108}$$

(see Problem 11.1 in [25]).

The situation of real Clifford algebras in general is more involved. It is not trivial any more to find an appropriate matrix representation for it. As we have already seen in Table 2.3, all real Clifford algebras are isomorphic to a matrix algebra of the form $K(2^n)$ or $K(2^n) \oplus K(2^n)$, where K either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The classification of the real Clifford algebras is a little more intricate. Nevertheless, knowing a representation of the first three real algebras, more precisely, of $Cl_{1,1}(\mathbb{R})$, $Cl_{2,0}(\mathbb{R})$ and $Cl_{0,2}(\mathbb{R})$, enable oneself to construct all real representations of $Cl_{p,q}(\mathbb{R})$ for $q \neq 0$. Such a construction one can find in the reference [28]. In fact, it leads directly to isomorphisms

$$Cl_{1,1}(\mathbb{R}) \otimes Cl_{p,q}(\mathbb{R}) \cong Cl_{p+1,q+1}(\mathbb{R}), \tag{2.109}$$
$$Cl_{0,2}(\mathbb{R}) \otimes Cl_{p,q}(\mathbb{R}) \cong Cl_{q,p+2}(\mathbb{R}), \tag{2.110}$$
$$Cl_{2,0}(\mathbb{R}) \otimes Cl_{p,q}(\mathbb{R}) \cong Cl_{q+2,p}(\mathbb{R}), \tag{2.111}$$

which allow to get representations of a large amount of Clifford algebras. An explicit construction of the rest of real representations can be proceeded by using the classification Table 2.3. Another strategy of the construction of a real representation of real Clifford algebras is also given in the paper of Okubo [30]. Some further information about real representations of Clifford algebras the reader can also find in [28, 27].

2.4.15 Spinors (fermions) as representation spaces of Clifford algebras

A spinor module S for the Clifford algebra $Cl_{2k}(\mathbb{C})$ is given by a choice of a $2k$-dimensional complex vector space S, together with an identification $Cl_{2k}(\mathbb{C}) = \mathrm{End}(S)$ of the Clifford algebra with the algebra of linear endomorphisms of S. So we can understand a spinor space as a complex vector space S, together with an instruction of how the $2k$ generators e_i of the Clifford algebra act as linear transformations on S.

In order to construct such a spinor module S, together with appropriate operators on it, we can make use of exterior algebra techniques. Considering the real exterior algebra $\Lambda^*(\mathbb{R}^n)$, we assign to a vector $v \in \Lambda^*(\mathbb{R}^n)$ an operator on $\Lambda^*(\mathbb{R}^n)$ given by exterior multiplication by v:

$$v\wedge : a \longrightarrow v \wedge a.$$

A chosen inner product on $\mathbb{R}^n$ always induces inner product on $\Lambda^*(\mathbb{R}^n)$, the one such that the wedge products of orthonormal basis vectors are orthonormal. Hence, with respect to this inner product there exists an adjoint operator to the operator $v\wedge$, which we denote by $\iota(v)$. So that $< v \wedge a, b >=< a, \iota(v)b >$.
We use here a physical notation for these operation:

$$\begin{aligned} a^\dagger(v) &= v\wedge, \quad a(v) = \iota(v) \quad \text{and} && (2.112)\\ a^\dagger(e_i) &= a_i^\dagger, \quad a(e_i) = a_i. && (2.113)\end{aligned}$$

The algebra of these operation is known as the algebra of Canonical Anticommutation Relations, since the $2n$ operators $a_i^\dagger$ and a_i satisfy

$$\{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0 \quad \text{and} \quad \{a_i, a_j^\dagger\} = \delta_{ij}. \tag{2.114}$$

Here we see the connection with the fermions: the $a_i^\dagger$ and a_i turn out to be the creation and annihilation operators.

By construction this algebra can be represented as operators on $\Lambda^*(\mathbb{R}^n)$. Further, we can identify this algebra with the Clifford algebra $Cl_n(\mathbb{R})$ as follows

$$Cl_n(\mathbb{C}) \ni v \longrightarrow a^\dagger(v) - a(v),$$

where one can easily see that

$$v^2 = (a^\dagger(v) - a(v))^2 = (a^\dagger(v))^2 + (a(v))^2 - \{a^\dagger(v), a(v)\} = -\|v\|^2 1. \tag{2.115}$$

For even dimension $n = 2k$ we can complexify $\Lambda^*(\mathbb{R}^n)$ to get a complex representation of $Cl_{2k}(\mathbb{C})$ on this space. However, this representation is of dimension 2^{2k} and not the 2^k-dimensional irreducible representation S we are searching for. For our aim we choose a complex structure J on $V = \mathbb{R}^{2k}$. Then $V \otimes \mathbb{C}$ can be decomposed as $V \otimes \mathbb{C} = W_J \oplus \overline{W}_J$, where W_J is the $+i$ eigenspace of J, $\overline{W}_J$ is the $-i$ eigenspace. In particular, we can pick an orthogonal complex structure J, in the sense that $< Jv, Jw >=< v, w >$. Moreover, we observe that the quadratic form coming from the inner product vanishes on W_J. Thus we conclude that the Clifford algebra $Cl(W_J)$ can be identified with the exterior algebra $\Lambda^*(W_J)$. We remark that it has the right dimension 2^k.

Choosing the standard complex structure on $V = \mathbb{R}^{2k}$, by setting $w_j = e_{2j-1} + ie_{2j}$, $j = 1, ..., n$, we can identify

$$Cl_{2k}(\mathbb{C}) \quad = \quad \mathrm{End}(\Lambda^*(\mathbb{C}^k)). \tag{2.116}$$

In terms of the creation and annihilation operators $a_i^\dagger, a_j$ the generator of the Clifford algebra are given as follows

$$e_{2j-1} \quad = \quad a_j^\dagger - a_j, \quad e_{2j} = -i(a_j^\dagger + a_j),\ j = 1, ..., k. \tag{2.117}$$

One can prove that the Clifford algebra relations hold: $\{e_i, e_j\} = -2\delta_{ij}$.

In addition to this, the set of all spinors can be decomposed in two subsets $S_{\pm}$. It can be seen by considering the eigenvectors in $\Lambda^*(\mathbb{C}^k)$ of the operator $\frac{1}{2}e_{2j-1}e_{2j} = \frac{i}{2}[a_j, a_j^\dagger]$, which are $\pm\frac{1}{2}$. The subspaces of $\Lambda^*(\mathbb{C}^k)$ consisting of vectors which are simultaneously the eigenvectors of all $e_{2j-1}e_{2j}$ are called the weights of the representation S. Each of the weights corresponds to a set of k values $\frac{1}{2}$ or $-\frac{1}{2}$: $(\pm\frac{1}{2}, ..., \pm\frac{1}{2})$. The decomposition into half-spin representations $S = S_+ \oplus S_-$ then corresponds to the decomposition into weights with even or odd number of minus signs [5, 31, 32].

2.4.16 Group $\mathrm{Spin}(p, q)$

Let K be a field $\mathbb{R}$ or $\mathbb{C}$ and V a finite-dimensional vector space over K. Also let Φ be a quadratic form on V of signature (p, q), $< x, y >= \frac{1}{4}(\Phi(x+y) - \Phi(y-x))$ the associated bilinear form. We assume that $< \cdot, \cdot >$ is non-degenerate, i.e., $< x, y_0 >= 0$, for all $x \in V$, implies $y_0 = 0$. We denote by $Cl(V)$ the Clifford algebra associated to V and Φ. Since the Clifford algebra $Cl(V) = T(V)/A$ can be considered as a quotient space of $T(V)$, the grading of the tensor algebra $T(V)$ induce a grading on $Cl(V)$ in a natural way:

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V) \quad \Longrightarrow \quad Cl(V) = \bigoplus_{k=0}^{\infty} Cl^k(V), \tag{2.118}$$

where $Cl^k(V)$ denotes the subset of the elements γ of the order k.

If $\gamma \in Cl(V)$ and the degree $\deg \gamma = k$, then we define the grading map $\alpha : Cl(V) \longrightarrow Cl(V)$ as $\alpha(x) = (-1)^k x$. The set Γ of all u in Cl, such that u has an inverse u^{-1} and

$$uV\alpha(u)^{-1} \in V, \quad \text{i.e.,} \quad ux\alpha(u)^{-1} \in V \text{ for all } x \in V,$$

is a group under multiplication, which is called the Clifford group of Φ. If u belongs to the Clifford group Γ of Φ, then $s_u : x \longrightarrow ux\alpha(u)^{-1}$ is an orthogonal transformation, i.e., it preserves Φ, because

$$\begin{aligned} \Phi(ux\alpha(u)^{-1}) \cdot 1 &= \pm(ux\alpha(u)^{-1})^2 = \pm ux^2u^{-1} = u(\Phi(x) \cdot 1)u^{-1} \\ &= \Phi(x) \cdot 1. \end{aligned} \tag{2.119}$$

Hence, the correspondence $\chi : u \longmapsto s_u$ is a linear representation of Γ, which is called the vector representation of Γ [33].

Extending the transpose map ^{t} on $T(V)$, which assigns to each element $x_1 \otimes ... \otimes x_k \in T(V)$ the element $x_k \otimes ... \otimes x_1$, to the transpose on $Cl_{p,q}(V)$ we introduce the square-norm on $Cl_{p,q}(V)$ via $||x||^2 = x\bar{x}$, where we set $\bar{x} = \alpha(x^t)$. In these fashion we define the pin group as:

$$\mathrm{Pin}_{p,q}(V) := \{x \in \Gamma_{p,q}(V) : ||x||^2 = \pm 1\} \tag{2.120}$$

and the spin group as a subgroup of it,

$$\mathrm{Spin}_{p,q}(V) := \mathrm{Pin}_{p,q}(V) \cap \Gamma^{+}_{p,q}(V), \tag{2.121}$$

with $\Gamma^{+}_{p,q}(V) = \Gamma \cap Cl^{+}_{p,q}$, where we denote $Cl^{+}_{p,q}(V) = \{u \in Cl_{p,q}(V) : \text{degree of } u \text{ is even}\}$ [23, 8, 29, 26, 5].

2.4.17 Supersymmetry and superalgebra

For a long time it was a goal of many physicists to obtain a unified description of all basic interactions of nature, i.e., strong, electroweak, and gravitational interactions. Several attempts have shown that a necessary ingredient in any unifying approach is supersymmetry (SUSY) (for a physical introduction to supersymmetry see for example [34, 35, 36]). The algebra involved in SUSY is a $\mathbb{Z}_2$-graded Lie algebra which is closed under a combination of commutation and anti-commutation relations. Under the $\mathbb{Z}_2$-grading the algebra elements decompose into bosons, the even elements, and the fermions, the odd elements of the algebra. Such an algebra is called a Lie superalgebra. Supersymmetry transforms bosons into fermions and vice versa. The generators Q_α of these transformations should therefore have fermionic character, i.e., they are generated by objects that transform in the spinor representations [5].

In its general form a superalgebra is an $\mathbb{Z}_2$-graded algebra containing Poincaré algebra

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \quad [P_\rho, M_{\mu\nu}] = \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu, & (2.122) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}) & (2.123) \end{aligned}$$

as well as a internal symmetry generated by the algebra elements T_r:

$$[T_r, T_s] = f_{rst} T_t. \tag{2.124}$$

In particular the supersymmetry must be a direct sum of both algebras (this fact follows from the Coleman-Mandula theorem [37]). This is achieved by the requirement that

$$[T_r, P_\mu] = 0 = [T_t, M_{\mu\nu}]. \tag{2.125}$$

In addition, the supersymmetry includes a fermionic structure which can be given by the set of anticommuting generators Q^i_α $(i = 1, ..., N)$ of the Lorentz algebra. One can choose Q^i_α belonging to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation, i.e.,

$$[Q^i_\alpha, M_{\mu\nu}] = \frac{1}{2} (\sigma_{\mu\nu})_\alpha^\beta Q^i_\beta. \tag{2.126}$$

The $\mathbb{Z}_2$-graduation means that the commutator of two even (bosonic) generators (P_μ, $M_{\mu\nu}$, T_r) has bosonic nature, the commutator of two odd (fermionic) generators Q^i_α has bosonic character and the commutator of an odd and an even generators is again fermionic. This implies that

$$[Q^i_\alpha, T_r] = (l_r)^i_j Q^j_\alpha + (t_r)^i_j (i\gamma_5)^\beta_\alpha Q^j_\beta, \tag{2.127}$$

where the summands $(l_r)^i_j + i\gamma_5 (t_r)^i_j$ define a representation of the Lie algebra of the inner symmetry, and

$$[Q^i_\alpha, P_\mu] = 0. \tag{2.128}$$

These equations are consequences of the generalized Jacobi identities. The last remaining relation is the relation between two fermionic generators which in a SUSY algebra must have the form

$$[Q^i_\alpha, Q^j_\beta] = 2(\gamma_\mu C)_{\alpha\beta}\delta^{ij}P^\mu + C_{\alpha\beta}U^{ij} + (\gamma_5 C)_{\alpha\beta}V^{ij}, \tag{2.129}$$

where U^{ij} and V^{ij} are even antisymmetric generators which commute with all generators included with itself. We call them center charges. The matrix $C_{\alpha\beta}$ is the charge conjugation matrix (see for example [37]).

The supersymmetry algebra, in its simplest form ($N = 1$) involves the generators of the Poincaré algebra together with a self-conjugate spin $\frac{1}{2}$ generator Q_α, $\alpha = 1, 2$. Recall that the generator of the Poincaré algebra in tensor representation are given by the space-time rotations $M^{\mu\nu}$ and translations P^μ. The fermionic generators Q_α and Q^*_α can be expressed as the Weyl spinors in the $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ representations of the spinor group $SL(2, \mathbb{C})$, respectively. The algebra that defines supersymmetry then is given by [5, 38, 35]

$$[Q_\alpha, P_\mu] = [Q^*_\alpha, P_\mu] = 0, \tag{2.130}$$

$$\{Q_\alpha, Q^*_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu, \tag{2.131}$$

$$\{Q_\alpha, Q_\beta\} = \{Q^*_\alpha, Q^*_\beta\} = 0, \tag{2.132}$$

where the indices take the values $\alpha, \beta = 1, 2$ and σ^μ denotes the Pauli matrices, and by the commutation relation of the Poincaré group (2.122-2.123). From the analysis of this section, we see that the SUSY algebra with the supercharges Q_α and Q^*_α is equivalent to a Clifford algebra $Cl_2(\mathbb{C})$. (For supersymmetry in 4-dimensional Euclidean see [39].) This is the simplest supersymmetric extension of the Poincaré group known as Wess-Zumino model.

Another models of supersymmetric algebras such as superconformal algebra and (anti-)deSitter algebras one can find in reference [37].

2.5 Theory of general Lie algebras

This section we devote to an introduction of Lie algebras, probably the most important algebras which one can find in a large spectrum of possible applications.

2.5.1 Lie algebras

Lie algebras get their name after Sophus Lie. They are algebras which have the most practical and theoretical use in studying geometrical objects such as Lie groups and differential manifolds. In this section we summarize the main notions and properties concerning Lie algebras. We begin with the general definition.

A Lie algebra is a vector space A over a field K equipped with a bilinear Lie bracket product $[\cdot, \cdot]$ which is anti-commutative and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \tag{2.133}$$

for all $X, Y, Z \in A$.

For a given associative algebra A with multiplication $*$, a Lie algebra $L(A)$ can be constructed by setting a commutator in A equal to:

$$[X, Y] = X * Y - Y * X. \tag{2.134}$$

The Jacobi identity for commutators in $L(A)$ is a consequence of the associativity of the multiplication $*$. The associative algebra A is called an enveloping algebra of the Lie algebra $L(A)$. In this fashion one can define the general linear Lie algebra $\mathfrak{gl}(n, K)$ as a Lie algebra of all endomorphisms of K^n (i.e., represented as $n \times n$ matrices over K), which together with the ordinary matrix product composes an associative algebra over K. We note that every subalgebra of $\mathfrak{gl}(n, K)$ is called a linear Lie algebra over K. It can be shown that every Lie algebra can be considered as an embedding into one that arises from an associative algebra [5].

Two Lie algebras L, L' over K are isomorphic if there exists a vector space isomorphism $\phi : L \longrightarrow L'$ satisfying

$$\phi([X, Y]) = [\phi(X), \phi(Y)], \quad \forall X, Y \in L. \tag{2.135}$$

Then ϕ is called an isomorphism of Lie algebras. The notion of (Lie) subalgebra of L is defined similarly: A subspace A of L is called a Lie subalgebra if $[x, y] \in A$ whenever $x, y \in A$; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any non-zero element $x \in L$ defines a one dimensional subalgebra Kx, with trivial multiplication, because any element of L commute with itself [40].

As examples of Lie algebras we can mention any commutative group, the cross product of 3-dimensional vectors, Lie algebras of Lie groups, Lie algebras generated by the commutator of vector fields, Lie derivatives and Kac-Moody algebras as an example of infinite-dimensional Lie algebras [5].

2.5.2 Matrix Lie algebras

As we already mentioned the space of $n \times n$ matrices over K forms a Lie algebra, because the matrix product is associative. Hence, it becomes apparent that every subgroup of $\mathrm{Mat}(n, K)$ closed under the action of the matrix commutator defines a Lie subalgebra of $\mathfrak{gl}(n, K)$. Here we are interested in the case when $K = \mathbb{R}$ or $\mathbb{C}$. We summarize the most important matrix Lie algebras in the list below. We notice that these algebras are identified as tangent spaces at the identity of corresponding matrix Lie groups.

1. The general linear Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ ($\mathfrak{gl}(n, \mathbb{C})$) of all real (complex) square $n \times n$ matrices is the endomorphisms group $\mathrm{Mat}(n, \mathbb{R})$ ($\mathrm{Mat}(n, \mathbb{C})$) endowed with the ordinary matrix commutator.

2. The special linear Lie algebras $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{C})$ are the spaces of all $n \times n$ matrices with zero trace in $\mathrm{Mat}(n, \mathbb{R})$, $\mathrm{Mat}(n, \mathbb{C})$, respectively.

3. The Lie algebras $\mathfrak{so}(n, \mathbb{R})$, $\mathfrak{so}(n, \mathbb{C})$ are the algebras of skew-symmetric $n \times n$ orthogonal matrices A with
$$A^T = -A, \tag{2.136}$$
where $A \in \mathfrak{so}(n, \mathbb{R})$ or $\mathfrak{so}(n, \mathbb{C})$.

4. The pseudo-orthogonal Lie algebra $\mathfrak{o}(p, q)$, $0 \leq p, q$, is the algebra of all real matrices A satisfying
$$Ag + gA^T = 0, \tag{2.137}$$
where $g = (g_{ij})$ is a real metric of type (p, q).

5. The unitary Lie algebra $\mathfrak{u}(n)$ is the algebra of all skew-Hermitian matrices U:
$$U^\dagger := \bar{U}^T = -U. \tag{2.138}$$

6. The special unitary Lie algebra $\mathfrak{su}(n)$ is the algebra consisting of all skew-symmetric Hermitian matrices with trace zero:
$$U^\dagger := -U, \qquad \mathrm{tr} U = 0. \tag{2.139}$$

7. The pseudo-unitary Lie algebra $\mathfrak{u}(p, q)$ is the algebra of complex $n \times n$ matrices U obeying:
$$Ag + g\bar{A}^T = 0,$$
where $g = (g_{ij})$ is a pseudo-Hermitian metric of signature (p, q).

8. The special unitary Lie algebra $\mathfrak{su}(p, q)$ is the algebra of matrices in $\mathfrak{u}(p, q)$ with zero trace [6].

2.5.3 Structure constants, Bianchi identities, Killing-Cartan metric

Structure constants

Let L be a finite dimensional Lie algebra and take a basis $\{e_1, e_2, ..., e_n\}$ for (the vector space) L. By bilinearity the $[\cdot,\cdot]$-operation in L is completely determined once the values $[e_i, e_j]$ are known. Since $[e_i, e_j] \in L$, we can write them as linear combinations of the basis vectors $\{e_i\}$.

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k = c_{ij}^k e_k. \tag{2.140}$$

The coefficients c_{ij}^k in these relations are called the structure constants of L (relative to the given basis).

The antisymmetry $c_{ij}^k = -c_{ji}^k$ and the validity of the Jacobi identity of the Lie algebra L imply the Bianchi identities:

$$c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l \quad = \quad 0. \tag{2.141}$$

Under a change of basis the structure constants transform as a tensor of type $(2,1)$: if $e'_j = a_j^i e_i$, then $c'^k_{ij} = c_{rs}^l a_i^r a_j^s (a^{-1})_l^k$.

We note that systems, which form the structure constants of some Lie algebra over a field K, form an algebraic set S, defined by the above linear and quadratic equations that correspond to the skew-symmetry and the Jacobi identity. The general linear group $GL(n,K)$ acts on S by the formulae above. The set of all structure constants relative to all bases of a given Lie algebra forms an orbit (set of all transforms of one element) under this operation. Conversely, the systems of structure constants in an orbit corresponds to unique Lie algebra up to an isomorphism. Thus, there is a natural bijection between orbits (of systems of structure constants) and isomorphism classes of Lie algebras (of dimension n) [41].

Killing-Cartan metric

Consider a Lie algebra $\mathfrak{g}$ over a field K. Every element x of $\mathfrak{g}$ defines the adjoint endomorphism $\mathrm{adj}(x)$ (also written as adj_x) of $\mathfrak{g}$ based on the Lie bracket via

$$\mathrm{adj}_x(y) = [x, y]. \tag{2.142}$$

Now, suppose $\mathfrak{g}$ is of finite dimension. The trace of the composition of two such endomorphisms defines a symmetric bilinear form

$$B(x, y) = \mathrm{tr}(\mathrm{adj}(x)\mathrm{adj}(y)), \tag{2.143}$$

with values in K, the Killing form on $\mathfrak{g}$. In terms of the coordinates in some basis the Cartan form is expressed as

$$B(x, y) = \mathrm{tr}((\mathrm{adj}_x)_k^i)(\mathrm{adj}_y)_i^s) = c_{lk}^i x^l c_{si}^k y^s = B_{ls} x^l y^s, \quad \text{so} \quad B_{ls} = c_{lk}^i c_{si}^k \tag{2.144}$$

The Killing form of $\mathfrak{g}$ is invariant, in the sense that it satisfies

$$B([x,y],z) + B(y,[x,z]) = 0, \qquad \forall x,y,z \in \mathfrak{g}. \tag{2.145}$$

It is in particular invariant under any automorphism of $\mathfrak{g}$ (for the proof see [42], Lemma 3.1). The Killing form is defined as bilinear symmetric form. If additionally the Killing form is non-degenerate, it can be used as a metric. In particular, it can be shown that (Cartans second criterion) the Killing form of a Lie algebra $\mathfrak{g}$ is non-degenerate if and only if the Lie algebra $\mathfrak{g}$ is semisimple [6] [43, 42].

Remark: The map $x \longmapsto \mathrm{adj}_x$ assigns to each x in $\mathfrak{g}$ an operator in $\mathrm{End}(\mathfrak{g})$ preserving the linearity and the bracket, in the sense that it holds:

$$([\mathrm{adj}_x, \mathrm{adj}_y])(z) = \mathrm{adj}_{[x,y]}(z), \quad \forall x,y,z \in \mathfrak{g}. \tag{2.146}$$

The proof can be easily obtained using the Jacobi identity. That is why the adjoint endomorphism adj is a representation of a Lie algebra $\mathfrak{g}$ on a vector space $\mathfrak{g}$. For $\mathfrak{g} = K^n$ the adj_x are matrices; expressed in terms of a basis $\{e_1,...,e_n\}$ of $\mathfrak{g}$ we can write for $a \in \mathfrak{g}$

$$[a, e_j] = \sum_{k=1}^{n} (\mathrm{adj}_a)_{kj} e_k. \tag{2.147}$$

The set of the matrices adj_a generates a n-dimensional representation of $\mathfrak{g}$, the adjoint representation of $\mathfrak{g}$ [41].

2.5.4 Roots and weights

It turns out that the studying of Lie algebras and corresponding semisimple Lie groups as well as their classification is profoundly connected with the structures called root systems. In this section we only briefly touch the notion root and give a first outlook of the topic.

To introduce the definition of roots of a Lie algebra we need some further concepts. At first we give the notion of the Cartan subalgebra. Let $\mathfrak{g}$ be a Lie algebra. Then a Cartan subalgebra $\mathfrak{h}$ is the maximal subalgebra of $\mathfrak{g}$ which is self-normalizing, that is, if $[g,h] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$, then $g \in \mathfrak{h}$ as well. Any Cartan subalgebra $\mathfrak{h}$ is nilpotent, i.e.,

$$\mathcal{D}_k \mathfrak{h} = 0, \quad \text{for} \quad k \geq 0, \tag{2.148}$$

where $\mathcal{D}$ is defined inductively as

$$\mathcal{D}_1 \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}], \qquad \mathcal{D}_k \mathfrak{h} = [\mathfrak{h}, \mathcal{D}_{k-1}\mathfrak{h}]. \tag{2.149}$$

[6] That is if $\mathfrak{g}$ is a direct sum of simple Lie algebras $\mathfrak{g}_i$, that means $\mathfrak{g}_i$ should be non-abelian Lie algebras whose only ideals are $\{0\}$ and $\mathfrak{g}_i$ itself.

A subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is called Cartan algebra if $\mathfrak{h}$ is abelian, i.e., $[\mathfrak{h}, \mathfrak{h}] = \{0\}$ and $\mathfrak{h} = \{X \in \mathfrak{g} : [X, H] = 0, \forall H \in \mathfrak{h}\}$. adj_H is for each $H \in \mathfrak{h}$ diagonalizable. It can be shown that every semisimple Lie algebra owns a Cartan algebra [44]. The dimension of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$. It can be proved that for any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ of a Lie algebra $\mathfrak{g}$ there exists an automorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Now we can proceed to the notion of a root. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the complex numbers and let $\mathfrak{h}$ be its Cartan subalgebra. It can be shown that then there exists a finite subset of Φ consisting of $\alpha \in \mathfrak{h}^* \backslash \{0\}$ such that $\mathfrak{g}$ is a direct sum of $\mathfrak{h}$-invariant[7] vector spaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h}, \tag{2.150}$$

satisfying the property that for each $\alpha \in \Phi$ the space $\mathfrak{g}_\alpha$ contains a common eigenvector of $\mathfrak{h}$ with eigenvalue $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$, and this is the only eigenvalue of $\mathfrak{h}$ on $\mathfrak{g}_\alpha$. Moreover, if we set $g_\alpha = 0$ for $\alpha \notin \Phi \cup \{0\}$, then for any $\alpha, \beta \in \mathfrak{h}^*$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (the proof can be found in [42], Theorem 4.5). In other words $\mathfrak{g}_\alpha$ can be written as

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

The elements of Φ are called the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and the $\mathfrak{g}_\alpha$ are called the root spaces.

The root decomposition (2.150) behaves nicely with respect to the Killing form, in the sense that if $\alpha, \beta \in \Phi \cup \{0\}$ are such that $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ and for $x, y \in \mathfrak{h}$ we have

$$B(x, y) = \sum_{\alpha \in \Phi} \alpha(x)\alpha(y) \dim \mathfrak{g}_\alpha. \tag{2.151}$$

For the proof look in [42], Lemma 4.7.

The root decomposition (2.150) has especially nice properties when $\mathfrak{g}$ is a semisimple Lie algebra. So from here for the rest of the section we take $\mathfrak{g}$ semisimple. Then the Killing form is non-degenerate. This has many consequences:

1. Φ spans $\mathfrak{h}^*$;
2. $\dim \mathfrak{g}_\alpha = 1$, $\forall \alpha \in \Phi$;
3. the restriction of $B(\cdot, \cdot)$ to $\mathfrak{h}$ is non-degenerate; for each linear form α on $\mathfrak{h}$ there exists a unique element h_α, called root vector, such that $B(h, h_\alpha) = \alpha(h)$, $\forall h \in \mathfrak{h}$;

[7] That is, if $[\mathfrak{h}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$.

4. $\Phi = -\Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$, $\alpha(h_\alpha) \neq 0$;
5. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is an orthogonal direct sum;
6. for each $\alpha \in \Phi$ the subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ obeys $\mathfrak{h}_\alpha \cap \ker(\alpha) = 0 \subset \mathfrak{h}$;

(see [42, 41, 12]). Moreover, the following statements ([42], Lemma 5.4) hold:

- If $\alpha, \beta \in \Phi$, then $\beta(H_\alpha) \in \mathbb{Z}$, and $\beta - \beta(H_\alpha)\alpha \in \Phi$, where $H_\alpha = \frac{2h_\alpha}{B(h_\alpha h_\alpha)}$, $\alpha \in \Phi$.
- If $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- For all $\alpha \in \Phi$ and $t \in \mathbb{Z}$ we have $t\alpha \in \Phi$ if and only if $t = \pm 1$.

We now summarize some of the previous constructions. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\alpha \in \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$, that is, the eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. The non-degeneracy of the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ allows us to define $h_\alpha \in \mathfrak{h}$ such that $\alpha(h) = B(h_\alpha, h)$, for any $h \in \mathfrak{h}$. It can be shown ([42], Lemma 5.4) that for any $\alpha, \beta \in \Phi$ the Cartan numbers, defined as

$$n_{\beta,\alpha} = 2\frac{B(h_\alpha, h_\beta)}{B(h_\alpha h_\alpha)}, \tag{2.152}$$

are integers. Moreover, one can prove that for each $\alpha \in \Phi$ a vector $X_\alpha \in \mathfrak{g}_\alpha$ can be chosen such that for all $\alpha, \beta \in \Phi$

- $[X_\alpha, X_{-\alpha}] = h_\alpha$, $[h, X_\alpha] = \alpha(h)X_\alpha$ for $h \in \mathfrak{h}$;
- $[X_\alpha, X_\beta] = 0$, if $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \Phi$;
- $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$,

where the constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. For any such choice $N^2_{\alpha,\beta} = \frac{q(1-p)}{2}\alpha(h_\alpha)$, where $\beta + n\alpha$, $p \leq n \leq q$, $n \in \mathbb{Z}$, is the series of roots containing β ([12], Theorem 5.5).

We define $\mathfrak{h}_\mathbb{R}$ as the vector space spanned by H_α defined above. Since $n_{\alpha,\beta} \in \mathbb{Z}$ any root defines a linear map $\mathfrak{h}_\mathbb{R} \longrightarrow \mathbb{R}$ and $\Phi \subset \mathfrak{h}^*_\mathbb{R}$. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}_\mathbb{R}$ defines a metric on $\mathfrak{h}_\mathbb{R}$ ([42], Lemma 6.1). As a consequence, one can get $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$. Let us write $(\cdot,\cdot)$ for the positive definite symmetric bilinear form on $\mathfrak{h}_\mathbb{R}$ which corresponds to the Killing form:

$$(\alpha, \beta) = B(\mathfrak{h}_\alpha, \mathfrak{h}_\beta) = \alpha(\mathfrak{h}_\beta) = \beta(\mathfrak{h}_\alpha). \tag{2.153}$$

Hence, we obtain

$$2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = \beta(H_\alpha) = n_{\alpha,\beta} \in \mathbb{Z}. \tag{2.154}$$

The pair $(\mathfrak{h}^*_{\mathbb{R}}, \Phi)$ is an example of an abstract pair (V, R) consisting of a finite set R, called roots system, of non-zero vectors spanning a real vector space$V$ equipped with a positive-definite symmetric bilinear form $(\cdot,\cdot)$, such that any two distinct proportional vectors of R are negatives of each other, the reflections with respect to the elements of R preserve R and $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, for any $\alpha, \beta \in \Phi$. The dimension of V is called the rank of R. The reflections in the roots of R generate a group of transformations called Weyl group, $W = W(R)$ [42].

The importance of the concept of a root system is based on the fact that the isomorphism classes of complex semisimple Lie algebras bijectively correspond to equivalence classes of root systems[8]. A basis $S \subset R$ of a root system R is a subset such that it is also a basis of the vector field V of R and every root of R is an integral linear combination of the elements of S all of whose coefficients have the same sign. The elements of S are called simple roots, and the elements of R that can be written as linear combinations of simple roots with positive coefficients are called positive roots (denoted as R^+). The matrix of size $r \times r$, where $r = |S|$ is the rank of R, whose entries are the Cartan numbers $n_{\alpha,\beta}$, is called the Cartan matrix of R.

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation. $\lambda \in \mathfrak{h}^*$ is called a weight of ρ relative to $\mathfrak{h}$ if

$$V^\lambda \quad := \quad \{v \in V : \rho(H)(v) = \lambda(H)v \quad \forall \quad H \in \mathfrak{h}\} \tag{2.155}$$

does not consist only of the zero vector. Non-zero vectors of V^λ are called weight vectors, V^λ is called weight space [5]. All weights of a representation ρ compose a set $\Gamma(\rho)$ with $\Gamma(\rho) \neq 0, |\Gamma(\rho)| < \infty$. If V is the adjoint representation of $\mathfrak{g}$, by the definition its weights except zero are roots of $\mathfrak{g}$, the weight spaces are root spaces, and weight vectors are root vectors. We call $\lambda \in \Gamma$ the highest weight if $\lambda + \alpha \notin \Gamma$ for all $\alpha \in R^+$. Each $v \in V^\lambda \backslash \{0\}$ then is called a maximal vector. It can be shown that for each representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ there exists a highest weight. Furthermore, we can write [45]

$$V \quad = \quad \bigoplus_{\lambda \in \Gamma} V^\lambda; \tag{2.156}$$

$$\rho(X_\alpha)(V^\lambda) \quad \subset \quad V^{\lambda+\alpha}\, \forall\, \alpha \in R^+, \lambda \in \mathfrak{h}^*. \tag{2.157}$$

The root system $R \subset V$ is irreducible if V cannot be written as an orthogonal direct sum $V = V_1 \oplus V_2$ such that $R = R_1 \subset R_2$, where $R_i \in V_i, i = 1, 2$, is a root system.

It can be shown that any root system is uniquely determined by its Dynkin diagram (complete classification see in [12]). Any irreducible root system is one

[8]We call two root systems (V_1, R_1) and (V_2, R_2) equivalent if there exists an isomorphism $\phi : V_1 \longrightarrow V_2$ such that $\phi(R_1) = R_2$ and for some constant $c \in \mathbb{R}^*$ we have $(\phi(x), \phi(y)) = c(x, y)$ for any $x, y \in V_1$.

of the classical root systems $A_n = \mathfrak{sl}(n)$, $n \geq 1$, $B_n = \mathfrak{o}(2n+1)$, $n \geq 2$, $C_n = \mathfrak{sp}(2n)$, $n \geq 2$, $D_n = \mathfrak{o}(2n)$, $n \geq 3$, or one of the exceptional root systems G_2, F_4, E_6, E_7, E_8 ([42], Theorem 6.14). Complete consideration of the classical root systems can be found in [43]. For classical root systems we can assign a basis in terms of root vectors as follows:

- $A_n : e_1 - e_2, e_2 - e_3, ..., e_n - e_{n+1}$;
- $B_n : e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, e_n$;
- $C_n : e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, 2e_n$;
- $D_n : e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n, e_{n-1} + e_n$.

2.5.5 Some elements of representation theory

Recall that a linear transformation $\phi : L \longrightarrow L'$ (L, L' Lie algebras over a field K) is called a homomorphism if $\phi([x, y]) = [\phi(x), \phi(y)]$, for all $x, y \in L$. A representation of a Lie algebra L is a homomorphism $\phi : L \longrightarrow \mathfrak{gl}(n, V)$ (V is vector space over K), the number n is called the dimension of the representation. Although we require L to be finite dimensional, it is useful to allow V to be of arbitrary dimension, $\mathfrak{gl}(n, V)$ makes sense in any case. The vector space V, together with the representation ϕ, is called an L-module. A representation is called faithful if its kernel is 0. A representation $L \longrightarrow \mathfrak{gl}(n, V)$ is called irreducible if the only L-invariant subspace $W \subset V, W \neq V$, is $W = 0$. Recall that a subspace $W \subset V, W \neq V$ is L-invariant if $LW \subset W$ [42, 40]. Any subalgebra is equipped with a natural representation of dimension $n = \dim \mathfrak{gl}(L)$, namely with the adjoint representation already mentioned previous sections.

2.5.6 Definitions of affine and Kac-Moody algebras

Let us briefly discuss the main concepts of the theory of infinite-dimensional Lie algebras, so-called Kac-Moody algebras. We begin with the theory of affine Lie algebras. They represent a class of infinite-dimensional Lie algebras which can be canonically constructed out of a finite-dimensional semisimple Lie algebra. Each affine Lie algebras is a Kac-Moody algebra whose generalized Cartan matrix is positive semi-definite and has corank 1. As we already mentioned every (untwisted) affine Lie algebra may be constructed from a finite-dimensional semisimple Lie algebra. By the definition semisimple Lie algebras are direct sums of simple Lie algebras. Therefore, affine Lie algebras built of semisimple Lie algebra are also direct sums of the affine Lie algebras constructed on these simple Lie algebras. In oder to simplify our discussion it will suffice to consider the construction for simple Lie algebras [5].

The way we construct the affine Lie algebra associated to a finite dimensional simple Lie algebra is the following. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over complex numbers $\mathbb{C}$. We repeat that it is a non-Abelian finite-dimensional Lie algebra with no non-trivial ideals. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]_0$. Let $\mathbb{C}[t,t^{-1}]$ denote the ring of Laurent polynomials in $t \in \mathbb{C}$ and $t^{-1} \in \mathbb{C}$, i.e., it consists of elements of the form

$$a = \sum_{n=-\infty}^{\infty} a_n t^n,\ a_n \in \mathbb{C}. \tag{2.158}$$

We put

$$\tilde{\mathfrak{g}}_t = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]. \tag{2.159}$$

If $\{f_1,...,f_a\} \in \mathfrak{g}$ denotes a basis for $\mathfrak{g}$, then we denote the element $f_s \otimes t^n$ by $f_s(n)$, where $s = 1,...,a$ and $t^n \in \mathbb{C}[t,t^{-1}], n \in \mathbb{Z}$, is in the canonical basis for $\mathbb{C}[t,t^{-1}]$. Then the set of vectors given by

$$\{f_s(n) = f_s \otimes t^n : s = 1,...,a;\ n \in \mathbb{Z}\} \tag{2.160}$$

generates a basis of $\tilde{\mathfrak{g}}_t$.

We are interesting in the algebra $\tilde{\mathfrak{g}}$ appearing as a restriction of $\tilde{\mathfrak{g}}_t$ to $t \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$. In this case we can write $t = e^{i\theta}$, where $\theta \in [0,2\pi]$, and $\tilde{\mathfrak{g}}$ then is spanned by $\{f_s(n) = f_s \otimes e_n | s = 1,...,a;\ n \in \mathbb{Z}\}$, where $e_n = e^{in\theta}$. The commutator of the Lie algebra $\tilde{\mathfrak{g}}$ is declared in natural way as

$$[e_s(m), e_p(n)]_\sim = [e_s, e_p]_0 \otimes t^{m+n}. \tag{2.161}$$

The algebra $\tilde{\mathfrak{g}}$ has the name loop algebra. The reason for that is that $\tilde{\mathfrak{g}}$ can be identified with the space of smooth mappings from S^1 to the Lie algebra $\mathfrak{g}$, which we denote by $L\mathfrak{g}$ and call loops. Moreover, we may identify $\tilde{G} \in \tilde{\mathfrak{g}}$ with $\sum_{n\in\mathbb{Z}} x_n e^{in\theta}$, where x_n in $\mathfrak{g}$ tends to zero as $|n|$ approaches infinity. Hence, we may write $\tilde{G}$ in $\tilde{\mathfrak{g}}$ as a Fourier series with coefficients x_n in $\mathfrak{g}$ [46].

Define $\tilde{\mathfrak{g}}'_t = \tilde{\mathfrak{g}}_t \oplus \mathbb{C} \cdot c$ and $\tilde{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C} \cdot c$ as the central extensions of $\tilde{\mathfrak{g}}_t$ and $\tilde{\mathfrak{g}}$, respectively. With respect to the bracket given by

$$[x(m), y(n)] = [x(m), y(n)]_\sim + < x, y > m\delta_{m+n,0}c, \quad [x, c] = 0, \tag{2.162}$$

where $x(m), y(n) \in \tilde{\mathfrak{g}}_t$ and $\tilde{\mathfrak{g}}$ respectively and $< \cdot,\cdot >$ denotes the usual Cartan-Killing form on $\mathfrak{g}$, the algebras $\tilde{\mathfrak{g}}_t$ and $\tilde{\mathfrak{g}}$ are Lie algebras. Expressed in original notation, the commutator can be given as

$$[x(m), y(n)] = [x(m), y(n)]_0 \otimes t^{m+n} + < x, y > m\delta_{m+n,0}c, \tag{2.163}$$

where x, y are basis vectors of $\mathfrak{g}$ and $n, m \in \mathbb{Z}$. The element commuting with all elements of $\tilde{\mathfrak{g}}'_t$ is called central and the Lie algebras $\tilde{\mathfrak{g}}'_t$ and $\tilde{\mathfrak{g}}'$ are called affine Kac-Moody algebras associated with $\mathfrak{g}$ [5, 46]. We note that the algebra $\tilde{\mathfrak{g}}'_t$ is generated by the elements $\{x(-1), x(0), x(1) : x \in \mathfrak{g}\}$ and that the mapping $x \longrightarrow x(0)$ is a Lie algebra isomorphism from $\mathfrak{g}$ into $\tilde{\mathfrak{g}}'_t$, i.e., it is invertible, linear and conserves the bracket operation. Since $\tilde{\mathfrak{g}}'$ is the restriction of $\tilde{\mathfrak{g}}'_t$ this naturally also holds for $\tilde{\mathfrak{g}}'$.

Now for generalization of the notion of affine Lie algebras we turn to Kac-Moody algebras from which the affine algebras can be derived. Let $A = (a_{ij})$ be a generalized Cartan matrix[9], i.e., a square $n \times n$-matrix such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. The associated Kac-Moody algebra $\mathfrak{g}(A)$ is a complex Lie algebra constructed on $3n$ generators e_i, f_i, h_i ($i = 1, ..., n$) and the following relations ($i, j = 1, ..., n$):

$$[h_i, h_j] = 0, [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \quad \text{if } i \neq j, \tag{2.164}$$

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{2.165}$$

$$(\mathrm{adj} e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{adj} f_i)^{1-a_{ij}} f_j = 0 \quad \text{if } i \neq j \tag{2.166}$$

where adj is again the adjoint representation of $\mathfrak{g}$ [47, 48].

Another, but equivalent way to define a Kac-Moody algebra, is by using the so-called realization of a generalized matrix A and the notion of roots and dual roots, known as coroots. It turns out that both definition coincide, hence we only note that in the terms of the second definition the generator h_i can be considered as basis vectors of the Cartan subalgebra $\mathfrak{h}$. The dual space $\mathfrak{h}^*$ then is spanned by the vectors h_i^* defined via $h_i^*(h_j) = a_{ij}$. They compose a root system of $\mathfrak{h}$. In this sense applying concepts of the foregoing sections the Kac-Moody algebra $\mathfrak{g}$ can be diagonalized into weight eigenvectors which we define as earlier:

$$\forall x \in \mathfrak{h} : [g, x] = \omega(x) g, \tag{2.167}$$

where ω is an element of $\mathfrak{h}^*$. Then $g \in \mathfrak{g}$ is said to have weight ω. The relations (2.164) yield that the Cartan subalgebra $\mathfrak{h}$ has weight zero, e_i has weight a_i^* and f_i has weight $-a_i^*$. If the Lie bracket of two weight eigenvectors is non-zero, then its weight is the sum of their weights [5].

The classification of different types of Kac-Moody algebras breaks up into three subclasses. It is convenient to describe them assuming that the matrix A is symmetrizable, i.e., A can either be decomposed into DS, where D is diagonal with positive integer entries and S is a symmetric matrix, or not, i.e., there is no partition of the set $1, ..., n$ into two non-empty subsets so that $a_{ij} = 0$ whenever j belongs to the first subset, while j belongs to the second.Then there are the following three possibilities [5, 47]:

- There is a vector v of positive integers such that all the coordinates of the vector Av are positive. In such case all the principal minors of the matrix A are positive and the Lie algebra $\mathfrak{g}(A)$ is finite-dimensional.

[9] Note that finite-dimensional semisimple Lie algebras are Kac-Moody algebras since the Cartan matrix $n_{\alpha\beta}$ of a semisimple Lie algebra is obviously a generalized Cartan matrix.

- There is a vector v of positive integers such that $Av = 0$. In such a case all the principal minors of the matrix A are non-negative and $\det A = 0$; the algebra $\mathfrak{g}(A)$ is infinite-dimensional. The Lie algebras of this subclass are called affine Lie algebras introduced in the first part of this section.

- There is a vector v of positive integers such that all the coordinates of the vector Av are negative. In such a case the Lie algebra $\mathfrak{g}(A)$ is of indefinite type.

2.6 Elements of Lie group theory

Here we consider the Lie theory from geometrical point of view and show explicitly the relation of Lie algebras with geometrical objects such as manifolds.

2.6.1 Matrix Lie groups and their relation with Lie algebras

We begin with a very important point concerning the theory of Lie algebras. Next we consider matrix groups which are tightly connected with the definition of any Lie algebra. It turns out that the structure of Lie groups is more complicated then the structure of the corresponding Lie algebra. However, it is possible to know the specialties of the underlying Lie group from the knowledge of the properties of the Lie algebra.

All the Lie groups studying in this section are all subgroups (of a certain sort) of the general linear groups. Let $\mathrm{Mat}(n, \mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries and let $\mathrm{Gl}(n, \mathbb{C})$ denote invertible matrices in $\mathrm{Mat}(n, \mathbb{C})$.

Definition 2. *Let A_m be a sequence of complex matrices in $Mat(n, \mathbb{C})$. We say that A_m converges to a matrix A if each entry of A_m converges (as $m \longrightarrow \infty$) to the corresponding entry of A (i.e., if $(A_m)_{kl}$ converges to A_{kl} for all $1 < k, l < n$).*

Definition 3. *A matrix Lie group G is any subgroup of $GL(n, \mathbb{C})$ with the following property: If A_m is any sequence of matrices in G and A_m converges to some matrix A, then either $A \in G$, or A is not invertible. In other words, a matrix Lie group is a closed subgroup of $GL(n, \mathbb{C})$ (most of the interesting matrix Lie groups G have the stronger condition - G is closed in $Mat(n, \mathbb{C})$).*

Let us begin with the most important examples of matrix Lie groups. It turns out that we have already take a contact with them in another context as with transformation groups preserving a certain property or bilinear form.

- The general linear group $\mathrm{GL}(n, K)$ consists of all invertible $n \times n$ matrices over K (we will assume that $K = \mathbb{R}$ or $\mathbb{C}$).

- The special linear group $\mathrm{SL}(n, K)$ consists of the elements of $\mathrm{GL}(n, K)$ with determinant 1.

- The (real) orthogonal group $\mathrm{O}(n,\mathbb{R})$ or just $\mathrm{O}(n)$ consists of the real $n \times n$ matrices M with $M^T M = \mathrm{Id}$; for the complex orthogonal group $\mathrm{O}(n,\mathbb{C})$ we replace real by complex in the definition.

- The special (real) orthogonal group $\mathrm{SO}(n,\mathbb{R}) = \mathrm{SO}(n)$ is $\mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$; similarly for $\mathrm{SO}(n,\mathbb{C})$.

- The unitary group $\mathrm{U}(n)$ consists of all the (complex) matrices M with $M^\dagger M = \mathrm{Id}$; the special unitary group $\mathrm{SU}(n)$ is $\mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$.

- The symplectic group $\mathrm{Sp}(n,K)$ consists of all $2n \times 2n$ matrices over K with $M^T JM = J$ (where J is a matrix with $J^2 = -\mathrm{Id}$); such matrices automatically have $\det M = 1$. The symplectic group $\mathrm{Sp}(n)$ is $\mathrm{Sp}(n,C) \cap \mathrm{U}(2n)$.

- Finally the Lorentz group consists of all real 4×4 matrices M with $M^T \mathrm{Id}_{3,1} M = \mathrm{Id}_{3,1}$ ($\mathrm{Id}_{3,1}$ is the identity matrix of signature $(3,1)$) as an example of indefinite real orthogonal Lie group $O(3,1)$.

The set of all $n \times n$ matrices over K can be identified with the standard vector space of dimension n^2 over K. Therefore, the groups defined above are subsets of various spaces $\mathbb{R}^m$ or $\mathbb{C}^m$, defined by a finite number of simple equations. Mathematically, they are algebraic varieties (except for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ [41]) and consequently they are all topological differentiable manifolds.

We now come to the relation of these groups with the corresponding Lie algebras. It is customary to use lowercase Gothic characters such as $\mathfrak{g}$ for a Lie algebra to refer to the corresponding Lie group G. Briefly we can formulate the correspondence between Lie algebras and Lie groups by stating that a Lie algebra is the tangent space of a Lie group at the unit element [41, 6]. In other words, for a given a Lie group a Lie algebra can be associated to it by endowing the tangent space to the identity with the differential of the adjoint map. So the elements of the corresponding Lie algebra are tangent vectors to the curves through the identity of G.

There are two deep theorems which express the connection of Lie algebras and Lie groups and their general structure. The first one is called Ado's theorem. One version of it says that every finite-dimensional Lie algebra is isomorphic to a matrix Lie algebra [49, 45]. For every finite-dimensional matrix Lie algebra, there is a linear group (matrix Lie group) with this algebra as its Lie algebra. So every abstract Lie algebra is the Lie algebra of some (linear) Lie group. This is in contrast to the situation for Lie groups, where most, but not all, Lie groups are matrix Lie groups. The second important theorem is Lie's fundamental theorem. It describes a relation between Lie groups and Lie algebras. In particular, any Lie group gives rise to a canonically determined Lie algebra, and conversely, for any Lie algebra there is a corresponding connected Lie group (Lie's third theorem, the prove is given in [45]).This Lie group is not determined uniquely, however, any two connected Lie groups with the same Lie algebra are locally isomorphic.

The correspondence between Lie algebras and Lie groups can be used in the classification of Lie groups and in the related question of the representation theory of Lie groups. Every representation of a Lie algebra determines uniquely a representation of the corresponding connected simply connected Lie group, and conversely every representation of any Lie group induces a representation of its Lie algebra; the representations are in one-to-one correspondence [5].

2.6.2 Generic definition of groups

We used in previous sections the notion of a group. We have defined the orthogonal, linear etc. groups and meant always a set whose elements are united in a certain way. It is time to give a more general definition of this object.

A group G is a set of elements together with a map $\cdot : G \times G \longrightarrow G$ that combines any two elements a and b of G into another element denoted by $a \cdot b$. The operation "$\cdot$" is an arbitrary map specified for any given group. For example one can take addition for the group of integer. The only condition on this operation is that the pair $(G, \cdot)$ must satisfy four requirements known as the group axioms:

- Closure: if $a, b \in G$ then $a \cdot b \in G$;
- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds for all $a, b, c \in G$;
- Existence of identity: there exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$, $\forall a \in G$;
- Existence of inverse: for any $a \in G$ there is in G an element a^{-1}, which obeys $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Examples are the real numbers with the ordinary addition $(\mathbb{R}, +)$, symmetry groups, cyclic groups etc. (see [5]).

For a given group G together with a binary operation "$\cdot$", we say that some subset H of G is a subgroup of G if H also forms a group under the operation "$\cdot$".

2.6.3 Topological groups and smooth Lie groups

We have mentioned that any Lie algebra is a tangent space of a Lie group at a given point. The concept of tangents spaces is related with topological objects called manifolds. To go stepwise to the consideration of Lie groups as manifolds we first of all endow the group with certain topological properties.

Definition 4. *A topological group is a group G together with a topology on G (see 2.1.9) that satisfies the following two properties:*

- *The map $p : G \times G \longrightarrow G$ defined by $p(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology.*[10]

- *The map* $\mathrm{inv} : G \longrightarrow G$ *defined by* $\mathrm{inv}(g) = g^{-1}$ *is continuous.*

The first property is equivalent to the statement that, whenever $U \subseteq G$ is open, and $g_1 g_2 \in U$, there exist open sets V_1, V_2 such that $g_1 \in V_1$, $g_2 \in V_2$ and $V_1 V_2 = \{h_1 h_2 : h_1 \in V_1, h_2 \in V_2\} \subseteq U$. The second property is equivalent to that, whenever $U \subseteq G$ is open, the set $U^{-1} = \{g^{-1} : g \in U\}$ is open. Moreover, we remark that every subgroup of a topological group, endowed with the subspace topology, is a topological group [50, 51]. At this place we give some examples of topological groups: $(\mathbb{R}, +)$, $(\mathbb{R}\backslash\{0\}, \cdot)$, $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ [50].

Now we turn to the idea of the Lie groups. A Lie group is a group whose group structure and differentiable structures are compatible, i.e., the multiplication map and the inversion map are differentiable. In this sense a Lie group is a differentiable manifold[11], on which the group operations of multiplication and inversion are differentiable maps. Differentiability of the group multiplication means that the map $\mu : G \times G \longrightarrow G$, $\mu(x, y) = xy$ with $x, y \in G$ is a differentiable mapping from the product manifold $G \times G$ into G. It is convenient to assume that Lie group is an infinite times continuously differentiable manifold. We then say G is smooth. All of the previous examples of groups fall within the class of smooth Lie groups. We do not list them here.

The representation of a Lie group G on finite dimensional vector space V (real or complex) is a homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ of Lie groups. The adjoint representation of a Lie group G is the natural representation of G on its own Lie algebra given by the map $\mathrm{adj}' : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$, $g \longmapsto (\mathrm{adj}_g(e))'$ sending each element g of the group to the derivative of the adjoint map at the identity.

2.6.4 Left, right and adjoint action of a Lie group G on itself

Let G be a group. Define $L_a : G \longrightarrow G$ by $L_a(g) = ag$ and $R_a : G \longrightarrow G$ by $R_a(g) = ga$. We call L_a the left action of G on itself and R_a the right action of G on itself given by the element a. The adjoint action on G is defined as combination of both these actions, precisely as $\mathrm{adj}_g : G \longrightarrow G$, $\mathrm{ad}_g(h) = ghg^{-1}$.

We call a vector field X on G left invariant if it is invariant under any left action L_a of G (respectively right action). This can be written with the help of pushforward transformation: $(L_a)_* X_{a'} = X_{aa'}$ for all $a, a' \in G$. Here $(L_a)_*$ is equal to the differential of L_a. A left (right) invariant vector field is always

[10] Open sets on $G \times G$ are explained intuitively: that are subsets of $G \times G$ which can be written as $O \times O'$, where $O, O' \in G$ are open sets relative to the topology on G.

[11] A differentiable manifold is a topological, paracompact, Hausdorff space which is locally homeomorphic to Euclidean space and whose transition maps are all differentiable (more information can be found for example in [5, 6, 14].

differentiable. The Lie algebra of G can then be identified with the set of all left invariant vector fields together with usual addition, scalar multiplication and usual bracket operation. As a vector space the algebra $\mathfrak{g}$ then is isomorphic to the tangent space at the identity T_e.

2.7 Homogeneous spaces

In this section we concentrate on a special type of Lie groups, namely on the homogeneous spaces. Furthermore, we investigate and get closer with the notion of nearly Kähler coset spaces which are of a particular interest of this thesis.

2.7.1 Closed subgroup H of a Lie group G

We consider a subset H of a topological group G which is closed as a subset, i.e., the complement of H is open relative to the topology on G. Let now G be a Lie group. A submanifold H of G is called a Lie subgroup if the following conditions hold:

1. H is a subgroup of the group G;

2. H is a topological group.

It can be shown that a Lie subgroup itself is a Lie group. Moreover, if H is a Lie subgroup of a Lie group G, then the Lie algebra $\mathfrak{h}$ of H (i.e., it is the tangent space of H at the identity) is a subalgebra of $\mathfrak{g}$ which is the Lie algebra of G (see [12], Theorem 2.1).

Remark: It can be immediately seen that H is a closed subgroup of G and it is a Lie group. The converse to this result is also true, but harder to prove: every closed subgroup H of a Lie group G is a submanifold of G [52, 45].

2.7.2 Homogeneous spaces G/H and $H\backslash G$ of a Lie group as equivalence classes

An action of a Lie group G on itself defines of course an action of any subgroup $H \subset G$. In particular, the left and right actions define an action of any subgroup $H \subset G$ on the whole group G. The orbits of the transformation group obtained in this way are called respectively the left and the right cosets of G under H. Thus, a left (right) coset consists of all elements of the form hg (gh), where $g \in G$ is some fixed element, and h takes all possible values in H. We denote the cosets respectively by

$$Hg = \{hg : h \in H\} \quad \text{and} \quad gH = \{gh : h \in H\}. \tag{2.168}$$

The general property of the left (right) cosets spaces follows from the fact that each element of G is in only one left (right) coset. Hence, the distribution of the elements among the left (right) coset spaces is disjoint.

Furthermore, cosets can be considered as equivalence classes. That means that any two elements x, y of a left coset of H in G are related in a way that also $x^{-1}y \in H$ (that holds if and only if the elements x, y are in the same left coset). We then say that the two elements x, y of G are equivalent [12] and write $x \sim y$. It is straightforward to see that this equivalence relation is an equivalent definition of left cosets. In this sense we say that the left cosets gH of H in G form equivalence classes relative to the above equivalence relation. We denote the set of all left (right) cosets by G/H ($H\backslash G$).

We call a subgroup N of G normal if for all $n \in N$ and all $g \in G$ also $gng^{-1} \in N$, i.e., in particular, if $gN = Ng$ for all $g \in G$. In this case the set of all cosets form a group which we call the quotient group G/N [15, 5]. For a Lie subgroup N which is not normal, the space G/N of left cosets does not have the structure of a group, but simply is a differentiable manifold on which G acts.

Suppose G is a Lie group and H is a closed subgroup of G. Let $\pi : G \longrightarrow G/H$ denote the natural mapping $g \longmapsto gH$ of G onto G/H. The mapping π endows the set G/H with the natural topology coming from G, since π is continuous and open.[13] Hence G/H is a smooth manifold called coset manifold. Furthermore, the map π has the property that any two points in G/H can be joined by the action of G, i.e., the action is transitive (for all $A, A' \in G/H$ there exists a $g \in G$ such that $gA = A'$). We call the resulting manifold furnished with this transitive action a homogeneous space (see [12, 53]). In general a homogeneous space is a manifold M with a transitive action of a Lie group G.

An equivalent definition is that a homogeneous space is a manifold of the form G/H, where G is a Lie group and H is a closed subgroup of G. It can be shown that the dimension of G/H is $\dim G - \dim H$ [53]. Examples are the sphere $\mathbb{S}^n \simeq \mathrm{O}(n+1)/\mathrm{O}(n)$ and the complex projective space $\mathbb{C}P^n \simeq \mathrm{U}(n+1)/\mathrm{U}(n)\times\mathrm{U}(1)$. Independently of the fact that the definition of homogeneous spaces was introduced by us for left coset spaces, it is quite obviously that the analogous consideration can be done also for the right cosets.

2.7.3 Splitting of the Lie algebra $\mathrm{Lie}\, G = \mathrm{Lie}\, H \oplus (\mathrm{Lie}\, H)^\perp$

Let G be a Lie group, H its closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H, respectively. Due to the Cartan theorem, every closed subgroup H of a Lie group G is a Lie subgroup. That means in particular that H is a smooth manifold embedded in G. Since the Lie algebra $\mathfrak{g}$ assigned to G

[12]In general an equivalence "$\sim$" relation is a binary operation between two elements x, y of a set G which is reflexive $x \sim y$, symmetric $x \sim y \Rightarrow y \sim x$ and transitive $x \sim y$, $y \sim z \Rightarrow x \sim z$.

[13]This means that images of open sets in G are open in G/H.

is the tangent space T_eG at the identity e of G and since that identity e is also the identity of the subgroup H, one can immediately see that the corresponding algebra $\mathfrak{h}$ of H is a subspace of the tangential space of G: $\mathfrak{h} = T_eH \subset T_eG = \mathfrak{g}$. Consequently the Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Considering $\mathfrak{h}$ and $\mathfrak{g}$ as vector spaces this relation can be expressed as a direct sum:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2.169}$$

where $\mathfrak{m}$ is a complement of $\mathfrak{h}$ in $\mathfrak{g}$. The homogeneous space G/H is called reductive if the Lie algebra $\mathfrak{g}$ can be decomposed into a vector space direct sum of the $\mathfrak{h}$ and an adj(H)-invariant subspace $\mathfrak{m}$, that is, if

- $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$; $\mathfrak{h} \cap \mathfrak{m} = 0$;
- adj(H)$\mathfrak{m} \subset \mathfrak{m}$ (i.e., $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$).

The space $\mathfrak{m}$ can be identified with the tangent space $T_e(G/H)$. It can be shown that if G is compact or connected and semisimple then G/H is reductive [54, 55, 56]. In particular, if G is semisimple the Lie algebra $\mathfrak{g}$ can be decomposed $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ with $\mathfrak{m} = \mathfrak{h}^{\perp}$, where $\mathfrak{h}^{\perp}$ is the orthogonal complement of $\mathfrak{h}$ relative to the Killing form of $\mathfrak{g}$ [57].

2.7.4 Splitting of commutators

Suppose, we have a fixed decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of a compact semisimple Lie group G in its Lie subalgebra $\mathfrak{h}$ of a Lie subgroup $H \subset G$ and the complement $\mathfrak{m}$. Then the original commutator of $\mathfrak{g}$ can be splitted in two subsets in the following manner. The generators $\{e_A\}$, $A = 1, ..., \dim \mathfrak{g}$, of the algebra $\mathfrak{g}$ split in $\{e_i\}$, $i = 1, ..., \dim \mathfrak{h}$, and $\{e_a\}$, $a = 1, ..., \dim \mathfrak{m}$, in such way that the generators $\{e_i\}$ span the subalgebra $\mathfrak{h}$ and the generators $\{e_i\}$ span the complement $\mathfrak{m}$. In this fashion the commutation relations

$$[e_A, e_B] = f^C_{AB} e_C \tag{2.170}$$

can also be rewritten in the explicit form

$$[e_i, e_j] = f^k_{ij} e_k, \quad [e_i, e_a] = f^k_{ia} e_k + f^c_{ia} e_c, \quad [e_a, e_b] = f^i_{ab} e_i + f^c_{ab} e_c. \tag{2.171}$$

In a special case when the coset space G/H is a reductive homogeneous space the structure constants f^k_{ia} vanish and the above relations simplify to

$$[e_i, e_j] = f^k_{ij} e_k, \quad [e_i, e_a] = f^c_{ia} e_c, \quad [e_a, e_b] = f^i_{ab} e_i + f^c_{ab} e_c. \tag{2.172}$$

2.7.5 The Maurer-Cartan equations

If G is a Lie group and $\gamma = \gamma(t)$ is a curve with $\gamma(0) = A \in G$, then the curve $A^{-1}\gamma$ passes through the identity at $t = 0$. Hence, we conclude that the derivative $A^{-1}\gamma'(0)$ is a tangent vector at the identity, i.e., an element of the Lie algebra $\mathfrak{g}$, assigned to G. In this way we obtain a linear differential form, $\theta : G \longrightarrow \mathfrak{g}$, which assigns to every element in G an element in $\mathfrak{g}$. In case if G is a subgroup of the general linear group $\mathrm{Gl}(n)$, the form θ can be written as

$$\theta = A^{-1}\mathrm{d}A. \tag{2.173}$$

For simplicity we will work with this case, although the main theorem, the Maurer-Cartan equation, holds for any arbitrary Lie group.

We note that the definition of the Lie groups amounts to constraints on A. The equation (2.173) gives a description of the Lie algebra $\mathfrak{g}$. For example, $G = \mathrm{O}(n)$ yields $AA^T = \mathrm{Id}$. Differentiating we obtain $A^{-1}\mathrm{d}A + (A^{-1}\mathrm{d}A)^T = 0$. Hence, $\mathfrak{g}$ consists of antisymmetric matrices.

Returning to our consideration, we take the derivative of the equation (2.173). For that purpose we have to compute $\mathrm{d}(A^{-1})$: using $\mathrm{d}(AA^{-1}) = 0$, we obtain

$$\mathrm{d}AA^{-1} + A\mathrm{d}(A^{-1}) = 0 \tag{2.174}$$
$$\Rightarrow \quad \mathrm{d}(A^{-1}) = -A^{-1}\mathrm{d}AA^{-1}. \tag{2.175}$$

Putting this in the equation (2.173), we get

$$\mathrm{d}\theta = \mathrm{d}(A^{-1}) \wedge \mathrm{d}A = (-A^{-1}\mathrm{d}AA^{-1}) \wedge \mathrm{d}A = -A^{-1}\mathrm{d}A \wedge A^{-1}\mathrm{d}A, \tag{2.176}$$

which yields the formula known as the Maurer-Cartan equation

$$\mathrm{d}\theta = -\theta \wedge \theta. \tag{2.177}$$

This relation can also be formulated with the help of a left invariant 1-form ω on G. First of all, a differential form is called left invariant if $(L_a)^*\omega = \omega$, for all $a \in G$, or equivalently $(L_a)^*\omega_{a'} = \omega_{a'^{-1}a}$, where $(L_a)^*\omega$ is the pullback of ω defined via $(L_a^*\omega)_x(X) = \omega)_{ax}((\mathrm{d}L_a)_x X)$, $\forall x \in G, X \in T_xG$. Moreover, the set of all left invariant 1-forms constitutes the dual vector space of the algebra $\mathfrak{g}$. Then the Maurer-Cartan equation can be written as $\mathrm{d}\omega(X,Y) = -\frac{1}{2}\omega([X,Y])$ for all $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$ [5, 58, 43, 59].

2.7.6 Metric, Levi-Civita connection, affine connection and torsion

Metric

Since a Lie group G is a smooth manifold, we can endow G with a Riemannian metric. Among all Riemannian metrics on a Lie group, those for which the left translations (or the right translations) preserve the metric play an important role, because they respect the group structure G.

Let g be a Riemannian metric on a Lie group G, i.e., a positive definite non-degenerate symmetric 2-form $TG \times TG \longrightarrow \mathbb{R}$. The metric g is called left-invariant (resp. right-invariant) iff

$$g(x,y)_a = g((\mathrm{d}L_a)_b x, (\mathrm{d}L_a)_b y)_{ab}, \tag{2.178}$$

$$(\text{resp.}\quad g(x,y)_a = g((\mathrm{d}R_a)_b x, (\mathrm{d}R_a)_b y)_{ba}), \tag{2.179}$$

for all $a, b \in G$ and all $x, y \in T_b G$. If a Riemannian metric is both left- and right-invariant, then we call it biinvariant. It can be shown that left-invariant metrics on G correspond bijectively to scalar products on $\mathfrak{m}$ (we again assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$). Since left-invariant translations are isometries and act transitively on G, the space G equipped with a left-invariant metric is called a homogeneous Riemannian space [12, 58]. It was proven by Élie Cartan that for any compact Lie group G and any compact subgroup $H \subset G$ for the homogeneous space G/H there exists a left-invariant Riemannian metric d, i.e., such that $d(x,y) = d(gx, gy), \forall x, y \in G/H, g \in G$, where G acts naturally on G/H [60, 58]. In this case the group G itself admits a biinvariant metric.

Affine connection

An affine connection on a manifold M is an operation ∇ which assigns to each tangent vector $X \in TM$ a linear map ∇_X of the vector space TM into itself, satisfying the following conditions:

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \quad \nabla_X(fY) = f\nabla_X(Y) + (Xf)Y. \tag{2.180}$$

We call the expression $\nabla_X Y$ the covariant derivative of Y in X direction. Let us look at how connection can be viewed locally. For this reason we rewrite the defining equation (2.180) for basis vectors $\{e_a\}$ of TM:

$$\nabla e_a = dx^i \otimes \omega^b_{ia} e_b, \tag{2.181}$$

where $\omega^b_{\alpha a}(x)$ are some functions and x^i are the local coordinates on M. Consider the one-forms

$$\omega^b_a = dx^i \omega^b_{ia}. \tag{2.182}$$

Then

$$\nabla e_a = \omega^b_a \otimes e_b. \tag{2.183}$$

It can be shown that any transformation of the basis vectors assigned to every point of M of the form $e'_a = {S^{-1}}^b{}_a e_b$ leads to the transformation of the one-forms ω, in particular,

$$\omega'^b_a = {S^{-1}}^c{}_a \omega^d_c S^b_d + \mathrm{d}{S^{-1}}^c{}_a S^b_c, \tag{2.184}$$

Omitting the indices in the last equation and taking the exterior derivative of it give the transformation formula

$$\mathrm{d}\omega' - \omega' \wedge \omega' = S^{-1}(\mathrm{d}\omega - \omega \wedge \omega)S. \tag{2.185}$$

Hence, we see that the form $\mathrm{d}\omega - \omega \wedge \omega$ transforms as a tensor. We call the corresponding 2-form $R_a^b = \mathrm{d}\omega_a^b - \omega_a^c \wedge \omega_c^b$, i.e., a form with values in $T^*M \otimes TM$, the local curvature form of connection ∇.

We are interested in affine connections on Lie groups. That is why we define a left-invariant connections as next step. Let G be a Lie group together with ∇, an affine connection on it. ∇ is said to be left-invariant if each left transition L_g ($g \in G$) is an affine transformation[14] on G [12]. In the same way we can introduce a left-invariant connection ∇ on a reduced homogeneous space G/H if for all $a \in G$ the functions $\tau(a) : G/H \longrightarrow G/H$, $xH \longmapsto axH$ are affine transformations [57]. The most important example of an affine connection is the Levi-Civita connection. It is the only connection which we say preserves the metric, i.e., it obeys

$$X(g(Y,Z)) = g(\nabla_X^{LC} Y, Z) + g(Y, \nabla_X^{LC} Z), \tag{2.186}$$

where g is a metric on G and $X, Y, Z \in TG$, and is torsion-free:

$$\nabla_X^{LC} Y - \nabla_Y^{LC} X - [X, Y] = 0, \tag{2.187}$$

where the left hand side of the equation (2.187) defines the torsion $T \in \Gamma(T^*M \otimes T^*M \otimes TM)$: $T = T^i \otimes \frac{\partial}{\partial x^i}$, where $T^i = \frac{1}{2} T^i_{jk} dx^j \wedge dx^k$ in some holonomic basis. In some non-holonomic basis $\{e^a\}$ the relation of T^a to the connection 1-form ω can be expressed by the following relation

$$de^a + \omega_b^a \wedge e^b = T^a. \tag{2.188}$$

The torsion vanishes for the Levi-Civita connection. So that explicitly the Levi-Civita connection can be given by the formula:

$$\begin{aligned} g(\nabla_X^{LC} Y, Z) &= \frac{1}{2}\{X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,W],X)\}. \end{aligned} \tag{2.189}$$

In holonomic basis the components of the corresponding 1-forms ω, which we denote by Γ, can be expressed in the terms of the metric components:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}), \tag{2.190}$$

where $g_{ij,l} = \frac{\partial g_{ij}}{\partial x^l}$ [5].

[14] A diffeomorphism Φ of G is called an affine transformation of G if ∇ is invariant under Φ, i.e., if $\nabla_X(Y) = (\Phi^{-1})^*(\nabla_{\Phi^* X}(\Phi^* Y))$, for all $X, Y \in TG$ [12].

It can be shown that on a reductive homogeneous spaces G/H with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ there exists one and only one G-invariant connection $\nabla^{\mathfrak{m}}$ which is torsion-free and such that the curves $\gamma(t) = \exp(tX)$ on G, for all $X \in \mathfrak{m}$, project by $\pi : G \longrightarrow G/H$ into geodesics[15] $\gamma^*(t) = \pi(\gamma(t))$ in G/H. This connection is called the canonical connection of the first kind [57] or the natural torsion-free connection. It coincides with the Levi-Civita connection on the coset space G/H. Using the above notation it can be given by $\nabla^{\mathfrak{m}}_X Y = \frac{1}{2}[X,Y]_{\mathfrak{m}}$ for all $X, Y \in \mathfrak{m} \cong T_e(G/H)$ [54, 58].

Another important kind of affine connection on reductive homogeneous spaces is the Riemannian connection. It provides the same geodesics as the natural torsion-free connection, but has non-vanishing torsion. It can be expressed in terms of a adj$(H)-$invariant non-degenerate symmetric bilinear form B on $\mathfrak{m}$ and the corresponding G-invariant metric g on G/H which is given by $B(X,Y) = g(X,Y)_0$ for $X, Y \in \mathfrak{m}$. In this fashion the Riemannian connection can be written as

$$\nabla'_X Y = \frac{1}{2}[X,Y] + T(X,Y), \tag{2.191}$$

where $T(X,Y)$ is the symmetric bilinear map from $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$, defined by $2B(T(X,Y),Z) = B(X,[Z,Y]_{\mathfrak{m}}) + B([Z,X]_{\mathfrak{m}},Y)$, for $X, Y, Z \in \mathfrak{m}$. This connection matches with the natural torsion-free connection if $B(X,[Z,Y]_{\mathfrak{m}}) + B([Z,X]_{\mathfrak{m}},Y) = 0$ [58].

2.7.7 Kähler coset spaces G/H

Let us start with a connected Lie group G. Let be H a closed subgroup of G and G/H be the space of left cosets of G modulo H, on which G acts by the left transitions. We assume that G is effective on H, i.e., H contains no non-trivial subgroup invariant in G. We define a complex smooth manifold as a manifold whose charts are homeomorphic to the open subsets of $\mathbb{C}^n$ and such that the transition functions are holomorphic. We can define a Kähler manifold M as an n-dimensional complex manifold, i.e., a $2n$-dimensional real manifold with a complex structure J, together with a Riemannian metric g on M, which is Hermitian[16] in the sense that it satisfies:

$$g(JX, JY) = g(X,Y) \qquad \forall X, Y \in TM \tag{2.192}$$

and whose Kähler form $\omega(X,Y) = g(JX,Y)$ is closed: $\mathrm{d}\omega = 0$ [14, 61].[17]

[15] A geodesic on a smooth manifold M with an affine connection ∇ is defined as a curve $\gamma(t)$ such that parallel transport along the curve preserves the tangent vector to the curve, so $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all t. $\dot\gamma$ denotes here the derivative of γ with respect to the curve parameter t. Is γ_v the unique geodesic, which for a given point $p \in M$ and for a given tangential vector $v \in T_pM$ satisfies $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$, then the exponential map is defined as $\exp_p(v) = \gamma_v(1)$.

[16] A complex manifold always admits a Hermitian metric [14].

[17] Setting $Z = X + iY, W = U + iV \in T_pM^{\mathbb{C}}$, one can extend the Riemannian metric g to

Now we can consider a complex manifold which is additionally a homogeneous space. We say that a coset space G/H is homogeneous complex (resp. homogeneous Kähler) if it carries a complex analytic structure J (resp. Kähler structure ω) invariant under the group structure G. For a compact connected Lie group it can be shown that "homogeneous complex and Kählerian" implies "homogeneous Kählerian" [61].

2.7.8 Nearly Kähler coset spaces in $d = 6$

Let us consider a manifold with a weaker condition than existence of a complex structure on it. More concretely, we introduce an almost complex structure. An almost complex structure over a manifold is a smooth tensor field J of rank $(1,1)$ such that $J^2 = -\mathrm{Id}$, where we regard J as a vector bundle isomorphism $J : TM \longrightarrow TM$. Then an almost Hermitian manifold (M^{2n}, g, J) is a $2n$-dimensional real manifold, which admits an almost complex structure J compatible with a Riemannian metric g, i.e $\forall X, Y \in TM$, $g(JX, JY) = g(X, Y)$. In particular, every complex manifold is almost complex.

A nearly Kähler manifold is a special type of almost Hermitian manifolds. Namely, it is an almost Hermitian manifold such that the pair (g, J) satisfies the equation $(\nabla_X J)X = 0$, $\forall X \in TM$, where ∇ denotes the Levi-Civita connection associated to metric g. It is called strict nearly Kähler if $(\nabla_X J) \neq 0$ whenever $X \in TM$, $X \neq 0$. An important observation is that in this definition the last condition can be exchanged with one of following equivalent statements:

- The canonical Hermitian connection $\overline{\nabla}$ uniquely defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY, \tag{2.193}$$

  for all $X, Y \in TM$, has totally antisymmetric torsion.

- The Kähler form ω related to g and J, defined via $\omega(X, Y) = g(JX, Y)$, obeys

$$\forall X \in TM, \quad \nabla_X \omega = \frac{1}{3}\iota_X \mathrm{d}\omega, \tag{2.194}$$

  where ι denotes the inner product operating on the differential forms of a manifold as $\iota_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$, $\iota_X\omega(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1})$.

- The form $\mathrm{d}\omega$ is of type (3,0)+(0,3) and the Nijenhuis tensor N, defined by $NJ(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]$, is totally skew-symmetric.

$$g(Z, W) = g(X, U) - g(Y, V) + i(g(X, V) + g(U, Y)).$$

With respect to the basis $\{\frac{\partial}{\partial z^i}\}$ and $\{\frac{\partial}{\partial \bar{z}^i}\}$ the metric components obey $\overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}$ and $\overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}$. If g is Hermitian, then it has the form $g = g_{\mu\bar{\nu}}dz^\mu \otimes dz^{\bar{\nu}} + g_{\bar{\mu}\nu}dz^{\bar{\mu}} \otimes dz^\nu$.

We again are interested in coset spaces. So we say that a coset space G/H is homogeneous nearly Kähler if the nearly Kähler structure is invariant under the action of G. A classification of homogeneous nearly Kähler manifolds is not an easy exercise, nevertheless in the recent times there were reached some interesting results. At this point we refer the reader to [62, 5]. We note that it was proven that for nearly Kähler manifolds a splitting theorem holds which states that any nearly Kähler manifold is locally a Riemannian product of a Kähler manifold and a strictly nearly Kähler one [63, 64].

Nearly Kähler manifolds in 6 dimensions have a particular nature. First of all, for each almost Hermitian 6-manifold (M, J, ω), with $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, the following conditions are equivalent:

1. The form $\nabla\omega \in \Lambda^1(M) \otimes \Lambda^2(M)$ is non-zero and totally skew-symmetric (that is, $\nabla\omega$ is a 3-form).

2. The structure group of M admits a reduction to SU(3), that is, there is a $(3,0)$-form Ω with

$$|\Omega| = 1, \quad \mathrm{d}\omega = 3\lambda\mathrm{Re}\Omega, \quad \mathrm{d}\Omega = -2\lambda\omega^2, \tag{2.195}$$

   where λ is a non-zero real constant.

Furthermore, any nearly Kähler manifold in 6 dimensions is isomorphic to one of the 4 homogeneous nearly Kähler manifolds: $S^3 \times S^3$ $=\mathrm{SU}(2)^3/\mathrm{SU}(2)$, $G_2/\mathrm{SU}(3) \simeq S^6$, $\mathrm{Sp}(2)/\mathrm{SU}(2)\times\mathrm{U}(1) \simeq \mathbb{C}P^3$, $\mathrm{SU}(3)/\mathrm{U}(1)\times \mathrm{U}(1) \simeq \mathbb{F}^3$. Their special property is also that they own a Killing spinor, i.e., on every nearly Kähler 6-manifold M there exists a Killing spinor Ψ in the bundle of spinors $\mathfrak{G}$ which satisfies

$$\nabla_X\Psi = \lambda X\Psi, \qquad \forall X \in TM. \tag{2.196}$$

Here we used the Clifford multiplication map $TM \otimes \mathfrak{G} \longrightarrow \mathfrak{G}$.

The consequence of the existence of a Killing spinor is that the manifolds listed above are Einstein manifolds, i.e., their Ricci tensor is proportional to the metric with Einstein constant $|\lambda|^2 > 0$. Furthermore, nearly Kähler manifolds provide a natural example of almost Hermitian manifolds admitting a Hermitian connection with totally skew symmetric torsion. From this point of view they are of interest in string theory [65, 62].

Chapter 3

Yang-Mills-theory on nearly Kähler manifolds

3.1 Yang-Mills equation on manifold M

We will now discuss the main aspects of connection on fibre bundles and finally turn to the Yang-Mills theory which as we will see appears in a natural way if we restrict ourselves to a discussion of the topology of principal bundles. The major ideas as well as advanced treatments of this topic can be found in references [14, 58, 66, 67, 68, 69].

3.1.1 Principal fibre bundle

Let M be a manifold and G an arbitrary Lie group. We define a (differentiable) principal fibre bundle over M with G as a pair of a manifold P and an action of G on P obeying the following conditions:

1. G acts freely on P on the right: $(u, a) \in P \times G \longrightarrow ua = R_a u \in P$;

2. M is the quotient space P/G of P given by the equivalence relation induced by G, and the canonical projection $\pi : P \longrightarrow M$ is differentiable;

3. P is locally trivial, that is if every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times G$, such that $\psi(u) = (\pi(U), \phi(u))$, where ϕ is a mapping of $\pi^{-l}(U)$ into G satisfying $\phi(ua) = (\phi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

The principal fibre bundle is usually denoted by $P(M, G)$. P is called the total space or the bundle space, M the base space, G the structure group and π the projection. In particular, for each point $x \in M$, the set $\pi^{-1}(x)$ is a closed submanifold of P, called the fibre over x. One can see that every fibre $\pi^{-1}(x)$ is diffeomorphic to G.

3.1.2 Transition functions

Consider now an open covering $\{U_\alpha\}$ of M. Then the concept of transition functions makes it possible to consider a principal bundle $P(M,G)$, which until now was a local object, as a global construction. Namely, because of the local triviality each $\pi^{-1}(U_\alpha)$ is endowed with a diffeomorphism $u \longrightarrow (\pi(u), \phi_\alpha(u))$ of $\pi^{-1}(U_\alpha)$ on $U_\alpha \times G$ such that $\phi_\alpha(ua) = (\phi(u))a$. This means that if the image of $u \in P$ under the projection π is in $U_\alpha \cap U_\beta$, then $\phi_\beta(ua)(\phi_\alpha(ua))^{-1} = \phi_\beta(a)(\phi_\alpha(a))^{-1}$. The last equation shows that the term $\phi_\beta(a)(\phi_\alpha(a))^{-1}$ depends only on $\pi(u)$ and not on u. Thus, the right multiplication is defined without reference to the local trivializations. Setting $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ as $\psi_{\alpha\beta}(\pi(u)) = \phi_\beta(u)(\phi_\alpha(u))^{-1}$, we define a family of transition functions $\psi_{\alpha\beta}$ of the bundle $P(M,G)$ corresponding to the open covering $\{U_\alpha\}$ of M. It can be easily verified that

$$\psi_{\gamma\alpha} = \psi_{\gamma\beta}\psi_{\beta\alpha} \quad \text{for} \quad x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{3.1}$$

Finally, let $U \subset M$ be an open set of M. A smooth map $s : U \longrightarrow P(M,G)$ from the base space M into the bundle space P is called a local cross section if $\pi(s(x)) = x$, for all $x \in U$. If a section can be defined on the whole manifold M, then we call it a global cross section. The set of all cross sections of P is denoted by $\Gamma(M,P)$.

3.1.3 Associated fibre bundle

Given a principal fibre bundle $P(M,G)$ and F a manifold, on which G acts on the left: $G \times F \ni (a, f) \longrightarrow af \in F$, we may construct an associated fibre bundle as follows. Define an action of $g \in G$ on $P \times F$ by

$$(u, f) \longrightarrow (ug, g^{-1}f), \tag{3.2}$$

where $u \in P$ and $f \in F$. Then the quotient space $P \times_G F := (P \times F)/G$ with respect to this group action is the associated fibre bundle $E(M,F,G,P)$.

Assume now that $P(M,G)$ is a principal fibre bundle and H a closed subgroup of G. The action of G on the coset space G/H on the left is defined in a natural way. So let us consider the associated fibre bundle $E(M,G/H,G,P)$ with standard fibre G/H. Since H is a subgroup of G, it acts on P on the right. Denoting the quotient space relative to this action P/H, one can show that it can be identified with the associated fibre bundle $E = P \times_G (G/H)$.

3.1.4 Associated vector bundle

Let us specify the case when F is a k-dimensional vector space V. Let ρ be the k-dimensional representation of G on V. Then we define the associated vector bundle $P \times_\rho V$ as an equivalence class $(P \times V)/G$, where we identify the points (u, v) and $(ug, \rho(g)^{-1}v)$ of $P \times V$, for each $g \in G$.

3.1.5 Connection

Let us now devote ourself to the theory of connections on a principal bundle. For that purpose we use the splitting of tangent the space $T_u(P)$ into vertical and horizontal subspaces.

For a given principal bundle $P(M, G)$ over a manifold M and a structure group G let $T_u(P)$ be a tangent space of P at $u \in P$ and V_u be the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through u. A connection Γ on P is an assignment of a subspace H_u of $T_u(P)$ to $u \in P$ such that

1. $T_u(P) = V_u \oplus H_u$ is a direct sum;

2. $H_{ua} = (R_a)_* H_u$ for each $u \in P$ and $a \in G$, where R_a is the right action on P induced by $a \in G$: $R_a u = ua$;

3. H_u depends differentiably on u.

H_u is called horizontal subspace at u and has the same dimension as the base space M. V_u is called vertical subspace at u and has the same dimension as the structure group G.

At this point we look at how the vertical space $V_u(P)$ can be constructed. For an element A of the Lie algebra $\mathfrak{g}$ of a given Lie group G the right action

$$R_{\exp(tA)} u = u \exp(tA) \tag{3.3}$$

defines a curve through u in P. Since $\pi(u) = \pi(u \exp(tA)) = p$, for some $p \in M$, this curve lies within the fibre $\pi^{-1}(p)$. Then we can define the fundamental vector field $A^* \in TP$ corresponding to A by

$$A^* f(u) = \frac{\mathrm{d}}{\mathrm{d}t} f(u \exp(tA))|_{t=0}, \tag{3.4}$$

where $f : P \longrightarrow \mathbb{R}$ is an arbitrary smooth function and $u \in P$. Obviously, the vector A^* is tangent to V at every u, so that $A^* \in V_u P$. We see that the fundamental vector fields associated to generators of the Lie algebra give a basis of the vertical subspace VP.

3.1.6 Holonomy group

Before going further, we give at this point a brief discussion about holonomy group. The main core of theory is that any kind of connection on a manifold gives rise to some notion of holonomy.

The definition for holonomy of connections on principal bundles proceeds in the following fashion. Let G be a Lie group and $\pi : P \longrightarrow M$ a principal G-bundle over a smooth manifold M which is paracompact. Let some connection be defined on P. For a given a piecewise smooth loop $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = x$ for some $x \in M$ and a point p in the fibre over x, i.e., $\pi(p) = x$, the connection defines a unique horizontal lift $\tilde{\gamma} : [0,1] \longrightarrow P$ such that

$$\tilde{\gamma}(0) = p \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}(t) \in H_{\tilde{\gamma}(t)} \subset T_{\tilde{\gamma}(t)}P. \tag{3.5}$$

The end point of the horizontal lift over a closed path, will not necessarily be the same point p but rather some other point $p \cdot g$ in the fibre over x. Therefore, we can introduce the notion of a holonomy group for fibre bundles as follows:
Let $\pi : P \longrightarrow M$ be a principal G-bundle, $p \in P_x, x \in M$. Then the holonomy group Hol_p of P in p is the following subgroup of G:

$$Hol_p = \{g \in G : \exists\, \text{loop}\, \gamma : [0,1] \longrightarrow M\, \gamma(0) = p,\ \text{so that}\, \tilde{\gamma}(1) = p \cdot g\}. \tag{3.6}$$

The restricted holonomy group based at p is the subgroup Hol_p^0 coming from horizontal lifts of contractible loops γ.

3.1.7 Connection one-form

A usual approach of how to separate T_uP into V_uP and H_uP is by introducing a Lie algebra valued one-form $\omega \in \mathfrak{g} \otimes T^*P$ called the connection one-form. For each $X \in T_uP$ the value of $\omega(X)$ is defined as a unique $A \in \mathfrak{g}$, such that $(A^*)_u$ is the vertical component of X. This projection property can be summarized by the following requirements,

i) $\omega(A^*) = A$ for any $A \in \mathfrak{g}$;

ii) $(R_a)^*\omega = \mathrm{adj}(a^{-1})\omega$, i.e., $(R_a)^*\omega_{ua}(X) = \omega_{ua}(R_a^*(X)) = a^{-1}\omega_u(X)a$, for all $X \in T_uP$ and $a \in \mathfrak{g}$.

Conversely, for a connection one-form ω satisfying i) and ii) we define the horizontal subspace H_uP as the kernel of ω:

$$H_u = \{X \in T_uP | \omega(X) = 0\}. \tag{3.7}$$

It is clear that the canonical projection $\pi : P \longrightarrow M$ maps the horizontal space $H_u(P)$ onto $T_{\pi(u)}M$ isomorphically.

Let $\{U_\alpha\}$ be an open covering of M and let σ_α be a local section defined on each U_α. For each α we define a Lie algebra valued one-form A_α on U_α, by

$$A_\alpha = \sigma_\alpha^* \omega. \tag{3.8}$$

The form A_α is also called local connection one-form or gauge potential. In case, a Lie algebra valued one-form A_α on U_α is given, it can be shown that one can uniquely reconstruct the connection one-form $\omega \in \mathfrak{g} \otimes \Omega^1(U_\alpha)$ on $\pi^{-1}(U_\alpha)$ whose pullback by σ_α is A_α.

For an open covering $\{U_\alpha\}$ of M with a family of isomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ we can choose in the principal bundle P a preferred set of local sections σ_α defined via $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$, where $x \in U_\alpha$ and e is the identity in G. Then the local connection one-forms $A_\alpha = \sigma_\alpha^* \omega$ satisfy on the intersection $U_\alpha \cap U_\beta$ the compatibility condition

$$\begin{aligned} A_\beta &= \mathrm{adj}_{\psi_{\alpha\beta}^{-1}}(A_\alpha) + \psi_{\alpha\beta}^{-1} \mathrm{d}\psi_{\alpha\beta} && (3.9) \\ &= \psi_{\alpha\beta}^{-1} A_\alpha \psi_{\alpha\beta} + \psi_{\alpha\beta}^{-1} \mathrm{d}\psi_{\alpha\beta}. && (3.10) \end{aligned}$$

3.1.8 Curvature 2-form

In oder to go over to the notion of the curvature of a connection, we recall that a connection ω on a principal bundle $P(M, G)$ separates $T_u P$ into $H_u P \oplus V_u P$. Accordingly, each vector X of $T_u P$ decomposes as $X = X^H + X^V$, where $X^H \in H_u P$ and $X_V \in V_u P$. Thus, we can define the concept of covariant derivative on the principal fibre bundle P as follows. Let ϕ be a Lie algebra valued r-form, $\phi \in \Lambda^r(P) \otimes \mathfrak{g}$ and let $X_1, ..., X_{r+1} \in T_u P$ be $r+1$ tangent vectors on P. The (exterior) covariant derivative of ϕ is defined via

$$D\phi(X_1, ..., X_{r+1}) = \mathrm{d}_P(X_1^H, ..., X_{r+1}^H), \tag{3.11}$$

where $\mathrm{d}_P \phi = \mathrm{d}\phi^a \otimes T_a$, $\phi = \phi^a \otimes T_a$ and T_a form a basis of the Lie algebra $\mathfrak{g}$.

The definition of a curvature form now is quite simple. For a given local connection one-form ω on P it is the covariant derivative of it:

$$\Omega = D\omega \in \Lambda^2 P \otimes \mathfrak{g}. \tag{3.12}$$

For any curvature form Ω the following theorem called the Cartan structure equation holds:

Theorem 1. *Let ω be a connection form and Ω its curvature form. Then*

$$d\omega(X, Y) = -\frac{1}{2}[\omega, \omega](X, Y) + \Omega(X, Y), \tag{3.13}$$

for $X, Y \in T_u(P)$, $u \in P$.

The structure equation (3.13) can be expressed as

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega, \tag{3.14}$$

or

$$d\omega = -\omega \wedge \omega + \Omega \tag{3.15}$$

Here we use the Lie bracket defined for two Lie algebra valued forms $\alpha \in \Lambda^p \otimes \mathfrak{g}$ and $\beta \in \Lambda^q \otimes \mathfrak{g}$ as

$$\begin{aligned} [\alpha, \beta] &= \alpha \wedge \beta - (-1)^{pq}\beta \wedge \alpha & (3.16)\\ &= T_a T_b \alpha^a \wedge \beta^b - (-1)^{pq} T_b T_a \beta^b \wedge \alpha^a & (3.17)\\ &= [T_a, T_b]\alpha^a \wedge \beta^b = f^c_{ab} T_c \alpha^a \wedge \beta^b, & (3.18) \end{aligned}$$

where f^c_{ab} are structure constants of the Lie algebra $\mathfrak{g}$ and T_a are its generators. Furthermore, it can be shown that every curvature form Ω fulfills the Bianchi identity: $D\Omega = 0$.

3.1.9 Bianchi equation

Analogously to the definition of the local connection one-form A_α on M we use a local section σ_α to pull back a global curvature form Ω on P to a local object F_α living on a patch of M. Namely, for a given section σ_α on $U_\alpha \subset M$ we define the local (Yang-Mills) field strength by

$$F_\alpha = \sigma^*_\alpha \Omega \in \Lambda^2(U_\alpha) \otimes \mathfrak{g}. \tag{3.19}$$

One can show that F is formally expressed in terms of the gauge potential as

$$F_\alpha = \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha. \tag{3.20}$$

Locally we can write the connection and curvature forms as $A = A_\mu \mathrm{d}x^\mu$ and $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ (here we omit the subscript α). Then we can find that the components of F are related to those of A in the following way,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.21}$$

With this relation one can obtain that on intersection of two local trivializations $\{U_\alpha, \phi_\alpha\}$ and $\{U_\beta, \phi_\beta\}$ the transformation of the curvature form has a simply form

$$F_\beta = g^{-1}_{\alpha\beta} F_\alpha g_{\alpha\beta}, \tag{3.22}$$

where $g_{\alpha\beta}$ is the corresponding transition function. Moreover, since $\mathrm{d}^2 = 0$, one finds that

$$\begin{aligned} \mathrm{d}F_\alpha &= \mathrm{d}A_\alpha \wedge A_\alpha - A_\alpha \wedge \mathrm{d}A_\alpha = F_\alpha \wedge A_\alpha - A_\alpha \wedge F_\alpha & (3.23)\\ &= -[A_\alpha, F_\alpha]. & (3.24) \end{aligned}$$

This equation results in the local identity called the Bianchi equation:

$$\mathcal{D}_\alpha F_\alpha \quad = \quad \mathrm{d}dF_\alpha + [A_\alpha, F_\alpha] = 0, \tag{3.25}$$

where the covariant derivative $\mathcal{D}$ is defined by

$$\mathcal{D}_\alpha \quad = \quad \mathrm{d} + [A_\alpha, \cdot]. \tag{3.26}$$

3.2 Yang-Mills equations on G/H

We have already gathered all needed ingredients to start our consideration of the Yang-Mills theory on coset spaces in order to finally go over to the concrete study of its equations on nearly Kähler manifolds. We begin with the most important things and end up with some solutions of the Yang-Mills equations to the issue of the chapter.

3.2.1 Coset space G/H

In the following we want to concretize our consideration supposing that M is a reductive homogeneous space of the form G/H, where G is a Lie group and H is its closed subgroup. Let $\{I_A\}$, $A = 1, ..., \dim G$, be the generators of the Lie group G and f^C_{AB} the corresponding structure constants,

$$[I_A, I_B] = f^C_{AB} I_C. \tag{3.27}$$

Since the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, as we have already seen in section 2.7.4 the generators $\{I_A\}$ split in two subsets $\{I_i\}$, $i = 1, ..., \dim H$, and $\{I_a\}$, $a = 1, ..., \dim G - \dim H$, of the generators of $\mathfrak{h}$ and $\mathfrak{m}$, respectively. As usual we identify the subspace $\mathfrak{m}$ of $\mathfrak{g}$ with the tangent space of G/H at any given point $x \in G/H$. The commutator relations for reductive homogeneous space satisfy (see equations (2.172))

$$[I_i, I_j] = f^k_{ij} I_k, \quad [I_i, I_a] = f^c_{ia} I_c, \quad [I_a, I_b] = f^i_{ab} I_i + f^c_{ab} I_c. \tag{3.28}$$

The Killing-Cartan metric on the Lie algebra $\mathfrak{g}$ can be taken to have the form

$$g_{AB} = f^C_{AD} f^D_{CB} = \delta_{AB} \tag{3.29}$$

after an appropriate normalization of the generators I_A. Explicitly one obtains

$$g_{ij} \quad = \quad f^k_{il} f^l_{kj} + f^b_{ia} f^a_{bj} = \delta_{ij}, \quad g_{ia} = 0, \tag{3.30}$$

$$g_{ab} \quad = \quad 2 f^i_{ad} f^d_{ib} + f^c_{ad} f^d_{cb} = \delta_{ab}. \tag{3.31}$$

On the coset space G/H we use the induced metric g_{ab} coming from g_{AB} on G.

3.2.2 Connection on G/H

The basis elements I_A of the Lie algebra $\mathfrak{g}$ can be represented by left-invariant vector fields $\hat{E}_A$ on the Lie group G. We denote by $\hat{e}^A$ the set of left-invariant one-forms dual to $\hat{E}_A$. The space G/H consists of left cosets gH and the natural projection $\pi : G \to G/H$, $g \mapsto gH$. Over a small contractible open subset U of G/H, one can choose a map $L : U \to G$ such that $\pi \circ L$ is the identity, i.e., L is a local section of the principal bundle $G \to G/H$. The pullbacks of $\hat{e}^A$ by L we denote e^A. In particular, e^a form an orthonormal frame for $T^*(G/H)$ over U, so we can write $e^i = e^i_a e^a$ with real functions e^i_a. The dual frame for $T(G/H)$ will be denoted by E_a.

For a given connection 1-form on G/H, which is eventually torsion-full, the following equation holds (compare with (2.7.6):

$$\mathrm{d}e^a + \omega^a_b \wedge e^b = T^a, \tag{3.32}$$

where $T^a = \frac{1}{2} T^a_{bc} e^b \wedge e^c$. The application examples, which are the most interesting for us, allow to choose the torsion components proportional to the structure constants,

$$T^a_{bc} = \kappa f^a_{bc}, \tag{3.33}$$

where κ is an arbitrary real parameter.

If we look from the other point of view, we can use the Mauer-Cartan equation to get similar equation. Namely, the 1-forms $\theta^A := e^A \otimes I_A$ satisfy

$$\mathrm{d}\theta = -\theta \wedge \theta. \tag{3.34}$$

This leads to the relations

$$\mathrm{d}e^a = -f^a_{ib} e^i \wedge e^b - \frac{1}{2} f^a_{bc} e^b \wedge e^c, \qquad \mathrm{d}e^i = -\frac{1}{2} f^i_{bc} e^b \wedge e^c - \frac{1}{2} f^i_{jk} e^j \wedge e^k. \tag{3.35}$$

Plugging (3.32), (3.33) and (3.35) together we find that [70, 71]

$$\omega^a_b = f^a_{ib} e^i + \frac{1}{2}(\kappa + 1) f^a_{cb} e^c =: \omega^a_{cb} e^c, \tag{3.36}$$

where we define

$$\omega^a_{cb} = f^a_{ib} e^i_c + \frac{1}{2}(\kappa + 1) f^a_{cb}. \tag{3.37}$$

3.2.3 Yang-Mills equations

Consider the principal fibre bundle $P(G/H, G)$ over G/H with the structure group G. As we learned from the previous section a $\mathfrak{g}$-valued connection, one-form $\mathcal{A}$

on P and the corresponding field strength $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, which can locally be expressed in the basis $\{e^a\}$ as

$$\mathcal{A} = A_a e^a \quad \text{and} \quad \mathcal{F} = \frac{1}{2} F_{ab} e^a \wedge e^b, \tag{3.38}$$

fulfill the Yang-Mills equation. In components this equation is given by

$$\mathcal{D}_a F^{ab} = \nabla_a F^{ab} + [A_a, F^{ab}] \tag{3.39}$$
$$= E_a F^{ab} + \omega^d_{da} F^{ab} + \omega^b_{ad} F^{ad} + [A_a, F^{ab}] = 0. \tag{3.40}$$

Note that since we work on a non-holonomic basis $\{E_a\}$, we must use at this point the covariant derivative ∇ associated to the torsion-full connection ω on G/H from (3.32) (see [70, 71] and references therein).

3.2.4 Hermitian-Yang-Mills equations

In this section we discuss a special class of solutions to the Yang-Mills equations (3.25). This class includes Hermitian-Yang-Mills connections on Kähler manifolds.

Let $\pi : E \longrightarrow M_{2m}$ be a unitary bundle of complex rank r, i.e., a vector bundle arises from a unitary representation of a structure group G (the simplest case is of cause $G =$U(r), so that $E = M \times U(r)$). Let M be a complex m-dimensional Kähler manifold with local real coordinates $x = (x^i)$ and the tangent space basis $\partial_i := \frac{\partial}{\partial x^i}$ for $i, j = 1, ..., 2m$. Then a metric g and the corresponding Kähler two-form ω read $\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\omega = \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$, respectively. We consider on the complex vector bundle E a gauge connection $\mathcal{A}$ and the curvature two-form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ corresponding to $\mathcal{A}$. For any connection $\mathcal{A}$ on E over M the components A_i and F_{ij} take values in $\mathfrak{u}(r)$. Moreover, the 2-form $\mathcal{F}$ decomposes as

$$\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}, \tag{3.41}$$

where $\mathcal{F}^{2,0}$ denotes the $(2,0)$-part of $\mathcal{F}$ ($\mathcal{F}^{0,2} = -(\mathcal{F}^{2,0})^*$), and $\mathcal{F}^{1,1}$ denotes the $(1,1)$-part of $\mathcal{F}$ with respect to the complex coordinates: $z^i = x^i + ix^{m+i}$, $\bar{z}^i = x^i - ix^{m+i}$, $i = 1, ..., m$. We call $\mathcal{A}$ a Hermitian-Yang-Mills connection on E if $\mathcal{A}$ is unitary and

$$\mathcal{F}^{1,1} \cdot \omega = \lambda \mathrm{Id}, \qquad \mathcal{F}^{0,2} = 0, \tag{3.42}$$

where $\lambda = \frac{m(C_1(E)[\omega]^{m-1}}{r[\omega]^m}$ is a constant factor, $C_1(E)$ the first Chern class of E and $[\omega]$ denotes the cohomology class represented by ω. In particular, after simple changing of the field strength $\mathcal{F}$ by performing the shift $\mathcal{F} \longrightarrow \mathcal{F} - \frac{1}{\mathrm{rank}E}\mathrm{tr}(\mathcal{F})$ one can assume that $\lambda = 0$ [72]. Equations (3.42) are known as Donaldson-Uhlenbeck-Yau or Hermitian-Yang-Mills equations [73, 74, 75]. In the special case of an almost complex 4-manifold with a metric g they coincide with the anti-self-dual Yang-Mills (ASDYM) equations

$$*\mathcal{F} = -\mathcal{F}, \tag{3.43}$$

where $*$ is the Hodge operator.

3.2.5 Some explicit solutions of the Hermitian-Yang-Mills equations

Solutions of the HYM equation on $\mathbb{C}P^n$

In [75] it was shown that some explicit solutions of the Hermitian-Yang-Mills equations on $\mathbb{C}P^n$ can be obtained in the commutative limit from non-commutative instanton-type solutions on $\mathbb{C}P^n$. We briefly sketch the ansatz and the construction for the commutative case.

We consider the group $\mathrm{U}(n+1)$. The complex projective space $\mathbb{C}P^n$ can be considered as the quotient space $\mathrm{U}(n+1)/\mathrm{U}(1)\times U(n)$ with the projection

$$\mathrm{U}(n+1) \xrightarrow{\mathrm{U}(1)\times\mathrm{U}(n)} \mathbb{C}P^n \tag{3.44}$$

and fibres $\mathrm{U}(1)\times\mathrm{U}(n)$. For $g\in\mathrm{U}(n+1)$ the canonical one-form $\Omega = g^\dagger \mathrm{d}g$ on $\mathrm{U}(n+1)$ takes values in the Lie algebra $\mathfrak{u}(n+1)$ and satisfies the Maurer-Cartan equation.

We define a $(n+1)\times(n+1)$ matrix $V\in\mathrm{U}(n+1)$ as the matrix

$$V = \begin{pmatrix} \gamma^{-1} & -\gamma^{-1}Y^\dagger \\ Y\gamma^{-1} & \Lambda \end{pmatrix}, \tag{3.45}$$

where

$$Y := \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \tag{3.46}$$

is a complex vector,

$$\gamma := \sqrt{y^a \bar{y}^a + 1} \tag{3.47}$$

and

$$\Lambda = \mathrm{Id}_n - Y\frac{1}{\gamma(\gamma+1)}Y^\dagger. \tag{3.48}$$

With this construction we can build a connection 1-form

$$\mathcal{A} = V^\dagger \mathrm{d}V \tag{3.49}$$

with the vanishing curvature two-form $\mathcal{F} = \mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$.

The flat connection form $\mathcal{A}$ can be rewritten in a block form

$$\mathcal{A} = \begin{pmatrix} a & -\phi^\dagger \\ \phi & A \end{pmatrix} \tag{3.50}$$

with $a \in \mathfrak{u}(1)$ and $A \in \mathfrak{u}(n)$. The values of ϕ, a and A can be obtained from the definition of $\mathcal{A}$ (3.49). Finally, one founds that the two-forms

$$\begin{aligned} F_{\mathfrak{u}(1)} &= \mathrm{d}a + a \wedge a = \phi^\dagger \wedge \phi = \delta_{\bar{a}b}\bar{\phi}^{\bar{a}} \wedge \phi^b, &(3.51)\\ F_{\mathfrak{u}(n)} &= \mathrm{d}A + A \wedge A = \phi \wedge \phi^\dagger = (\phi^a \wedge \bar{\phi}^{\bar{b}}) &(3.52)\end{aligned}$$

obey the Hermitian-Yang-Mills equation with respect to the basis $\{\phi^a \wedge \bar{\phi}^{\bar{b}}\}$ of $(1,1)$-forms:

$$F_{ab} = 0 = F_{\bar{a}\bar{b}}, \qquad F_{a\bar{a}} = F_{1\bar{1}} + ... + F_{n\bar{n}} = \mathrm{Id}_n \tag{3.53}$$

and

$$f_{ab} = 0 = f_{\bar{a}\bar{b}}, \qquad f_{a\bar{a}} = f_{1\bar{1}} + ... + f_{n\bar{n}} = n, \tag{3.54}$$

where $F_{ab}, F_{a\bar{a}}$ and $f_{ab}, f_{a\bar{a}}$ are components of $F_{\mathfrak{u}(n)}$ and $F_{\mathfrak{u}(1)}$, respectively.

The matrix V from (3.45) defines an embedding of $\mathbb{C}P^n$ into $\mathrm{U}(n+1)$. For this embedding the one-form Ω on $\mathbb{C}P^n$ matches with the flat connection $\mathcal{A}$ on $U(n+1)$ given by (3.49). The splitting of $\mathcal{A}$ in components a, A, ϕ corresponds to the decomposition of $\mathcal{A}$ in one-forms on $\mathbb{C}P^n$ with values in $\mathfrak{u}(1) \oplus \mathfrak{u}(n)$ and into one-forms lying in $T(\mathbb{C}P^n)$. The gauge potential A defining $F_{\mathfrak{u}(n)}$ and satisfying the Hermitian-Yang-Mills equation coincides with the canonical connection on $\mathbb{C}P^n$.

Anti-self-dual gauge fields on $M^4 = S^4$

As we have already mentioned, on four dimensional manifolds the DUY equations are equivalent to the anti-self-dual Yang-Mills equation $*\mathcal{F} = -\mathcal{F}$. We give here an explicit solution of the ASD for the special example $M^4 = S^4 = \mathrm{Sp}(2)/\mathrm{Sp}(1){\times}\mathrm{Sp}(1)$. Let start with a flat connection $\mathcal{A}$ on the trivial vector bundle $M^4 \times \mathbb{C}^4 \longrightarrow M^4$ given by the one-form

$$\mathcal{A} = Q^{-1}\mathrm{d}Q =: \begin{pmatrix} A^- & -\phi \\ \phi^\dagger & A^+ \end{pmatrix}, \tag{3.55}$$

where Q's are 4×4 matrices in $\mathrm{Sp}(2){\subset}\mathrm{SU}(4)$. We consider the local sections Q of the form

$$Q := f^{-\frac{1}{2}} \begin{pmatrix} \mathrm{Id}_2 & -x \\ x^\dagger & \mathrm{Id}_2 \end{pmatrix} \quad \text{and} \quad Q^{-1} = Q^\dagger = f^{-\frac{1}{2}} \begin{pmatrix} \mathrm{Id}_2 & x \\ -x^\dagger & \mathrm{Id}_2 \end{pmatrix}, \tag{3.56}$$

where

$$x = x^\mu \tau_\mu, \quad x^\dagger = x^\mu \tau_\mu^\dagger, \quad f := 1 + x^\dagger x = 1 + \delta_{\mu\nu}x^\mu x^\nu, \tag{3.57}$$

and

$$(\tau_\mu) = (-i\sigma^i, \mathrm{Id}_2) \quad \text{and} \quad (\tau^\dagger_\mu) = (i\sigma^i, \mathrm{Id}_2) \tag{3.58}$$

with

$$\tau^\dagger_\mu \tau_\nu =: \delta_{\mu\nu}\mathrm{Id}_2 + \eta_{\mu\nu} \tag{3.59}$$

Here x^μ are local coordinates on $U \subset S^4$. From (3.55) and (3.56) we obtain

$$A^- = \frac{1}{f}\bar{\eta}_{\mu\nu}x^\mu \mathrm{d}x^\nu \in \mathfrak{su}(2), \tag{3.60}$$

$$A^+ = \frac{1}{f}\eta_{\mu\nu}x^\mu \mathrm{d}x^\nu \in \mathfrak{su}(2), \tag{3.61}$$

$$\phi = \frac{1}{f}\mathrm{d}x = \frac{-i}{f}\begin{pmatrix} \mathrm{d}x^3 + i\mathrm{d}x^4 & \mathrm{d}x^1 - i\mathrm{d}x^2 \\ \mathrm{d}x^1 + i\mathrm{d}x^2 & -(\mathrm{d}x^3 - i\mathrm{d}x^4) \end{pmatrix} = \tag{3.62}$$

$$= \frac{-i}{f}\begin{pmatrix} \mathrm{d}z & \mathrm{d}\bar{y} \\ \mathrm{d}y & -\mathrm{d}\bar{z} \end{pmatrix} =: \frac{-i}{2\Lambda}\begin{pmatrix} \theta^2 & \theta^{\bar{1}} \\ \theta^1 & -\theta^{\bar{2}} \end{pmatrix}, \tag{3.63}$$

with

$$\theta^1 := \frac{2\Lambda \mathrm{d}y}{1+r^2} = \overline{(\theta^{\bar{1}})}, \qquad \theta^2 := \frac{2\Lambda \mathrm{d}z}{1+r^2} = \overline{(\theta^{\bar{2}})}, \tag{3.64}$$

where the bar denotes complex conjugation and the real parameter Λ is the radius of S^4. The connection (3.55) is flat, so we get

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{3.65}$$

$$= \begin{pmatrix} F^- - \phi \wedge \phi^\dagger & -(\mathrm{d}\phi + A^- \wedge \phi + \phi \wedge A^+) \\ \mathrm{d}\phi^\dagger + A^+ \wedge \phi^\dagger + \phi^\dagger \wedge A^- & F^+ - \phi^\dagger \wedge \phi \end{pmatrix} \tag{3.66}$$

$$= 0, \tag{3.67}$$

where $F^\pm = \mathrm{d}A^\pm + A^\pm \wedge A^\pm$ (3.65) implies

$$F^- = \phi \wedge \phi^\dagger = \frac{1}{4\Lambda^2}\begin{pmatrix} \theta^1 \wedge \theta^{\bar{1}} - \theta^2 \wedge \theta^{\bar{2}} & 2\theta^{\bar{1}} \wedge \theta^2 \\ -2\theta^1 \wedge \theta^{\bar{2}} & -\theta^1 \wedge \theta^{\bar{1}} + \theta^2 \wedge \theta^{\bar{2}} \end{pmatrix}, \tag{3.68}$$

$$F^+ = \phi^\dagger \wedge \phi = \frac{1}{4\Lambda^2}\begin{pmatrix} \theta^1 \wedge \theta^{\bar{1}} + \theta^2 \wedge \theta^{\bar{2}} & 2\theta^{\bar{1}} \wedge \theta^{\bar{2}} \\ -2\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^{\bar{1}} - \theta^2 \wedge \theta^{\bar{2}} \end{pmatrix}. \tag{3.69}$$

It can easily be shown that F^- is anti-self-dual (ASD) gauge field on a rank-2 complex vector bundle $E \longrightarrow M^4$, i.e., $*F^- = -F^-$. This also means that F^- obeys the HYM equation as well.

These solution of ASD equations on S^4 can be used in further applications for instance for finding some solutions of HYM equations on twistor space of S^4. This is a special sort of spaces which is in case of S^4 in particular Kähler and homogeneous. The complete information about solutions to HYM on S^4 and some other four dimensional manifolds as well as their twistor spaces can be found in reference [72].

Solutions of HYM equations on Calabi-Yau cones

We do not give here the explicit solutions of HYM equation on Calabi-Yau cones, but refer the reader to the paper of Correia [76], where the author approaches $2d^{\mathbb{C}}$-dimensional Calabi-Yau cones and suggests explicit solutions of the flow equations that correspond to non-trivial $\mathrm{SU}(d^{\mathbb{C}})$ HYM instantons.

3.3 Manifolds with G_2-structure

At first we are going to introduce the notion of a G_2-structure and in this context take a look at the compact 6-dimensional nearly Kähler coset spaces G/H. Such manifolds are important examples of SU(3)-structure manifolds used in flux compactifications of string theories.

3.3.1 Seven-dimensional manifolds with G_2-structure

In this section we give a brief introduction to G_2-geometry, following [63]. Beginning with a 7-dimensional vector space V^7, we consider the group $\mathrm{GL}(7,\mathbb{R})$ acting on $\Lambda_3(V^7)$ with two open orbits. For v in one of these orbits the stabilizer[1] $\mathrm{St}(v) \in \mathrm{GL}(7,\mathbb{R})$ is 14-dimensional. It can easily be shown that $\mathrm{St}(v)$ is a real representation of a Lie group G_2. Depending on in which orbit v is, the stabilizer is either compact or non-compact form of G_2. A 7-manifold M equipped with a 3-form $\phi \in \Lambda_3(V^7)$ is called a G_2-manifold if ϕ is stable everywhere in M. In this case we say that the structure group of M is reduced to G_2. Moreover, on M there can be declared a natural Riemannian structure g_ϕ:

$$x, y \longrightarrow \int_M (\phi \lrcorner x) \wedge (\phi \lrcorner y) \wedge \phi, \qquad \forall x, y \in TM. \tag{3.70}$$

One says that G_2-manifold is parallel if $\nabla\phi = 0$, where ∇ is the Levi-Civita connection associated with this Riemannian structure.

In local coordinates $\{x^1, ..., x^7\}$ of $\mathbb{R}^7$ the 3-form ϕ can be given by

$$\phi = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}, \tag{3.71}$$

with exterior form $\mathrm{d}x^{123} = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ on $\mathbb{R}^7$. In each case a G_2-structure fixes the 4-form $*\phi$, the Euclidean metric $g = (\mathrm{d}x^1)^2 + ...(\mathrm{d}x^7)^2$ and orientation of M [77].

A G_2-manifold may contain isolated singularities whose simplest examples are conical singularities. In general, a metric space X with fixed points $x^1, ..., x^n$ is called a space with isolated singularities if $X\backslash\{x^1, ..., x^n\}$ is a Riemannian manifold.

---

[1] For every x in a set X and a group G acting on X the stabilizer subgroup of X is defined as the set G_x of all elements of X fixing x, $G_x\{g \in G | gx = x\}$.

Suppose, there is only one singular point $x \in X$. Then the singularity is called conical if X is equipped with a flow acting on X by homotheties and contracting X to x. In this case, X is isomorphic to a Riemannian cone of a Riemannian manifold M.

As it will be discussed in the next section, a cone $C(M)$ of a nearly Kähler manifold is always equipped with a parallel G_2-structure. The converse statement is also true: every conical singularity of a parallel G_2-manifold is obtained as $C(M)$, for some nearly Kähler manifold M (see for example [63]).

In general there exist exactly 16 classes of G_2-structures. For more information see for instance [78]. Moreover, we remark that G_2 manifold is always Ricci-flat, and a G_2 structure ϕ is torsion-free if and only if it is both closed and coclosed: $\mathrm{d}\phi = 0$ and $\mathrm{d}\psi = \mathrm{d}(*_{g_\phi}\rho) = 0$ [79].

3.3.2 Cones over nearly Kähler 6-manifolds

First of all we repeat what we have already learned about nearly Kähler manifolds in previous chapter. An almost Hermitian manifold (M, g, J), with fundamental two-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ and Levi-Civita connection ∇ is called nearly Kähler if the corresponding canonical Hermitian connection $\bar{\nabla}$ has a totally skew-symmetric torsion. It is also known that nearly Kähler manifolds with integrable almost complex structure are necessarily Kähler. The non-trivial case is then the so-called strict nearly Kähler manifold, when $\mathrm{d}\omega = \nabla\omega = 3\nabla J \neq 0$ at every point of M.

It was shown by A. Gray and P.A. Nagy (see [80] and references therein) that any nearly Kähler manifold is locally a product of 6-dimensional strict nearly Kähler manifolds, locally homogeneous manifolds and twistor spaces of positive quaternionic Kähler manifolds. Due to this result, the most interesting objects remaining to be studied are strict nearly Kähler structures in dimension 6. We remark that 4-dimensional nearly Kähler manifolds are automatically Kähler. On the other hand, the dimension 6 is particularly important because of some amount of properties which we discuss in this section.

One of the important results is the fact that nearly Kähler structures are closely related to G_2 structures via the cone construction. To see this at first we give a description of the kind of objects cones are.

Definition 5. *Let (M, g) be a Riemannian manifold. The Riemannian cone of M is*

$$C(M) := (M \times \mathbb{R}_{>0}, t^2 g + dt^2),$$

where t denotes the coordinate on the half-line $\mathbb{R}_{>0}$.

Depending on the cone structure of a manifold M one can make some conclusions about the geometry of the basis manifold M. If $C(M)$ has special holonomy, then the geometry of M can be given in the following table. Furthermore, im-

Table 3.1: Riemannian cones with special holonomy

Holonomy of $C(M)$	Geometry of $C(M)$	Geometry of M
SO(n)	Riemannian	-
U(n)	Kähler	Sasakian
SU(n)	Calabi-Yau	Sasaki-Einstein
Sp(n)	hyper-Kähler	3-Sasakian
Sp(n)Sp(1)	quaternionic-Kähler	-
G_2	G_2-manifolds	nearly Kähler
Spin(7)	Spin(7)-manifolds	nearly G_2-manifolds

portant statements [80, 81] are:

- A Riemannian manifold (M^6, g) carries a nearly Kähler structure if and only if its cone $(M^6 \times \mathbb{R}_{>0}, t^2 g + \mathrm{d}t^2)$ has holonomy contained in G_2.
- Real Killing spinors on M correspond uniquely to parallel spinors on $C(M)$. This implies that then $C(M)$ is Ricci-flat. G_2-manifolds also belongs to the Ricci-flat manifolds and as follows admit parallel spinors.

More information can for instance be found in [63]. As an example of cone over nearly Kähler manifold we mention $\mathbb{R}^7$ which is the cone over the sphere S^6.

3.3.3 Special G_2-manifolds of type $\mathbb{R}\times G/H$ with nearly Kähler coset spaces G/H

In this section we consider a special type of G_2-manifolds, namely the cylinder spaces $\mathbb{R} \times G/H$ over nearly Kähler coset spaces G/H. So suppose that the coset space G/H is nearly Kähler. This means that the manifold G/H admits a Riemannian metric g, an almost complex structure J, a real two-form ω and a complex three-form Ω reducing the structure group of the tangent bundle to SU(3) and satisfying the compatibility condition $g(J\cdot,\cdot) = \omega(\cdot,\cdot)$. With respect to J, the forms ω and Ω are of type $(1,1)$ and $(3,0)$, respectively. We fix their normalization by the volume form V_g of g so that

$$\omega \wedge \omega \wedge \omega = 6V_g \quad \text{and} \quad \Omega \wedge \bar{\Omega} = -8iV_g. \tag{3.72}$$

This SU(3)-structure on the coset space is nearly Kähler if additionally

$$\mathrm{d}\omega = 3\rho \mathrm{Im}\Omega \quad \text{and} \quad \mathrm{d}\Omega = 2\rho\omega \wedge \omega \tag{3.73}$$

for some real non-zero constant ρ (in case $\rho = 0$ the manifold is Calabi-Yau).

Moreover, an important property of the compact nearly Kähler manifolds in general is a 3-symmetry. It manifests itself in the existence of an automorphism

s on the structure group G with the features s^3 =Id and $s(H) = H$. This automorphism is special, because it gives rise to definition of a canonical almost complex structure on the coset space itself. This proceeds as follows.

Since s acts on G/H, there is a corresponding automorphism $S : \mathfrak{g} \longrightarrow \mathfrak{g}$ on the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of G, which in particular is trivial on the $\mathfrak{h}$, the Lie algebra of the fixed subgroup $H \subset G$. It can be shown that on $\mathfrak{m}$ the map S defines an almost complex structure $J : \mathfrak{m} \longrightarrow \mathfrak{m}$ via

$$S_{|\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J = \exp(\frac{2\pi}{3} J) \tag{3.74}$$

where J satisfies $J^2 = -\mathrm{Id}$. Therefore, the G-invariant metric g, the almost complex structure and the nearly Kähler form ω on the coset G/H have in terms of the local orthogonal frame $\{e^a\}$ and the corresponding dual frame E_a the following form

$$g = \delta_{ab} e^a e^b, \quad J = J_a^b e^a E_b \quad \text{and} \quad \omega = \frac{1}{2} J_{ab} e^a \wedge e^b. \tag{3.75}$$

The Mauerer-Cartan equations (3.35) imply for $\mathrm{d}\omega$ and $*\mathrm{d}\omega$ that

$$\mathrm{d}\omega = \frac{1}{2} J_{ab} (\mathrm{d}e^a \wedge e^b - e^a \wedge \mathrm{d}e^b) = -\frac{1}{2} \tilde{f}_{abc} e^a \wedge e^b \wedge e^c, \tag{3.76}$$

$$*\mathrm{d}\omega = \frac{1}{2} f_{abc} e^a \wedge e^b \wedge e^c, \tag{3.77}$$

where

$$\tilde{f}_{abc} := f_{abd} J_{dc} \tag{3.78}$$

are components of a totally antisymmetric tensor on G/H. Because of (3.73), we see that $\mathrm{Im}\Omega$ should be normalized so that $||\mathrm{Im}\Omega||^2 = (\mathrm{Im}\Omega)_{abc}(\mathrm{Im}\Omega)_{abc} = 4$. Besides, on any nearly Kähler coset the following identities between the components of J and the structure constants hold [2]

$$f_{aci} f_{bci} = f_{acd} f_{bcd} = \frac{1}{3} \delta_{ab}, \tag{3.79}$$

$$J_{cd} f_{adi} = J_{ad} f_{cdi} \quad \text{and} \quad J_{ab} f_{abi} = 0. \tag{3.80}$$

Hence, from (3.79) we obtain

$$||\omega||^2 = 3, \tag{3.81}$$

and consequently

$$\mathrm{Im}\Omega = -\frac{1}{\sqrt{3}} \tilde{f}_{abc} e^a \wedge e^b \wedge e^c, \; \mathrm{Re}\Omega = -\frac{1}{\sqrt{3}} f_{abc} e^a \wedge e^b \wedge e^c \text{ and } \rho = \frac{1}{2\sqrt{3}}. \tag{3.82}$$

Now we show that $\mathbb{R}\times G/H$ can easily be furnished with a G_2-structure. All what we need is already there. We use the forms defined above to introduce the three-forms

$$\Sigma = e^0 \wedge \omega + \mathrm{Im}\Omega \tag{3.83}$$

and

$$\Sigma' = e^0 \wedge \omega + \mathrm{Re}\Omega. \tag{3.84}$$

Both three-forms (3.83) and (3.84) define a G_2-structure on $\mathbb{R}\times G/H$, i.e., a reduction of the holonomy group SO(7) to a subgroup $G_2 \subset \mathrm{SO}(7)$. Moreover, rrom (3.83) and (3.84) one sees that both G_2-structures are induced by SU(3)-structure of G/H.

3.4 Yang-Mills on G_2-manifolds $\mathbb{R}\times G/H$

After a long preparation we now are able to apply the collected toolkits and try to search for solutions of the Yang-Mills theory on the nearly Kähler coset spaces explicitly. Although the finding of a general solution is highly intricate, it will be possible give some specific solutions, in particular those which follow from certain flow equations.

3.4.1 Yang-Mills equations on spaces $\mathbb{R}\times G/H$: G-invariant ansätze and reductions to kink-type equations

On a d-dimensional Riemannian manifold, the Yang-Mills equation with torsion has the form

$$\mathcal{D} * \mathcal{F} + *\mathcal{H} \wedge \mathcal{F} = 0, \tag{3.85}$$

where $\mathcal{H}$ is a three-form. In this section we consider this equation (3.85) on the special case of seven-dimensional manifolds $\mathbb{R}\times G/H$, with G/H a nearly Kähler coset space (see [2]). For the metric and the volume form we choose

$$g_7 = (e_0)^2 + g_6 \quad \text{and} \quad V_7 = e_0 \wedge V_6, \tag{3.86}$$

where $e_0 = \mathrm{d}\tau$ and τ is a coordinate on $\mathbb{R}$, while g_6 and V_6 are metric and volume form on G/H. Since $\mathcal{H}$ can be chosen freely, we suppose that

$$* \mathcal{H} = -\frac{1}{3}\kappa \mathrm{d}\tau \wedge \mathrm{d}\omega, \tag{3.87}$$

with real constant κ.

For the connection one-form $\mathcal{A} = \mathcal{A}_0 e^0 + \mathcal{A}_a e^a$ we make the most general G-invariant ansatz,

$$\mathcal{A} = e^i I_i + e^a X_a(\tau), \quad \text{with} \quad X_a(\tau) = \lambda_a I_a + \mu_a J_{ab} I_b, \tag{3.88}$$

where λ_a and μ_a are real functions of τ. We remark that the gauge freedom allows us to set here $\mathcal{A}_0 = 0$ without any lost of generality.

Computation of the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = F_{0a} e^0 \wedge e^a + \frac{1}{2} F_{ab} e^a \wedge e^b$ of this connection gives in components

$$F_{ab} = -(f^i_{ab} I_i + f^c_{ab} X_a - [X_a, X_b]) + e^i_b([I_i, X_a] - f^c_{ia} X_c) \quad \text{and} \quad F_{0a} = \dot{X}_a, \tag{3.89}$$

with $\dot{X}_a = \frac{\partial X_a}{\partial \tau}$. Due to the G-invariance the last term of F_{ab} in (3.89) vanishes and

$$[I_i, X_a] = f^c_{ia} X_c. \tag{3.90}$$

In order to write the Yang-Mills equation in components, it is necessary to introduce the torsionful spin connection on G/H. Recall that a linear connection is a matrix of one-forms $\Gamma^a_b = e^c \Gamma^a_{cb}$. The connection is metric compatible if $\Gamma^c_a g_{cb}$ is anti-symmetric. Its torsion $T^a = \frac{1}{2} T^a_{bc} e^b \wedge^c$ is given by the structure equation

$$\mathrm{d}e^a + \Gamma^a_b \wedge e_b = T^a. \tag{3.91}$$

Choosing

$$T^a = -e^a \lrcorner \mathcal{H} \quad \Longleftrightarrow \quad T^a_{bc} = \kappa f^a_{bc}, \tag{3.92}$$

the unique metric-compatible linear connection with this torsion reduces to

$$\Gamma^a_{cb} = e^i_c f^a_{ib} + \frac{1}{2}(\kappa + 1) f^a_{cb}. \tag{3.93}$$

The torsionful spin connection on $\mathbb{R} \times G/H$ is then given by Γ^a_{cb}, with additional condition

$$\Gamma^0_{0b} = \Gamma^a_{0b} = \Gamma^0_{cb} = 0. \tag{3.94}$$

Putting (3.91) in the Yang-Mills equation with torsion (3.85) one gets

$$\begin{aligned} E_0 F^{0b} + E_a F^{ab} + \Gamma^d_{da} F^{ab} + \Gamma^b_{cd} F^{cd} + [A_a, F^{ab}] &= 0, & (3.95)\\ E_a F^{a0} + \Gamma^d_{da} F^{a0} + [A_a, F^{a0}] &= 0. & (3.96) \end{aligned}$$

We are now going to evaluate these Yang-Mills equation in terms of our ansatz (3.88). To do this we need to take in consideration the structure constant identities coming from the Jacobi-identities between the generators I_A (see (A.3)):

$$f^i_{ba} f^c_{kb} + f^i_{cd} f^a_{kd} + f^i_{kj} f^j_{ca} = 0, \qquad f^d_{ba} f^c_{kb} + f^d_{cb} f^a_{kb} + f^d_{kb} f^b_{ca} = 0, \tag{3.97}$$

and that they additionally satisfy

$$f^i_{cd} f^a_{cd} = 0, \quad f^a_{cd} f^b_{cd} = \frac{1}{3} \delta^{ab}, \quad f^k_{ba} f^d_{kb} = \frac{1}{3} \delta^d_a. \tag{3.98}$$

These relations will enormously simplify the outgoing expressions. One more point is that the tensor f^c_{ab} is totally antisymmetric. This implies that for the following arising term vanishes:

$$\begin{aligned} e^k_d f^d_{kb}[X_b, X_a] &- e^k_c f^a_{kd}[X_c, X_d] - e^i_b f^c_{ia}[X_b, X_c] + e^i_b f^c_{ib}[X_a, X_c]) = \\ &= (e^k_d f^d_{kb} + e^k_c f^b_{kc})[X_b, X_a] + (e^k_d f^a_{kb} + e^k_c f^b_{ka})[X_c, X_b] = 0 \end{aligned} \tag{3.99}$$

Plugging now the ansatz (3.88) into (3.95), (3.96) we obtain at first

$$\begin{aligned} \ddot{X}_a &= e^i_d f^d_{ib}(f^j_{ba} I_j + f^c_{ba} X_c - [X_b, X_a]) \\ &\quad + (e^k_c f^a_{kd} + \frac{1}{2}(\kappa+1) f^a_{cd})(f^i_{cd} I_i + f^b_{cd} X_b - [X_c, X_d]) \\ &\quad + [e^i_b I_i + X_b, f^k_{ba} I_k + f^d_{ba} X_d - [X_b, X_a]] \end{aligned} \tag{3.100}$$

$$\begin{aligned} &= (f^i_{ba} f^c_{kb} + f^i_{cd} f^a_{kd} + f^i_{kj} f^j_{ca}) e^k_c I_i + (f^d_{ba} f^c_{kb} + f^d_{cb} f^a_{kb} + f^d_{kb} f^b_{ca}) e^k_c X_d \\ &\quad + \frac{1}{2} f^i_{cd} f^a_{cd}(\kappa+1) I_i + \frac{1}{2} f^a_{cd} f^b_{cd}(\kappa+1) X_b + f^k_{ba} f^d_{kb} X_d \\ &\quad - (e^k_d f^d_{kb}[X_b, X_a] - e^k_c f^a_{kd}[X_c, X_d] - e^i_b f^c_{ia}[X_b, X_c] + e^i_c f^c_{ib}[X_a, X_b]) \\ &\quad - \frac{1}{2}(\kappa+1) f^a_{cd}[X_c, X_d] + f^d_{ba}[X_b, X_d] - [X_b[X_b, X_a]]. \end{aligned} \tag{3.101}$$

Next we take (3.98) and (3.99) into account, pull down the indices and find that the Yang-Mills equations can be written in terms of X_a as

$$\ddot{X}_a = \frac{\kappa - 1}{6} X_a - \frac{\kappa+3}{2} f_{acd}[X_c, X_d] - [X_b, [X_b, X_a]] \tag{3.102}$$

and

$$[X_a, \dot{X}_a] = 0, \tag{3.103}$$

respectively.

3.4.2 General solution of the G-invariance condition

In this section we consider how the G-invariance condition

$$[I_i, X_a] = f^b_{ia} X_b \tag{3.104}$$

influences solutions X_a of the Yang-Mills equations (3.102) and (3.103), i.e., how it determines a specific form of the solutions.

First of all (3.88) and (3.104) directly imply

$$X_{ab} f_{bci} = f_{iab} X_{bc}. \tag{3.105}$$

The general solution of (3.104) reads

$$Z^{(p)}_{\alpha_p} = \phi_p I^{(p)}_{\alpha_p}, \quad \text{with} \quad \phi_p \in \mathbb{C};\ p = 1, \dots, q;\ \alpha_p = 1, \dots, d_p, \tag{3.106}$$

where $\sum_{p=1}^{q} d_p = 3$ and $\{I^{(p)}_{\alpha_p}\}$ are complex linear combinations of $\{I_a\}$ such that

$$[I_i, I^{(p)}_{\alpha_p}] = f^{\beta_p}_{i\alpha_p} I^{(p)}_{\beta_p}. \tag{3.107}$$

Notice that for different p each group of $\{I^{(p)}_{\alpha_p}\}$ is closed for itself. The regulation of the grouping to different p is done according to decomposition of the adjoint representation of the gauge group G into the irreducible adjoint representation H and the rest,

$$\mathrm{adj}(G) = \mathrm{adj}(H) \oplus \sum_{p=1}^{q} R_p \oplus \sum_{p=1}^{q} \bar{R}_p. \tag{3.108}$$

For all nearly Kähler cases we are interested in representations R_p are complex. Due to the fact that both

$$Z^{(p)}_{\alpha_p} = \phi_p I^{(p)}_{\alpha_p} \quad \text{and} \quad \bar{Z}^{(p)}_{\alpha_p} = \bar{\phi}_p \bar{I}^{(p)}_{\alpha_p} \tag{3.109}$$

are solutions of (3.104), for each value of (p, α_p) the antihermitian solutions X_a can be constructed as

$$X^{(1)}_{\alpha_p} = \frac{1}{2}(Z^{(p)}_{\alpha_p} + \bar{Z}^{(p)}_{\alpha_p}) = \mathrm{Re}\phi_p \mathrm{Re} I^{(p)}_{\alpha_p} - \mathrm{Im}\phi_p \mathrm{Im} I^{(p)}_{\alpha_p}, \tag{3.110}$$

$$X^{(2)}_{\alpha_p} = \frac{1}{2i}(Z^{(p)}_{\alpha_p} - \bar{Z}^{(p)}_{\alpha_p}) = \mathrm{Im}\phi_p \mathrm{Re} I^{(p)}_{\alpha_p} + \mathrm{Re}\phi_p \mathrm{Im} I^{(p)}_{\alpha_p}. \tag{3.111}$$

For further computation, such as in calculations of the potential, we will need the trace $\mathrm{tr}_R(I_a I_b)$ over the chosen representation R of G. Although the gauge group G can in general be represented arbitrarily, in order to work with $\mathcal{A}$ and $\mathcal{F}$ taking values in the adjoint representation of G, we should chose the smallest non-trivial matrix representation of G to describe the generators I_i, I_a and solutions X_a. In the following we devote ourself to consideration of relevant cases.

$G_2/\mathrm{SU}(3)$

The smallest non-trivial matrix representation $14(G_2)$ of the group G_2 decomposes in

$$14(\mathrm{G}_2) = 8_{\mathrm{adj}} \oplus 3(\mathrm{SU}(3)) \oplus \bar{3}(\mathrm{SU}(3)), \tag{3.112}$$

so that $q = 1$ and $d_1 = 3$. $I^{(1)}_1, I^{(1)}_2, I^{(1)}_3$ and $\bar{I}^{(1)}_1, \bar{I}^{(1)}_2, \bar{I}^{(1)}_3$ generate $3(\mathrm{SU}(3))$ and $\bar{3}(\mathrm{SU}(3))$, respectively, and are linear combinations of $\{I_1, ..., I_6\}$. Since there is only one group, $q = 1$, we have only one complex parameter ϕ:

$$Z^{(1)}_{\alpha_1} = \phi I^{(1)}_{\alpha_1}, \quad \text{with} \quad \alpha_1 = 1, 2, 3. \tag{3.113}$$

As an appropriate matrix representation one may take the 7-dimensional fundamental of G_2. This case was already treated in [2].

SU(3)/U(1) × U(1)

In this case we have

$$\begin{aligned} 8(\mathrm{SU}(3)) &= ((0,0) \oplus (0,0))_{\mathrm{adj}} \oplus (2,0) \oplus (-2,0) \oplus \\ &\oplus (1,3) \oplus (-1,-3) \oplus (-1,3) \oplus (1,-3), \end{aligned} \tag{3.114}$$

where the corresponding generators are $I_7, I_8, I^{(1)}, \bar{I}^{(1)}, I^{(2)}, \bar{I}^{(2)}, I^{(3)}, \bar{I}^{(3)}$ (notice that there is no index α_p, because of the representation of H is one-dimensional) and the notation is according to: (q_7, q_8) has under the action of (I_7, I_8) the charge $(\frac{-1}{2\sqrt{3}}q_7, \frac{-1}{6}q_8)$. Since $q = 3$, there are three complex parameter ϕ_1, ϕ_2, ϕ_3 and

$$Z^{(p)} = \phi_p I^{(p)}, \quad \bar{Z}^{(p)} = \bar{\phi}_p \bar{I}^{(p)}, \quad \text{with} \quad p = 1, 2, 3. \tag{3.115}$$

Using the structure constants given in [82] we furnish the generators I_A with an additional factor $\frac{1}{\sqrt{3}}$ and respectively get $f^A_{CD} \longrightarrow \frac{1}{\sqrt{3}} f^A_{CD}$ (see also 3.159-3.160). Then due to

$$[I_7, I_1 + iI_2] = \frac{-i}{\sqrt{3}}(I_1 + iI_2), \qquad [I_8, I_1 + iI_2] = 0, \tag{3.116}$$

$$[I_7, I_3 + iI_4] = \frac{-i}{\sqrt{3}}(I_3 + iI_4), \qquad [I_8, I_3 + iI_4] = \frac{-i}{2}(I_3 + iI_4), \tag{3.117}$$

$$[I_7, I_5 + iI_6] = \frac{i}{\sqrt{3}}(I_5 + iI_6), \qquad [I_8, I_5 + iI_6] = \frac{-i}{2}(I_5 + iI_6), \tag{3.118}$$

one obtains

$$I^{(1)} = I_1 + iI_2, \quad I^{(2)} = I_3 + iI_4, \quad I^{(3)} = I_5 + iI_6. \tag{3.119}$$

Then we have

$$X_1 = \frac{1}{2}(Z^{(1)} + \bar{Z}^{(1)}) = \mathrm{Re}(\phi_1) I_1 + \mathrm{Im}(\phi_1) I_2, \tag{3.120}$$

$$X_3 = \frac{1}{2i}(Z^{(1)} - \bar{Z}^{(1)}) = \mathrm{Im}(\phi_1) I_1 - \mathrm{Re}(\phi_1) I_2, \tag{3.121}$$

and analogously for X_3, X_4 and X_5, X_6.

Sp(2)/Sp(1) × U(1)

Here we have the following decomposition of the group Sp(2),

$$10(\mathrm{Sp}(2)) = (3_0 \oplus 1_0)_{\mathrm{adj}} \oplus 2_1 \oplus 2_{-1} \oplus 1_2 \oplus 1_{-2}, \tag{3.122}$$

where in the designation of n_q the number n denotes the dimension of the corresponding irreducible part in the representation and q is the double charge under the action of U(1). Note that $\bar{n}_q = n_{-q}$. The corresponding generators are I_8, I_9, I_{10} and I_7 of $(3_0 \oplus 1_0)_{\text{adj}}$, $I_1^{(1)}, I_2^{(1)}$ of (2_1) and $I^{(2)}$ of 1_2 (and the complex conjugate generators for $2_{-1}, 1_{-2}$, respectively). Since $q = 2$, we get two parameter ϕ_1 and ϕ_2, so that

$$Z^{(1)} = \phi_1 I^{(1)}, \quad Z^{(2)} = \phi_2 I^{(2)}, \quad \text{and} \quad c.c. \tag{3.123}$$

Choosing the structure constants used in [82] and rescaling the generators I_A with a factor $\frac{1}{\sqrt{3}}$, we have to rescale $f^A_{BC} \longrightarrow \frac{1}{\sqrt{3}} f^A_{BC}$ (see also section 3.4.5). Then we get (notation: $\text{adj}(A) \cdot B = [A, B]$)

$$\begin{aligned}
\text{adj}(I_8, I_9, I_10) \cdot (\alpha(I_1 + iI_2) + \beta(I_3 + iI_4)) &= \gamma(I_1 + iI_2) + \delta(I_3 + iI_4) && (3.124)\\
\rightsquigarrow n &= 2 &&\\
\text{adj} \cdot (I_8, I_9, I_10)(I_5 + iI_6) &= 0 \rightsquigarrow n = 1 && (3.125)\\
[I_7, I_1 + iI_2] &= \frac{i}{2}(I_1 + iI_2) \rightsquigarrow q = 1, && (3.126)\\
[I_7, I_3 + iI_4] &= \frac{i}{2}(I_3 + iI_4) \rightsquigarrow q = 1, && (3.127)\\
[I_7, I_5 + iI_6] &= i(I_5 + iI_6) \rightsquigarrow q = 2. && (3.128)
\end{aligned}$$

Consequently,

$$I_1^{(1)} = I_1 + iI_2,\ I_2^{(2)} = I_3 + iI_4,\ I^{(3)} = I_5 + iI_6, \quad \text{and} \quad c.c. \tag{3.129}$$

Then solutions of the G-invariance condition for corresponding $Z^{(i)}$ are

$$\begin{aligned}
X_1 &= \frac{1}{2}(Z_1^{(1)} + \bar{Z}_1^{(1)}) \qquad X_2 = \frac{1}{2i}(Z_1^{(1)} - \bar{Z}_1^{(1)}), && (3.130)\\
X_3 &= \frac{1}{2}(Z_2^{(1)} + \bar{Z}_2^{(1)}) \qquad X_4 = \frac{1}{2i}(Z_2^{(1)} - \bar{Z}_2^{(1)}), && (3.131)\\
X_5 &= \frac{1}{2}(Z^{(2)} + \bar{Z}^{(2)}) \qquad X_6 = \frac{1}{2i}(Z^{(2)} - \bar{Z}^{(2)}). && (3.132)\\
& && (3.133)
\end{aligned}$$

3.4.3 Calculation of traces over $I_A I_B$

At first, we consider the traces tr_R in an arbitrary representations R of a simple group G. Let T_A, $A = 1, ..., d = \dim(G)$, be the generators of G and κ_{AB} the Killing form. Then,

$$\text{tr}_R T_A T_B = \chi_R \kappa_{AB}, \tag{3.134}$$

where χ_R is known as the (2nd-order) Dynkin index of R. Any rescaling of the T_A changes the structure constants and κ_{AB}, but the ratio to the adjoint representation $R = \text{adj}$ (as a reference) does not change. Therefore,

$$\text{tr}_R T_A T_B = \frac{\chi_R}{\chi_{\text{adj}}} \text{tr}_{\text{adj}} T_A T_B. \tag{3.135}$$

The Dynkin index is related to the value of the quadratic Casimir in the given representation (of dimension d_R):

$$d_{\text{adj}}\chi_R = d_R C_R, \tag{3.136}$$

where C_R is defined via

$$\sum_{A,B} \kappa^{AB} T_A T_B = C_R \text{Id}_R \tag{3.137}$$

and κ^{AB} is the inverse of κ_{AB}.

In this thesis we choose the following normalization

$$\text{tr}_{\text{adj}} T_A T_B = f^D_{AC} f^C_{BD} = -\delta_{AB}. \tag{3.138}$$

Hence,

$$\sum_A (T_A)^2 = (f^D_{AC} f^B_{AD}) = -(\delta^B_C). \tag{3.139}$$

Therefore, comparing (3.134) and (3.138), we find that κ_{AB} is proportional to δ_{AB}, and ratios C_R/C_{adj} of quadratic Casimirs do not change if we replace κ^{-1}_{AB} by $-\delta_{AB}$ in (3.137), i.e.,

$$\sum_A (T_A)^2 = -C'_R (\text{Id})_R. \tag{3.140}$$

In particular, $C'_{\text{adj}} = 1$.

Therefore, combining (3.135) and (3.136) we get (set $d_{\text{adj}} = d$)

$$\begin{aligned} \text{tr}_R T_A T_B &= -\frac{d_R C_R}{d C_{\text{adj}}}\delta_{AB} = -\frac{d_R C'_R}{d C'_{\text{adj}}}\delta_{AB} = -\frac{d_R}{d} C'_R \delta_{AB} \\ &=: -\chi'_R \delta_{AB}, \end{aligned} \tag{3.141}$$

defining the proportionality

$$\chi'_R = \frac{d_R}{d} C'_R = \frac{\text{tr}_R}{\text{tr}_{\text{adj}}}. \tag{3.142}$$

which is independent of normalization. In particular, $\chi'_{\text{adj}} = 1$. It remains to compute C'_R from (3.142).

Case SU(3)

The 3-dimensional representation R = '3' (see (3.4.4)) yields

$$C'_3 = \frac{4}{9}, \quad d_3 = 3, \quad d = 8 \quad \Longrightarrow \quad \chi'_3 = \frac{4}{9}\frac{3}{8} = \frac{1}{6}, \tag{3.143}$$

thus, $\text{tr}_3 = \frac{1}{6}\text{tr}_8$ for G =SU(3).

Case Sp(2)

In the 4-dimensional representation $R =$ '4' (3.4.5) we get

$$C_4' = \frac{5}{12}, \quad d_4 = 4, \quad d = 10 \implies \chi_4' = \frac{5}{12}\frac{4}{10} = \frac{1}{6}, \tag{3.144}$$

thus, $\mathrm{tr}_4 = \frac{1}{6}\mathrm{tr}_{10}$ for $G =$Sp(2).

3.4.4 Yang-Mills on $\mathbb{R} \times \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$

For the most general G-invariant ansatz (3.88) on $\mathbb{R}\times$SU(3)/U(1)$\times$U(1) the functions λ_i and μ_i, $i = 1, ..., 6$, have to satisfy the G-invariance condition (3.104). One possible solution to the G-invariant ansatz (3.88) is

$$\lambda_1 = \lambda_2, \quad \lambda_3 = \lambda_4, \quad \lambda_5 = \lambda_6, \tag{3.145}$$

$$\mu_1 = \mu_2, \quad \mu_3 = \mu_4, \quad \mu_5 = \mu_6. \tag{3.146}$$

Hence we can define three complex functions

$$\phi_1 = \lambda_1 + i\mu_1, \quad \phi_2 = \lambda_2 + i\mu_2, \quad \phi_3 = \lambda_3 + i\mu_3, \tag{3.147}$$

and consequently with (see [82])

$$J_{12} = -1, \quad J_{34} = 1, \quad J_{56} = -1 \tag{3.148}$$

obtain the following expressions for X_a (compare with section 3.4.2)

$$X_1 = \mathrm{Re}(\phi_1)I_1 + \mathrm{Im}(\phi_1)I_2, \qquad X_2 = \mathrm{Im}(\phi_1)I_1 - \mathrm{Re}(\phi_1)I_2, \tag{3.149}$$

$$X_3 = \mathrm{Re}(\phi_2)I_3 + \mathrm{Im}(\phi_2)I_4, \qquad X_4 = -\mathrm{Im}(\phi_2)I_3 + \mathrm{Re}(\phi_2)I_4, \tag{3.150}$$

$$X_5 = \mathrm{Re}(\phi_3)I_5 - \mathrm{Im}(\phi_3)I_6, \qquad X_6 = \mathrm{Im}(\phi_3)I_5 + \mathrm{Re}(\phi_3)I_6. \tag{3.151}$$

Now we have to choose a matrix representation for the generators I_A of the group SU(3). An appropriate basis is the 3-dimensional fundamental representation given by the Gell-Mann-matrices, which we rescale with the factor $\frac{-i}{\sqrt{3}}$:

$$I_1 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_2 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.152}$$

$$I_3 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad I_4 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{3.153}$$

$$I_5 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad I_6 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{3.154}$$

$$I_7 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_8 = \frac{-i}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.155}$$

For the matrices X_a this means

$$X_1 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & \phi_1 & 0 \\ \bar{\phi}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & -i\phi_1 & 0 \\ i\bar{\phi}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.156}$$

$$X_3 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & \bar{\phi}_2 \\ 0 & 0 & 0 \\ \phi_2 & 0 & 0 \end{pmatrix}, \qquad X_4 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & -i\bar{\phi}_2 \\ 0 & 0 & 0 \\ i\phi_2 & 0 & 0 \end{pmatrix}, \tag{3.157}$$

$$X_5 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \phi_3 \\ 0 & \bar{\phi}_3 & 0 \end{pmatrix}, \qquad X_6 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\phi_3 \\ 0 & i\bar{\phi}_3 & 0 \end{pmatrix}. \tag{3.158}$$

For the structure constants we use those given in [82] with a rescaling $f_{ABC} \longmapsto \frac{1}{\sqrt{3}} f_{ABC}$:

$$f_{127} = \frac{1}{\sqrt{3}}, \qquad f_{136} = f_{154} = f_{235} = f_{246} = \frac{1}{2\sqrt{3}} = f_{347} = f_{657}, \tag{3.159}$$

$$f_{568} = f_{348} = \frac{1}{2}. \tag{3.160}$$

Therefore, in order to calculate the equation of motion (3.102) for X_1 we take advantage of the fact that the most of structure constants are zero, especially this means that

$$\begin{aligned} f^1_{cd}[X_c, X_d] &= 2(f^1_{36}[X_3, X_6] + f^1_{54}[X_5, X_4]) \\ &= \frac{1}{\sqrt{3}}([X_3, X_6] + [X_5, X_4]). \end{aligned} \tag{3.161}$$

Evaluating this we get

$$f^1_{cd}[X_c, X_d] = \frac{-i}{6\sqrt{3}} \begin{pmatrix} 0 & \bar{\phi}_2\bar{\phi}_3 & 0 \\ \phi_2\phi_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.162}$$

Further, we need the terms of the form $[X_a, [X_a, X_1]]$:

$$[X_2, [X_2, X_1]] = \frac{i}{6\sqrt{3}} \begin{pmatrix} 0 & \phi_1|\phi_1|^2 & 0 \\ \phi_1|\phi_1|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.163}$$

$$[X_3, [X_3, X_1]] = [X_4, [X_4, X_1]] = \frac{i}{12\sqrt{3}} \begin{pmatrix} 0 & \phi_1|\phi_2|^2 & 0 \\ \bar{\phi}_1|\phi_2|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.164}$$

$$[X_5, [X_5, X_1]] = [X_6, [X_6, X_1]] = \frac{i}{12\sqrt{3}} \begin{pmatrix} 0 & \phi_1|\phi_3|^2 & 0 \\ \bar{\phi}_1|\phi_3|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.165}$$

If we now plug above results in our e.o.m. (3.102) and consider only the (1,2) component, we get

$$6\ddot{\phi}_1 \quad = \quad (\kappa-1)\phi_1 - (\kappa+3)\bar{\phi}_2\bar{\phi}_3 + \phi_1(2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2). \tag{3.166}$$

Because of the symmetry in ϕ_1, ϕ_2 and ϕ_3 the e.o.m. for ϕ_2 and ϕ_3 can be obtained via cyclic permutation of indices.

It remains to derive an equation corresponding to the constraints (3.103). Using expressions for X_a, we easily obtain

$$0 = [X_a, \dot{X}_a] = \frac{-1}{12}\begin{pmatrix} Y_1 - Y_2 & 0 & 0 \\ 0 & Y_3 - Y_1 & 0 \\ 0 & 0 & Y_2 - Y_3 \end{pmatrix}, \tag{3.167}$$

where $Y_i = \phi_i\dot{\bar{\phi}}_i - \dot{\phi}_i\bar{\phi}_i$. Therefore, we have

$$Y_1 - Y_2 = Y_3 - Y_1 = Y_2 - Y_3 = 0. \tag{3.168}$$

3.4.5 Yang-Mills on $\mathbb{R} \times Sp(2)/Sp(1) \times U(1)$

Analogously to the case on $\mathbb{R}\times SU(3)/U(1)\times U(1)$ we proceed the calculations for $\mathbb{R}\times Sp(2)/Sp(1)\times U(1)$. Here the G-invariance condition 3.113for the almost complex structure given in (3.148) has a solution of the form (see section 3.4.2)

$$X_1 = \mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2, \qquad X_2 = \mathrm{Im}(\phi_1)I_1 + \mathrm{Re}(\phi_1)I_2, \tag{3.169}$$

$$X_3 = \mathrm{Re}(\phi_1)I_3 - \mathrm{Im}(\phi_1)I_4, \qquad X_4 = \mathrm{Im}(\phi_1)I_3 + \mathrm{Re}(\phi_1)I_4, \tag{3.170}$$

$$X_5 = \mathrm{Re}(\phi_2)I_5 - \mathrm{Im}(\phi_2)I_6, \qquad X_6 = \mathrm{Im}(\phi_2)I_5 + \mathrm{Re}(\phi_2)I_6, \tag{3.171}$$

with complex valued functions ϕ_1, ϕ_2. As a matrix representation of the group Sp(2) we choose the four-dimensional fundamental representation given by

$$I_1 = \frac{-i}{2\sqrt{6}}\sigma_2 \times \sigma_2 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.172}$$

$$I_2 = \frac{-i}{2\sqrt{6}}\sigma_2 \times \sigma_1 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.173}$$

$$I_3 = \frac{-i}{2\sqrt{6}}\sigma_2\times\sigma_3 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\\ i & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{pmatrix}, \tag{3.174}$$

$$I_4 = \frac{-i}{2\sqrt{6}}\sigma_1\times \mathrm{Id}_2 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}, \tag{3.175}$$

$$I_5 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\times\sigma_2 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}, \tag{3.176}$$

$$I_6 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\times\sigma_1 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}, \tag{3.177}$$

$$I_7 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\times\sigma_3 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \tag{3.178}$$

$$I_8 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\times\sigma_2 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{3.179}$$

$$I_9 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\times\sigma_1 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{3.180}$$

$$I_{10} = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\times\sigma_3 = \frac{-i}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{3.181}$$

In this way we obtain

$$X_1 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & -\bar\phi_1\\ 0 & 0 & \phi_1 & 0\\ 0 & \bar\phi_1 & 0 & 0\\ -\phi_1 & 0 & 0 & 0\end{pmatrix},\quad X_2 = \frac{-i}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & -i\bar\phi_1\\ 0 & 0 & -i\phi_1 & 0\\ 0 & i\bar\phi_1 & 0 & 0\\ i\phi_1 & 0 & 0 & 0\end{pmatrix}, \tag{3.182}$$

$$X_3 = \frac{-i}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & -i\phi_1 & 0 \\ 0 & 0 & 0 & i\bar{\phi}_1 \\ i\bar{\phi}_1 & 0 & 0 & 0 \\ 0 & -i\phi_1 & 0 & 0 \end{pmatrix}, \quad X_4 = \frac{-i}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & -\phi_1 & 0 \\ 0 & 0 & 0 & -\bar{\phi}_1 \\ -\bar{\phi}_1 & 0 & 0 & 0 \\ 0 & -\phi_1 & 0 & 0 \end{pmatrix}, \tag{3.183}$$

$$X_5 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\phi_2 \\ 0 & 0 & i\bar{\phi}_2 & 0 \end{pmatrix}, \quad X_6 = \frac{-i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 \\ 0 & 0 & \bar{\phi}_2 & 0 \end{pmatrix} \tag{3.184}$$

For the structure constants we use those given in [82] rescaling $f \longrightarrow \frac{1}{\sqrt{3}}f$. So that the only non-vanishing components are $f^5_{41} = f^5_{32} = f^6_{13} = f^6_{42} = \frac{1}{2\sqrt{3}}$.

In order to evaluate the Yang-Mills equations (3.102) and (3.103), we remark that there is no symmetry in ϕ_1 and ϕ_2, so that consequently the equations of motions and constrains for X_a, $a = 1, ..., 4$ and $X_b, b = 5, 6$, should be considered separately. But at first we calculate the relevant terms:

$$\begin{aligned} f^1_{cd}[X_c, X_d] &= 2(f^1_{36}[X_3, X_6] + f^1_{54}[X_5, X_4]) &\quad (3.185) \\ &= \frac{1}{\sqrt{3}}([X_3, X_6] + [X_5, X_4]) &\quad (3.186) \\ &= \frac{-i}{6\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -\phi_1\phi_2 \\ 0 & 0 & \bar{\phi}_1\bar{\phi}_2 & 0 \\ 0 & \bar{\phi}_1\bar{\phi}_2 & 0 & 0 \\ -\phi_1\phi_2 & 0 & 0 & 0 \end{pmatrix}. &\quad (3.187) \end{aligned}$$

Further, we need cubic terms of type $[X_a, [X_a, X_1]]$:

$$[X_2, [X_2, X_1]] = \frac{i}{4\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -\bar{\phi}_1|\phi_1|^2 \\ 0 & 0 & \phi_1|\phi_1|^2 & 0 \\ 0 & \phi_1|\phi_1|^2 & 0 & 0 \\ -\bar{\phi}_1|\phi_1|^2 & 0 & 0 & 0 \end{pmatrix}, \tag{3.188}$$

$$\begin{aligned} [X_3, [X_3, X_1]] &= [X_4, [X_4, X_1]] = 2[X_5, [X_5, X_1]] = 2[X_6, [X_6, X_1]] &\quad (3.189) \\ &= \frac{i}{36\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -\bar{\phi}_1|\phi_2|^2 \\ 0 & 0 & \phi_1|\phi_2|^2 & 0 \\ 0 & \phi_1|\phi_2|^2 & 0 & 0 \\ -\bar{\phi}_1|\phi_2|^2 & 0 & 0 & 0 \end{pmatrix}. &\quad (3.190) \end{aligned}$$

Plugging all the terms together in (3.102) and focusing on the $(2,3)$ component, we arrive at

$$6\ddot{\phi}_1 = (\kappa - 1)\phi_1 - (\kappa + 3)\bar{\phi}_1\bar{\phi}_2 + \phi_1(3|\phi_1|^2 + |\phi_2|^2). \tag{3.191}$$

Calculation for X_6 gives then with

$$f^6_{cd}[X_c, X_d] = f^6_{13}[X_1, X_3] + f^6_{42}[X_4, X_2] \tag{3.192}$$

$$= \frac{-i}{6\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\phi}_1^2 \\ 0 & 0 & \phi_1^2 & 0 \end{pmatrix}, \tag{3.193}$$

$$[X_1, [X_1, X_6]] = [X_2, [X_2, X_6]] = [X_3, [X_3, X_6]] = \tag{3.194}$$

$$= [X_4, [X_4, X_6]] = \frac{i}{6\sqrt{3}}\begin{pmatrix} 0 & \bar{\phi}_1^2\bar{\phi}_2 & 0 & 0 \\ \phi_1^2\phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2|\phi_1|^2 \\ 0 & 0 & \bar{\phi}_2|\phi_1|^2 & 0 \end{pmatrix}, \tag{3.195}$$

$$[X_2, [X_2, X_6]] = \frac{i}{6\sqrt{3}}\begin{pmatrix} 0 & -\bar{\phi}_2\bar{\phi}_1^2 & 0 & 0 \\ -\phi_1^2\phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2|\phi_1|^2 \\ 0 & 0 & \bar{\phi}_2|\phi_1|^2 & 0 \end{pmatrix}, \tag{3.196}$$

$$[X_3, [X_3, X_6]] = \frac{i}{6\sqrt{3}}\begin{pmatrix} 0 & \phi_1^2\phi_2 & 0 & 0 \\ \bar{\phi}_1^2\bar{\phi}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2|\phi_1|^2 \\ 0 & 0 & \bar{\phi}_2|\phi_1|^2 & 0 \end{pmatrix}, \tag{3.197}$$

$$[X_4, [X_4, X_6]] = \frac{i}{6\sqrt{3}}\begin{pmatrix} 0 & -\phi_1^2\phi_2 & 0 & 0 \\ -\bar{\phi}_1^2\bar{\phi}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2|\phi_1|^2 \\ 0 & 0 & \bar{\phi}_2|\phi_1|^2 & 0 \end{pmatrix}, \tag{3.198}$$

$$[X_5, [X_5, X_6]] = \frac{i}{6\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2|\phi_2|^2 \\ 0 & 0 & \bar{\phi}_2|\phi_2|^2 & 0 \end{pmatrix}, \tag{3.199}$$

the following result

$$\ddot{\phi}_2 = \frac{\kappa-1}{6}\phi_2 - \frac{\kappa+3}{6}\bar{\phi}_1^2 + \frac{1}{3}\phi_2(|\phi_1|^2 + |\phi_2|^2). \tag{3.200}$$

Constraints

$$[X_a, \dot{X}_a] = 0, \tag{3.201}$$

also easily obtained with

$$[X_1, \dot{X}_1] = [X_2, \dot{X}_2] = \frac{-1}{24}\begin{pmatrix} Y_1 & 0 & 0 & 0 \\ 0 & -Y_1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & -Y_1 \end{pmatrix}, \tag{3.202}$$

$$[X_3, \dot{X}_3] = [X_4, \dot{X}_4] = \frac{-1}{24}\begin{pmatrix} -Y_1 & 0 & 0 & 0 \\ 0 & Y_1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & -Y_1 \end{pmatrix}, \tag{3.203}$$

$$[X_5, \dot{X}_5] = [X_6, \dot{X}_6] = \frac{-1}{12}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -Y_2 & 0 \\ 0 & 0 & 0 & Y_2 \end{pmatrix} \tag{3.204}$$

with

$$Y_i = \dot{\phi}_i\bar{\phi}_i - \phi_i\dot{\bar{\phi}}_i. \tag{3.205}$$

Therefore, from the constraint (3.201) can be written as

$$Y_1 - Y_2 = 0 \quad \text{or} \quad \dot{\phi}_1\bar{\phi}_1 - \phi_1\dot{\bar{\phi}}_1 = \dot{\phi}_2\bar{\phi}_2 - \phi_2\dot{\bar{\phi}}_2. \tag{3.206}$$

Remark: We observe that the equation of motion (3.191) for the case $\mathbb{R} \times \mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$ can be derived from e.o.m. computed for $\mathbb{R}\times\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$ via identification $\phi_i \longrightarrow \phi_2$, $\phi_j \longrightarrow \phi_1$ and $\phi_k \longrightarrow \phi_1$ for $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$.

3.4.6 Action

In the same way as it was presented in [2], we will now see that the Yang-Mills equations with torsion (3.263) and (3.191) can be derived from an action functional, concretely of the form

$$S = \int_{\mathbb{R}\times G/H} \mathrm{tr}\mathcal{F} \wedge *\mathcal{F} + \frac{1}{3}\mathrm{tr}(\kappa e^0 \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F}). \tag{3.207}$$

First of all our goal is to evaluate this action in terms of X_a. For that reason rewrite it as follows,

$$S = S_1 + \frac{\kappa}{3}S_2, \tag{3.208}$$

with

$$S_1 = \int_{\mathbb{R}\times G/H} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}), \tag{3.209}$$

$$S_2 = \int_{\mathbb{R}\times G/H} \mathrm{tr}(e^0 \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F}). \tag{3.210}$$

Then we have

$$S_1 = \int\limits_{\mathbb{R}\times G/H} \frac{1}{2}\mathrm{tr}(F_{ab}F^{ab} + 2F_{0a}F_{0a})e^0\wedge e^1\wedge e^2\wedge e^3\wedge e^4\wedge e^5\wedge e^6 \quad (3.211)$$

$$= \frac{1}{2}\mathrm{Vol}(G/H)\int\limits_{\mathbb{R}} \mathrm{d}\tau\mathrm{tr}(F_{ab}F^{ab} + 2F_{0a}F^{0a}). \quad (3.212)$$

Recall that according to (3.89)

$$F_{ab} = -(f^i_{ab}I_i + f^c_{ab}X_c - [X_a, X_b]),\quad F_{0a} = \dot{X}_a \quad \text{and} \quad (3.213)$$

$$A_0 = 0,\quad A_a = e^i_a I_i + X_a. \quad (3.214)$$

Then we have

$$S_1 = \frac{1}{2}\mathrm{Vol}(G/H)\int\limits_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(2\dot{X}_a\dot{X}_a + f^i_{ab}f^j_{ab}I_iI_j + f^c_{ab}f^d_{ab}X_cX_d + +[X_a,X_b][X_a,X_b] - 2f^i_{ab}f^c_{ab}I_iX_c + 2f^i_{ab}I_i[X_a,X_b] + +2f^c_{ab}X_c[X_a,X_b]), \quad (3.215)$$

where we used the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Then, since

$$f^c_{ab}f^d_{ab} = f^c_{ib}f^d_{ib} = \tfrac{1}{3}\delta_{cd}, \quad (3.216)$$

$$\mathrm{tr}(I_i[X_a,X_b]) = -\mathrm{tr}([I_i,X_b]X_a) = -f^c_{ib}\mathrm{tr}(X_cX_a), \quad (3.217)$$

and

$$\mathrm{tr}(I_iX_a) = (\lambda_a\delta_{ab} + \mu_aJ_{ab})\mathrm{tr}(I_iI_b) = 0, \quad (3.218)$$

because of $I_i \in \mathfrak{h}\perp\mathfrak{m}$ we get

$$S_1 = \frac{1}{2}\mathrm{Vol}(G/H)\int\limits_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(2\dot{X}_a\dot{X}_a + f^i_{ab}f^j_{ab}I_iI_j - \frac{1}{3}X_cX_c - -2f^c_{ab}X_c[X_a,X_b] + [X_a,X_b][X_a,X_b]) \quad (3.219)$$

In order to calculate S_2 we recall that $\omega\wedge\omega\wedge\omega = 6\mathrm{vol}_6$ and $\omega = \frac{1}{2}J_{ab}e^a\wedge e^b$. Thus,

$$6\mathrm{vol}_6 = \frac{1}{8}J_{ab}J_{cd}J_{ef}e^a\wedge e^b\wedge e^c\wedge e^d\wedge e^e\wedge e^f = \quad (3.220)$$

$$= \frac{1}{8}J_{ab}J_{cd}J_{ef}\epsilon^{abcdef}\mathrm{vol}_6. \quad (3.221)$$

The relevant term appearing in the integral S_2 is $\omega_{ab}\epsilon^{abcdef} = \frac{1}{2}J_{ab}\epsilon^{abcdef}$. So we extract it from (3.221) taking in mind that it is total antisymmetric in indices. We find

$$\frac{1}{2}J_{ab}\epsilon^{abcdef} = J^{(cd}J^{ef)}, \quad (3.222)$$

where denote by $J^{(cd}J^{ef)}$ the totally antisymmetric combination $J^{cd}J^{ef} - J^{ce}J^{df} - J^{cf}J^{ed}$. A proof confirms (compare with (3.221))

$$\begin{aligned}
\frac{1}{4}J_{cd}J_{ef}J_{(cd}J_{ef)}\mathrm{vol}_6 &= \frac{1}{4}J_{cd}J_{ef}(J_{cd}J_{ef} - J_{ce}J_{df} - J_{cf}J_{ed}) = && (3.223)\\
&= \frac{1}{4}(\delta^c_c\delta^e_e - \delta^c_f\delta^c_f - \delta^e_d\delta^e_d) = && (3.224)\\
&= \frac{1}{4}(36 - 6 - 6) = 6\mathrm{vol}_6. && (3.225)
\end{aligned}$$

To evaluate the under integral expression of S_2 we substitute (3.222) and (3.89) into it:

$$\begin{aligned}
\mathrm{tr}(d\tau \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F}) &= \mathrm{tr}(\frac{\kappa}{8}J_{ab}\epsilon^{abcdef}F_{cd}F_{ef})\mathrm{vol}_7 = && (3.226)\\
&= \frac{1}{4}\mathrm{tr}(J_{(cd}J_{ef)}F_{cd}F_{ef})\mathrm{vol}_7. && (3.227)
\end{aligned}$$

Because of $J_{ab}f^i_{ab} = 0$, $J_{ab}f^i_{cb} = J_{cb}f^i_{ab}$ and the antisymmetry of $\tilde{f}_{abc} = f_{abd}J_{dc}$ and of J we obtain

$$J_{cd}J_{ef}F_{cd}F_{ef} = 0 \quad \text{and} \quad J_{ce}J_{df}F_{cd}F_{ef} = J_{cf}J_{ed}F_{cd}F_{ef}. \qquad (3.228)$$

Therefore,

$$\begin{aligned}
\mathrm{tr}(d\tau \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F}) &= -\frac{1}{2}\mathrm{tr}(J_{ce}J^{df}(f^i_{ab}I_i + f^c_{ab}X_c - [X_a, X_b])(f^i_{cd}I_i + f^a_{cd}X_a - \\
&\quad -[X_c, X_d])(f^k_{ef}I_k + f^b_{ef}X_b - [X_e, X_f]))\mathrm{vol}_7 = && (3.229)\\
&= -\frac{1}{2}J_{ce}J_{df}(f^i_{cd}f^k_{ef}\mathrm{tr}(I_iI_k) - 2f^i_{cd}\mathrm{tr}(I_i[X_e, X_f]) - \\
&\quad -2f^b_{cd}\mathrm{tr}(X_b[X_e, X_f]) + f^a_{cd}f^b_{ef}\mathrm{tr}(X_aX_b) + \\
&\quad +\mathrm{tr}([X_c, X_d][X_e, X_f]))\mathrm{vol}_7. && (3.230)
\end{aligned}$$

At this point we calculate

$$\begin{aligned}
J_{ce}J_{df}f^i_{cd}f^k_{ef} &= f^i_{cd}f^k_{cd}, \quad J_{ce}J_{df}f^i_{cd} = f^i_{ef}, && (3.231)\\
J_{ce}J_{df}f^b_{cd} &= -f^b_{ef}, \quad J_{ce}J_{df}f^a_{cd}f^b_{ef} = -\frac{1}{3}\delta^{ab}, && (3.232)\\
J_{(cd}J_{ef)}\mathrm{tr}([X_c, X_d][X_e, X_f]) &= -J_{(cd}J_{ef)}\mathrm{tr}(X_c[[X_e, X_f], X_d]) = && (3.233)\\
&= J_{cd}J_{ef}(\mathrm{tr}(X_c[[X_e, X_f], X_d] - \\
&\quad -X_c[[X_d, X_f], X_e] - X_c[[X_f, X_e], X_d]) = && (3.234)\\
&= J_{cd}J_{ef}(\mathrm{tr}(X_c[[X_e, X_f], X_d] + \\
&\quad +X_c[[X_f, X_d], X_e] + X_c[[X_d, X_e], X_f]) = && (3.235)\\
&= 0, && (3.236)\\
f^i_{ef}\mathrm{tr}(I_i[X_e, X_f]) &= -f^i_{ef}\mathrm{tr}([I_i, X_f]X_e) = -f^i_{ef}f^b_{if}\mathrm{tr}(X_bX_e) = && (3.237)\\
&= \frac{1}{3}\delta_{eb}\mathrm{tr}(X_bX_e) = \frac{1}{3}\mathrm{tr}(X_bX_b). && (3.238)
\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{tr}(\mathrm{d}\tau\wedge\omega\wedge\mathcal{F}\wedge\mathcal{F}) &= -\frac{1}{2}(f^i_{cd}f^k_{cd}\mathrm{tr}(I_iI_k)+2f_{abc}\mathrm{tr}(X_a[X_b,X_c])-\\ &\quad -\mathrm{tr}(X_aX_b))\mathrm{vol}_7. \end{aligned}\tag{3.239}$$

Plugging (3.239) in the expression for S_2, we get

$$S_2=\frac{1}{2}\mathrm{Vol(G/H)}\int_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(-f^i_{cd}f^k_{cd}I_iI_k-2f_{aef}X_a[X_e,X_f]+X_aX_a). \tag{3.240}$$

So that for the action functional it follows

$$\begin{aligned} S &= S_1+\frac{\kappa}{3}S_2=\mathrm{Vol}(G/H)\int_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(\dot{X}_a\dot{X}_a+\frac{1}{2}f^i_{ab}f^j_{ab}I_iI_j-\\ &\quad -\frac{1}{6}X_cX_c-f^c_{ab}X_c[X_a,X_b]+\frac{1}{2}[X_a,X_b][X_a,X_b])+\\ &\quad +\frac{\kappa}{6}\mathrm{Vol}(G/H)\int_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(X_bX_b-2f_{abc}X_a[X_b,X_c]-f^i_{cd}f^k_{cd}I_iI_k) &(3.241)\\ &= \mathrm{Vol}(G/H)\int_{\mathbb{R}}\mathrm{d}\tau\mathrm{tr}(\dot{X}_a\dot{X}_a+\frac{\kappa-1}{6}X_bX_b-\frac{\kappa+3}{3}f^c_{ab}X_c[X_a,X_b]+\\ &\quad +\frac{1}{2}[X_a,X_b][X_a,X_b])+\frac{3-\kappa}{6}f^i_{ab}f^j_{ab}I_iI_j). &(3.242)\end{aligned}$$

Varying this action, one can see that the Euler-Lagrange equations for it are (3.102).

Remark: The action, in particular the integral S_2, can also be evaluated by using the Chern-Simon form of $\mathcal{F}\wedge\mathcal{F}$ (see for example [14, 5]).

3.4.7 Action for the case $\mathbb{R}\times\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$

To get the action in terms of ϕ_1, ϕ_2 and ϕ_3, let us substitute the exact form of X_a for the case SU(3)/U(1)×U(1) the into (3.242). We take the matrices given in (3.156-3.158). Then we obtain

$$\begin{aligned} \mathrm{tr}(X_aX_a) &= -\frac{1}{3}(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2), &(3.243)\\ f^a_{bc}(X_a[X_b,X_c]) &= -\frac{1}{6}(\phi_1\phi_2\phi_3+\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3), &(3.244)\\ \mathrm{tr}([X_a,X_b][X_a,X_b]) &= -\frac{1}{9}(|\phi_1|^4+|\phi_2|^4+|\phi_3|^4+\\ &\quad +|\phi_1|^2|\phi_2|^2+|\phi_1|^2|\phi_3|^2+|\phi_2|^2|\phi_3|^2), &(3.245)\\ \frac{1}{2}f^i_{ab}f^j_{ab}\mathrm{tr}(I_iI_j) &= a^2. &(3.246)\end{aligned}$$

The constant term a^2 can be evaluated according to the previous section (3.4.3), where we find that in case of the gauge group SU(3) the trace over $I_A I_B$ satisfies

$$\mathrm{tr} I_A I_B = -\chi'_R \delta_{AB} = -\frac{1}{6}\delta_{AB}, \tag{3.247}$$

so that

$$a^2 = \frac{1}{2} f^i_{ab} f^j_{ab} \mathrm{tr}(I_i I_j) = -\frac{1}{12} f^i_{ab} f^j_{ab} \delta^{ij} = -\frac{1}{12} f^i_{ab} f^i_{ab} = -\frac{1}{12}\frac{1}{3}\delta^{aa} = -\frac{1}{6}. \tag{3.248}$$

Substituting (3.243)-(3.247) into (3.242) we obtain

$$\begin{aligned} S &= \mathrm{Vol}(G/H)\int_{\mathbb{R}} \mathrm{d}\tau\Big(-\frac{1}{3}\sum_i |\dot{\phi}_i|^2 - \frac{\kappa - 1}{18}\sum_i |\phi_i|^2 + \\ &\quad + \frac{\kappa+3}{18}(\phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3) - \frac{1}{18}\sum_i |\phi_i|^4 - \\ &\quad - \frac{1}{18}(|\phi_1|^2|\phi_2|^2 + |\phi_1|^2|\phi_3|^2 + |\phi_2|^2|\phi_3|^2\Big) + \frac{\kappa - 3}{6}. \end{aligned} \tag{3.249}$$

Writing this action in terms of a potential $V(\phi)$,

$$S = \mathrm{Vol}(G/H)\int_{\mathrm{R}} \mathrm{d}\tau \mathrm{tr}\left(-\frac{1}{3}\sum_i |\dot{\phi}|^2 - \frac{1}{2}V(\phi)\right), \tag{3.250}$$

with

$$\begin{aligned} V(\phi) &= \frac{\kappa - 1}{9}\sum_i |\phi_i|^2 - \frac{\kappa+3}{9}(\phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3) + \frac{1}{9}\sum_i |\phi_i|^4 + \\ &\quad + \frac{1}{9}(|\phi_1|^2|\phi_2|^2 + |\phi_1|^2|\phi_3|^2 + |\phi_2|^2|\phi_3|^2) - \frac{\kappa - 3}{3}, \end{aligned} \tag{3.251}$$

one finds that the e.o.m (3.263) then reads

$$6\ddot{\phi}_i = 9\frac{\partial V}{\partial \bar{\phi}_i}. \tag{3.252}$$

3.4.8 Action for the case $\mathbb{R} \times \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$

Again using the explicit form of the matrices X_a, we compute the action but now for the case $\mathbb{R}\times\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$. Plugging the corresponding matrices (3.182-3.184) into the action integral (3.242), we obtain that

$$\mathrm{tr}(X_a X_a) = -\frac{1}{3}(2|\phi_1|^2 + |\phi_2|^2), \tag{3.253}$$

$$f^a_{bc}(X_a[X_b, X_c]) = -\frac{1}{6}(\bar{\phi}_1^2\bar{\phi}_2 + \phi_1^2\phi_2), \tag{3.254}$$

$$\mathrm{tr}([X_a, X_b][X_a, X_b]) = -\frac{1}{9}(3|\phi_1|^4 + |\phi_2|^4 + 2|\phi_1|^2|\phi_2|^2), \tag{3.255}$$

$$\frac{1}{2} f^i_{ab} f^j_{ab}\mathrm{tr}(I_i I_j) = a'^2, \tag{3.256}$$

with

$$a'^2 = -\frac{1}{2} f^i_{ab} f^j_{ab} \chi'_R \delta_{ij} = -\frac{1}{2} f^i_{ab} f^j_{ab} \frac{1}{6} \delta_{ij} = -\frac{1}{6}. \qquad (3.257)$$

Consequently we arrive at

$$\begin{aligned} S &= \mathrm{Vol}(G/H) \int_{\mathbb{R}} \mathrm{d}\tau \big(-\frac{1}{3}(2|\dot\phi_1|^2 + |\dot\phi_2|^2) - \\ &\quad -\frac{\kappa-1}{18}(2|\phi_|^2 + |\phi_2|^2) + \frac{\kappa+3}{18}(\bar\phi_1^2\bar\phi_2 + \phi_1^2\phi_2) - \\ &\quad -\frac{1}{18}(3|\phi_1|^4 + |\phi_2|^4 + 2|\phi_1|^2|\phi_2|^2) + \frac{\kappa-3}{6}. \end{aligned} \qquad (3.258)$$

Decomposing again the integrand in kinetic and potential terms

$$S = \mathrm{Vol}(G/H) \int_{\mathbb{R}} \mathrm{d}\tau \big(-\frac{1}{3}(2|\dot\phi_|^2 + |\dot\phi_2|^2) - \frac{1}{2} V(\phi_1, \phi_2)\big), \qquad (3.259)$$

with

$$\begin{aligned} V(\phi) &= \frac{\kappa-1}{9}(2|\phi_|^2 + |\phi_2|^2) - \frac{\kappa+3}{9}(\bar\phi_1^2\bar\phi_2 + \phi_1^2\phi_2) - \\ &\quad -\frac{1}{9}(3|\phi_1|^4 + |\phi_2|^4 + 2|\phi_1|^2|\phi_2|^2) - \frac{\kappa-3}{3}, \end{aligned} \qquad (3.260)$$

one gets

$$4\ddot\phi_1 = 3\frac{\partial V}{\partial\bar\phi_1}, \qquad 6\ddot\phi_2 = 9\frac{\partial V}{\partial\bar\phi_2}. \qquad (3.261)$$

3.4.9 Yang-Mills on $\mathbb{R} \times G_2/\mathrm{SU}(3)$

The most general G-invariant ansatz (3.88) for the connection one-form $\mathcal{A}$ on $\mathbb{R} \times G_2/SU(3)$ has the form (see 3.104)

$$\mathcal{A} = e^i I_i + e^a X_{ab}(\tau) I_b \quad \text{with} \quad X_{ab} = \phi_1 \mathrm{Id}_{ab} + \phi_2 J_{ab}. \qquad (3.262)$$

We identify X with the complex valued function $\phi = \phi_1 + i\phi_2$. This case was particularised in [2], but can also be derived directly from the already discussed case $\mathbb{R}\times$SU(3)/U(1)$\times$ U(1). For this purpose we should only put equal the functions $\phi := \phi_1 = \phi_2 = \phi_3$. In this way we obtain from (3.263) the Yang-Mills equation for $\mathbb{R} \times G_2/\mathrm{SU}(3)$, namely:

$$6\ddot\phi = (\kappa-1)\phi - (\kappa+3)\bar\phi^2 + 4\phi|\phi|^2. \qquad (3.263)$$

The constrains (3.168) then are fulfilled identically. For the corresponding potential we obtain

$$V(\phi) = \frac{\kappa-1}{3}|\phi|^2 - \frac{\kappa+3}{9}(\phi^3 + \bar\phi^3) + \frac{2}{3}|\phi|^4 - \frac{\kappa-3}{3} \qquad (3.264)$$

and hence,

$$6\ddot\phi = 3\frac{\partial V}{\partial\bar\phi}. \qquad (3.265)$$

3.4.10 Yang-Mills on $\mathbb{R} \times S^3 \times S^3$

In this work we will not discuss the Yang-Mills equations on $\mathbb{R} \times S^3 \times S^3$, the last relevant case for possible nearly Kähler structure, but instead refer the reader to [83]. There one can find the complete discussion this case.

3.4.11 Solutions on $\mathbb{R} \times \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$

The following sections we dedicate to solutions of the Yang-Mills equations (3.263), which can be carried out thanks to a possibility in some certain cases to introduce a "superpotential", due to which the second order differential equations deduce from first order differential equation. The latter, which we call the gradient or Hamiltonian flow equations, can be solved much easier.

Critical points of the potential

Let us consider the equation of motion found for the case $\mathbb{R} \times \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$:

$$6\ddot{\phi}_1 \quad = \quad (\kappa - 1)\phi_1 - (\kappa + 3)\bar{\phi}_2\bar{\phi}_3 + \phi_1(2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2) = 9\frac{\partial V}{\partial \bar{\phi}_1}. \quad (3.266)$$

We observe that this equations describe the motion of a particle moving in a three-dimensional complex space under the action of the potential $-V$. Note that only for real κ the potential is real and this mechanical interpretation is just straightforward.

First of all we are interested in instanton solutions corresponding to a particle moving form critical points of V to critical points. Due to conservation of energy, such a trajectory can exist only if $V(\tau \longrightarrow \infty) = V(\tau \longrightarrow -\infty)$. Searching for the extrema $\hat{\phi}$ of the potential V we find the following points:
where $\nu = \frac{1}{4}(\kappa - 1)$. The continuous symmetry (let us call it the 3α-symmetry)

$(\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3)$	$(0,0,0)$	$(0,0,\sqrt{\frac{1-\kappa}{2}}e^{i\alpha})$	$(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$	(ν,ν,ν)
V	$\frac{2}{9}(1-2\nu)$	$-\frac{(\kappa+1+2\sqrt{3})(\kappa+1-2\sqrt{3})}{36}$	0	$\frac{2}{9}(1+\nu)(1-\nu)^3$

$\phi_i \longrightarrow \phi_i e^{i\alpha}$, where $\sum_i \alpha_i = 0 \bmod 2\pi$, as well as the permutation symmetry applied on the given $(\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3)$ produce further extreme points. Remark that $(0, 0, \sqrt{\frac{1-\kappa}{2}}e^{i\alpha})$ is a local maximum for $\kappa < 1$ with respect to ϕ_3, since $\partial_{\phi_3}\partial_{\bar{\phi}_3} V(0, 0, e^{i\alpha}\sqrt{(1-\kappa)/2}) < 0$ for $\kappa < 1$.

Following the notation in [2], we differ between transverse trajectories, connecting one of the critical point with that critical point descending from the first one by the 3-symmetry $\phi \longmapsto \exp(\frac{2}{3}\pi i)$, and the radial trajectories, connecting two of the critical points with $\hat{\phi}$ on the real axis degenerated in energy. As we have found there, similar to the case $\mathbb{R} \times G_2/\mathrm{SU}(3)$ discussed in [2], the radial trajectories are possible for $\kappa = -3, -1 \pm 2\sqrt{3}, 1, 3, 9$. The transverse trajectories occur for $\kappa = -1, -7$ (i.e., for $\nu = -\frac{1}{2}, -2$).

The action of an instanton configurations is finite only if $V(\tau = \pm\infty) = 0$. Comparing with the results of [2], this is the case for $\kappa = -3, -1-2\sqrt{3}, 3$ (i.e., for $\nu = -1, -\frac{1}{2}, \frac{1}{2}$).

Physical meaning of the constraints

Let us take a look at the constraints

$$Y_i - Y_j \;=\; 0, \quad \forall i, \tag{3.267}$$

where $Y_i = \phi_i\dot{\bar{\phi}}_i - \dot{\phi}_i\bar{\phi}_i$. First of all we observe that due to (3.251)

$$\begin{aligned} \dot{Y}_i &= \phi_i\ddot{\bar{\phi}}_i - \ddot{\phi}_i\bar{\phi}_i = && (3.268)\\ &= \frac{3}{2}\left(\phi_i\frac{\partial V}{\partial\phi_i} - \bar{\phi}_i\frac{\partial V}{\partial\bar{\phi}_i}\right) = \frac{\kappa+3}{2}(\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 - \phi_1\phi_2\phi_3) && (3.269)\end{aligned}$$

$\dot{Y}_i$ is index independent. It follows that $\dot{Y}_i - \dot{Y}_j = 0$ and $Y_i - Y_j$ is always conserved. Consequently, also conserved is $\tilde{Y}_i = Y_i - \frac{1}{3}(Y_1+Y_2+Y_3)$ with $\sum_i \tilde{Y}_i = 0$ or $Y_i = Y_j$. Suppose that $Y_1 + d = Y_2$, $Y_2 + \tilde{d} = Y_3$. However, the constraints (3.267) mean that there are no additional constants $d, \tilde{d}$. Therefore, they fix the symmetry between the ϕ_i, $i = 1, 2, 3$.

Duality

The surprising duality appearing in the case $\mathbb{R}\times G_2/\mathrm{SU}(3)$ and discussed in [2] can also be observed on the solutions of the equations of motion (3.263). In order to see that we write the equations in terms of ν:

$$\begin{aligned} 6\ddot{\phi}_i &= 4\nu\phi_i - 4(\nu+1)\bar{\phi}_j\bar{\phi}_k + \phi_i(2|\phi_i|^2 + |\phi_j|^2 + |\phi_k|^2), && (3.270)\\ 3V(\phi_1,\phi_2,\phi_3) &= \frac{2}{3}(1-2\nu) + \frac{4}{3}\sum_n |\phi_n|^2 - \frac{8}{3}(1+\nu)\mathrm{Re}\phi_1\phi_2\phi_3 + \\ &\quad + \frac{2}{3}\sum_n |\phi_n|^4, && (3.271)\end{aligned}$$

with $i \neq j \neq k$. The duality consists in the correspondence between solutions of (3.270) connected via the map

$$(\nu, \phi_1(\tau), \phi_2(\tau), \phi_3(\tau)) \longmapsto \left(\frac{1}{\nu}, \frac{1}{\nu}\phi_1\left(\frac{\tau}{\nu}\right), \frac{1}{\nu}\phi_2\left(\frac{\tau}{\nu}\right), \frac{1}{\nu}\phi_3\left(\frac{\tau}{\nu}\right)\right). \tag{3.272}$$

Gradient flow

We consider the case $\kappa = 3$. The duality transformation makes it possible to apply the result also to $\kappa = 9$. Moreover, accomplishments obtained in [2] can be used here. Concretely, we also find out that the special case $\phi_1 = \phi_2 = \phi_3$ of the e.o.m. (3.263) are implied by the first order gradient flow equations expressed by a superpotential W (they also resolve the constrains (3.267)):

$$\begin{aligned} \pm\sqrt{3}\dot{\phi}_i &= \bar{\phi}_i^2 - \phi_i = \frac{\partial W}{\partial \bar{\phi}_i}, && (3.273)\\ \phi_1 &= \phi_2 = \phi_3, && (3.274) \end{aligned}$$

where

$$W = \phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 - \sum_i |\phi_i^2|. \quad (3.275)$$

Note that W and the flow equation (3.273) are invariant under 3α-symmetry. The finite-action kink solutions then are

$$(\phi_1, \phi_2, \phi_3) = \frac{1}{2}\left(1 \pm \tanh\frac{\tau-\tau_0}{2\sqrt{3}}, 1 \pm \tanh\frac{\tau-\tau_0}{2\sqrt{3}}, 1 \pm \tanh\frac{\tau-\tau_0}{2\sqrt{3}}\right), \quad (3.276)$$

with the collective coordinate τ_0. Note that the 3α-symmetry gives further solutions.

The explicit infinite-action solutions of the gradient flow were already given in [2]. So we only briefly give the final outcome. For the polar decomposition $\phi_1(\tau) = r(\tau)\exp(i\varphi(\tau))$ and $|\phi_1| = |\phi_2| = |\phi_1|$ with the corresponding $3\alpha-$symmetry the solution is

$$r(\varphi) = -\frac{1}{3}(\sin 3\varphi)^{-1/3}[\cos 3\varphi\, {}_2F_1(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2 3\varphi) + C], \quad (3.277)$$

for some real constant C and the hypergeometric functions ${}_2F_1$. This trajectories are unbounded.

Continuous symmetry

We consider the case $\kappa = -3$. It is special due to the continuous symmetry arising from the fact that $\kappa = -3$ is fixing point of the duality transformation and from the $U(1)^3$ symmetry of the solutions of the e.o.m. (3.263). As before, we found a superpotential W only for solutions of a special type, $|\phi_1| = |\phi_2| = |\phi_3|$. Namely,

$$W = 2|\phi_1||\phi_2||\phi_3| - 2\sum_i |\phi_i|. \quad (3.278)$$

With that W the second order equations of motion for ϕ_i follow from the gradient flow equations

$$\pm\sqrt{3}\dot{\phi}_i = \frac{\phi_i}{|\phi_i|}(|\phi_j||\phi_k| - 1) = \frac{\partial W}{\partial \bar{\phi}_i}, \ \phi_2 = e^{i\alpha}\phi_1, \ \phi_3 = e^{i\beta}\phi_1, \quad (3.279)$$

with arbitrary $\alpha,\beta\in[0,2\pi]$. Finite-action solutions we extract from [2] are

$$(\phi_1,\phi_2,\phi_3)=\left(e^{i\alpha}\tanh\frac{\tau-\tau_0}{\sqrt{3}},e^{i\beta}\tanh\frac{\tau-\tau_0}{\sqrt{3}},e^{i\gamma}\tanh\frac{\tau-\tau_0}{\sqrt{3}}\right), \tag{3.280}$$

where α,β,γ are arbitrary angles. The infinite-action solution have also been given. The corresponding trajectories connect the critical $S^1\times S^1\times S^1$ torus with the point at the infinity (∞,∞,∞):

$$(\phi_1,\phi_2,\phi_3)=\mp(e^{i\alpha}\coth\frac{\tau-\tau_0}{\sqrt{3}},e^{i\beta}\coth\frac{\tau-\tau_0}{\sqrt{3}},e^{i\gamma}\coth\frac{\tau-\tau_0}{\sqrt{3}}), \tag{3.281}$$

with arbitrary α,β,γ. There are only these radial solutions of the gradient flow equations.

Now we turn to an other case. We consider solutions of (ϕ_1,ϕ_2,ϕ_3), where some two ϕ_i vanish identically. Suppose, it is ϕ_1 which remains in the differential equations. So the equation of motion takes the form

$$6\ddot{\phi}_1=(\kappa-1)\phi_1+2\phi_1|\phi_1|^2,\quad \phi_2=\phi_3\equiv 0. \tag{3.282}$$

In particular, there is no coupling between ϕ_i in the differential equation. Comparing with the previous solution one finds that the superpotential for this special case is

$$W=\frac{\sqrt{2}}{3}\sum_i|\phi_i|^3+\frac{1}{\sqrt{2}}(\kappa-1)\sum_i|\phi_i|, \tag{3.283}$$

then the e.o.m. (3.282) follows from the gradient flow equation

$$\pm\sqrt{3}\dot{\phi}_1=\frac{1}{\sqrt{2}}\frac{\phi_1}{|\phi_1|}\left(|\phi_1|^2+\frac{\kappa-1}{2}\right)=\frac{\partial W}{\partial\bar{\phi}_1}. \tag{3.284}$$

Supposing $\phi_1=\sqrt{\frac{1-\kappa}{2}}\tilde{\phi}_1$, we again come back to (3.279) and obtain as a radial trajectory solution

$$\tilde{\phi}_1=e^{i\alpha}\tanh\left(\sqrt{\frac{1-\kappa}{12}}(\tau-\tau_0)\right),\quad \alpha\in[0,2\pi], \tag{3.285}$$

or

$$\phi_1=e^{i\alpha}\sqrt{\frac{1-\kappa}{2}}\tanh\left(\sqrt{\frac{1-\kappa}{12}}(\tau-\tau_0)\right),\quad \alpha\in[0,2\pi],\quad \phi_2=\phi_3\equiv 0. \tag{3.286}$$

For $\kappa>1$ the trajectory is periodic and non-continuous. Since $V(e^{i\alpha}\sqrt{\frac{1-\kappa}{2}},0,0)=-\frac{(\kappa+1+2\sqrt{3})(\kappa+1-2\sqrt{3})}{36}$ the solutions (3.286) have finite action only if

$$\kappa=-1-2\sqrt{3}. \tag{3.287}$$

Another solution of the gradient flow equation (3.284) is

$$\phi_1 = \mp \sqrt{\frac{1-\kappa}{2}} e^{i\alpha} \coth\left(\sqrt{\frac{1-\kappa}{12}} (\tau - \tau_0) \right), \quad \alpha \in [0, 2\pi], \quad \phi_2 = \phi_3 \equiv 0. \tag{3.288}$$

For $\kappa < 1$ it corresponds to a infinite-action solution and to a trajectory connecting the critical circle $(e^{i\alpha}, 0, 0)$ with the point at the infinity $(\infty, 0, 0)$. For $\kappa > 1$ this solution yields a periodic infinite-action solution if $\kappa \neq -1 + 2\sqrt{3}$, otherwise the action is finite and the trajectory is periodic.

At last we consider the case $\kappa = 1$. Then equation of motion takes the form

$$6\ddot{\phi}_1 = 2\phi_1|\phi_1|, \quad \phi_2 = \phi_3 \equiv 0. \tag{3.289}$$

Using the same superpotential (3.283), we obtain the gradient flow equation

$$\pm\sqrt{3}\dot{\phi}_1 = \frac{1}{\sqrt{2}}\phi_1|\phi_1| = \frac{\partial W}{\partial \bar{\phi}_1}. \tag{3.290}$$

In polar representation $\phi_1 = r(\tau)e^{i\varphi(\tau)})$ we then have

$$\pm\sqrt{6}(\dot{r} + ir\dot{\varphi}) = r^2. \tag{3.291}$$

Hence, $\pm\sqrt{6}\dot{r} = r^2$, $\dot{\varphi} = 0$ and φ is constant. We find a solution

$$r(\tau) = \frac{\sqrt{6}}{|\tau| + \tau_0}, \quad \tau_0 > 0. \tag{3.292}$$

We remark that this solution is not differentiable at the origin. The trajectory of this radial infinite-action solution $\phi_1(\tau) = \frac{\sqrt{6}}{|\tau|+\tau_0}e^{i\varphi}$, $\phi_2 = \phi_3 = 0$ connects the origin with itself.

Note that for all cases mentioned above further solutions are obtained via 3α-symmetry.

Hamiltonian flow

Now let us consider transverse trajectories. For that goal we again use solutions presented in [2]. Concretely, for ϕ_i connected by the 3α-symmetry and $|\phi_1| = |\phi_2| = |\phi_3|$ and for $\kappa = -1$ ($\kappa = 7$ yields a solution, which can be obtained by the duality transformation) there is a superpotential

$$W = \phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3 - \sum_i |\phi_i|^2, \tag{3.293}$$

such that e.o.m. (3.263) are a conclusion of the first-oder differential equations

$$\begin{aligned} \pm\sqrt{3}\dot{\phi} &= (\bar{\phi}^2 - \phi_i) = i\frac{\partial W}{\partial \bar{\phi}}, && (3.294)\\ \phi \equiv \phi_1 &= \phi_2 = \phi_3. && (3.295) \end{aligned}$$

The finite-action solutions are

$$\phi = \frac{1}{2}\left(1 \pm i\sqrt{3}\tanh\left(\frac{\tau-\tau_0}{2}\right)\right), \quad \forall i, \tag{3.296}$$

and those obtained via 3α symmetry. The trajectory describing the solution (3.296) connects the points $(e^{\frac{4}{3}\pi i}, e^{\frac{4}{3}\pi i}, e^{\frac{4}{3}\pi i})$ and $(e^{\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i})$.

The infinite-action solution is in implicit form can be given by the cubic equation (see [2])

$$\cos 3\varphi = \frac{1}{2}\frac{r^2+C}{r^3}, \tag{3.297}$$

where we use the polar representation $\phi = r\exp(i\varphi)$ ($\phi_1 = \phi_2 = \phi_3$ plus 2α-symmetric solutions) and C is an invariant preserved by the flow equation (3.294), $C = W((\phi_1,\phi_2,\phi_3)(\tau))$. Periodic trajectories exists for $|\frac{1}{2}\frac{r^2+C}{r^3}| \leq 1$, i.e., for $-\frac{10}{27} < C < 0$.

3.4.12 Solutions on $\mathbb{R}\times \mathrm{Sp}(2)/\mathrm{Sp}(1)\times \mathrm{U}(1)$

Critical points

The equations of motion on $\mathbb{R}\times \mathrm{Sp}(2)/\mathrm{Sp}(1)\times \mathrm{U}(1)$ are

$$6\ddot{\phi}_1 = (\kappa-1)\phi_1 - (\kappa+3)\bar{\phi}_1\bar{\phi}_2 + \phi_1(3|\phi_1|^2+|\phi_2|^2) = \frac{9}{2}\frac{\partial V}{\partial\bar{\phi}_1}, \tag{3.298}$$

$$6\ddot{\phi}_2 = (\kappa-1)\phi_2 - (\kappa+3)\phi_1^2 + 2\phi_2(|\phi_1|^2+|\phi_2|^2) = 9\frac{\partial V}{\partial\bar{\phi}_2}, \tag{3.299}$$

with

$$\begin{aligned} V(\phi) &= \frac{\kappa-1}{9}(2|\phi_1|^2+|\phi_2|^2) - \frac{\kappa+3}{9}(\bar{\phi}_1^2\bar{\phi}_2+\phi_1^2\phi_2) \\ &\quad -\frac{1}{9}(3|\phi_1|^4+|\phi_2|^4+2|\phi_1|^2|\phi_2|^2) - \frac{\kappa-3}{3}, \end{aligned} \tag{3.300}$$

We remember that there is a correspondence to the e.o.m. on $\mathbb{R}\times\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$ via the projection $\phi_1 \longrightarrow \phi_1$, $\phi_2 \longrightarrow \phi_1$ and $\phi_3 \longrightarrow \phi_2$. So that the solution space of (3.298) and (3.299) is a subspaces of solutions of (3.263). Therefore we can read off the critical points $(\hat{\phi}_1,\hat{\phi}_2)$ of the potential V from the previous section: where $2\alpha_1+\alpha_2 = 0 \mod 2\pi$ and $\nu = \frac{1}{4}(\kappa-1)$. The 3α-symmetry as well as

$(\hat{\phi}_1,\hat{\phi}_2)$	$(0,0)$	$(0,\sqrt{\frac{1-\kappa}{2}}e^{i\alpha})$	$(e^{i\alpha_1},e^{i\alpha_2})$	$(\nu e^{i\alpha_1},\nu e^{i\alpha_2})$
V	$\frac{2}{9}(1-2\nu)$	$-\frac{(\kappa+1+2\sqrt{3})(\kappa+1-2\sqrt{3})}{36}$	0	$\frac{2}{9}(1+\nu)(1-\nu)^3$

permutation symmetry are now missing, but instead we have a symmetry, let call it 2α-symmetry, transforming solutions ϕ_1, ϕ_2 respectively to the solutions $e^{i\alpha_1}\phi_1$ and $e^{i\alpha_2}\phi_2$, with $2\alpha_1+\alpha_2 = 0$.

Gradient and Hamiltonian flows

Obviously, any solution of (3.298) and (3.299) is also a solution of (3.263), so it is a subcase that we discuss here. At first we observe that the simplest way to get some solutions for (3.298) and (3.299) is to narrow the solution set of the e.o.m. for the case $\mathbb{R} \times SU(3)/U(1) \times U(1)$, which we constructed sections. It is done by reducing the 3α-symmetry to the 2α-symmetry, because then we have the same differential equations (3.298) and (3.299). It is clear that for the relatively trivial case, when $|\phi_1|$ and $|\phi_2|$ are equal, the second order Yang-Mills equation are consequences of the same flow equations as before for three functions ϕ_i. Nevertheless, here appears a question whether the flow equations are reasoned by the corresponding superpotentials. It is easy to see that it indeed the fact, but only with a little alteration

$$\pm\sqrt{3}\phi_1 = \frac{1}{2}\frac{\partial W}{\partial\bar{\phi}_1}, \quad \pm\sqrt{3}\phi_2 = \frac{\partial W}{\partial\bar{\phi}_2}. \tag{3.301}$$

The superpotential remains unchanged, one has only to identify $\phi_1 \longrightarrow \phi_1$, $\phi_2 \longrightarrow \phi_1$ and $\phi_3 \longrightarrow \phi_2$.

3.4.13 Instanton-anti-instanton chains and dyons

Periodic solutions

Looking for periodic solutions, we again assume that $\phi = \phi_1 = \phi_2 = \phi_3$ so that the equations of motions take the simplest form

$$6\ddot{\phi} \quad = \quad (\kappa-1)\phi - (\kappa+3)\bar{\phi}^2 + 4\phi|\phi|^2. \tag{3.302}$$

After writing the function ϕ as $\phi =: \varphi_1 + i\varphi_2$, we may consider trajectories of $\varphi_1(\tau)$ unchanging with τ. Then we have to find appropriate values for the constants φ_1 and κ so that also $\ddot{\varphi}_1 = 0$ is satisfied for all φ_2. From (3.302) we obtain

$$6\ddot{\varphi}_1 = (\kappa-1)\varphi_1 - (\kappa+3)\varphi_1^2 + 4\varphi_1^3 + [(\kappa+3) + 4\varphi_1]\varphi_2^2 \tag{3.303}$$

and get exactly three solutions

$$(\varphi_1, \kappa) = (0, -3),\ (-\frac{1}{2}, -1) \quad \text{and} \quad (1, -7)\,. \tag{3.304}$$

Substituting these values into (3.302), we obtain the equations for φ_2:

$$\begin{array}{lll} \text{a)} & (\varphi_1,\kappa) = (0,-3): & 2\ddot{\varphi}_2 = -(1-\varphi_2^2)\varphi_2\,, \\ \text{b)} & (\varphi_1,\kappa) = (-\frac{1}{2},-1): & 2\ddot{\varphi}_2 = -(\frac{3}{4}-\varphi_2^2)\varphi_2\,, \\ \text{c)} & (\varphi_1,\kappa) = (1,-7): & 2\ddot{\varphi}_2 = -(3-\varphi_2^2)\varphi_2\,. \end{array} \tag{3.305}$$

Here, we rescaled $\tau \rightarrow \frac{2}{\sqrt{3}}\tau$.

We can use solutions of this type of equations, which we already treated in section 3.4.12, in order to obtain a periodic kind of solutions (sphalerons). For that purpose we require ϕ to be cyclic:

$$\varphi_1(\tau + L) = \varphi_1(\tau) \quad \text{and} \quad \varphi_2(\tau{+}L) = \varphi_2(\tau)\,. \tag{3.306}$$

This condition is equivalent to the consideration of function ϕ on a circle S^1 with circumference L. Sphaleron solutions to three equations (3.305) can be described simultaneously by

$$\varphi_2 = \gamma\, 2k\, b(k)\, \mathrm{sn}[b(k)\gamma\tau; k] \quad \text{with} \quad \gamma = 1, \frac{\sqrt{3}}{2}, \sqrt{3}\,, \tag{3.307}$$

respectively, where $b(k) = (2 + 2k^2)^{-1/2}$ and $0 \leq k \leq 1$. The Jacobi elliptic function $\mathrm{sn}[u; k]$ has a period of $4K(k)$ (see appendix A.2), the condition (3.306) is satisfied for

$$b(k)\, L = 4K(k)\, n \quad \text{for} \quad n \in \mathbb{N}\,, \tag{3.308}$$

which fixes $k = k(L, n)$ so that $\varphi_2(\tau; k) =: \varphi_2^{(n)}(\tau)$. Solutions (3.307) exist if $L \geq L_n := 2\pi\sqrt{2}\, n$ [84].

By virtue of the periodic boundary conditions (3.306), the topological charge of the sphaleron

$$\phi_n = \beta + \mathrm{i}\,\gamma\, 2k\, b(k)\, \mathrm{sn}[b(k)\gamma\tau; k], \tag{3.309}$$

with $(\beta, \gamma) = (0, 1), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(1, \sqrt{3})$ is zero. In fact, the configuration (3.309) can be interpreted as a chain of n kinks and n antikinks, alternating and equally spaced around the circle [84]. The energy of the sphaleron is

$$E = \int_0^L \mathrm{d}\tau \left\{ |\dot{\phi}|^2 + V(\phi, \bar{\phi}) \right\} \tag{3.310}$$

and, e.g., for the case a) in (3.305) we obtain

$$E(\phi_n) = \frac{2n}{3\sqrt{2}} \left[8(1 + k^2)\, E(k) \;-\; (1 - k^2)(5 + 3k^2)\, K(k) \right]\,, \tag{3.311}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively [84].

Substituting the non-BPS solutions (3.309) into (3.88), (3.156)-(3.158) or (3.182)-(3.184), we obtain a finite-action Yang-Mills configuration which is interpreted as a chain of n instanton-anti-instanton pairs sitting on $S^1 \times G/H$, where G/H is a six-dimensional nearly Kähler coset space.

We note that the last preceding sections deal with the special sorts of solutions of the Yang-Mills equations on the manifolds over the nearly Kähler manifolds. The general case with different ϕ_1, ϕ_2 and ϕ_3 can be considered by constructing a numerical solution. Finding of an analytical solution to the e.o.m. for the general case is rather problematical because of the non-linearity and derivatives of the second order.

Dyonic solutions

In the final step we consider dyonic solutions of (3.302) characterized by introducing Lorentzian metric on $\mathbb{R} \times G/H$. That is done by changing the signature of the τ-coordinate of $\mathbb{R}$, i.e., by modification $t = -\mathrm{i}\tau$ so that $\tilde{e}^0 = \mathrm{d}t = -\mathrm{i}\mathrm{d}\tau$. The metric on $\mathbb{R} \times G/H$ is then given by

$$\mathrm{d}s^2 = -(\tilde{e}^0)^2 + \delta_{ab}e^a e^b \; . \tag{3.312}$$

Since the G-invariant ansätze (3.156) and (3.182) for matrices X_a do not change their form, substituting them into the Yang-Mills equations on $\mathbb{R}\times G/H$ we arrive at the same second-order differential equations as in the Euclidean case, except for the replacement

$$\ddot{\phi}_i \;\longrightarrow\; -\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} \quad \text{and hence,} \quad V(\phi_1,\phi_2,\phi_3) \;\longrightarrow\; -V(\phi_1,\phi_2,\phi_3) \tag{3.313}$$

for the kinetic terms and the potential energy. This alteration implies a sign change of the left hand side relative to the right hand side in (3.263) and (3.298), (3.299). Moreover, the Lorentzian variant of equations (3.305) can be not derived from first-order equations. Nevertheless, they can be integrated explicitly so that we obtain as a solution

$$\phi(t) = \beta + \mathrm{i}\,\sqrt{2}\,\gamma\, \cosh^{-1}\frac{\gamma t}{\sqrt{2}} \; , \tag{3.314}$$

with $(\beta,\gamma) = (0,1), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(1, \sqrt{3})$. Any such configuration is a bounce in an inverted double-well potential, which from the asymptotic local minimum briefly explores the unstable region.

After inserting (3.314) into the gauge potential (3.88), we arrive at dyon-type configurations with smooth non-vanishing 'electric' and 'magnetic' field strength F_{0a} and F_{ab}, respectively. The full energy, given by

$$-\,\mathrm{tr}(2F_{0a}F_{0a} + F_{ab}F_{ab}) \times \mathrm{Vol}(G/H), \tag{3.315}$$

for these configurations is finite, while the action is not.

To the Lorentzian equations for ϕ_1, ϕ_2, ϕ_3 one can also seek a numerical solutions. However, we leave that task to the prospective research papers and theses.

Appendix A

Appendix

A.1 Jacobi identities

As elements of a Lie algebra the generators I_A satisfy the Jacobi identity

$$[I_b, [I_j, I_a]] + [I_j, [I_a, I_b]] + [I_a, [I_b, I_j]] = 0. \tag{A.1}$$

The commutation relation between the elements

$$[I_i, I_j] = f_{ij}^k I_k, \quad [I_i, I_a] = f_{ia}^c I_c, \quad [I_a, I_b] = f_{ab}^i I_i + f_{ab}^c I_c \tag{A.2}$$

allow us to rewrite it in terms of the structure constants:

$$(f_{ja}^c f_{bc}^d + f_{ab}^c f_{jc}^d - f_{jb}^c f_{ac}^d) I_d + (f_{ja}^c f_{bc}^i + f_{ab}^k f_{jk}^i - f_{jb}^c f_{ac}^i) I_i = 0. \tag{A.3}$$

Since the metric has the simple form $g_{AB} = \delta_{AB}$, we can pull down the indices. Moreover, the generators I_A are linear independent so that each of both summands in (A.3) has to vanish. Thus,

$$f_{jac} f_{dbc} + f_{abc} f_{jcd} + f_{jbc} f_{adc} = 0, \qquad f_{jac} f_{ibc} + f_{kb} f_{ijk} + f_{jbc} f_{iac} = 0. \tag{A.4}$$

A.2 Jacobi elliptic functions

The Jacobi elliptic functions arise from the inversion of the elliptic integral of the first kind,

$$u = F(\xi, k) = \int_0^{\xi} \frac{\mathrm{d}x}{\sqrt{1 - k^2 \sin x}}, \qquad 0 \leq k^2 < 1, \tag{A.5}$$

where $k = \operatorname{mod} u$ is the elliptic modulus and $\xi = \operatorname{am}(u, k) = \operatorname{am}(u)$ is the Jacobi amplitude, given by

$$\xi = F^{-1}(u, k) = \operatorname{am}(u, k). \tag{A.6}$$

Then the three basic functions sn, cn and dn are defined by

$$\mathrm{sn}[u;k] = \sin(\mathrm{am}(u,k)) = \sin\xi, \tag{A.7}$$
$$\mathrm{cn}[u;k] = \cos(\mathrm{am}(u,k)) = \cos\xi, \tag{A.8}$$
$$\mathrm{dn}[u;k] = \sqrt{1-k^2\sin^2(\mathrm{am}(u,k))} = \sqrt{1-k^2\sin^2\xi}. \tag{A.9}$$

These functions are periodic in $K(k)$ and $\tilde{K}(k)$,

$$\mathrm{sn}[u+2mK+2ni\tilde{K};k] = (-1)^m\mathrm{sn}[u;k], \tag{A.10}$$
$$\mathrm{cn}[u+2mK+2ni\tilde{K};k] = (-1)^{m+n}\mathrm{cn}[u;k], \tag{A.11}$$
$$\mathrm{dn}[u+2mK+2ni\tilde{K};k] = (-1)^n\mathrm{dn}[u;k], \tag{A.12}$$

where $\tilde{K}(k) := K(\sqrt{1-k^2})$ and $K(k)$ is the complete elliptic integral of the first kind, $K(k) := F(\frac{\pi}{2},k)$. In the following we eventually omit the parameter k and write $\mathrm{sn}[u;k] = \mathrm{sn}(u)$ etc.

The Jacobi elliptic functions satisfy some identities, including

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1, \tag{A.13}$$
$$k^2\mathrm{sn}^2u + \mathrm{dn}^2u = 1, \tag{A.14}$$
$$\mathrm{cn}^2u + \sqrt{1-k^2}\,\mathrm{sn}^2u = 1 \tag{A.15}$$

and

$$\mathrm{sn}[u;0] = \sin u, \tag{A.16}$$
$$\mathrm{cn}[u;0] = \cos u, \tag{A.17}$$
$$\mathrm{dn}[u;0] = 1, \tag{A.18}$$

so they are a generalization of the trigonometric functions.

One may also define cn, dn and sn as solutions to the differential equations

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2-k)^2y + y^3, \tag{A.19}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -(1-2k^2)y + 2k^2y^3, \tag{A.20}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -(1+k^2)y + 2k^2y^3, \tag{A.21}$$

respectively.

List of Tables

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