Sasakian and 3-Sasakian Quiver Gauge Theories and Instantons on Homogeneous Spaces

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Kurzzusammenfassung

In dieser Arbeit konstruiere ich neue Sasaki-Köcher-Eichtheorien auf konkreten homogenen Räumen G/H mit Sasaki-Einstein- oder 3-Sasaki-Struktur und untersuche die resultierenden Instanton-Matrixgleichungen auf den metrischen Kegeln über G/H. Die betrachteten Quotientenräume umfassen die Mannigfaltigkeit $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$, die runde 7-Sphäre $\mathrm{SU}(4)/\mathrm{SU}(3)$, die gequetschte 7-Sphäre $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ sowie den Aloff-Wallach-Raum $X_{1,1}$, wobei die beiden letzteren 3-Sasaki-Mannigfaltigkeiten sind.

Köcher-Eichtheorien treten im Zusammenhang mit äquivarianter Dimensionsreduktion von Eichtheorien auf Räumen der Form $M^d \times G/H$ auf und ermöglichen einen diagrammatischen Zugang zum Feldinhalt der Theorie: Für eine gewählte G-Darstellung erfordert die Äquivarianz eine isotopische Zerlegung der beteiligten Vektorbündel, bei der jeder Summand einer Darstellung von H innerhalb der G-Darstellung entspricht und zu einem Vertex des Köcherdiagramms führt. Die Wirkung der gesamten Gruppe G verbindet solche Unterdarstellungen und erzeugt dadurch die Pfeile des Köcherdiagramms; daher kann die Eichtheorie als Darstellung eines Köcherdiagramms betrachtet werden.

Diese Konstruktion kann auch verwendet werden, um eine Invarianzbedingung an Eichzusammenhänge auf homogenen Räumen zu erfüllen, und liefert allgemeinere Lösungen als ein skalarer Ansatz, der für viele Instantonstudien angewandt worden ist. Für alle vier Quotientenräume werde ich die Äquivarianzbedingung diskutieren und explizite Beispiele der resultierenden Köcherdiagramme geben, wobei ich die Ergebnisse untereinander und mit Literaturergebnissen zu verwandten geometrischen Räumen vergleiche.

Verallgemeinerte Instantonen lassen sich für Mannigfaltigkeiten mit reellen Killing-Spinoren in Form einer verallgemeinerten Selbstdualitätsbedingung definieren und implizieren bekanntermaßen die normale Yang-Mills-Gleichung. Instantonen sind als Lösungen der Gaugino-Gleichung, die zu den BPS-Bedingungen gehört, ein wichtiger Bestandteil der heterotischen Supergravitation. Nach der Konstruktion von äquivarianten Zusammenhängen, die durch die Köcherdiagramme beschrieben werden, untersuche ich Instantonen auf den Calabi-Yau- und Hyperkähler-Kegeln (bzw. -Zylindern) über den homogenen Räumen. Dies führt zu bestimmten Matrixgleichungen, die nur von der Kegelrichtung abhängen und deren Form durch die Köcherdiagramme bestimmt ist.

Die Instantonbedingungen auf den Calabi-Yau-Kegeln über $T^{1,1}$ und S^7 liefern das erwartete Ergebnis aus der Literatur, und daher kann der Modulraum durch koadjungierte Orbits oder als Kählerquotient beschrieben werden. Instantonen auf den Hyperkähler-Kegeln über $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ und $X_{1,1}$ können auf den Schnitt von drei Hermiteschen Yang-Mills-Gleichungen zurückgeführt werden, wobei andere algebraische Bedingungen auftreten, die die unterschiedliche Bündelstruktur von Sasaki- und 3-Sasaki-Mannifaltigkeiten widerspiegeln. Ich diskutiere die Folgen dieser veränderten Instantonbedingungen für die Modulräume.

Schlagwörter: Verallgemeinerte Instantonen, Köcher-Eichtheorie, Sasaki-Einstein-Mannigfaltigkeit.

Abstract

In this thesis I construct new Sasakian quiver gauge theories on concrete homogeneous spaces G/H endowed with Sasaki-Einstein or 3-Sasakian structures and study the resulting instanton matrix equations on the metric cones over G/H. The coset spaces taken into account are the Romans space $T^{1,1}$ in dimension 5, the round seven-sphere SU(4)/SU(3), the squashed seven-sphere Sp(2)/Sp(1) as well as the Aloff-Wallach space $X_{1,1}$, where the latter two examples are 3-Sasakian manifolds.

Quiver gauge theories arise in the context of equivariant dimensional reduction of gauge theories on $M^d \times G/H$ and allow a diagrammatic approach to the field content of the gauge theory: for a fixed G-representation on the fibers, equivariance requires an isotopical decomposition of the vector bundles involved where each summand corresponds to an H-representation inside the G-representation and gives rise to a vertex of a quiver diagram. The entire group action connects different subrepresentations and therefore induces arrows; thus, the gauge theory can be considered as representation of a quiver.

This construction can also be applied for satisfying an invariance condition on gauge connections over homogeneous spaces and provides more general solutions than the usual scalar approach, which has been employed for various instanton studies. For all four cosets, I will discuss the equivariance condition and give explicit examples of the resulting quiver diagrams, comparing the findings among each other and also with literature results for related geometries.

Generalized instantons can be defined on manifolds with real Killing spinors in terms of a generalized self-duality equation, and they are known to imply the usual Yang-Mills equation. Instantons are solutions of the gaugino equation, which is part of the BPS conditions, and therefore constitute important ingredients of heterotic supergravity. Having constructed equivariant gauge connections encoded in the quiver diagrams, I will study instantons on the Calabi-Yau and hyper-Kähler cones (or cylinders) over the homogeneous spaces. The quiver approach then yields instanton matrix equations which depend on the cone direction and whose structure is determined by the quiver diagrams.

The instanton conditions on the Calabi-Yau cones over $T^{1,1}$ and S^7 match the expected results from the literature, and therefore the moduli space can be described in terms of coadjoint orbits or as a Kähler quotient. Instantons on the hyper-Kähler cones over Sp(2)/Sp(1) and $X_{1,1}$ can be traced back to the intersection of three Hermitian Yang-Mills conditions, but the algebraic conditions are changed, reflecting the different bundle structure of Sasakian and 3-Sasakian manifolds. I will discuss the impact of these different conditions on the moduli spaces.

Keywords: Generalized Instantons, Quiver Gauge Theory, Sasaki-Einstein Manifolds.

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Contents

1 Introduction

String theory – the most promising candidate for a unified description of gravity and quantum theory – comprises an astonishing mixture of theoretical physics and various fields of mathematics such as differential geometry, algebraic geometry and number theory. This unique mélange often reveals surprising links between different fields, inspiring both mathematicians and physicists: mirror symmetry, the web of dualities between the different types of string theory and M-theory as well as the AdS/CFT correspondence have emerged in this context, to name a few.

Compactification and Killing spinors. One of the most striking aspects of string theory is the need for – or the prediction of – a ten-dimensional spacetime for superstrings or an eleven-dimensional background for M-theory. Actually, the idea of considering theories in dimensions higher than four is older than string theory and goes back to Kaluza's and Klein's attempt to unify gravity and electromagnetism in terms of a 5-dimensional theory [1, 2]. The arising extra-dimensions of string theory can be accounted for by applying the *compactification* ansatz: the higher-dimensional spacetime is written as a (possibly warped) product of a Lorentzian manifold $\mathbb{X}_{D-d}$ and a compact Riemannian manifold M^d, assumed to be so tiny that it is beyond our perception. Constructing a theory that leads to phenomenologically reasonable physical results on the external spacetime $\mathbb{X}_{D-d}$ then requires finding a suitable *internal manifold* M^d. The external manifold is usually a highly symmetric space like Minkowski or Anti-de-Sitter (AdS) spaces.

The supersymmetry of string theory is described by field variations which twist bosonic and fermionic degrees of freedom, and these conditions strongly restrict the possible geometric structure of the internal manifold M^d. In the simplest case, the manifold M^d has to admit parallel spinor(s), $\nabla_X \psi = 0$, so that it is forced to have reduced holonomy. According to Berger's list [3], only the Lie groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(m)$, $\mathrm{Sp}(m)\mathrm{Sp}(1)$ and the exceptional cases G_2 and $\mathrm{Spin}(7)$ can occur as special holonomy (of non-symmetric spaces) [4], which corresponds to Kähler, Calabi-Yau, hyper-Kähler, quaternionic Kähler, G_2 and Spin(7) manifolds, respectively. A prototypical example of this ansatz is compactification of type IIA and IIB string theory on Calabi-Yau 3-folds.

More complicated scenarios comprise so-called *fluxes* [5,6], certain p-form contributions, and the Killing spinor equations, ensuring supersymmetry of the background geometry, require the manifold to carry a G-structure, i.e. to admit a reduction of the *structure* group, see for example [7]. Many compactification setups are now described by *real* Killing spinors, i.e. spinors ψ satisfying

$$\nabla_X \psi = \alpha X \cdot \psi \ , \tag{1.1}$$

with $\alpha \in \mathbb{R} \setminus \{0\}$. Bär [8] observed that the metric cone over a manifold with real Killing spinors is endowed with parallel spinors and therefore special holonomy, which yielded a classification: besides the round spheres, only Sasaki-Einstein, 3-Sasakian, nearly-parallel G_2 and nearly-Kähler manifolds admit real Killing spinors. All of them have been intensively studied as compactification spaces in the literature, see for example the summary of brane configurations and compactification manifolds in [9] and the references therein. Sasaki-Einstein manifolds can be used as (relatively simple) solutions in backgrounds of the form $\mathrm{AdS}_{D-d} \times M^d$ in string and M-theory and therefore play a crucial role in the AdS/CFT correspondence.

Instantons. A physical theory which extends the Standard Model of particle physics naturally has to incorporate gauge theory; in particular, in type I and heterotic supergravity the bosonic degrees of freedom include a gauge field $\mathcal{A}$. Therefore, an understanding of higher-dimensional gauge theories [10] is necessary. The breakthrough of gauge theory in four dimensions has been the development of Yang-Mills theory and the construction of the Standard Model of particle physics as a gauge theory with gauge group $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. On Riemannian 4-manifolds, the understanding of physical and mathematical properties of gauge theories was deepened by the study of gauge connections $\mathcal{A}$ whose curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ obeys

$$\star \mathcal{F} = \pm \mathcal{F} \qquad (\Longleftrightarrow \ \mathcal{F}_{12} = \pm \mathcal{F}_{34}, \ \mathcal{F}_{13} = \mp \mathcal{F}_{24}, \ \mathcal{F}_{14} = \pm \mathcal{F}_{23}). \tag{1.2}$$

Such (anti-)self-dual connections are called *instantons* [11]. They satisfy the Yang-Mills equation $D_{\mathcal{A}} \star \mathcal{F} = 0$ as equation of motion of the Yang-Mills functional thanks to the Bianchi identity $D_{\mathcal{A}} \mathcal{F} = 0$. In fact, instantons minimize the action functional and, consequently, play an important role as non-perturbative tools in field theory [12–14]. Moduli spaces of such gauge configurations turned out to yield fruitful new insights, in both physics and mathematics [15].

Generalized instantons and heterotic supergravity. The concept of instantons can be extended to higher-dimensional manifolds [16–19] in several (more or less equivalent) ways. From the point of view of heterotic supergravity (cf. [7]), an instanton is a gauge connection that satisfies – as part of the supersymmetry / BPS conditions – the *gaugino equation*

$$\mathcal{F} \cdot \epsilon = 0, \tag{1.3}$$

where the spinor ϵ is the supersymmetry generator. In terms of manifolds with G-structures, the instanton definition restricts the curvature $\mathcal{F}$ to lie in the Lie algebra of the reduced structure group, and this can rephrased

$$\star_d \mathcal{F} = -\mathcal{F} \wedge \star_d Q, \tag{1.4}$$

where Q is an invariant 4-form on M^d. On manifolds with Killing spinors this form can be constructed as bilinear of the spinors [19], and by virtue of the Killing spinor equations the generalized self-duality condition (1.4) then implies the Yang-Mills equation $D_{\mathcal{A}} \star_d \mathcal{F} = 0$, analogously to the 4-dimensional case. Hence, a *first*-order equation (in terms of the gauge connection $\mathcal{A}$) yields solutions to an intricate *second*-order equation of motion.

As an essential part of the supersymmetry equations, instanton solutions may serve as building blocks for full solutions of heterotic supergravity, as pursued e.g. in [19,20], and therefore the construction of instantons attracts interest of string theorists.

Instantons on homogeneous spaces. A large number of compactification spaces in string theory and supergravity are coset spaces G/H [21, 22]. Due to their relatively easy description, their well-understood properties and the manifest symmetries, they appear as typical model spaces in many applications, also for dimensional reduction of gauge theories [23, 24].

Many studies, for example [19,25–33], constructed generalized instantons and Yang-Mills connections on manifolds with Killing spinors or their metric cones/cylinders, for instance on G_2 manifolds, nearly Kähler manifolds and on Sasakian manifolds, typically on explicit homogeneous spaces G/H. The standard approach consists in expressing the gauge connection locally as $\mathcal{A} = \Gamma + X_\mu \otimes e^\mu$, where Γ is the canonical connection [19] adapted to the geometry, e^μ are a basis of 1-forms on the cotangent space of the coset, and X_μ are endomorphisms valued in the Lie algebra of the structure group of the gauge bundle. For invariance of the connection, one imposes the *equivariance condition* [34]

$$[I_j, X_\mu] = f_{j\mu}^{\nu} X_\nu, \tag{1.5}$$

where I_j are the generators of the subgroup H. In most cases, this condition is solved by applying the *scalar ansatz* $X_\mu = \lambda_\mu(x) I_\mu$ with functions $\lambda_\mu(x)$ depending on coordinates of an external manifold only. Often, the metric cone/cylinder $\mathbb{R}^+ \times G/H$ is included in this way and the instanton equations then reduce to ordinary differential equations on a set of functions which depend on the cone coordinate only.

Quiver gauge theory. Studying more general solutions of the equivariance condition (1.5) naturally leads to quiver gauge theories as they occur in the context of equivariant dimensional reduction of gauge theories on spaces $M^d \times G/H$ [34–38].

The central idea is to start from an arbitrary G-representation $\mathcal{D}$ which decomposes under restriction to the subgroup into a direct sum of several H-representations. The generators I_j as well as the canonical connection Γ split in the same way, and one can depict each contribution as vertex of an oriented graph, called *quiver diagram*. The group action of G (and equivalently the matrices X_μ) connect different subrepresentations and therefore can be represented by arrows in the diagram. Similarly, this decomposition induces a breaking of the structure group of the gauge bundle, which makes equivariant dimensional reduction an interesting realization for gauge theories with symmetry breaking.

When applied to the construction of instantons, this ansatz yields matrix equations rather than conditions on functions, and the occurring contributions are depicted in the quiver diagrams associated to chosen G-representations. Such quiver gauge theories have been derived for the homogeneous spaces $\mathbb{C}P^1$, $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathrm{SU}(3)/(\mathrm{U}(1) \times \mathrm{U}(1))$, endowed with Kähler or nearly-Kähler structures [35–43].

In [44] this concept has been extended to the Sasaki-Einstein space $\mathrm{SU}(2)/\Gamma$ and was named *Sasakian quiver gauge theories*. Thereafter, the round five-sphere S^5, the odd-dimensional counterpart of the space $\mathbb{C}P^2$, has been included into this framework [45, 46]. In both cases, the motivation (see [45, Ch. 7]) was to gain further insights into the interplay between different quiver gauge theories in related geometric setups, by using the “bridging property” of Sasakian geometry.

This thesis is devoted to constructing new examples of Sasakian quiver gauge theories on 5– and 7-dimensional coset spaces as well as to studying the moduli space of instantons on the metric cones over these cosets, endowed with Calabi-Yau and hyper-Kähler structures. It includes the results from [47–49], which are extended and related to each other.

Outline and results

Since the mathematical framework and the general ansatz for instantons on homogeneous spaces have already been established, the focus of this thesis is placed on studying *explicit examples* of Sasakian quiver gauge theories and on closing gaps in the literature by including some well-known homogeneous spaces in five and seven dimensions. The motivation arises from their application as possible compactification manifolds in string theory (see e.g. [50, Fig. 1]), in particular with respect to heterotic supergravity, and their well-understood mathematical properties.

More precisely, we will study the homogeneous spaces $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$ and $T^{1,1} := \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$, endowed with their Sasaki-Einstein structures, as well as the 3-Sasakian manifolds $S^7 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)$ and $X_{1,1} \cong \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$. Considering these spaces is a very natural choice: the space $T^{1,1}$ completes the list of relevant structures studied in dimension 5, and the round seven-sphere is the prototype of a 7-dimensional Sasaki-Einstein manifold. The squashed seven-sphere and the space $X_{1,1}$ exhaust – up to isometry and under certain regularity conditions – the list of 3-Sasakian manifolds in dimension seven.

Results. Besides giving a self-contained overview over Sasakian quiver gauge theories on various coset spaces G/H, the aims of this thesis can be divided into two parts: (i) completing the picture of Sasakian quiver gauge theories (in dimensions 5 and 7) and (ii) including 3-Sasakian manifolds into the framework of quiver gauge theories. The main results can be summarized as follows:

- In five dimensions the study of the space $T^{1,1}$, the only other complete homogeneous Sasaki-Einstein manifold besides S^5, closes a gap in the existing literature by providing another rank-2 quiver gauge theory. We derive the generic description for all representations of G and compare the results to those on S^5. Moreover, the reduction of the quiver gauge theory to the underlying Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ correctly reproduces the results from [43].
- Sasakian quiver gauge theories on the round sphere S^7 are constructed. We obtain results analogous to those on the five-sphere in [46] for low-dimensional examples of SU(4)-representations. As a by-product, one derives a further quiver gauge theory on a Kähler manifold by taking the limit $\mathbb{C}P^3$. In addition, aspects of the generalization to *any* odd-dimensioanl sphere $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ are sketched.
- Manifolds with 3-Sasakian structure are included into the framework of quiver gauge theories over coset spaces, and the main features, also in comparison with Sasaki-Einstein cosets, are elaborated. As a prototype the *squashed* seven-sphere is discussed by considering some explicit examples for low-dimensional Sp(2)-representations. We briefly comment on extensions to any squashed sphere $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ as well.
- 3-Sasakian quiver gauge theories on $X_{1,1}$ yield results similar to those for the squashed seven-sphere. By emphasizing the entire 3-Sasakian structure, the discussion here significantly extends [48], which only focussed on the Sasaki-Einstein geometry.
- We describe the moduli spaces of instantons on the Calabi-Yau and hyper-Kähler cones over the cosets G/H. While the instanton conditions on the cones over S^7 and $T^{1,1}$ lead to Nahm-type equations and correspond to the general results for Calabi-Yau cones [51], the hyper-Kähler case can be discussed as the intersection of three Hermitian Yang-Mills equations, which causes some changes.

Structure of the thesis. This thesis is organized as follows: the next two chapters review the mathematical basics that are relevant for the discussion of the various gauge theories in the remainder. Chapter 2 introduces the geometric setups of interest, *Sasaki-Einstein* and *3-Sasakian* manifolds, as well as the closely related Kähler and hyper-Kähler spaces, recalling the most important properties. Moreover, the notion of *generalized* or *higher-dimensional instantons* on these structures is reviewed, making use of the existence of Killing spinors. The chapter concludes with a short section on the conceptionally similar idea of calibrations.

The subsequent Chapter 3 reviews quiver bundles and diagrams in the context of equivariant dimensional reduction in the first section. Section 3.2 relates this construction to an invariance condition on gauge connections over homogeneous spaces. The expressions here constitute the basis for the studies of explicit examples of quiver gauge theories. The last section summarizes the basic properties, provides a list of typical examples in the literature, and reviews the prototypical equivariant dimensional reduction on $\mathbb{C}P^1$.

The following parts are the core of this work: we study Sasakian quiver gauge theories on the round seven-sphere (Chapter 4) and on the five-dimensional space $T^{1,1}$ (Chapter 5) as well as 3-Sasakian quiver gauge theories on the squashed seven-sphere (Chapter 6) and on the Aloff-Wallach space $X_{1,1}$ (Chapter 7). In all cases we proceed in the same way: based on a local section, the geometric structure of the coset space is described and one constructs the canonical connection. Then we discuss the equivariance conditions, derive the action functional and consider some explicit examples of quiver diagrams as well as the resulting instanton matrix equations for certain representations of the groups G. For the manifold $T^{1,1}$ we provide a discussion of quiver diagrams for generic G-representations. Reductions to quiver gauge theories on similar spaces – typically the underlying Kähler or quaternionic manifolds – are investigated as well. For all four examples we discuss the instanton equations on their metric cones, applying an established approach for the Sasaki-Einstein case and commenting on the differences in the 3-Sasakian case.

Chapter 8 summarizes the results and contains open questions which are left for future work. Appendix A provides some details on instantons and their generalization, whereas the Appendices B – E collect details on the geometric structures and representations of the relevant Lie groups $G = \mathrm{SU}(4), \mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{Sp}(2)$ and $\mathrm{SU}(3)$.

Part I

Review of mathematical preliminaries

2 Sasaki-Einstein geometry and generalized instantons

This chapter discusses the relevant geometric properties of Sasakian manifolds and their relation to Kähler manifolds. We review the definition of higher-dimensional instantons in terms of a generalized self-duality equation, which comprises the Hermitian Yang-Mills equation in the case of Kähler manifolds. The last section briefly comments on the relationship between instantons and generalized calibrations.

The basic definitions regarding Kähler and Sasakian manifolds can be found in standard textbooks on geometry and review articles like [52]; we refer to some more specialized references below. A discussion of higher-dimensional instantons in terms of a generalized self-duality equation, whose notation we follow, is given in [19, 20]. The geometric properties and the instanton construction are based on the existence of Killing spinors, and a good description of the spinor geometry of Sasakian manifolds can be found in [53].

2.1 Sasakian and 3-Sasakian geometry

A fundamental property of Sasakian manifolds is that they are the odd-dimensional counterparts of Kähler manifolds. Therefore, the usual presentation of Sasakian manifolds starts with a review of Kähler manifolds, and we will do the same here. Among the large number of articles and books on Sasakian geometry we refer in particular to [54–56].

2.1.1 Kähler and Sasakian manifolds

We start by recalling the basic definitions in the context of complex geometry (from the viewpoint of differential geometry). An *almost complex structure* on a manifold M^{2n} of real dimension $2n$ is an endomorphism J on the tangent bundle such that $J_x^2 = -\mathrm{id}_x$ for all points $x \in M^{2n}$.

An almost complex structure J is called *integrable* or *complex* if its *Nijenhuis tensor*

$$N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

identically vanishes, $N \equiv 0$, where $[\cdot,\cdot]$ is the Lie bracket of vector fields on M^{2n}. An integrable complex structure allows the introduction of holomorphic forms on the manifold via

$$J\Theta \coloneqq J(e^1 - \mathrm{i}e^2) = \mathrm{i}(e^1 - \mathrm{i}e^2) = \mathrm{i}\Theta. \tag{2.2}$$

A Riemannian manifold (M^{2n}, g) equipped with an almost complex structure J is called *almost Hermitian* if the almost complex structure is compatible with the metric, i.e. $g(X,Y) = g(JX, JY)$ for all vector fields X, Y on M^{2n}.

Kähler geometry. A *Kähler manifold* (M^{2n}, g, J) is an almost Hermitian manifold with integrable complex structure such that the *Kähler form* $\Omega(X,Y) \coloneqq g(JX,Y)$ is closed. Hence, a Kähler manifold is an almost Hermitian manifold characterized by the two conditions $N \equiv 0$ and $\mathrm{d}\Omega = 0$.

In terms of holonomy groups and Berger's list [3], a Kähler manifold is equivalently defined as a Riemannian manifold (M^{2n}, g) such that its holonomy group is contained in the unitary group $U(n)$. This point of view will be advantageous for one of the definitions of instantons later on.

A very important aspect of Kähler manifolds is that they are not only Hermitian but also symplectic [57] due to $(\Omega)^n \neq 0$. In many relevant situations, for instance in the context of moduli spaces, one may use *Kähler quotients*. Let G be a compact Lie group acting on a Kähler manifold (M^{2n}, g, Ω) such that it preserves the Kähler structure, i.e. $\mathcal{L}_{X^\sharp} g = 0 = \mathcal{L}_{X^\sharp}\Omega$ for the induced vector field $X^\sharp$. Then a moment map [57, 58] is defined as a G-equivariant map $\mu : M^{2n} \longrightarrow \mathrm{Lie}(G)^*$ such that

$$\mathrm{d}\langle \mu(p), X \rangle = \iota_{X^\sharp}\Omega. \tag{2.3}$$

For elements ξ in the center of the Lie algebra of G, one constructs the well-defined quotient

$$\mu^{-1}(\xi)/G, \tag{2.4}$$

which inherits a Kähler structure from (M^{2n}, g, J) and which is known as Kähler quotient. In the context of generalized instantons on Calabi-Yau cones, it appears as possible description of the moduli space of Hermitian Yang-Mills connections [51], which will be applied in Section 4.5.2.

Sasakian manifolds. Being odd-dimensional counterparts of Kähler spaces, Sasakian manifolds can be defined by certain conditions on their metric cones. Recall that the metric cone $(\widetilde{M} = \mathbb{R}^+ \times M, g_c)$ over a manifold (M, g) is given by the warped metric

$$g_c = r^2 g + \mathrm{d}r \otimes \mathrm{d}r = r^2(g + \mathrm{d}\tau \otimes \mathrm{d}\tau) =: r^2 g_{cyl}, \tag{2.5}$$

where $\tau \coloneqq \ln(r)$ denotes the rescaled cone-coordinate and the second equality establishes a conformal equivalence[1] between the metric cone and the cylinder over M.

[1]The conformal factor does not matter for the self-duality equation and one can use both cone and cylinder interchangeably. However, metric cones and cylinders lead to different Yang-Mills

A *Sasakian manifold* is a Riemannian manifold such that its metric cone is Kähler. Equivalently (see e.g. [52]), a Sasakian manifold is given by a Riemannian manifold (M^{2n+1}, g) endowed with a vector field ξ, its dual 1-form η, and a tensor field $\varphi : TM^{2n+1} \to TM^{2n+1}$ satisfying:

(i) $\varphi^2 = -\mathbf{1} + \eta \otimes \xi \quad$ and $\quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\, \eta(Y)$,

(ii) $d\eta = 2F$ with fundamental form $F(X, Y) \coloneqq g(X, \varphi(Y))$,

(iii) and the vanishing of the Nijenhuis tensor

$$N(X, Y) \coloneqq [\varphi(X), \varphi(Y)] - \varphi[\varphi(X), Y] - \varphi[X, \varphi(Y)] + \varphi^2[X, Y] + d\eta(X, Y)\xi.$$

The vector field ξ is referred to as *characteristic* or *Reeb vector field*, and its dual 1-form η is a contact form, i.e. $\eta \wedge (d\eta)^n \neq 0$. From the defining properties above, it follows directly that the metric cone is Kähler. Conversely, the leaf spaces from the foliation $\mathcal{D} \coloneqq \ker(\eta)$ along the Reeb vector field yield transverse Kähler spaces with fundamental form $d\eta$ in dimension $2n$ [54]. Thus, Sasakian manifolds are in a way sandwiched between Kähler spaces, and the typical geometric structure one should keep in mind is that of U(1)-bundles over Kähler manifolds.

Of particular interest in this thesis are *Sasaki-Einstein manifolds* [55], that is Sasakian manifolds $(M^{2n+1}, g, \xi, \eta, \varphi)$ whose metric g is Einstein. This additional condition on the Ricci tensor implies that the metric cone is a Ricci-flat Kähler manifold, i.e. it is *Calabi-Yau*. Therefore, the holomomy of the metric cone $\widetilde{M}^{2n+2}$ is contained in the subgroup $\mathrm{SU}(n+1) \subset \mathrm{U}(n+1) \subset \mathrm{SO}(2n+2)$, and the reduction on the Lie-algebra level,

$$\mathfrak{so}(2n+2) = \mathfrak{u}(n+1) \oplus \mathfrak{k} = \mathfrak{su}(n+1) \oplus \mathfrak{u}(1) \oplus \mathfrak{k}, \tag{2.6}$$

can be obtained by the closure of some defining forms. More precisely, the closure of the fundamental form associated to the (integrable) complex structure reduces the holonomy to $\mathrm{U}(n+1)$, while the closure of the holomorphic top-degree form $\Omega^{n+1,0}$ yields the further reduction to $\mathrm{SU}(n+1)$; this criterion will prove useful for the explicit examples later.

Standard results on Sasaki-Einstein manifolds M^{2n+1} state that their scalar curvature is given by $s = 2n(2n+1)$ and that they admit $\mathrm{SU}(n)$ structures. Simply-connected Sasaki-Einstein manifolds are endowed with (at least) two Killing spinors [59].

Typical examples of homogeneous Sasaki-Einstein manifolds are the round spheres $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ and the Stiefel manifolds $\mathrm{SO}(m+1)/\mathrm{SO}(m-1)$. Furthermore, a new class of non-homogeneous Sasaki-Einstein manifolds, named $Y^{p,q}$, has been constructed on $S^2 \times S^3$ [60], and this construction was generalized to all dimensions [61].

equations; see also the remark in Appendix A.1.2.

2.1.2 Hyper-Kähler and 3-Sasakian structures

The close relationship between Kähler and Sasakian manifolds finds an analogue in the interplay of hyper-Kähler and 3-Sasakian manifolds. The latter ones are a restrictive subclass of Sasaki-Einstein manifolds in dimension $4m+3$ and constitute the second type of geometries we are particularly interested in.

Hyper-Kähler manifolds. In Berger's list there appears the case of holonomy contained in the subgroup[2] $\mathrm{Sp}(m) \subset \mathrm{U}(2m) \subset \mathrm{SO}(4m)$, and manifolds with this holonomy are referred to as *hyper-Kähler manifolds* [4, 62, 63].

Equivalently, a hyper-Kähler manifold (M^{4m}, g) is equipped with a triple of covariantly constant endomorphisms $J_\alpha = I, J, K : TM^{4m} \to TM^{4m}$, $\nabla J_\alpha = 0$, which satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathbf{1}. \tag{2.7}$$

Considering $J = s^\alpha J_\alpha$ with $s_1^2 + s_2^2 + s_3^2 = 1$, they give rise to a whole $\mathbb{C}P^1$-family of complex structures (and similarly of Kähler forms) and lead to the so-called *twistor spaces*. Analogously to the Kähler quotient, one can now take the *hyper-Kähler quotient* [58] which employs a triplet of moment maps μ_α (see also Appendix A.2).

Because of the quaternionic structure the group $\mathrm{Sp}(m)$ acts on, typical examples of hyper-Kähler manifolds are given by the quaternionic spaces $\mathbb{H}^m$ and metric cones over 3-Sasakian manifolds (see below). Moreover, the famous *quiver varieties* [64] introduced by Nakajima are hyper-Kähler varieties, constructed as hyper-Kähler quotients on the space of linear maps in a representation of a given quiver diagram.

3-Sasakian manifolds. A manifold is called *3-Sasakian* if its metric cone is hyper-Kähler. This definition is equivalent to describing [53, 65] a 3-Sasakian manifold as Riemannian manifold (M^{4m+3}, g) which admits a triplet of Sasakian structures such that the characteristic vector fields ξ_α are orthonormal, $g(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta}$, and satisfy the $\mathfrak{su}(2)$ commutation relations

$$[\xi_\alpha, \xi_\beta] = 2\epsilon_{\alpha\beta}{}^{\gamma}\xi_\gamma. \tag{2.8}$$

A 3-Sasakian manifold (M^{4m+3}, g) is automatically Einstein (with scalar curvature $s = (4m+2)(4m+3)$) and carries an $\mathrm{Sp}(m)$ structure. Simply-connected spin manifolds with 3-Sasakian structure admit (at least) three independent Killing spinors [59, Thm. 6].

Homogeneous 3-Sasakian manifolds have been completely classified, revealing a correspondence between simple Lie algebras and 3-Sasakian structures [65, Thm. 3.2.6].

[2] The notation $\mathrm{Sp}(m)$ always refers to the compact form of the symplectic group, which is also known as $\mathrm{USp}(2m)$ in some literature.

Examples include the squashed spheres $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ of dimension $4m+3$, the Aloff-Wallach space $X_{1,1} = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ in dimension 7 and some exceptional cosets like $G_2/\mathrm{Sp}(1)$, for example. Note that every compact simply-connected 7-dimensional spin manifold with regular 3-Sasakian structure is isometric to either the squashed seven-sphere $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ or the Aloff-Wallach space $X_{1,1}$ [53, Sec. 4.4., Cor. 4].

While the properties of Sasakian manifolds are dominated by being U(1)-bundles over Kähler manifolds, 3-Sasakian manifolds are either Sp(1)-bundles (in the case of the squashed spheres) or SO(3)-bundles over quaternionic Kähler spaces. This property will become eminent in the discussion of the instanton equations on metric cones over these spaces in Chapters 6 and 7.

Related geometries. 3-Sasakian manifolds, in particular in dimension 7, admit various closely related geometric structures, which are summarized in [54, Sec. 8] for instance. Inter alia, every 3-Sasakian manifold is endowed with a second Einstein metric of positive scalar curvature [65]. Furthermore, in dimension 7 the 3-Sasakian metric is also a nearly-parallel G_2 metric (also known as manifold with *weak G_2 holomomy*). Therefore, 3-Sasakian manifolds also occur [66] in the context of G_2 geometry, and one can study Spin(7) instantons on the metric cones over 3-Sasakian manifolds [28].

2.2 Higher-dimensional instantons

Before considering instantons in higher dimensions, let us recall the properties of 4-dimensional instantons. In four Euclidean dimensions the Hodge star operator $\star_4$ squares to the identity when acting on 2-forms, so that the 6-dimensional space $\Lambda^2 T^* M^4$ of 2-forms splits into two eigenspaces with eigenvalues ± 1,

$$\Lambda^2 T^* M^4 = \Lambda^+ \oplus \Lambda^- . \tag{2.9}$$

This decomposition motivates the study of connections with curvature valued in one of these eigenspaces, $\star \mathcal{F}_\pm = \pm \mathcal{F}_\pm$, which are referred to as *(anti-)self-dual connections* or *instantons* [11]. Such connections automatically satisfy the Yang-Mills equation $D_\mathcal{A} \star \mathcal{F} = 0$ by virtue of the Bianchi identity $D_\mathcal{A}\mathcal{F} = 0$. Morevover, it can be shown that the Yang-Mills action is bounded from below by a topological term and that the bound is saturated if the connection is an instanton. These basic properties are reviewed in Appendix A.1 in more detail.

The self-duality equations and the structure of their moduli spaces played a crucial role for the characterization of 4-manifolds [15, 67] and also provided important insights in related geometric questions, see [68] for instance. In physics, instantons are of importance as non-perturbative tools in field theory, see e.g. [12–14], and are closely related to branes and other solitonic objects like magnetic monopoles.

2.2.1 Generalized self-duality condition

In order to formulate an analogue of the original self-duality condition $\star\mathcal{F} = \pm\mathcal{F}$ in higher dimensions [16–18], one has to compensate the different form degrees of the curvature 2-form $\mathcal{F}$ and its Hodge dual $\star_n\mathcal{F}$ of degree $n-2$ by extending the $\star$-operator with a suitable differential form. On manifolds with Killing spinors, one can construct an invariant 4-form Q and consider the action of the operator $\star(\star Q \wedge \cdot)$, so that generalized instantons can be defined (see e.g. [69]) as connections $\mathcal{A}$ such that their curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ satisfies

$$\star_d \mathcal{F} = -\mathcal{F}\wedge\star_d Q. \tag{2.10}$$

The construction of the 4-form Q for different geometries of manifolds with Killing spinors is described in [19,20] (see also [7]), and we apply their formulation for Sasaki-Einstein manifolds and 3-Sasakian manifolds here. Definition (2.10) implies – as in four dimensions – a variant of the Yang-Mills equations: taking the derivative of the self-duality equation yields the Yang-Mills equation with torsion term,

$$D_{\mathcal{A}}\star\mathcal{F} - \mathcal{F}\wedge\star H = 0, \tag{2.11}$$

where one has defined $\star H = \mathrm{d}\star Q$, see [19,25] for example. For manifolds with Killing spinors the torsion term vanishes: on manifolds with *real* Killing spinors the vanishing of the torsion term is proven in [19]. Their metric cones admit *parallel* Killing spinors and therefore special holonomy, hence the form Q (on the cone) is both closed and co-closed, and the statement is clear. Thus, the *first-order* equation (2.10) implies the usual *second-order* Yang-Mills equation $D_{\mathcal{A}}\star\mathcal{F} = 0$ without torsion.

The Yang-Mills equation with torsion (2.11) can be shown to be the equation of motion of the action functional

$$S = \int_{M^d} \mathrm{Tr}\left[\mathcal{F}\wedge\star\mathcal{F} + (-1)^d\mathcal{F}\wedge\mathcal{F}\wedge\star Q\right] , \tag{2.12}$$

which is the usual Yang-Mills action plus a Chern-Simons-type term, see e.g. [20,25, 70,71] and the detailed discussion in Appendix A.1.2. Solutions to the generalized self-duality equation (2.10) minimize this functional.

Heterotic supergravity. The interest in higher-dimensional instantons in gauge theories arises not only as mathematical topic in its own right but also in the context of heterotic supergravity [7]. Besides the metric g, the dilaton Φ and the Kalb-Ramond field B – present in the common sector of type II string theories – the bosonic degrees of freedom in heterotic supergravity contain a gauge field $\mathcal{A}$. Supersymmetric backgrounds require the vanishing of the field variations [19,20,72,73],

$$\delta\Psi \propto \nabla^-\epsilon = 0, \tag{2.13a}$$

$$\delta\lambda \propto (\mathrm{d}\Phi - H)\cdot\epsilon = 0, \tag{2.13b}$$

$$\delta\chi \propto \mathcal{F}\cdot\epsilon = 0, \tag{2.13c}$$

constituting the so-called *BPS equations* which comprise the *gravitino*, *dilatino* and *gaugino equation*, respectively. Here ∇^- denotes a metric connection with torsion $T = -H$. This provides a physical motivation for studying connections with skew-torsion [74]. The forms in (2.13b) and (2.13c) act on the supersymmetry generator ϵ according to Clifford multiplication.

The definition of instantons in terms of the generalized self-duality equation (2.10) implies the definition as solution to the gaugino equation (2.13c) [7, 19]. Therefore, finding gauge connections satisfying (2.11) serves as a starting point for the construction of solutions of heterotic supergravity [19, 20]. The notation $\star H = \mathrm{d} \star Q$ in the generalized Yang-Mills equation (2.11) emphasizes the setting of having a metric connection with skew-symmetric torsion H in heterotic supergravity. However, one can also study (2.11) in its own right, by allowing for a torsion form H that does not arise from the BPS equation (2.10), and this leads to non-BPS Yang-Mills connections with torsion, see for instance [28, 33].

Canonical connection. Note that the generalized self-duality equation ensures the curvature of a generalized instanton to take values in the subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$, where $\mathfrak{g}$ is the Lie algebra of the reduced structure group. This is an even more general definition of instantons since it does not need Killing spinors but only a reduction of the structure group. In particular, the curvature of generalized instantons on Sasaki-Einstein manifolds M^{2n+1} must be valued in $\mathfrak{su}(n)$, and that of 3-Sasakian manifolds M^{4m+3} is further restricted to lie in $\mathfrak{sp}(m) \subset \mathfrak{su}(2m) \subset \mathfrak{so}(4m)$.

Following [19], one can introduce the so-called *canonical connection* on manifolds with G-structures, in particular those with Killing spinors. It is defined as a metric connection with holonomy G and skew-torsion with respect to a G-compatible metric. This connection is a generalized instanton and serves as starting point in many constructions of instantons, e.g. in [34], and also in this thesis. As clarified in [19, Sec. 3.5], this connection differs from the *characteristic connection* of [74] in the case of Sasaki-Einstein manifolds.

Instantons on Sasaki-Einstein manifolds. For Sasaki-Einstein manifolds (M^{2n+1}, g) one can construct the defining forms as [19]

$$\eta = (\epsilon^\dagger \gamma_\mu \epsilon)\; e^\mu , \quad \omega = -\tfrac{\mathrm{i}}{2} (\epsilon^\dagger \gamma_{\mu\nu} \epsilon)\; e^{\mu\nu} , \tag{2.14}$$

where the Killing spinor ϵ satisfies $\nabla_\mu \epsilon = \frac{1}{2} \mathrm{i} \gamma_\mu \epsilon$ and the 1-forms e^μ are an orthonormal basis with respect to the Sasaki-Einstein metric, $g = \delta_{\mu\nu} e^\mu \otimes e^\nu$. Moreover, one defines the spinor bilinears

$$-\tfrac{\mathrm{i}}{3!} (\epsilon^\dagger \gamma_{\mu\nu\rho} \epsilon)\; e^{\mu\nu\rho} =: P = \eta \wedge \omega , \quad -\tfrac{1}{4!} (\epsilon^\dagger \gamma_{\mu\nu\rho\sigma} \epsilon)\; e^{\mu\nu\rho\sigma} =: Q = \tfrac{1}{2} \omega \wedge \omega , \tag{2.15}$$

where the shorthand notation $e^{i_1\cdots i_k} := e^{i_1} \wedge \ldots \wedge e^{i_k}$ is used for wedge products of forms. The invariant 4-form Q is the square of the fundamental form ω of the underlying Kähler manifold, and the generalized instanton equation takes the form $\star\mathcal{F} = -\frac{1}{(n-2)!}\eta \wedge \omega^{n-2} \wedge \mathcal{F}$. Using the differentials of the forms obtained by the Killing spinor equation, it is shown [19] that this generalized self-duality equation implies the usual Yang-Mills equation without torsion term.

The canonical connection associated to the SU(n)-structure of the Sasaki-Einstein manifold is uniquely determined by introducing its torsion components

$$T^{2n+1} = P_{2n+1\mu\nu}e^{\mu\nu} = 2\omega = \mathrm{d}\eta, \qquad T^a = \frac{n+1}{2n}P_{a\mu\nu}e^{\mu\nu}, \tag{2.16}$$

where $e^{2n+1} \equiv \eta$ denotes the contact form. We will use this canonical connection as starting point for quiver gauge theories on the Sasaki-Einstein manifolds $T^{1,1}$ and S^7.

Instantons on Calabi-Yau cones. On the Calabi-Yau cone (and the conformally equivalent cylinder) over a Sasaki-Einstein manifold M^{2n+1}, the generalized self-duality condition can be formulated in terms of the 4-form [19]

$$Q_Z := \mathrm{d}\tau \wedge P + Q. \tag{2.17}$$

Recalling the Kähler form on the cone, $\Omega^{1,1} = r^2(\mathrm{d}\tau \wedge \eta + \omega)$, the 4-form can also be written as $r^4 Q_Z = \frac{1}{2}\Omega^{1,1} \wedge \Omega^{1,1}$, which immediately shows the closure of the form on the cone. Taking the Hodge star thereof yields again a power of the Kähler form, so that $r^4 Q_Z$ is indeed closed and co-closed, as expected for a manifold with special holonomy [19].

Instantons on 3-Sasakian manifolds. Given a 3-Sasakian manifold M^{4m+3} with the triple of contact forms η^α and writing their differentials (which describe the associated Kähler forms) as

$$\mathrm{d}\eta^\alpha = \epsilon^\alpha{}_{\beta\gamma}\eta^{\beta\gamma} + \omega^\alpha, \tag{2.18}$$

with ω^α 2-forms on the underlying quaternionic manifold, the relevant quantities for the instanton equation read [19]

$$Q := \tfrac{1}{6}\sum_\alpha \omega^\alpha \wedge \omega^\alpha, \qquad P = \tfrac{1}{3}\eta^{123} + \tfrac{1}{3}\sum_\alpha \eta^\alpha \wedge \omega^\alpha. \tag{2.19}$$

The canonical connection of the 3-Sasakian structure is defined by the torsion components

$$T^\alpha = 3P_{\alpha\mu\nu}e^{\mu\nu}, \qquad T^a = \tfrac{3}{2}P_{a\mu\nu}e^{\mu\nu}, \tag{2.20}$$

where α again denotes the three indices of the contact directions, a those of the underlying quaternionic manifold, and the 1-forms e^μ are orthonormal with respect to the 3-Sasakian metric, $g = \delta_{\mu\nu}e^\mu \otimes \mathrm{e}^\nu$.

Instantons on hyper-Kähler cones. The canonical connection of the 3-Sasakian manifold can be lifted to an instanton on the metric cone. The generalized self-duality condition involves the 4-form [19]

$$Q_Z := \frac{1}{6}(\omega^\alpha \wedge \omega^\alpha + \epsilon_{\alpha\beta\gamma}\omega^\alpha \wedge \eta^{\beta\gamma} + 2\mathrm{d}\tau \wedge \eta^\alpha \wedge \omega^\alpha + 6\mathrm{d}\tau \wedge \eta^{123}). \tag{2.21}$$

Since the three Kähler forms on the cone are given by $\Omega^{1,1}_\alpha = \omega^\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}\eta^{\beta\gamma} + \mathrm{d}\tau \wedge \eta^\alpha$, the invariant 4-form can be written as

$$Q_Z = \tfrac{1}{3}\sum_\alpha \Omega^{1,1}_\alpha \wedge \Omega^{1,1}_\alpha. \tag{2.22}$$

As in the case of metric cones over Sasaki-Einstein manifolds, this expression shows the closure and co-closure of the relevant form on the cone, and therefore the Yang-Mills equation without torsion follows from the generalized self-duality condition in this case as well.

Moreover, the form of (2.22) suggests that a description of instantons on hyper-Kähler cones can be related to three single instanton equations known from Calabi-Yau cones. With the results of [75] one can indeed trace back the moduli space in this way [49], which will be used for the discussion of instanton moduli spaces of quiver gauge theories on the cones over $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ and $X_{1,1}$.

2.3 Hermitian Yang-Mills equation

On Kähler manifolds (M^{2n}, Ω) – in the case at hand Calabi-Yau cones over Sasaki-Einstein manifolds – there is a description of instantons in terms of the *Hermitian Yang-Mills (HYM)* or *Donaldson-Uhlenbeck-Yau (DUY) equations* [69,73,76,77]. Splitting the curvature $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$ according to the complex structure induced by Ω, the Hermitian Yang-Mills equations are given by

$$\mathcal{F}^{2,0} = 0 = \mathcal{F}^{0,2} \qquad \text{(“holomorphicity condition”)}, \tag{2.23a}$$

$$0 = \Omega \lrcorner \mathcal{F} \qquad \text{(“stability condition”)}, \tag{2.23b}$$

where $\lrcorner$ denotes the contraction of forms. These equations can be considered as definition of instantons in the spirit of G-structures: since Kähler manifolds carry (integrable) $U(n)$ structures, the generic structure group of the frame bundles can be decomposed as $\mathfrak{so}(2n) = \mathfrak{u}(1) \oplus \mathfrak{su}(n) \oplus \mathfrak{k}$. The first equation (2.23a), the holomorphicity condition, forces the curvature to take values in the subalgebra $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. The condition (2.23b) constrains the $\mathfrak{u}(1)$-part further. It is called *stability-like condition*, since the formulation of the DUY equations [76,77] deals with the stability of holomorphic vector bundles.

In the remainder of this thesis, we will apply the HYM equations as an equivalent formulation of the instanton equations on the metric cones (cylinders) carrying Kähler structures (here: the Calabi-Yau cones). It turns out [51] (cf. Section 4.5.2) that evaluating the Hermitian Yang-Mills equations in our setups leads to Nahm-type equations. The holomorphicity conditions $\mathcal{F}_{\alpha\beta} = 0$ (with α, β denoting holomorphic forms) gives rise to a version of the complex equation (A.21a), while the stability-like condition $\Omega^{\alpha\bar{\beta}}\mathcal{F}_{\alpha\bar{\beta}} = 0$ yields an analogue of the real equation (A.21b).

2.4 Calibrations

The previous sections reviewed generalized instantons as certain BPS configurations which are defined by first-order equations and imply second-order equations of motion. Another prominent example of evaluating first-order equations in order to obtain solutions to a second-order problem are *calibrations* [78] for minimal submanifolds and the extension to supersymmetric cycles wrapped by branes, which are described by *generalized calibrations* [7, 79, 80]. We review this concept here due to its analogy as a first-order BPS equation constructed from Killing spinors, as can be seen in [7] for instance. Applications of calibrations in gauge theory and the relationship between both fields are also discussed in [10, 69, 81].

Calibrated submanifolds. According to [78], a *calibration form* ϕ on a Riemannian manifold (M^d, g) is a closed p-form $(p < d)$ which satisfies the inequality

$$\phi|_S \leq \mathrm{vol}_S \tag{2.24}$$

for all oriented tangent p-planes S on M^d. Here vol_S denotes the induced volume form and the inequality has to be read as an inequality on the scalar function in front of the volume form $\mathrm{d}\xi^1 \wedge \ldots \wedge \mathrm{d}\xi^p$ on both sides of the expression.

A p-dimensional submanifold Σ that saturates the inequality (2.24) in each point,

$$\phi|_\Sigma = \mathrm{vol}_\Sigma \ , \tag{2.25}$$

is called a *calibrated submanifold*. The crucial point is that calibrated submanifolds minimize the volume within their homology class, as the following standard argument [78, 80] shows. Let Σ' be another manifold in the same homology class, i.e. the manifolds only differ by a boundary term, $\Sigma' = \Sigma + \partial B$. Then one obtains

$$\begin{aligned} \mathrm{Vol}(\Sigma') = \int_{\Sigma'} \mathrm{vol}_{\Sigma'} \geq \int_{\Sigma'} \phi|_{\Sigma'} &= \int_\Sigma \phi|_\Sigma + \int_B \mathrm{d}\phi|_B \\ &= \int_\Sigma \phi|_\Sigma = \int_\Sigma \mathrm{vol}_\Sigma = \mathrm{Vol}(\Sigma), \end{aligned} \tag{2.26}$$

where one has first used the definition of the calibration form (2.24), then Stokes' Theorem and the closure of ϕ, and finally the saturation of the inequality (2.25) for the calibrated submanifold Σ.

Recall that the embedding of a p-dimensional submanifold into M^d is described by

$$X : S \to M^d, \qquad (\xi_1, \dots \xi_p) \longmapsto (X_1(\xi_i), \dots, X_d(\xi_i)), \tag{2.27}$$

constituting the typical setup of the worldsheet and branes in string theory. The induced metric on S is given by $g_{\alpha\beta} = \partial_\alpha X^m \partial_\beta X^n g_{mn}$, and finding a minimum of the induced volume $\mathrm{vol}_S = \sqrt{\det(g_{\alpha\beta})}\mathrm{d}\xi^1 \wedge \dots \wedge \xi^p$ amounts to solving a complicated *second-order equation* in the embedding functions X_μ. The *first*-order equation (2.25), however, implies for calibrated submanifolds the minimization of the volume, in analogy to the implication of the generalized self-duality equation (2.10) for the Yang-Mills action.

Examples. The standard examples (see e.g. [78,82,83]) of calibrations and submanifolds comprise:

- complex submanifolds calibrated by $\phi := \frac{1}{p!}\Omega^p$ on Kähler manifolds (M^{2n}, g, Ω),
- *special Lagrangian submanifolds (SLAGs)* calibrated by the real part of the holomorphic top-degree form, $\phi := \mathrm{Re}(\Theta^1 \wedge \dots \Theta^n)$, on Calabi-Yau n-folds,
- submanifolds calibrated by the 3-form ϕ defining the group G_2 or its dual 4-form $\star_7\phi$ on G_2-manifolds,
- *Cayley planes* calibrated by the self-dual 4-form ϕ on manifolds admitting a Spin(7)-structure.

This intimate relation between special holonomy manifolds in Berger's list and calibrations is caused by their common origin from Killing spinors, as explained in the references. It also holds for the generalized case which describes supersymmetric cycles.

Generalized calibrations. The concept of calibrations for minimal submanifolds can be extended to the study of supersymmetric brane configurations in string theory, which leads to the notion of *generalized calibrations* [80,82–84]. The action functional of branes embedded in a certain background is not only given by the volume but also includes fluxes – the NS 3-form H, the field in the *Dirac-Born-Infeld* (DBI) action and Ramond-Ramond (RR) fluxes of the background – so that the definition has to be slightly adapted.

The flux contributions obstruct the closure of the usual calibration form and therefore have to be compensated by including suitable potential terms in the calibration form. The closure of the generalized calibration form again follows from the Killing spinor equations of the background geometry, which are often already formulated such that the calibration form is evident, see e.g. the supersymmetry equations in [85, Sec. 3.2].

The analogue of the inequality (2.24) is derived from the supersymmetry condition imposed on the cycles wrapped by branes, such as the so-called κ-symmetry condition. Contracting this equation then yields an inequality on a certain differential form which is saturated if and only if the cycle is supersymmetric; this has been applied in [86] for instance. Since type II string theory can be formulated in terms of generalized geometry, the calibration form involves the pure spinors that describe the supersymmetric background in these setups [87].

To name a few examples related to our geometries of interest, we consider AdS-compactifications of type IIB string theory and M-theory on Sasaki-Einstein (SE) manifolds. It is known that on backgrounds $\mathrm{AdS}_5 \times \mathrm{SE}_5$ the 3-form $\eta \wedge \omega$ calibrates D3-branes [88], while M5-branes in $\mathrm{AdS}_4 \times \mathrm{SE}_7$ are calibrated by the 5-form $\frac{1}{2}\eta \wedge \omega \wedge \omega$ [89]. More generally, the vector structures of *exceptional Sasaki-Einstein structures* constitute generalized calibrations [90, 91].

Turning to the metric cones with special holonomy and taking up the discussion from Section 2.2.1, the interplay between calibrations and instantons is illuminated in [7, Sec. 3]: the flux equation

$$\star H = \mathrm{e}^{2\Phi}\mathrm{d}(\mathrm{e}^{-2\Phi}\Xi) \tag{2.28}$$

can be regarded as a generalized calibration for supersymmetric cycles wrapped by fivebranes, collected in [7, Table 1]. In heterotic supergravity, the occurring form Ξ is exactly the tensor that governs the generalized instanton equation $\star\mathcal{F} = \Xi \wedge \mathcal{F}$. More precisely, for manifolds M^d with holomomy SU(3), SU(4) and Sp(2) (with respect to the connection ∇ with torsion H), the expressions read [7]

$$-\,\Xi = \begin{cases} \Omega^{1,1}, & \mathrm{SU}(3)\ , d = 6, \\ \frac{1}{2}\Omega^{1,1} \wedge \Omega^{1,1}, & \mathrm{SU}(4)\ , d = 8, \\ \frac{1}{2}\Omega^{1,1}_{\alpha} \wedge \Omega^{1,1}_{\alpha}, \quad \alpha = 1, 2, 3, & \mathrm{Sp}(2)\ , d = 8\ . \end{cases} \tag{2.29}$$

Recalling that the notation of [7] is related by $\Xi = -\star Q$ to ours, one recognizes the forms introduced in Section 2.2.1. For the various calibrations in dimension 8, consult also the discussion in [92, Sect. 3.2]. Hence, the above forms occur both as ingredients of a higher-dimensional instanton equation and as generalized calibrations, so that they play a crucial role for defining BPS configurations in two (related) contexts.

Despite this digression into typical compactification setups *with* fluxes, let us emphasize that we always deal with the fluxless case in the remainder of this thesis, i.e. H vanishes, the dilaton is constant and the even-dimensional manifolds have special holonomy with respect to the Levi-Civita connection.

3 Equivariant bundles and quiver gauge theories

After reviewing the *universal* geometric properties of Sasakian and 3-Sasakian manifolds and the notion of generalized instantons, we now turn to the *concrete* form of the manifolds as homogeneous spaces G/H, which is employed for the construction of quiver bundles. A general introduction to homogeneous spaces can be found in [93] for instance.

The motivation for studying quiver gauge theories is twofold. On the one hand, quiver diagrams and quiver bundles arise naturally in the context of equivariant dimensional reduction of gauge theory on $M^d \times G/H$, which we will discuss in Section 3.1, following [35, 37, 38]. On the other hand, invariance of gauge connections over homogeneous spaces [94], expressed by the *equivariance condition* (1.5), can be encoded in quiver diagrams [34, Sec. 4], which will be the content of the second section. This approach yields gauge connections which are parametrized by more degrees of freedom than a certain scalar ansatz which has been employed in many studies on instantons. Section 3.3 summarizes quiver gauge theory on homogeneous spaces and reviews a prototypical example of equivariant dimensional reduction.

3.1 Quivers and equivariant vector bundles

This section describes the arising of quiver gauge theories from equivariant vector bundles in gauge theories, following [35, 37, 38]. We consider coset space dimensional reduction [23], reducing a Yang-Mills theory over $M^d \times G/H$ to a Yang-Mills-Higgs theory on M^d, where the potential terms depend on the geometry of the coset space. In our setup, M^d denotes a Riemannian manifold of dimension d, and we consider Hermitian vector bundles which the gauge connection takes values in. Although the natural objects in the context of gauge theory are *principal* bundles, we will be working with *vector* bundles, associated to the relevant principal fiber bundles.

Quivers. We start by recalling the basic definitions regarding quivers [95, 96]: a *quiver* $\mathcal{Q}$ is a directed graph, i.e. formally a quiver is a pair $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ of a (finite) set $\mathcal{Q}_0$ of *vertices* and a (finite) set $\mathcal{Q}_1$ of *arrows* such that the maps $h, t : \mathcal{Q}_1 \to \mathcal{Q}_0$ assign starting and ending point (tail and head) to each arrow in $\mathcal{Q}_1$.

The importance of quiver diagrams arises due to their relation to representation theory of algebras. A *representation of a quiver* $(\mathcal{Q}_0, \mathcal{Q}_1)$ is a collection of vector spaces and linear maps among them obtained in the following way: to each vertex $v_i \in \mathcal{Q}_0$ one assigns a vector space E_i, and each arrow between vertices v_i and v_j represents a linear map $\rho_{ji} \in \mathrm{Hom}(E_i, E_j)$. A *relation on a quiver* is a formal sum of paths. For example, the conditions imposed by commutativity of a diagram are quiver relations.

Equivariant vector bundles. We now consider a Hermitian vector bundle of rank k, i.e. a bundle $\pi : \mathcal{E} \to M^d \times G/H$ with structure group $\mathrm{U}(k)$ and typical fiber $\mathbb{C}^k$. Let the group G act trivially on M^d and by its standard group action on the coset space. Such a bundle $\mathcal{E}$ is called *G-equivariant* if the G-action ρ on the base and that on the total space $\tilde{\rho}$, respectively, commute with the projection map, i.e. $\pi\tilde{\rho} = \rho\pi$,

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \tilde{\rho}\ } & \mathcal{E} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M^d \times G/H & \xrightarrow{\ \rho\ } & M^d \times G/H \end{array} \tag{3.1}$$

and if the G-action induces vector space isomorphisms between the fibers $\mathcal{E}_p$ and $\mathcal{E}_{g\cdot p}$ over points $p \in M^d \times G/H$. It can be shown [35] that G-equivariant bundles over $M^d \times G/H$ are in one-to-one correspondence to H-equivariant bundles $E \to M^d$ by

$$\mathcal{E} = G \times_H E. \tag{3.2}$$

From this observation, one derives [35] a couple of correspondences between equivariant vector bundles and representations of the Lie group G, finally leading to a description in terms of quiver diagrams and quiver bundles. This has been carried out for *parabolic* subgroups H, i.e. groups which contain a maximal torus of G and give rise to flag manifolds G/H, by using a Levi decomposition of H.

While most examples of quiver gauge theories involve flag manifolds, such as the prototype $\mathbb{C}P^1$, $\mathbb{C}P^2$ and $\mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$, Sasaki-Einstein coset spaces are different: due to their structure as $\mathrm{U}(1)$-bundle over a Kähler manifold, the subgroup H *cannot* contain a maximal torus. However, according to [35, Rem. 1.12], a quiver with relations can be associated to *any* algebraic subgroup H of G by employing a semidirect decomposition $H =: U \ltimes L$. Therefore, the procedure on flag manifolds [37, 38] can be extended to Sasakian manifolds. For the Sasaki-Einstein manifolds to be studied here the focus is placed on constructing explicit examples instead of providing a rigorous algebraic description comparable to that of [35].

Since the subgroup H acts trivially on the manifold M^d, equivariance implies that each fiber $E_p \cong \mathbb{C}^k$ must carry a representation of the subgroup H. In general, this representation on the fibers may consist of several smaller H-representations ρ_j (inside the representation of the structure group $\mathrm{U}(k)$), and one applies the ansatz [37] that the ρ_j stem from the restriction of a G-representation $\mathcal{D}$ to the subgroup H,

$$\mathcal{D}|_H =: \bigoplus_{j=1}^{N} \rho_j, \qquad \rho_j : H \to \mathrm{Aut}(V_j). \tag{3.3}$$

Therefore, each fiber admits an *isotopical decomposition* (cf. [35, 40, 43]) as

$$E_p \cong \bigoplus_{j=1}^{N} E_j \otimes V_j, \qquad \sum_{j=1} k_j d_j = k, \tag{3.4}$$

where we have split the summands into a vector space V_j carrying the H-representation ρ_j and a vector space E_j with trivial action of the subgroup, writing $k_j := \dim(E_j)$ and $d_j := \dim(V_j)$. This isotopical decomposition is accompanied by a breaking of the structure group (of the bundle E) as [37, 38, 42]

$$\mathrm{U}(k) \to \prod_{j=1}^{N} \mathrm{U}(k_j), \tag{3.5}$$

while the structure group of the bundle $\mathcal{E}$ is reduced to $H \times \prod_{j=1}^{N} \mathrm{U}(k_j)$. Therefore, the formalism of equivariant gauge theories incorporates the typical symmetry breaking of physical gauge theories [38,39]. A product of several groups, each of them attached to a vertex, as gauge group is exactly the situation of the commonly known *quiver gauge theories* for brane configurations (cf. Section 3.3.2) and also for Nakajima's quiver varieties [64].

Quiver diagrams. The quiver diagram associated to a chosen G-representation $\mathcal{D}$ is obtained by identifying each summand V_j in (3.4) with a vertex that is endowed with a vector space E_j. While, by construction, the subgroup H acts as an endomorphism on each vertex, the entire G-action connects different representations ρ_j and therefore yields arrows of the quiver diagram which represent linear maps $\phi_{ji} \in \mathrm{Hom}(E_i, E_j)$. Combining the fiber-wise decomposition (3.4) and using the induction (3.2) yields the equivariant bundle $\mathcal{E}$ as a *quiver bundle*. Hence, the structure of equivariant dimensional reduction of gauge theories on $M^d \times G/H$ is determined by the isotopical decomposition (3.4), which, in turn, is encoded in the representation $(\{E_i\}, \{\phi_{ji}\})$ of a quiver $(\mathcal{Q}_0, \mathcal{Q}_1)$ depending on a given G-representation $\mathcal{D}$.

In order to preserve equivariance, the linear maps ϕ_{ji} in the quiver bundle may depend on coordinates of the external space M^d but not on those of the coset G/H. In this way, a pure Yang-Mills theory on $M^d \times G/H$ induces a Yang-Mills-Higgs theory on M^d via dimensional reduction, where the form of the potential terms is depicted in the quiver diagrams and therefore depends on the structure of G/H as a homogeneous space.

Construction and properties. In order to derive the quiver diagram associated to a given G-representation $\mathcal{D}$, one first constructs the corresponding weight diagram. This weight diagram is collapsed along the action of the subalgebra, i.e. along the ladder operators of $\mathfrak{h}$, to identify the subrepresentations ρ_j in the decomposition $\mathcal{D}|_H = \bigoplus_{j=1}^N \rho_j$. The remaining vertices carry the representation spaces V_j and are therefore considered as the vertices $\mathcal{Q}_0$ of the quiver, which the spaces E_j are attached to. The allowed arrows $\mathcal{Q}_1$ of the quiver diagram are precisely the remaining arrows after collapsing.

Quiver diagrams arising in this way may be thought of as a natural extension of (collapsed) weight diagrams. A weight diagram is a trivial quiver in the sense that each node represents a 1-dimensional vector space isomorphic to $\mathbb{C}$, and arrows correspond to elements of $\mathbb{C}^*$ as homomorphisms $\mathbb{C} \to \mathbb{C}$. For quiver gauge theory, the 1-dimensional vector spaces are replaced by arbitrary ones, E_j, and compatibility with the structure of the homogeneous space G/H requires the collapsing along the ladder operators of $\mathfrak{h}$.

This construction of quiver bundles implies that in the special case of H being a maximal torus of G, the quiver diagram is simply the weight diagram of the chosen G-representation [35].

3.2 Invariant gauge connections on homogeneous spaces

After describing equivariant vector bundles in terms of quivers, we now focus on the costruction of invariant gauge connections and instantons over $M^d \times G/H$, following the typical approach explained in [25, 32–34, 70] and the references therein.

Often one studies instantons on metric cones or cylinders over manifolds with Killing spinors, in our case on Calabi-Yau or hyper-Kähler cones over Sasaki-Einstein or 3-Sasakian manifolds G/H. This approach resembles the situation of (equivariant) dimensional reduction on $M^d \times G/H$ for the special case of $M^d = \mathbb{R}^+$ being the cone direction. Imposing the instanton conditions then amounts to a set of algebraic relations as well as ODEs on matrix-valued functions.

Reductive homogeneous space. As in the previous section, let G be a compact semi-simple Lie group and H a closed subgroup of G. The reductive homogeneous space G/H admits an Ad(H)-invariant splitting of the Lie algebra as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.6}$$

The geometric structure of G/H is that of a principal H-bundle, and the complement $\mathfrak{m}$ can be identified with the tangent space of G/H at the identity in the canonical way, $\mathfrak{m} \cong T_e(G/H)$. We denote the generators of $\mathfrak{g}$ as $\{I_\mu\} = \{I_j, I_a\}$, where the I_j

generate the subalgebra $\mathfrak{h}$ and the I_a belong to the complement $\mathfrak{m}$. The corresponding structure constants are given by

$$[I_i, I_j] = f_{ij}^k \; I_k, \quad [I_j, I_a] = f_{ja}^b I_b, \quad [I_a, I_b] = f_{ab}^j I_j + f_{ab}^c I_c. \tag{3.7}$$

The 1-forms e^μ dual[1] to these generators satisfy the Maurer-Cartan structure equations

$$\mathrm{d}e^\mu = -\tfrac{1}{2} f^\mu_{\rho\sigma} e^\rho \wedge e^\sigma. \tag{3.8}$$

They split, according to (3.7), as

$$\mathrm{d}e^i = -\tfrac{1}{2} f^i_{jk} e^j \wedge e^k, \qquad \mathrm{d}e^a = -\tfrac{1}{2} f^a_{bc} e^b \wedge e^c - f^a_{ib} e^i \wedge e^b. \tag{3.9}$$

Invariant gauge connection. A gauge connection $\mathcal{A}$ on a Hermitian vector bundle $\mathcal{E} \to M^d \times G/H$ of rank k can be expressed as

$$\mathcal{A} = \Gamma + \sum_{a=1}^{\dim(\mathrm{G/H})} X_a \otimes e^a \tag{3.10}$$

where Γ is a "suitable starting point" in the space of connections (to be specified shortly), and X_a are matrices valued in the Lie algebra of the structure group, in our case $\mathfrak{u}(k)$. Therefore, the space of connections is parametrized by the endomorphisms X_a as an affine space with respect to the chosen connection Γ.

In order to yield a reasonable gauge theory on M^d, the connection $\mathcal{A}$ should be G-invariant. Furthermore, since we are interested in instanton solutions, the instanton equations shall be imposed on the invariant gauge connection. For a convenient description in terms of the matrices X_a, also the starting point Γ should satisfy these two conditions. That is, Γ should (i) be a G-invariant connection and (ii) satisfy the instanton equations. An obvious candidate obeying the first condition is the *canonical connection* in the sense of homogeneous spaces [94] which is characterized by having the torsion $T(X,Y) \coloneqq -[X,Y]_{\mathfrak{m}}$ for vector fields on $\mathfrak{m}$, where the subscript denotes the projection of the commutator to the space $\mathfrak{m}$. In index notation, one has $T^a = -\frac{1}{2} f^a_{\ bc} e^b \wedge e^c$, and the canonical connection is written as[2]

$$\Gamma = \sum_{j=1}^{\dim(H)} I_j \otimes e^j. \tag{3.11}$$

It is valued in the subalgebra $\mathfrak{h}$ only and therefore adapted to the geometry of G/H as principal H-bundle.

[1] The natural metric on the semi-simple Lie algebra $\mathfrak{g}$ is determined by the Killing form $B(X,Y) \propto \mathrm{Tr}(XY)$, and one takes the pullback to a left-invariant metric on $\mathfrak{m}$, which may be piecewise rescaled for obtaining the relevant geometries. See for instance the discussion of the 3-Sasakian case in Section 6.1 and in Appendix D.3.

[2] We will denote generators in a chosen representation on the fibers with the same symbols as the abstract generators of G throughout this text.

On the other hand, one can construct on G/H the canonical connection in the sense of manifolds with Killing spinors, as in Section 2.2.1. According to the results of [19], both notions of canonical connection coincide[3] for homogeneous spaces endowed with Killing spinors, as we will explicitly see for all later examples. Recall that the canonical connection of [19] is adapted to the reduction of the structure group of the frame bundle of G/H to $\mathrm{SU}(n)$ for $(2n+1)$-dimensional Sasaki-Einstein spaces or to $\mathrm{Sp}(m)$ for 3-Sasakian manifolds of dimension $4m+3$. Since the canonical connection Γ combines both desirable properties, it will be employed as starting point in the formulation (3.10). After fixing the offset in (3.10), one still has to impose a condition on the matrices X_a to ensure G-invariance of $\mathcal{A}$. For this purpose, consider the curvature of the gauge connection, ignoring possible dependence of the matrices X_a for the time being:

$$\mathcal{F} = (\mathrm{d}\Gamma + \Gamma \wedge \Gamma) + ([I_j, X_a] - f_{ja}^{b} X_b) e^j \wedge e^a + \tfrac{1}{2}([X_a, X_b] - f_{ab}^{c}) e^a \wedge e^b. \tag{3.12}$$

The first term consists of the curvature of the canonical connection, which is an instanton and G-invariant by construction. The second contribution in (3.12), however, contains mixed two-forms, taking values in $\mathfrak{h}^* \wedge \mathfrak{m}^*$, which spoil the invariance of the expression. Thus, one has to impose the following *equivariance condition* [34]:

$$[I_j, X_a] = f_{ja}^{b} X_b \qquad \forall I_j \in \mathfrak{h}. \tag{3.13}$$

This equation does not only arise from this consideration, but it is the consequence of a known invariance condition on connections over homogeneous spaces. According to [94, Ch.2, Sec.11], there is a one-to-one correspondence of G-invariant connections in the principal H-bundle G/H and linear maps $\Lambda : \mathfrak{m} \to \mathfrak{g}$ such that

$$\Lambda(\mathrm{Ad}(h)Y) = \mathrm{Ad}(h)\Lambda(Y) \quad \forall h \in H, \quad Y \in \mathfrak{m}. \tag{3.14}$$

As explained in [25, Sec. 2.5], setting $X_a = \Lambda(I_a)$ then yields the equivariance condition (3.13) as infinitesimal version thereof. The equivariance condition (3.13) fixes the commutation relations between the endomorphisms X_a and the generators $I_j = \bigoplus_l I_j^{(l)} \otimes \mathbf{1}_{k_l} \in \bigoplus_l V_l \otimes E_l$ of the subgroup H in a chosen G-representation $\mathcal{D}$, and therefore amounts to the construction of the quiver bundles from the previous section.

Many constructions of instantons on homogeneous spaces with Killing spinors have applied the *scalar ansatz* [19]: setting $X_a = \lambda_a(x) I_a$ automatically satisfies the equivariance condition (3.13), and the instanton equations reduce to equations on a set of functions. Typically the geometric structure of the Killing spinor manifold, e.g.

[3]For the 3-Sasakian cosets (cf. Chapters 6 and 7), one has to consider the canonical connection with respect to the 3-Sasakian structure rather than the canonical connection of one of the Sasaki-Einstein structures only.

Sasaki manifolds as U(1)-bundles over Kähler manifolds, restricts the number of independent fields further. Quiver gauge theory allows for more general solutions to the equivariance condition, since each arrow in the quiver diagram corresponds to an independent matrix-valued function. Imposing the instanton conditions then yields a set of ODEs on these bundle maps as well as certain quiver relations.

The equivariant gauge connection, subject to the equivariance condition (3.13), can also be considered (cf. [34, Sec. 4]) as twist and extension of the canonical flat connection for a G-representation by bundle maps. This will be clarified for the examples of quiver diagrams constructed in the remainder of this thesis.

3.3 Quiver gauge theories

We briefly summarize the definition and main properties of quiver gauge theories and equivariant dimensional reduction in the following.

Quiver gauge theory. In this thesis, the notion of *quiver gauge theory* refers to gauge theories on vector bundles $\mathcal{E} \to M^d \times G/H$ which admit a decomposition as quiver bundles due to G-equivariance. The quiver diagram associated to a chosen G-representation is obtained by collapsing the weight diagram and assigning vector spaces E_j and linear maps ϕ_{ji} to the remaining vertices and arrows, which constitutes a representation of the quiver. The quiver bundle is accompanied by a breaking of the structure group into a product of unitary groups acting on each vertex.

An equivariant gauge connection is compatible with this structure and can be parametrized by endomorphisms X_a as in (3.10), which are subject to the equivariance condition (3.13); the matrices X_a are referred to as *Higgs fields*. Aiming for instanton solutions, we always choose the canonical connection [19] as starting point Γ in the ansatz (3.10). A quiver gauge theory is called *Sasakian* or *3-Sasakian quiver gauge theory* if the coset space G/H admits the corresponding structure, and this name shall also distinguish the concept from other quiver gauge theories in physics.

The instanton conditions on the metric cones over G/H yield a system of equations on matrix-valued functions which depend on the cone direction and whose structure follows from the quiver diagrams. These equations comprise algebraic conditions, i.e. *quiver relations*, as well as ordinary differential (matrix) equations. The latter ones are also referred to as *flow equations* since they admit an interpretation as gradient flow equations (see e.g. [27, 71]).

Remark 1. It is worth recalling that the concept of quiver bundles [35] relies on the structure as homogeneous space G/H and that the canonical connection of manifolds with Killing spinors [19], albeit very useful for aiming at the construction of instantons, only enters by the coincidence of both definitions on homogeneous spaces.

On the other hand, by virtue of the canonical connection on manifolds X with real Killing spinors, a similar construction is applicable even if the manifold itself is not given as a homogeneous space [34]. The manifold X admits an H-structure, such as $H = \mathrm{SU}(n)$ or $\mathrm{Sp}(m)$ for Sasaki-Einstein manifolds X^{2n+1} or 3-Sasakian manifolds X^{4m+3}, and the canonical connection is adapted to it, i.e. it takes values in the Lie algebra of H. The metric cone over the Killing spinor manifold X has special holonomy G, with $G = \mathrm{SU}(n+1)$ or $G = \mathrm{Sp}(m+1)$ for Sasakian or 3-Sasakian manifolds, respectively. The structure group H is embedded into the holonomy group G, and one can mimick the approach for homogeneous spaces G/H by applying the same ansatz (3.10) for the gauge connection [34].

Remark 2. For Sasakian or 3-Sasakian quiver gauge theory, evaluating the equivariance condition (3.13) alone is not completely equivalent to constructing the quiver diagrams by collapsing the weight diagrams along the ladder operators of $\mathfrak{h}$ and twisting with bundle maps. Without further restrictions of the matrices $X_a \in \mathfrak{u}(k)$, condition (3.13) may admit additional contributions because H does not contain a maximal torus and therefore its generators I_j (in the fixed representation) do not determine the weight diagrams uniquely, in contrast to the case of flag manifolds. On the other hand, since the approach is based on the G-action with the representations descending from a G-representation $\mathcal{D}$ and with the induction $\mathcal{E} = G \times_H E$, it is reasonable to require the matrices to be of the form of the generators of G. This is ensured by imposing (3.13) for *all* Cartan generators additionally or by assuming $X_a \in \mathfrak{g} \subset \mathfrak{u}(k)$.

We will focus on the typical approach of quiver diagrams obtained by the collapsing procedure and comment on more general Higgs fields that are compatible with (3.13) only briefly.

3.3.1 Example: $\mathbb{C}P^1$

Let us illustrate the concept of quiver gauge theory and equivariant dimensional reduction by considering the guiding example $\mathbb{C}P^1 \cong \mathrm{SU}(2)/\mathrm{U}(1)$ [35, 36, 38, 39, 97]. This example illuminates the general concept but also serves as a building block of the quiver gauge theories for $\mathbb{C}P^1 \times \mathbb{C}P^1$ [43], $T^{1,1}$ (cf. Chapter 5), as well as for cosets involving Lie groups G of higher rank.

For the case of $G = \mathrm{SU}(2)$ and $H = \mathrm{U}(1)$, the structure of the equivariant vector bundles and the gauge connection follows almost immediately. The subgroup H is generated by the Cartan generator I_3, for which one can use the representation $I_3 = \frac{1}{2}(m, m-2, \ldots, -m)$ on $\mathbb{C}^{m+1}$ as well as the typical $\mathfrak{sl}(2, \mathbb{C})$ commutation relations

$$[I_3, I_\pm] = \pm 2 I_\pm, \qquad [I_+, I_-] = -2 I_3. \tag{3.15}$$

Since $H = \mathrm{U}(1)$ is abelian, the representation spaces $V_j \equiv S_j$ in the isotopical decomposition (3.4) are 1-dimensional, and the subgroup acts with eigenvalues m, $m-2$, $\ldots$,$-m$ on them. Equivariance then requires that the entire group action on the fibers is compatible with the commutation relations, so that the quiver diagram is given by the so-called *holormophic chain* [36]

$$\underset{(\mathbf{m})}{\bullet} \xrightarrow{\phi_0} \underset{(\mathbf{m-2})}{\bullet} \xrightarrow{\phi_1} \underset{(\mathbf{m-4})}{\bullet} \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{m-1}} \underset{(\mathbf{-m})}{\bullet} \tag{3.16}$$

where the arrows represent the homomorphisms $\phi_j : E_j \to E_{j-2}$, induced by the action of I_-. As claimed before, the quiver diagram simply consists of the weight diagram because H is a maximal torus of G. The isotopical decomposition of the bundle reads

$$\mathcal{E}_x = \sum_{j=0}^{m} \mathcal{E}_j \equiv \sum_{j=0}^{m} E_j \otimes \mathcal{L}_{m-2j}, \tag{3.17}$$

where $\mathcal{L}_{m-2j} := \mathrm{SU}(2) \times_{\mathrm{U}(1)} S_j$ is the monopole bundle (see e.g. [43]), which appears due to the Hopf fibration $\mathrm{SU}(2) \to \mathbb{C}P^1$. The equivariant gauge connection $\mathcal{A}$ takes the form [38]

$$\mathcal{A} = \begin{pmatrix} A_0 + a_m \mathbf{1}_{k_0} & -\phi_0^\dagger \beta & 0 & \ldots & 0 \\ \phi_0 \bar\beta & A_1 + a_{m-2}\mathbf{1}_{k_1} & -\phi_1^\dagger \beta & \ldots & 0 \\ 0 & \phi_1 \bar\beta & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & & -\phi_{m-1}^\dagger \beta \\ 0 & 0 & \ldots & \phi_{m-1}\bar\beta & A_m + a_{-m}\mathbf{1}_{k_m} \end{pmatrix}, \tag{3.18}$$

where the A_j constitute components of a gauge connection A on the bundle $E \to M^d$ and β is a basis 1-form on $\mathbb{C}P^1$. The monopole forms a_{m-2j} consist of the form part of the canonical connection $\Gamma = I_3 \otimes a$ (with a the 1-form dual to I_3) living on the representation spaces V_j of the vertices.

Based on this equivariant gauge connection one can construct the Yang-Mills action and reduce it to a Yang-Mills-Higgs theory on M^d. Since the connection is formulated in terms of the holormophic forms β, one can easily evaluate the HYM equations to obtain the so-called *chain vortex equations* [35, 36] as BPS conditions. We will encounter them as a limit of the BPS equations (5.47) and (5.42) on $T^{1,1,}$ if the reduction to the underlying Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ is considered and if one of the SU(2) factors is represented trivially.

3.3.2 Overview and different quiver gauge theories

Besides the prototypical manifold $\mathbb{C}P^1$, the following homogeneous spaces G/H have been included into the framework of quiver gauge theories from equivariant dimensional reduction on $M^d \times G/H$:

- $\mathbb{C}P^1 \times \mathbb{C}P^1$ with Kähler structure [35, 43],
- $\mathbb{C}P^2 \cong \mathrm{SU}(3)/(\mathrm{SU}(2) \times \mathrm{U}(1))$ with Kähler structure [35, 40, 42, 98],
- $\mathrm{SU}(3)/(\mathrm{U}(1) \times \mathrm{U}(1))$ with Kähler and nearly-Kähler structure [41, 42, 98],
- $S^3/\Gamma = \mathrm{SU}(2)/\Gamma$ with Sasaki-Einstein structure [44],
- $S^5/\mathbb{Z}_k = (\mathrm{SU}(3)/\mathrm{SU}(2))\,/\mathbb{Z}_k$ with Sasaki-Einstein structure [45, 46].

Due to the bundle structures of Sasakian and 3-Sasakian manifolds and their relation to Kähler manifolds, it is natural to compare the quiver gauge theories constructed in the following chapters to known results. For instance, since the space $T^{1,1}$ (cf. Chapter 5) is a U(1)-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$, it constitutes the obvious counterpart of the quiver gauge theory on $M^d \times \mathbb{C}P^1 \times \mathbb{C}P^1$ [43]. This possibility of relating quiver gauge theories in different dimensions by the "sandwiching property" of Sasakian manifolds motivated studying *Sasakian* quiver gauge theories on S^3 and S^5 [44–46].

Different notion of quiver gauge theories. Most prominently, the concept of *quivers* appears in Nakajima's *quiver varieties* [64] and in *quiver gauge theories* of brane configurations, which share the same basic idea with the Sasakian quiver gauge theories but arise in a different context. Nakajima's construction is based on a given quiver diagram whose vertices are endowed with two Hermitian vector spaces. The space of linear maps between the vector spaces, according to the arrow structure of the quiver, inherits a hyper-Kähler structure and one can define a compatible group action on it. The quiver variety is obtained by taking the hyper-Kähler quotient, and this construction generalizes the description of 4-dimensional instantons and their hyper-Kähler moduli space by the ADHM construction [99, 100].

The second notion of quiver gauge theories occurs for the description of brane configurations, beginning with the description of D-branes at the orbifold $\mathbb{C}^2/\mathbb{Z}_k$ and resolved ALE spaces [101], which is actually related to the ADHM construction and quiver varieties. This approach has been extended to other setups and has been established as a suitable tool to describe branes located at certain singularities. Roughly speaking, the quivers depict the breaking of the gauge group of a stack of branes into a product of smaller gauge groups, depending on the brane configuration. The field content of the theory then has to transform in bifundamental or adjoint representations of these unitary gauge groups.

The Sasakian quiver gauge theories we are studying here do not necessarily have a relation to branes, but they simply arise in the context of equivariant dimensional reduction. However, for some of the Kähler quiver gauge thories listed above an interpretation in terms of certain brane configurations was given [43, 102], once a noncommutative deformation of the external space M^d is applied. Detailed discussions illuminating both concepts – Sasakian quiver gauge theories and quiver gauge theories of branes placed at singularities – can be found in [44] and [45, Sec. 7].

Part II

Sasakian quiver gauge theories

4 Quiver gauge theory on the round seven-sphere

In this chapter we study Sasakian quiver gauge theories on orbifolds of the round seven-sphere $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$, which is the prototype of 7-dimensional Sasaki-Einstein manifolds. The seven-sphere and its orbifolds typically appear in the context of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality in M-theory and supergravity compactifications since, as Sasaki-Einstein manifolds, they realize Freund-Rubin solutions [22, 103]; a list of compact homogeneous spaces in supergravity compactifications can be found in [21] for example. The near-horizon geometry of these backgrounds is that of M2-branes placed at the conical singularity of $\mathbb{C}^4/\mathbb{Z}_k$, and the low energy effective theories can be described as 3-dimensional superconformal Chern-Simons theories with $\mathcal{N} = 6$ [104]. The setup $\mathrm{AdS}_4 \times S^7$ constitutes an effective $\mathcal{N} = 8$ supergravity, and deforming the round metric on S^7 to the squashed metric (cf. Chapter 6), which is also nearly parallel G_2, breaks the supersymmetry[1] to $\mathcal{N} = 1$ [105, 106]. The Killing-spinor equations for M-theory on backgrounds $\mathrm{AdS}_4 \times M_7$ with $\mathcal{N} = 2$ supersymmetry are given in [89,107]. More general solutions M_7 than Sasaki-Einstein manifolds are comprised in the notion of *exceptional Sasaki-Einstein structures* [90], which employ the formalism of exceptional generalized geometry.

We study Sasakian quiver gauge theories on S^7 following the analogous discussion of the five-sphere [45,46]. In contrast to the complete characterization in the case of S^5, using the Biedenharn basis for SU(3), we do not provide such a complete list because of the higher dimension of the group SU(4). Emphasis is placed on illuminating the basic features, and we therefore discuss explicit examples of the quiver diagrams and resulting instanton equations for low-dimensional representations of $G = \mathrm{SU}(4)$. It turns out that, due to the regularity of the construction of the round spheres, these examples are the higher-dimensional analogues of those obtained for (orbifolds of) the five-sphere $S^5 \cong \mathrm{SU}(3)/\mathrm{SU}(2)$ [46]. The chapter concludes by briefly sketching some generalizations to higher-dimensional spheres $S^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

The following discussion is an extended version of the content covered in [49], a collaboration with O. Lechtenfeld, A. D. Popov and R. J. Szabo.

[1] A 3-Sasakian manifolds admits 3/8 maximal supersymmetry while a nearly parallel G_2 structure breaks it to 1/8 [54, Sec. 10] (see also [9, Sec. 5]).

4.1 Geometric structure

In this section we review the Sasaki-Einstein geometry of the homogeneous space SU(4)/SU(3) and construct the associated canonical connection, which serves as starting point for the equivariant gauge connections.

4.1.1 Local section

The discussion of the geometric structure of the round seven-sphere $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$ is based on the fibration over the complex projective space $\mathbb{C}P^3$ as

$$\begin{array}{ccc} \mathrm{SU}(4) & \xrightarrow{\ \mathrm{SU}(3)\ } & S^7 \\ & \searrow^{S(U(3)\times U(1))} & \downarrow \mathrm{U}(1) \\ & & \mathbb{C}P^3 \end{array} \tag{4.1}$$

From this bundle one obtains a local section of $\mathrm{SU}(4) \longrightarrow \mathbb{C}P^3$ by following the procedure in [42, 46]. For a local patch $\mathcal{U}_0 = \{[w_0 : w_1 : w_2 : w_3] \in \mathbb{C}P^3 |\ w_0 \neq 0\}$ one introduces coordinates

$$Y \coloneqq (y_1, y_2, y_3)^T \coloneqq (w_0^{-1} w_1, w_0^{-1} w_2, w_0^{-1} w_3)^T, \tag{4.2}$$

and the matrix

$$V \coloneqq \frac{1}{\gamma} \begin{pmatrix} 1 & Y^\dagger \\ -Y & \Lambda \end{pmatrix} \quad \text{with} \quad \Lambda \coloneqq \gamma \mathbf{1}_3 - \tfrac{1}{1+\gamma} Y Y^\dagger, \quad \gamma \coloneqq \sqrt{1 + Y^\dagger Y}. \tag{4.3}$$

These definitions imply the properties $\Lambda Y = Y$, $Y^\dagger \Lambda = Y^\dagger$ and $\Lambda^2 = \gamma^2 \mathbf{1}_3 - Y Y^\dagger$, so that the matrix V is a local section of the bundle $\mathrm{SU}(4) \longrightarrow \mathbb{C}P^3$. The Maurer-Cartan form $A_0 \coloneqq V^{-1} \mathrm{d}V$ provides SU(4)-left-invariant 1-forms on $\mathbb{C}P^3$,

$$A_0 = V^\dagger \mathrm{d}V =: \begin{pmatrix} -a & \beta^\dagger \\ -\beta & B \end{pmatrix} \quad \text{with} \quad a = -\tfrac{1}{2\gamma^2}(Y^\dagger \mathrm{d}Y - \mathrm{d}Y^\dagger Y), \quad \beta = \tfrac{1}{\gamma^2} \Lambda \mathrm{d}Y,$$
$$B = \tfrac{1}{\gamma^2}(Y \mathrm{d}Y^\dagger + \Lambda \mathrm{d}\Lambda) - \tfrac{1}{2\gamma^2} \mathrm{d}(Y^\dagger Y), \tag{4.4}$$

which are the analogues of those in [42, 46]. The flatness of the connection A_0, $\mathrm{d}A_0 + A_0 \wedge A_0 = 0$, leads to the structure equations

$$\mathrm{d}a = -\beta^\dagger \wedge \beta, \quad \mathrm{d}\beta = \beta \wedge a - B \wedge \beta, \quad \mathrm{d}B = \beta \wedge \beta^\dagger - B \wedge B. \tag{4.5}$$

The section over $\mathbb{C}P^3$ can be promoted to a section of SU(4) over S^7 by including a U(1) factor as

$$S^7 \ni (y_1, y_2, y_3, \varphi) \mapsto \widetilde{V} \coloneqq V \times \mathrm{diag}(\mathrm{e}^{3\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}). \tag{4.6}$$

The resulting canonical flat connection $\widetilde{A}_0 = \widetilde{V}^\dagger \mathrm{d}\widetilde{V}$ of this fibration reads

$$\begin{aligned}\widetilde{A}_0 &= \begin{pmatrix} -a + 3\mathrm{i}\mathrm{d}\varphi & \mathrm{e}^{-4\mathrm{i}\varphi}\beta^\dagger \\ -\beta\mathrm{e}^{4\mathrm{i}\varphi} & B - \mathrm{i}\,\mathrm{d}\varphi\,\mathbf{1}_3 \end{pmatrix} \\ &=: \begin{pmatrix} 3\mathrm{i}\mu_7 e^7 & \zeta_1\Theta^1 & \zeta_2\Theta^2 & \zeta_3\Theta^3 \\ -\zeta_1\Theta^{\bar 1} & -\mathrm{i}\mu_7 e^7 + 2\mathrm{i}\mu_8 e^8 & \lambda_4\Theta^4 & \lambda_5\Theta^5 \\ -\zeta_2\Theta^{\bar 2} & -\lambda_4\Theta^{\bar 4} & -\mathrm{i}\mu_7 e^7 - \mathrm{i}\mu_8 e^8 - \mathrm{i}\mu_9 e^9 & \lambda_6\Theta^6 \\ -\zeta_3\Theta^{\bar 3} & -\lambda_5\Theta^{\bar 5} & -\lambda_6\Theta^{\bar 6} & -\mathrm{i}\mu_7 e^7 - \mathrm{i}\mu_8 e^8 + \mathrm{i}\mu_9 e^9 \end{pmatrix},\end{aligned} \tag{4.7}$$

and its flatness yields the structure equations (B.1). The real parameters ζ_1, ζ_2, ζ_3 and μ_7 will be fixed by the Sasaki-Einstein geometry in the following discussion. The other paramaters, describing the subgroup $H = \mathrm{SU}(3)$, can be arbitrarily chosen and henceforth we will set $\lambda_4 = \lambda_5 = \lambda_6 = 1$, $\mu_8 = \frac{1}{6}$ and $\mu_9 = \frac{1}{2}$ for simplicity.

4.1.2 Sasaki-Einstein structure

We now introduce real 1-forms $\mathrm{e}^{2\alpha-1} - \mathrm{i}\mathrm{e}^{2\alpha} := \Theta^\alpha$ for $\alpha = 1, 2, 3$ and consider the orthonormal metric $\mathrm{d}s^2 = \sum_{\mu=1}^7 e^\mu \otimes e^\mu$ on the tangent space $T_e S^7 \cong \mathfrak{m}$. It is convenient to verify the Sasaki-Einstein property of this metric by checking that its cone is Calabi-Yau, which is guaranteed by the closure of some defining tensors, following the procedure in [27] for instance.

With a fourth complex 1-form $\Theta^0 := \frac{\mathrm{d}r}{r} - \mathrm{i}e^7 =: \mathrm{e}^\tau - \mathrm{i}e^7$, one obtains an integrable[2] complex structure J on the metric cone by setting $J\Theta^\alpha = \mathrm{i}\Theta^\alpha$ for $\alpha = 0, 1, 2, 3$. The cone metric then takes the form $\mathrm{d}s_c^2 = r^2\delta_{\alpha\beta}\Theta^\alpha \otimes \Theta^{\bar\beta}$, and the corresponding Kähler form reads

$$\Omega^{1,1} = -\tfrac{\mathrm{i}}{2} r^2 (\Theta^{0\bar 0} + \Theta^{1\bar 1} + \Theta^{2\bar 2} + \Theta^{3\bar 3}). \tag{4.8}$$

Its closure requires $\zeta_1 = \zeta_2 = \zeta_3$ and $3\mu_7 = \zeta_1^2$, as the calculation (B.2) shows. For the further reduction of the holonomy from U(4) to SU(4), one needs the closure of the holomorphic top-degree form (ensuring the triviality of the canonical bundle)

$$\Omega^{4,0} = r^4 \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^0, \tag{4.9}$$

which yields the condition $\zeta^2 = 1$ and $\mu_7 = \frac{1}{3}$. The relevant structure equations therefore read

$$\begin{aligned} \mathrm{d}\Theta^1 &= -\tfrac{4}{3}\mathrm{i}e^7 \wedge \Theta^1 + \tfrac{1}{3}\mathrm{i}e^8 \wedge \Theta^2 + \Theta^{2\bar 4} + \Theta^{3\bar 5}, \\ \mathrm{d}\Theta^2 &= -\tfrac{4}{3}\mathrm{i}e^7 \wedge \Theta^2 - \tfrac{1}{6}\mathrm{i}e^8 \wedge \Theta^2 - \tfrac{1}{2}\mathrm{i}e^9 \wedge \Theta^2 - \Theta^{14} + \Theta^{3\bar 6}, \\ \mathrm{d}\Theta^3 &= -\tfrac{4}{3}\mathrm{i}e^7 \wedge \Theta^3 - \tfrac{1}{6}\mathrm{i}e^8 \wedge \Theta^3 + \tfrac{1}{2}\mathrm{i}e^9 \wedge \Theta^3 - \Theta^{15} - \Theta^{26}, \\ \mathrm{d}e^7 &= -\mathrm{i}(\Theta^{1\bar 1} + \Theta^{2\bar 2} + \Theta^{3\bar 3}) = 2\omega, \end{aligned} \tag{4.10}$$

where ω is the Kähler form on the leaf space of the Sasakian manifold, i.e. the Kähler form on $\mathbb{C}P^3$.

[2] The intergrability, i.e. the vanishing of the Nijenhuis tensor $N(X, Y)$, can be easily seen due to $J\Theta^\alpha = \mathrm{i}\Theta^\alpha$ and the relevant commutators (B.9).

4.1.3 Canonical connection

Based on the Sasaki-Einstein structure of S^7, we construct the associated canonical connection [19], as reviewed in Section 2.2.1. The 3-form P and 4-form Q (2.15) that govern the generalized self-duality condition read

$$P = \eta \wedge \omega = e^7 \wedge (e^{12} + e^{34} + e^{56}), \quad Q = \tfrac{1}{2}\omega \wedge \omega = e^{1234} + e^{1256} + e^{3456}, \tag{4.11}$$

so that the torsion components (2.16) are given by

$$\begin{aligned} T^1 = \tfrac{4}{3}e^{27}, \quad T^2 = -\tfrac{4}{3}e^{17}, \quad T^3 = \tfrac{4}{3}e^{47}, \quad T^4 = -\tfrac{4}{3}e^{37}, \\ T^5 = \tfrac{4}{3}e^{67}, \quad T^6 = -\tfrac{4}{3}e^{57}, \quad T^7 = 2(\mathrm{e}^{12} + e^{34} + e^{56}). \end{aligned} \tag{4.12}$$

Plugging this torsion into the Maurer-Cartan equation $\mathrm{d}e^\mu = -\Gamma^\mu_\nu \wedge \mathrm{e}^\nu + T^\mu$, one identifies the canonical connection

$$\Gamma = I_8 \otimes e^8 + I_9 \otimes e^9 + \sum_{\beta=4}^{6} (I^+_\beta \otimes \Theta^\beta + I^-_{\bar\beta} \otimes \Theta^{\bar\beta}) =: \sum_{\mu=8}^{15} I_\mu \otimes e^\mu, \tag{4.13}$$

where the definition of the generators is based on (4.7) and collected in Appendix B.2. In accordance with the general theory and as claimed above, the expression (4.13) coincides with the canonical connection of the homogeneous space G/H, which is obtained by setting the torsion to be $T(X,Y) = -[X,Y]_{\mathfrak{m}}$. The field strength $\mathcal{F}_\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma$ of (4.13) is given by

$$\begin{aligned} \mathcal{F}_\Gamma = I^+_4 \otimes \Theta^{\bar 1 2} + I^+_5 \otimes \Theta^{\bar 1 3} + I^+_6 \otimes \Theta^{\bar 2 3} + I^-_{\bar 4} \otimes \Theta^{1\bar 2} + I^-_{\bar 5} \otimes \Theta^{1\bar 3} + I^-_{\bar 6} \otimes \Theta^{2\bar 3} \\ + I_8 \otimes (2\mathrm{i}\Theta^{1\bar 1} - \mathrm{i}\Theta^{2\bar 2} - \mathrm{i}\Theta^{3\bar 3}) + I_9 \otimes (-\mathrm{i}\Theta^{2\bar 2} + \mathrm{i}\Theta^{3\bar 3}), \end{aligned} \tag{4.14}$$

which obeys the generalized instanton equation (2.10) with 4-form Q from (4.11).

4.2 Equivariance condition and instanton equation

We now consider the gauge theory on a Hermitian vector bundle $\mathcal{E}$ of rank k over the manifold $M^d \times G/H$, as discussed in Section 3.2. Since the canonical connection Γ is an instanton, we (locally) apply the typical ansatz (3.10) for invariant gauge connections on $M^d \times \mathrm{SU}(4)/\mathrm{SU}(3)$, writing

$$\mathcal{A} = \Gamma + A + \sum_{\mu=1}^{7} X_\mu \otimes e^\mu = \Gamma + A + \sum_{\alpha=1}^{3} (Y_\alpha \otimes \Theta^\alpha + Y_{\bar\alpha} \otimes \Theta^{\bar\alpha}) + X_7 \otimes e^7. \tag{4.15}$$

Here A denotes a connection on the vector bundle E over M^d which is compatible with the decomposition (3.3) of the H-representations on the fibers and which takes values in the Lie algebra of the broken structure group (3.5).[3] The matrices X_μ

[3] More precisely, the connection is of the form $A = \bigoplus_l \mathbf{1}_{d_l} \otimes A_l \in \bigoplus_l \mathfrak{u}(k_l) \subset \mathfrak{u}(k)$ [37, Eq. (5.1)]. Otherwise mixed contributions of Γ and components of A would arise, spoiling the equivariance. Recall that the generators of H, which enter Γ in (4.13), decompose as $I_j = \bigoplus_l I^{(l)}_j \otimes \mathbf{1}_{k_l}$, see for instance [34].

and $Y_\alpha := \frac{1}{2}(X_{2\alpha-1} + \mathrm{i}X_{2\alpha})$ denote the endomorphism part of the connection and are called *Higgs fields*. Being valued in $\mathfrak{u}(k)$, the matrices X_μ are skew-Hermitian, $X_\mu^\dagger = -X_\mu$, and we write $\phi^{(\alpha)} := Y_{\bar{\alpha}} := -Y_\alpha^\dagger$ for the complex combinations. According to the bundle structure of the Sasaki-Einstein manifold S^7 (over $\mathbb{C}P^3$), the field X_7 accompanying the contact form e^7 is referred to as *vertical* Higgs fields, while $\phi^{(\alpha)}$ are *horizontal* Higgs fields.

The canonical connection Γ lifts to an instanton on the metric cone, so that we will use the very same ansatz for connections on $\mathbb{R}^+ \times \mathrm{SU}(4)/\mathrm{SU}(3)$, including radial dependence of the endomorphisms, $X_\mu = X_\mu(\tau)$, and a possible eighth matrix X_τ associated to the cone direction. Therefore, the quiver diagrams for both the round sphere S^7 and its metric cone $\mathbb{R}^8$ take the same form.

4.2.1 Equivariance condition

With the explicit form of the canonical connection (4.13) and the structure constants (B.9) the equivariance condition (3.13) reads

$$\begin{aligned} &[\tilde{I}_8, \phi^{(1)}] = 2\phi^{(1)}, \quad [\tilde{I}_8, \phi^{(2)}] = -\phi^{(2)}, \quad [\tilde{I}_8, \phi^{(3)}] = -\phi^{(3)}, \quad [\tilde{I}_8, X_7] = 0, \\ &[\tilde{I}_9, \phi^{(1)}] = 0, \qquad [\tilde{I}_9, \phi^{(2)}] = -\phi^{(2)}, \quad [\tilde{I}_9, \phi^{(3)}] = \phi^{(3)}, \qquad [\tilde{I}_9, X_7] = 0, \end{aligned} \tag{4.16a}$$

with respect to the two Cartan generators of $\mathfrak{h}$, and the ladder operators require

$$\begin{aligned} &[I_4^+, \phi^{(2)}] = \phi^{(1)}, \qquad [I_4^-, \phi^{(1)}] = -\phi^{(2)}, \qquad [I_5^+, \phi^{(3)}] = \phi^{(1)}, \\ &[I_5^-, \phi^{(1)}] = -\phi^{(3)}, \qquad [I_6^+, \phi^{(3)}] = \phi^{(2)}, \qquad [I_6^-, \phi^{(2)}] = -\phi^{(3)}, \end{aligned} \tag{4.16b}$$

as well as the analogous expressions for the conjugated fields and the vanishing of all other commutators, in particular those involving X_7. The equations (4.16a) fix the action of the Higgs fields in the weight diagrams of SU(4) representations to be

$$\begin{aligned} &\phi^{(1)} : (\nu_7, \nu_8, \nu_9) \longmapsto (*, \nu_8 + 2, \nu_9) \qquad \phi^{(2)} : (\nu_7, \nu_8, \nu_9) \longmapsto (*, \nu_8 - 1, \nu_9 - 1) \\ &\phi^{(3)} : (\nu_7, \nu_8, \nu_9) \longmapsto (*, \nu_8 - 1, \nu_9 + 1) \quad X_7 : (\nu_7, \nu_8, \nu_9) \longmapsto (*, \nu_8, \nu_9)\,, \end{aligned} \tag{4.17}$$

while (4.16b) ensures compatibility with the ladder operators of the subgroup SU(3).

We encounter the typical feature of Sasakian quiver gauge theories: the equivariance with respect to H as in (3.13) does not necessarily fix the action of the Higgs fields to coincide with the action of the ladder operators in the weight diagrams but may allow for further contributions, since the action on ν_7 is undetermined.

As discussed in Section 3.3, our primary definition of quiver gauge theories involves the ansatz (3.3) with H-representations ρ_j, stemming from a G-representation $\mathcal{D}$, and the Higgs fields are given by the generators of $\mathfrak{g}$ after collapsing along the subalgebra $\mathfrak{h}$ and extending by bundle maps. Hence, we consider Higgs fields that act according to (B.11). Equivalently, this can be ensured by imposing equivariance also with respect to the third Cartan generator I_7, resembling the case of flag manifolds. Indeed, this approach yields the quiver associated to the underlying flag manifold modified by vertex loops due to the vertical Higgs field, as the examples in Section 4.3 will illustrate.

4.2.2 Yang-Mills functional and instanton equation on S^7

Having imposed the equivariance conditions (4.16), we now determine the Yang-Mills action of the invariant gauge connection (4.15),

$$S_{\mathrm{YM}} = -\tfrac{1}{4}\int_{M^d\times S^7} \mathrm{Tr}\ (\mathcal{F}\wedge\star\mathcal{F}). \tag{4.18}$$

With the Sasaki-Einstein metric $\mathrm{d}s^2 = \delta_{\alpha\beta}\Theta^\alpha\otimes\Theta^{\bar\beta} + e^7\otimes e^7$ in terms of the complex forms, the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}\mathrm{Tr}\ \mathcal{F}_{MN}\mathcal{F}^{MN} = -\tfrac{1}{4}\mathrm{Tr}\ \{&\mathcal{F}_{mn}\mathcal{F}^{mn} + 8g^{mn}\mathcal{F}_{m\alpha}\mathcal{F}_{n\bar\alpha} + 2g^{mn}\mathcal{F}_{m7}\mathcal{F}_{n7}\\ &+8\mathcal{F}_{\alpha\beta}\mathcal{F}_{\bar\alpha\bar\beta} + 8\mathcal{F}_{\alpha\bar\beta}\mathcal{F}_{\bar\alpha\beta} + 8\mathcal{F}_{\alpha7}\mathcal{F}_{\bar\alpha7}\},\end{aligned} \tag{4.19}$$

where capital indices refer to the entire space $M^d\times S^7$, the indices m and n to M^d only, and $\{\alpha,\bar\beta,7\}$ denote indices on S^7. Inserting the curvature components of the equivariant gauge connection (4.15) into the above expression yields

$$\begin{aligned}\mathrm{S}_{\mathrm{YM}} = \mathrm{Vol}(S^7)\int_{M^d}\mathrm{d}^dy\sqrt{g}\ \mathrm{Tr}\ \Big\{&\tfrac{1}{4}F_{mn}(F^{mn})^\dagger + 2\sum_{m,\alpha}\Big|D_m\phi^{(\alpha)}\Big|^2 + \tfrac{1}{2}\sum_m|D_mX_7|^2\\ &+2\Big|[\phi^{(1)},\phi^{(1)\dagger}] - \mathrm{i}X_7 + 2\mathrm{i}I_8\Big|^2 + 2\Big|[\phi^{(2)},\phi^{(2)\dagger}] - \mathrm{i}X_7 - \mathrm{i}I_8 - \mathrm{i}I_9\Big|^2\\ &+2\Big|[\phi^{(3)},\phi^{(3)\dagger}] - \mathrm{i}X_7 - \mathrm{i}I_8 + \mathrm{i}I_9\Big|^2 + 4\Big|[\phi^{(2)},\phi^{(1)\dagger}] + I_{\bar 4}^-\Big|^2\\ &+4\Big|[\phi^{(3)},\phi^{(1)\dagger}] + I_{\bar 5}^-\Big|^2 + 4\Big|[\phi^{(3)},\phi^{(2)\dagger}] + I_{\bar 6}^-\Big|^2 + 2\sum_\alpha\Big|[X_7,\phi^{(\alpha)}] + \tfrac{4}{3}\mathrm{i}\phi^{(\alpha)}\Big|^2\\ &+4\Big|[\phi^{(1)},\phi^{(2)}]\Big|^2 + 4\Big|[\phi^{(1)},\phi^{(3)}]\Big|^2 + 4\Big|[\phi^{(2)},\phi^{(3)}]\Big|^2\Big\},\end{aligned} \tag{4.20}$$

where $\mathcal{F}_{mn} = F_{mn} = (\mathrm{d}A + A\wedge A)_{mn}$ is the curvature of the gauge connection A over M^d. We have introduced the notation $|X|^2 \coloneqq XX^\dagger$ and defined the usual covariant derivatives

$$DX_\mu \coloneqq \mathrm{d}X_\mu + [A, X_\mu]. \tag{4.21}$$

Due to equivariance the matrices X_μ must not depend on coordinates of S^7, and therefore the integral over the seven-sphere in the action functional simply yielded its volume. That is, Yang-Mills theory on $M^d\times G/H$ has been reduced to a Yang-Mills-Higgs theory on M^d, where the contributions to the potential stem form the Sasaki-Einstein geometry of the coset space.

Instanton equation on S^7. The instanton equation (2.10) on the round seven-sphere takes the form $\star_7\mathcal{F} = -\omega\wedge\eta\wedge\mathcal{F}$, which yields the conditions

$$\begin{aligned} &\mathcal{F}_{13}=\mathcal{F}_{24}, \quad \mathcal{F}_{14}=-\mathcal{F}_{23}, \quad \mathcal{F}_{15}=\mathcal{F}_{26}, \quad \mathcal{F}_{12}+\mathcal{F}_{34}+\mathcal{F}_{56}=0, \\ &\mathcal{F}_{16}=-\mathcal{F}_{25}, \quad \mathcal{F}_{35}=\mathcal{F}_{46}, \quad \mathcal{F}_{36}=-\mathcal{F}_{45}, \quad \mathcal{F}_{\mu 7}=0 \quad \text{for} \quad \mu=1,\dots 6. \end{aligned} \tag{4.22}$$

Of course, these conditions are the Hermitian Yang-Mills equations on the underlying Kähler manifold $\mathbb{C}P^3$ with fundamental form ω, together with the additional condition

$$0=\mathcal{F}_{\mu 7}=[X_\mu, X_7]-f^{\nu}_{\mu 7}X_\nu. \tag{4.23}$$

Imposing these instanton equations leads to the vanishing of the last four potential terms in the action functional (4.20). The vanishing of the torsion term in the generalized Yang-Mills equation (2.11) is explicitly verified in Appendix B.1.1.

4.2.3 Hermitian Yang-Mills instantons on the cone

We are particularly interested in instantons on the metric cone over S^7, which can be obtained by evaluating the Hermitian Yang-Mills equations

$$\mathcal{F}_{\alpha\beta}=0 \quad \alpha,\beta=0,\dots,3 \qquad \text{and} \qquad \mathcal{F}_{0\bar{0}}+\mathcal{F}_{1\bar{1}}+\mathcal{F}_{2\bar{2}}+\mathcal{F}_{3\bar{3}}=0. \tag{4.24}$$

Recall from Section 2.2.1 that the HYM equations are equivalent to the generalised self-duality equation (2.10) with $Q_Z=\mathrm{d}\tau\wedge P+Q$ on the cylinder or the corresponding form $Q_C=\frac{1}{2}\Omega^{1,1}\wedge\Omega^{1,1}$ on the conformally equivalent cone. The torsion term in (2.11) vanishes in the latter case since Q_C is self-dual and therefore co-closed, as claimed for special holonomy manifolds [19].

However, here we focus on the first-order BPS equation itself rather than on the second-order Yang-Mills equation, so that cone and cylinder can be used interchangeably. With the structure equations (4.10) and the curvature components from (4.20), the HYM equations turn into the algebraic conditions

$$[\phi^{(1)},\phi^{(2)}]=[\phi^{(1)},\phi^{(3)}]=[\phi^{(2)},\phi^{(3)}]=0, \tag{4.25}$$

and the flow equations

$$\dot{\phi}^{(\alpha)}=-\tfrac{4}{3}\phi^{(\alpha)}-\mathrm{i}[\phi^{(\alpha)},X_7], \qquad \text{for} \qquad \alpha=1,2,3, \tag{4.26a}$$

$$\dot{X}_7=-6X_7-2\mathrm{i}[\phi^{(1)},\phi^{(1)\dagger}]-2\mathrm{i}[\phi^{(2)},\phi^{(2)\dagger}]-2\mathrm{i}[\phi^{(3)},\phi^{(3)\dagger}]. \tag{4.26b}$$

In order to emphasize the holomorphic picture, one may keep the endomorphism X_τ associated to the cone direction in the ansatz (4.15). Then one has simply to replace $-\frac{\mathrm{i}}{2}X_7$ by $Y_{\bar{0}}\coloneqq\frac{1}{2}(X_\tau-\mathrm{i}X_7)$ without any further changes, neither in the equivariance conditions nor in the instanton equations.

The algebraic conditions (4.25) impose commutativity as relations on the quiver diagram. The HYM instanton equations (4.25) and (4.26) on the metric cone over the seven-sphere correspond to the general results of [51] for $n = 3$. In particular, one can apply their discussion of the moduli space in terms of coadjoint orbits and Kähler quotients, which we will be the content of Section 4.5.

4.3 Examples of quiver diagrams

This section treats the quiver diagrams and instanton equations for the representations $\underline{\mathbf{4}}$, $\underline{\mathbf{6}}$, $\underline{\mathbf{10}}$, and $\underline{\mathbf{15}}$ of SU(4), whose weight diagrams and generators are collected in Appendix B.2. Placing the focus on explicit examples, we will not use the whole machinery of representation theory of SU(4) and $\mathrm{SL}(4,\mathbb{C})$, details of which are given in [108, Ch. 15] for instance.

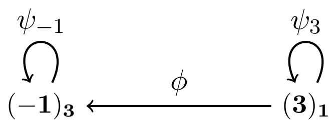
We will provide the quiver diagrams and the resulting instanton matrix equations, while the bundle structure (3.4) itself follows from the "dictionary" presented in Chapter 3. Recall that each example represents entire *families* of quiver gauge theories since the dimensions of the internal spaces E_j attached to the vertices are not further specified.

4.3.1 Fundamental representation $\underline{\mathbf{4}}$

The fundamental representation $\underline{\mathbf{4}}$ of SU(4), given by the generators (B.8) and the weight diagram (B.13), decomposes under restriction to the subgroup SU(3) into one trivial and one fundamental representation thereof,

$$\underline{\mathbf{4}}|_{\mathrm{SU}(3)} = \underline{(\mathbf{3},\mathbf{0},\mathbf{0})_1} \oplus \underline{(-\mathbf{1},-\mathbf{1},\mathbf{1})_3}. \tag{4.27}$$

The subscripts denote the dimension and the three numbers label the quantum numbers of the state which represents the corresponding subalgebra ρ_j. This decomposition provides the representation spaces V_j in (3.3) and (3.4). Both the collapsing procedure and the evaluation of the equivariance conditions (4.16) yield the quiver diagram



(4.28)

and the Higgs fields read

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & 0 \\ \phi \otimes I^{(\alpha)} & \mathbf{0}_3 \end{pmatrix} \quad \text{and} \quad X^7 = \begin{pmatrix} \psi_3 & 0 \\ 0 & \psi_{-1} \otimes \mathbf{1}_3 \end{pmatrix} \tag{4.29}$$

with

$$I^{(1)} \coloneqq \big(1,0,0\big)^{\mathrm{T}}, \ I^{(2)} \coloneqq \big(0,1,0\big)^{\mathrm{T}}, \ I^{(3)} \coloneqq \big(0,0,1\big)^{\mathrm{T}}. \tag{4.30}$$

The constant matrices $I^{(\alpha)}$ constitute the part acting on the representation spaces V_j in the isotopical decomposition (3.4). The two vertices of the quiver represent the two vector spaces E_j, and the arrows describe the allowed homomorphisms $\phi \in \mathrm{Hom}(E_3, E_{-1})$, $\psi_3 \in \mathrm{End}(E_3)$ and $\psi_{-1} \in \mathrm{End}(E_{-1})$. For a better readibility, we will always refrain from depicting also the adjoint maps $\phi^\dagger$ in the quiver diagrams. The equivariant connection (4.15) associated to the quiver (4.28) reads

$$\mathcal{A} = \Gamma + \left(\begin{array}{c|ccc} A_3 + \psi_3 \otimes e^7 & -\phi^\dagger \otimes \Theta^1 & -\phi^\dagger \otimes \Theta^2 & -\phi^\dagger \otimes \Theta^3 \\ \hline \phi \otimes \Theta^{\bar{1}} & & & \\ \phi \otimes \Theta^{\bar{2}} & & \left(A_{-1} + \psi_{-1} \otimes e^7\right) \mathbf{1}_3 & \\ \phi \otimes \Theta^{\bar{3}} & & & \end{array} \right) , \tag{4.31}$$

and one notices that the gauge connection is an extension of the canonical flat connection (4.7) by bundle maps. Taking up the motivation for dimensional reduction in Section 3.1, the fundamental representation can be used for modelling a breaking of the structure group (of the bundle $E \to M^d$) as

$$\mathrm{U}(k_3 + 3k_{-1}) \longrightarrow \mathrm{U}(k_3) \times \mathrm{U}(k_{-1}). \tag{4.32}$$

Imposing the Hermitian Yang-Mills equations on the equivariant gauge connection (4.31) restricts the matrices ϕ, ψ_3 and ψ_{-1}. While the algebraic conditions (4.25) are automatically satisfied, the flow equations (4.26) read

$$\dot{\phi} = -\tfrac{4}{3}\phi - \mathrm{i}\phi\psi_3 + \mathrm{i}\psi_{-1}\phi, \quad \dot{\psi}_3 = -6\psi_3 + 6\mathrm{i}\phi^\dagger\phi, \quad \dot{\psi}_{-1} = -6\psi_{-1} - 2\mathrm{i}\phi\phi^\dagger. \tag{4.33}$$

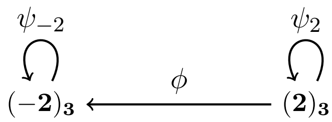
The quiver diagram (4.28) and the resulting instanton matrix equations (4.33) for the defining representation $\underline{\mathbf{4}}$ are the higher-dimensional analogue of the result for the fundamental representation of SU(3) on the five-sphere SU(3)/SU(2) in [46], which, in turn, consists of the results for $\mathbb{C}P^2$ modified by vertex loops. The flow equations for the entries of the vertical Higgs field, represented by the loops, show a coupling to terms of the form $\phi\phi^\dagger$ and $\phi^\dagger\phi$, which is a typical feature of *Sasakian* quiver gauge theory, induced by the stability-like condition (2.23b) of the HYM equations.

4.3.2 Representation $\underline{6}$

According to the weight diagram (B.14), the 6-dimensional representation (B.15) decomposes into one fundamental and one anti-fundamental representation of SU(3),

$$\underline{\mathbf{6}}|_{\mathrm{SU}(3)} = \underline{(\mathbf{2}, -\mathbf{1}, \mathbf{1})_3} \oplus \underline{(-\mathbf{2}, -\mathbf{2}, \mathbf{0})_3} \ . \tag{4.34}$$

Imposing the equivariance conditions or applying the collapsing procedure yield the 2-quiver



(4.35)

where the endomorphisms are given by

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & 0 \\ \phi \otimes I^{(\alpha)} & 0 \end{pmatrix} \quad \text{and} \quad X^7 = \begin{pmatrix} \psi_2 \otimes \mathbf{1}_3 & 0 \\ 0 & \psi_{-2} \otimes \mathbf{1}_3 \end{pmatrix}, \tag{4.36}$$

with the matrices $I^{(\alpha)}$ from (B.15) and the maps $\phi \in \mathrm{Hom}(E_2, E_{-2})$, $\psi_2 \in \mathrm{End}(E_2)$ and $\psi_{-2} \in \mathrm{End}(E_{-2})$. The quiver diagram has the same shape as that of the fundamental representation (4.28), but now the dimensions of the representation spaces V_j of the two vertices are equal. Therefore, the structure group of the equivariant bundle associated to $\underline{\mathbf{6}}$ is broken as

$$\mathrm{U}(3k_2 + 3k_{-2}) \longrightarrow \mathrm{U}(k_2) \times \mathrm{U}(k_{-2}). \tag{4.37}$$

As in the previous example, the algebraic conditions (4.25) are automatically satisfied, while the flow equations of the HYM instantons yield

$$\dot{\phi} = -\tfrac{4}{3}\phi - \mathrm{i}\phi\psi_2 + \mathrm{i}\psi_{-2}\phi, \quad \dot{\psi}_2 = -6\psi_2 + 4\mathrm{i}\phi^\dagger\phi, \quad \dot{\psi}_{-2} = -6\psi_{-2} - 4\mathrm{i}\phi\phi^\dagger. \tag{4.38}$$

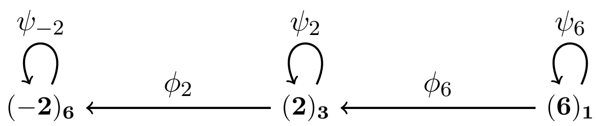
The 6-dimensional representation is based on an exceptional isomorphism of the relevant Dynkin diagrams, so that there is no analogue in the case of S^5. Since the quiver has the same shape as (4.28), the flow equation for ϕ is the same, while for the vertical Higgs field X_7 different numerical factors occur due to the different dimensions of the subrepresentations ρ_j in (4.27) and (4.34).

4.3.3 Representation $\underline{10}$

The representation $\underline{\mathbf{10}}$ with weight diagram (B.16) and generators (B.17) decomposes under restriction to the subgroup SU(3) as

$$\underline{\mathbf{10}}|_{\mathrm{SU}(3)} = \underline{(\mathbf{-2},\mathbf{-2},\mathbf{2})_6} \oplus \underline{(\mathbf{2},\mathbf{-1},\mathbf{1})_3} \oplus \underline{(\mathbf{6},\mathbf{0},\mathbf{0})_1} \tag{4.39}$$

and yields a modified holomorphic chain of length 3 as quiver diagram:



(4.40)

The explicit form of the Higgs fields is the following:

$$X_7 = \begin{pmatrix} \psi_6 & 0 & 0 \\ 0 & \psi_2 \otimes \mathbf{1}_3 & 0 \\ 0 & 0 & \psi_{-2} \otimes \mathbf{1}_6 \end{pmatrix}, \quad \phi^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 \\ \phi_6 \otimes I_1^{(\alpha)} & 0 & 0 \\ 0 & \phi_2 \otimes I_2^{(\alpha)} & 0 \end{pmatrix} \tag{4.41}$$

with the $I_1^{(\alpha)}$ from (4.29), and matrices $I_2^{(\alpha)}$ given by

$$I_2^{(1)} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{4.42}$$

The structure group of the SU(3)-equivariant bundle $E \to M^d$ now reduces to a product of three unitary groups, according to (3.5):

$$\mathrm{U}(6k_{-2} + 3k_2 + k_6) \to \mathrm{U}(k_{-2}) \times \mathrm{U}(k_2) \times \mathrm{U}(k_6). \tag{4.43}$$

The algebraic instanton conditions, $[\phi^{(\alpha)}, \phi^{(\beta)}] = 0$, are automatically satisfied, while the differential equations read

$$\begin{aligned} \dot{\phi}_6 &= -\tfrac{4}{3}\phi_6 - \mathrm{i}\phi_6\psi_6 + \mathrm{i}\psi_2\phi_6, & \dot{\phi}_2 &= -\tfrac{4}{3}\phi_2 - \mathrm{i}\phi_2\psi_2 + \mathrm{i}\psi_{-2}\phi_2, \\ \dot{\psi}_6 &= -6\psi_6 + 6\mathrm{i}\phi_6^\dagger\phi_6, & \dot{\psi}_2 &= -6\psi_2 - 2\mathrm{i}\phi_6\phi_6^\dagger + 8\mathrm{i}\phi_2^\dagger\phi_2, \\ \dot{\psi}_{-2} &= -6\psi_{-2} - 4\mathrm{i}\phi_2\phi_2^\dagger. \end{aligned} \tag{4.44}$$

The quiver diagram (4.40) with these instanton equations is the higher-dimensional analogue of the $C^{2,0}$-quiver in [46].

Remark. It is straighforward to generalize the discussion of the representations $\underline{\mathbf{4}}$ and $\underline{\mathbf{10}}$ to quivers associated to the representations $C^{l,0,0}$, whose highest weight state is given by applying l times one of the generators $I^-_{\bar{\alpha}}$, $\alpha = 1, 2, 3$; see the root system (B.12). Therefore, the weight diagram of the representation $C^{l,0,0}$ comprises a tetrahedron of length l which consists of triangles of increasing length $0, 1, \ldots, l$ as subrepresentations of SU(3). Consequently, the dimensions of the representation spaces V_j are given by $d_j = \frac{1}{2}j(j+1)$ and the total dimension of $C^{l,0,0}$ reads $D_l = \sum d_j = \frac{1}{6}(l+1)(l+2)(l+3)$. Under restriction to SU(3) this representation decomposes as

$$\underline{\mathbf{D}_l}|_{\mathrm{SU}(3)} = \bigoplus_{j=1}^{l+1} \underline{(\mathbf{3l + 4(1 - j), -j + 1, j - 1})_{d_j}}, \tag{4.45}$$

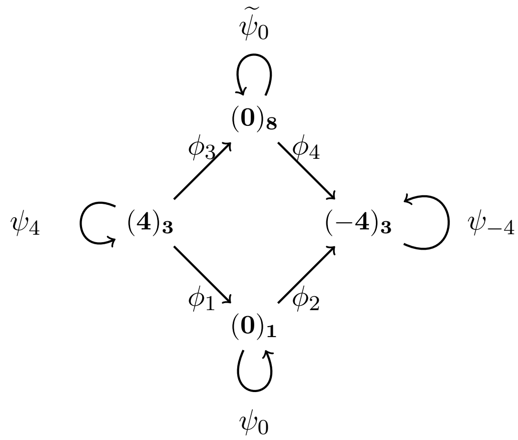
which reproduces the above results for the special cases of $l = 1$ and $l = 2$. The quiver diagrams are therefore modified holomorphic chains with $l + 1$ vertices, i.e. A_{l+1}-quivers with U(1) vertex loops. They yield instanton matrix equations of the typical form, such as (4.44), and only the numerical values of the coefficients are subject to changes.

4.3.4 Representation $\underline{15}$

Due to the large dimension of the adjoint representation $\underline{\mathbf{15}}$ of SU(4) we discuss the resulting instanton matrix equations only schematically, emphasizing the advantages of the diagrammatic approach in terms of quivers to construct the invariant gauge connection $\mathcal{A}$. From the structure constants (B.9) or the root system (B.12) one derives the weight diagram (B.18) and therefore obtains the splitting

$$\underline{\mathbf{15}}|_{\mathrm{SU}(3)} = \underline{(\mathbf{0},\mathbf{0},\mathbf{2})}_8 \oplus \underline{(\mathbf{4},\mathbf{1},\mathbf{1})}_3 \oplus \underline{(-\mathbf{4},-\mathbf{1},\mathbf{1})}_3 \oplus \underline{(\mathbf{0},\mathbf{0},\mathbf{0})}_1 \tag{4.46}$$

into adjoint, fundamental, anti-fundamental and trivial representations of $H = \mathrm{SU}(3)$. Collapsing the weight diagram along the ladder operators of $\mathfrak{h}$ yields the quiver



(4.47)

The Higgs fields are given by

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & \phi_1 \otimes I_1^{(\alpha)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_2 \otimes I_2^{(\alpha)} & 0 & 0 & \phi_4 \otimes I_4^{(\alpha)} \\ 0 & \phi_3 \otimes I_3^{(\alpha)} & 0 & 0 \end{pmatrix} \tag{4.48}$$

as well as $X_7 = \mathrm{diag}(\widetilde{\psi}_0, \psi_4 \otimes \mathbf{1}_3, \psi_{-4} \otimes \mathbf{1}_3, \psi_0 \otimes \mathbf{1}_8)$, where the concrete form of the matrices $I_j^{(\alpha)}$ follows from the generators in the adjoint representation. The quiver diagram (4.47) is the higher-dimensional analogue of the adjoint representation $C^{1,1}$ of SU(3) in [46].

This representation is our first example[4] for which the conditions (4.16a) and (4.16b) alone might yield further contributions since they are compatible with a homomorphism connecting the fundamental and anti-fundamental representations as well. Furthermore, the equivariance conditions do not obstruct the endomorphisms ψ_0 and $\widetilde{\psi}_0$, which would not follow from twisting the generator $\widetilde{I}_7$ with bundle maps because it acts with eigenvalue zero on the vertices $(\mathbf{0})_\mathbf{8}$ and $(\mathbf{0})_\mathbf{1}$. We keep these two

[4]We discuss a further example in Appendix B.2.5.

endomorphisms here (and also in later examples) for the symmetry of the exposition in the matrix equations; they can be set to zero consistently at any time.

Without the need of knowing the matrices $I_j^{(\alpha)}$ precisely, one obtains the flow equations for the horizontal Higgs fields:

$$\begin{aligned}\dot{\phi}_1 &= -\tfrac{4}{3}\phi_1 - \mathrm{i}(\phi_1\psi_4 - \widetilde{\psi}_0\phi_1), &\qquad \dot{\phi}_2 &= -\tfrac{4}{3}\phi_2 - \mathrm{i}(\phi_2\psi_0 - \psi_{-4}\phi_2),\\ \dot{\phi}_3 &= -\tfrac{4}{3}\phi_3 - \mathrm{i}(\phi_3\psi_{-4} - \psi_0\phi_3), &\qquad \dot{\phi}_4 &= -\tfrac{4}{3}\phi_4 - \mathrm{i}(\phi_4\psi_0 - \psi_{-4}\widetilde{\psi}_0).\end{aligned} \tag{4.49}$$

The general structure of the matrix equations for the entries of the vertical field X_7 and the non-trivial algebraic relation, which amounts to commutativity of the square (4.47), can be deduced from the quiver diagram as well. The explicit form of the generators is only necessary to derive the numerical constants therein.

4.4 Reduction to $\mathbb{C}P^3$ and orbifolding

This section deals with quiver gauge theory on two geometries intimately related to S^7, namely on orbifolds $S^7/\mathbb{Z}_{q+1}$ and on the underlying Kähler manifold $\mathbb{C}P^3$. In both cases an additional equivariance condition with respect to the third Cartan generator I_7 has to be imposed.

4.4.1 Reduction to $\mathbb{C}P^3$

Due to the construction of the local section (4.1) as a U(1)-bundle over $\mathbb{C}P^3$, it is natural to consider the reduction of the quiver gauge theory on S^7 to that on $\mathbb{C}P^3$. The complex projective space $\mathbb{C}P^3$ appears as compactification manifold in type IIA supergravity, which can be derived from the 11-dimensional setup [104, 109, 110].

In order to obtain the quiver gauge theory on $M^d \times \mathbb{C}P^3$, one sets $X_7 = I_7$ in the ansatz for the gauge connection (4.15) and has to impose, besides (4.16), the additional equivariance condition

$$[\tilde{I}_7, \phi^{(\alpha)}] = -4\phi^{(\alpha)} \qquad \text{for} \qquad \alpha = 1, 2, 3. \tag{4.50}$$

This further condition uniquely fixes the endomorphisms $\phi^{(\alpha)}$ to have the same action in the weight diagrams as the ladder operators $I_{\bar{\alpha}}^-$, i.e. the quiver diagram coincides with the collapsed weight diagram of the chosen representation, without ambiguity between the two approaches for the construction of the quivers. On the other hand, for Higgs fields acting according to the G-action (B.11), as have been applied throughout the previous discussion, it is automatically satisfied. Of course, the U(1)-loops in the quiver diagrams disappear because X_7 is not a degree of freedom any more.

Examples. Let us consider this limit explicitly for the examples of the previous section. In the fundamental representation $\underline{\mathbf{4}}$ of SU(4), we have to set $\psi_3 = \mathrm{i}\ \mathbf{1}_{k_3}$ and $\psi_{-1} = -\frac{\mathrm{i}}{3}\mathbf{1}_{k_{-1}}$, so that the instanton equations (4.33) reduce to

$$\dot{\phi} = 0, \qquad \mathbf{1}_{k_3} = \phi^\dagger \phi, \qquad \mathbf{1}_{k_{-1}} = \phi\phi^\dagger. \tag{4.51}$$

These equations correspond to the commutative version of the BPS equations [42, Eq. (4.24)] on $\mathbb{C}P^2$ for the fundamental representation, and therefore constitute their higher-dimensional counterpart on $\mathbb{C}P^3$.

For the exceptional representation $\underline{\mathbf{6}}$, taking the limit $\mathbb{C}P^3$ comprises the substitutions $\psi_2 = \frac{2}{3}\mathrm{i}\ \mathbf{1}_{k_2}$ and $\psi_{-2} = -\frac{2}{3}\mathrm{i}\ \mathbf{1}_{k_{-2}}$, and therefore the instanton equations (4.38) yield

$$\dot{\phi} = 0, \qquad \mathbf{1}_{k_2} = \phi^\dagger \phi, \qquad \mathbf{1}_{k_{-2}} = \phi\phi^\dagger. \tag{4.52}$$

Finally, the reduction for the representation $\underline{\mathbf{10}}$ requires $\psi_6 = 2\mathrm{i}\ \mathbf{1}_{k_6}$, $\psi_2 = \frac{2}{3}\mathrm{i}\ \mathbf{1}_{k_2}$, and $\psi_{-2} = -\frac{2}{3}\mathrm{i}\ \mathbf{1}_{k_{-2}}$, so that the instanton equations (4.44) turn into

$$\dot{\phi}_6 = 0 = \dot{\phi}_2, \quad \mathbf{1}_{k_6} = \tfrac{1}{2}\phi_6^\dagger\phi_6, \quad \mathbf{1}_{k_2} = -\tfrac{1}{2}\phi_6\phi_6^\dagger + 2\phi_2^\dagger\phi_2, \quad \mathbf{1}_{k_{-2}} = \phi_2\phi_2^\dagger, \tag{4.53}$$

which resemble the structure of the BPS equations [42, Eq. (4.25)].

As expected, the BPS condition for the examples of quiver diagrams on S^7 constructed here take the familiar form of instanton equations on complex projective spaces, in analogy to the reduction from S^5 to $\mathbb{C}P^2$ [46]. An agreement between this limit of the Sasakian quiver gauge theory and the known results on the underlying Kähler manifold will be obtained for the reduction from $T^{1,1}$ to $\mathbb{C}P^1 \times CP^1$ in the following chapter as well; this verifies consistency of our construction of Sasakian quiver gauge theories.

4.4.2 Orbifold $S^7/\mathbb{Z}_{q+1}$

A typical setup for compactifications in string theory is to consider orbifolds of highly symmetric spaces, which reduces the amount of supersymmetry of the theory in consideration. For implementing the orbifold action of the cyclic subgroup $\mathbb{Z}_{q+1}$ in our setup, we closely follow the exposition in [44, 46].

For having a well defined quotient, the cyclic group $\mathbb{Z}_{q+1}$ has to commute with the subgroup $H = \mathrm{SU}(3)$. Therefore it is embedded into the group U(1) generated by I_7, i.e. $\mathbb{Z}_{q+1}$ acts in the fundamental representation by elements

$$h = \mathrm{diag}(\zeta_{q+1}^3, \zeta_{q+1}^{-1}, \zeta_{q+1}^{-1}, \zeta_{q+1}^{-1}), \qquad \zeta_{q+1} := \exp(\tfrac{2\pi \mathrm{i}}{q+1}). \tag{4.54}$$

For a local section of the orbifold, the U(1)-factor in (4.6) has to be modified to

$$(y_1, y_2, y_3, \tfrac{\varphi}{q+1}) \mapsto \tilde{V} := V \times \exp\left(\mathrm{diag}(3\mathrm{i}\tfrac{\varphi}{q+1}, -\mathrm{i}\tfrac{\varphi}{q+1}, -\mathrm{i}\tfrac{\varphi}{q+1}, -\mathrm{i}\tfrac{\varphi}{q+1})\right). \tag{4.55}$$

Defining the 1-forms as in (4.7) still yields a Sasaki-Einstein space, where in all expressions the substitution $\varphi \mapsto \frac{\varphi}{q+1}$ has to be employed. For equivariant gauge connections on the homogeneous space $M^d \times S^7/\mathbb{Z}_{q+1}$, not only equivariance with respect to $H = \mathrm{SU}(3)$ has to be satisfied but also a condition with respect to the cyclic group $\mathbb{Z}_{q+1}$. It acts onto both the endomorphism part X_μ of a connection and the forms e^μ. More precisely, equivariance requires [44, 46]

$$\gamma(h) X_\mu \gamma(h)^{-1} = \pi(h)^{-1} X_\mu, \tag{4.56}$$

where $\gamma(h)$ is a representation of $\mathbb{Z}_{q+1}$ on the fibers of the gauge bundle and $\pi(h)$ denotes the action on the 1-forms. The representation $\gamma(h)$ follows from the embedding into the subgroup U(1) generated by I_7 in the chosen SU(4)-representation, i.e. it acts as $\gamma(h) = \zeta_{q+1}^{\nu_7} \mathbf{1}_l$ on each l-dimensional SU(3) subrepresentation $\underline{(\nu_{\mathbf{7}})_l}$ with (constant) quantum number ν_7. In order to determine $\pi(h)$, one considers the action of $\mathbb{Z}_{q+1}$ on vectors $w \equiv (w_0, w_1, w_2, w_3)^T \in \mathbb{C}^4$ according to (4.54). From the definition of the local 1-forms based on the $\mathbb{C}P^3$-quotient (4.2) or from the expression (4.7) one deduces the following action:

$$\pi(h)\Theta^\alpha = \zeta_{q+1}^{-4}\Theta^\alpha, \quad \pi(h)\Theta^{\bar\alpha} = \zeta_{q+1}^{4}\Theta^{\bar\alpha}, \quad \pi(h)e^7 = e^7. \tag{4.57}$$

Plugging this action into the additional equivariance condition (4.56) and imposing it for *all* values of q requires the matrices X_μ to act in the weight diagram as the generators of $\mathfrak{m}$ do; thus, it has the same effect as imposing equivariance also with respect to I_7. However, it is worth emphasizing that equivariance with respect to I_7, as imposed for $\mathbb{C}P^3$, is a *stronger* condition. The reason is that, for a fixed value q, equation (4.56) holds only modulo $q+1$ powers of ζ_{q+1}, so that there might be more general contributions for special values of q under this form of equivariance.

Since imposing equivariance on the orbifold amounts to basically the same additional condition on I_7 as the $\mathbb{C}P^3$ limit, it does not constrain the quiver diagrams discussed above further, as the arrows have already been chosen according to (B.11). In contrast to the quiver gauge theory on $M^d \times \mathbb{C}P^3$, there are still the vertex loops caused by the vertical Higgs field X_7 as degrees of freedom, of course.

4.5 Moduli space of HYM instantons

In this section we discuss the moduli space of the Hermitian Yang-Mills equations (4.25) and (4.26) on the Calabi-Yau cone over the seven-sphere by applying approaches from the literature.

4.5.1 Constant endomorphisms

Before we proceed with the general discussion of the moduli space of the Hermitian Yang-Mills equations under the contraints imposed by equivariance, we consider the special case of constant endomorphisms $\phi^{(\alpha)}$. Then the radial coordinate r enters the setup just as a label of foliations consisting of the underlying Sasaki-Einstein manifold along the cone direction. Applying the temporal gauge $X_\tau = 0$, one obtains from the flow equations (4.26a) the conditions

$$[X_7, \phi^{(\alpha)}] = -\tfrac{4}{3}\mathrm{i}\phi^{(\alpha)}, \qquad \text{for} \quad \alpha = 1,2,3, \tag{4.58}$$

which lead to the vanishing of the sum in the second line of (4.20). As we have seen before, the above condition can be satisfied, for instance, by the quiver gauge theory on the complex projective space $\mathbb{C}P^3$. Of course, due to the "bridging property" of Sasakian manifolds between two Kähler manifolds, this result is not unexpected

4.5.2 Moduli space of HYM equations

For the discussion of the flow equations under the given constraints, one can apply the results of [51] for Hermitian Yang-Mills instantons on Calabi-Yau cones over generic Sasaki-Einstein manifolds M^{2n+1}, based on an adaptation of Donaldson's and Kronheimer's studies of Nahm's equations [111, 112]. We review the main aspects of this description, referring to [51] and the references therein for details.

The space of gauge connections $\mathbb{A}$ over a Kähler manifold (M^{2n}, g, Ω) naturally inherits a Kähler structure from the underlying manifold. Explicitly, the metric in a point $\mathcal{A} \in \mathbb{A}$ is defined as [51]

$$\tilde{g}|_{\mathcal{A}}(X_1, X_2) = \int_{M^{2n}} \mathrm{Tr}(X_1 \wedge \star X_2), \tag{4.59}$$

and a Kähler form follows from the base manifold M^{2n} as

$$\tilde{\Omega}|_{A}(X_1, X_2) = \int_{M^{2n}} \mathrm{Tr}(X_1 \wedge X_2) \wedge \Omega^{n-1}. \tag{4.60}$$

The space of connections is therefore an (infinite-dimensional) Kähler manifold, on which an infinite-dimensional gauge group (see below) acts.

Coadjoint orbits. The HYM equations on the Calabi-Yau cone over S^7 yielded the set of equations (4.25) and (4.26). Rescaling the Higgs fields

$$\phi^{(\alpha)} =: \mathrm{e}^{-\frac{4}{3}\tau} W_\alpha \quad \text{for} \quad \alpha = 1,2,3 \quad \text{and} \quad Y_{\bar{0}} =: \mathrm{e}^{-6\tau} Z \tag{4.61}$$

and changing the argument to $s := -\frac{1}{6}\mathrm{e}^{-6\tau}$ leads to the flow equations

$$\frac{\mathrm{d}W_\alpha}{\mathrm{d}s} = 2[W_\alpha, Z], \quad \text{for} \quad \alpha = 1,2,3\ , \tag{4.62a}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}(Z + Z^\dagger) + 2[Z, Z^\dagger] + 2\sum_{\alpha=1}^{3} (-6s)^{-\frac{14}{9}} [W_\alpha, W_\alpha^\dagger] =: \mu(s), \tag{4.62b}$$

and the algebraic condition

$$[W_\alpha, W_\beta] = 0, \tag{4.63}$$

in correspondence with the general results of [51] for $n = 3$. The first equations, (4.62a), are referred to as *complex equations*, and (4.62b) is called the *real equation*, in analogy to Nahms's equations (cf. Appendix A.2). The further discussion of the moduli space of these Nahm-type equations is based on the invariance of the complex equations under the gauge transformation

$$W_\alpha \mapsto W_\alpha^g \coloneqq g(s) W_\alpha g(s)^{-1}, \quad Z \mapsto Z^g \coloneqq g(s) Z g(s)^{-1} - \frac{1}{2}\left(\frac{\mathrm{d}g(s)}{\mathrm{d}s}\right) g(s)^{-1}, \tag{4.64}$$

for $g \in \mathcal{G}^{\mathbb{C}} \subset C^\infty\left((-\infty, 0], \mathrm{GL}(\mathbb{C}, k)\right)$. For compatibility with the equivariance conditions, the gauge transformations have to be restricted such that they preserve the structure of the quiver diagrams, which corresponds to the breaking of the structure group (3.5). Moreover, note that the real equation is only invariant under the real gauge group $\mathcal{G} \subset C^\infty\left((-\infty, 0], \mathrm{U}(k)\right)$, i.e. $g^{-1} = g^\dagger$.

By virtue of the gauge invariance (4.64), Kronheimer's and Donaldson's techniques can be applied here. Consider a local gauge in which Z^g vanishes, i.e. $Z = \frac{1}{2} g^{-1} g'$ is pure gauge. Then the complex equations imply that the matrices W_α are constant, i.e. one has the local solution [51]

$$Z = \tfrac{1}{2} g^{-1} g' \quad \text{and} \quad W_\alpha = \mathrm{Ad}(g^{-1}) T_\alpha \quad \text{with} \quad [T_\alpha, T_\beta] = 0, \tag{4.65}$$

for constant matrices T_α, which can be chosen from a Cartan subalgebra. Hence, over some interval $\mathcal{I} \subset (-\infty, 0]$, solutions to the complex equations are described as orbits of a tuple $(T_1, T_2, T_3, 0)$ under the adjoint action of the complex gauge group $\mathcal{G}^{\mathbb{C}}$. The real equation can be interpreted as the equation of motion, $\delta\mathcal{L} = 0$, of a suitably constructed Lagrangian [51, 111],

$$\mathcal{L}[g] = \int_{\mathcal{I}} \mathrm{d}s \ \mathrm{Tr} \left\{ |Z^g + Z^{g\dagger}|^2 + 2\,(-6s)^{-\frac{14}{9}} \sum_{\alpha=1}^{3} |W_\alpha^g|^2 \right\}. \tag{4.66}$$

Using $\delta W_\alpha = [\delta g, W_\alpha]$ and $\delta Z = [\delta g, Z] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(\delta g)$ for $g = 1 + \delta g$ (with variation $(\delta g)^\dagger = \delta g$), one verifies that the real equation (4.62b) indeed follows from the variation of (4.66). Rewriting the Lagrangian in terms of the local solution (4.65) to the complex equations casts the real equation into a variational problem (with a non-negative potential) on gauge transformations $h \coloneqq g^\dagger g$, and the existence of a solution is guaranteed by standard arguments concerning variational problems.

For uniqueness of the solution and in order to apply this approach on the entire range of $s \in \mathbb{R}_{\leq 0}$, one restricts to *framed*[5] instantons and has to impose certain boundary conditions for $s \to -\infty$, i.e. for the conical singularity at $r = 0$, [51]

$$\exists\ g_0 \in \mathcal{G}^{\mathbb{C}} \ \text{s.t.} \lim_{s \to -\infty} W_\alpha(s) = \mathrm{Ad}(g_0) T_\alpha. \tag{4.67}$$

[5] This means that one considers real gauge transformations $h \coloneqq g^\dagger g$ with $h = 1$ at the boundaries [51].

With these boundary conditions and assuming that the elements T_α are regular, it is shown [51] that the boundary conditions for $s \to 0$ are determined by (4.67), and that the solutions are described as coadjoint orbit of the regular tuple. To sum it up, the gauge transformation (4.65) allowed finding a local solution of the complex equations, which also solves the real equation and extends to the entire range of s if suitable boundary conditions are imposed; for more details of this approach, see [46, 51, 111, 112].

Kähler quotient. The moduli space also admits a description as a Kähler quotient [51]. The space of solutions to the complex equations and the equivariance conditions, denoted as

$$\mathbb{A}^{1,1} := \{\text{equivariant } \mathcal{A} \mid \mathcal{F}^{(2,0)} = 0 = \mathcal{F}^{(0,2)}\}, \tag{4.68}$$

inherits a Kähler structure from $\mathbb{A}$, which is preserved by the action of the (real) gauge group $\mathcal{G}$. The real equation (4.62b) can be interpreted as a moment map $\mu: \ \mathbb{A}^{1,1} \longrightarrow \mathrm{Lie}(\mathcal{G})$, and the moduli space can be written as the Kähler quotient

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G}. \tag{4.69}$$

It turns out that this quotient can be related to the set of stable points as well [51].

4.6 Translationally invariant instantons on $\mathbb{R}^8/\mathbb{Z}_k$

The concept of Sasakian quiver gauge theories has been introduced [44] on orbifolds $\mathrm{SU}(2)/\Gamma$, with a discrete subgroup $\Gamma \subset \mathrm{SU}(2)$, so that the metric cones were given by $\mathbb{C}^2/\Gamma$. The known brane configurations at orbifold singularities of type ADE and constructions of ALE gravitational instantons in these settings [64, 100, 101] motivated considering translationally invariant gauge connections and instantons, whose structure is determined by the discrete subgroup only. Similarly, translationally invariant instantons have been studied on $\mathbb{C}^3/\mathbb{Z}_k$ as the metric cone over the lens spaces $S^5/\mathbb{Z}_k$ [45, 46]. Here we briefly sketch the analogous situation of translationally invariant instantons on $\mathbb{C}^4/\mathbb{Z}_k$, closely following [44–46, 113], for SU(4)-representations $\Gamma^{l,0,0}$.

4.6.1 Equivariant connections

Let us consider a vector bundle $E \to \mathbb{C}^4/\mathbb{Z}_k$ of rank r and use the coordinates (z_1, z_2, z_3, z_4) on $\mathbb{C}^4$, which is equipped with the standard metric and complex structure $J\, \mathrm{d}z_\alpha = \mathrm{i}\, \mathrm{d}z_\alpha$. The differentials of the coordinates provide a translationally invariant basis of 1-forms, and any connection can be written as

$$\mathcal{A} = \sum_{\alpha=1}^{4} (Y_\alpha \otimes \mathrm{d}z^\alpha + Y_{\bar\alpha} \otimes \mathrm{d}z^{\bar\alpha}) \tag{4.70}$$

with the Higgs fields $Y_{\bar\alpha} = -Y_\alpha^\dagger$ describing the endomorphism part of the connection (acting on the fibers $\mathbb{C}^r$ of the vector bundle E), as before. Translational invariance of this connection, i.e. $\mathrm{d}\mathcal{A} = 0$, then implies

$$\mathrm{d}Y_\alpha = 0 \ = \ \mathrm{d}Y_{\bar\alpha} \ . \tag{4.71}$$

The form of the endomorphisms is determined by equivariance with respect to the discrete subgroup $\Gamma = \mathbb{Z}_k$ only. As before, the condition of equivariance reads [44, 46]

$$\gamma(h)\, Y_{\bar\alpha}\, \gamma(h)^{-1} = \pi(h)^{-1} Y_{\bar\alpha} \ , \tag{4.72}$$

but now the action both on the fibers and on the form part is different compared to (4.56). When introducing the orbifold in Section 4.4, the action π on 1-forms followed from the construction of the local section: it was induced by the fundamental action of SU(4) on $\mathbb{C}^4$ in (4.55) and the ensuing quotient leading to the local patch in $\mathbb{C}P^3$. Now, however, we directly start from the space $\mathbb{C}^4$ and one can define for $h \in \mathbb{Z}_k$, according to the action of I_7 in the fundamental representation,

$$\pi(h) : \mathrm{d}z_a \ \longmapsto \ \zeta_k^{-1}\, \mathrm{d}z_a \ , \ a \ = \ 1,2,3 \ , \qquad \mathrm{d}z_4 \ \longmapsto \ \zeta_k^3\, \mathrm{d}z_4 \tag{4.73}$$

with ζ_k a primitive k-th root of unity. In [46, Section 6.1] there is a detailed discussion of the choices for the $\mathbb{Z}_k$-action $\gamma(h)$ on the fibers, explaining the differences between H-equivariant connections and the translationally-invariant case. Following this reference, we do not consider the weights associated to the generator I_7, which has been used for our discussion of SU(4)-equivariant instantons on orbifolds of S^7 above, but consider the weights pertaining to the other Cartan generators, as the examples below will clarify.

Hermitian Yang-Mills equations. With the standard metric and complex structure on $\mathbb{C}^4$, the Kähler form is given by $\Omega \ = \ -\frac{\mathrm{i}}{2}\sum_{\alpha=1}^4 \mathrm{d}z^\alpha \wedge \mathrm{d}z^{\bar\alpha}$ and the holomorphicity condition $\mathcal{F}_{\alpha\beta} = 0 = \mathcal{F}_{\bar\alpha\bar\beta}$ of the Hermitian Yang-Mills equation yields commutativity,

$$[Y_\alpha, Y_\beta] \ = \ 0 \ = \ [Y_{\bar\alpha}, Y_{\bar\beta}] \qquad \text{for} \quad \alpha,\beta \ = \ 1,2,3,4. \tag{4.74}$$

due to (4.71). On the space $\mathbb{C}^4/\mathbb{Z}_k$, one can include a Fayet-Iliopoulos (FI) term Ξ in the stability-like condition $\Omega \lrcorner \mathcal{F} = \Xi$,

$$\sum_{\alpha=1}^{4} [Y_\alpha, Y_{\bar\alpha}] \ = \ \Xi \ , \tag{4.75}$$

where Ξ is a constant element in the center of the Lie algebra $\mathfrak{u}(r)$ of the structure group. This parameter determines the properties of the Kähler quotient, which can be applied for the description of the moduli space, and the mimimal resolution of the singularity; see for instance [113] and the references therein.

4.6.2 Examples of quiver diagrams

For choosing an action $\gamma(h)$ on the fibers of E, we keep the decompositions of the SU(4)-representations from the previous discussion, but now assign to the subspaces (each of them carrying a constant quantum number ν_7) an action with respect to the quantum number ν_8 of the weight state that labels the subrepresentation[6]. Note that this choice of $\gamma(h)$ is neither unique nor necessary, but just a *possible* choice for the action on the fibers which allows us to obtain solutions to equation (4.72), as in [46].

Representation $\underline{\mathbf{4}}$. For the fundamental representation of SU(4), the generator of $\mathbb{Z}_k$ is chosen based on the decomposition (4.27) and the quantum numbers with respect to $\widetilde{I}_8$ as

$$\gamma(h) = \begin{pmatrix} \zeta_k^{-1}\,\mathbf{1}_3 \otimes \mathbf{1}_{r_{-1}} & 0 \\ 0 & 1 \otimes \mathbf{1}_{r_3} \end{pmatrix} , \tag{4.76}$$

where r_j denotes the dimension of the vector spaces E_j attached to the two vertices. The equivariance condition (4.72) then yields the 3-Kronecker quiver

$$(-\mathbf{1},-\mathbf{1},\mathbf{1})_3 \;\overset{\Phi_\alpha}{\Lleftarrow\!\!\leftarrow}\; (\mathbf{3},\mathbf{0},\mathbf{0}) \tag{4.77}$$

and the Higgs fields take the form

$$Y_{\bar{\alpha}} = \begin{pmatrix} 0 & \Phi_\alpha \\ 0 & 0_3 \end{pmatrix} \qquad \text{for} \quad \alpha \;=\; 1,2,3 \qquad \text{and} \qquad Y_4 = 0\ . \tag{4.78}$$

As expected, this is the higher-dimensional analogue of the quiver for the fundamental representation of SU(3) in [46]. The holomorphicity condition $[Y_\alpha, Y_\beta] = 0$ is trivially satisfied, while the stability condition (4.75) yields

$$\begin{aligned} \Phi_1\,\Phi_1^\dagger + \Phi_2\,\Phi_2^\dagger + \Phi_3\,\Phi_3^\dagger &= -\xi_{-1}\,\mathbf{1}_3 \otimes \mathbf{1}_{r_{-1}}\ , \\ \Phi_1^\dagger\,\Phi_1 + \Phi_2^\dagger\,\Phi_2 + \Phi_3^\dagger\,\Phi_3 &= \xi_3\,1 \otimes \mathbf{1}_{r_3}\ , \end{aligned} \tag{4.79}$$

where ξ_i are the components of Ξ, decomposed according to (4.76).

Representation $\underline{\mathbf{10}}$. For the 10-dimensional representation we use the embedding

$$\gamma(h_k) = \begin{pmatrix} \zeta_k^{-2}\,\mathbf{1}_6 \otimes \mathbf{1}_{r_{-2}} & 0 & 0 \\ 0 & \zeta_k^{-1}\,\mathbf{1}_3 \otimes \mathbf{1}_{r_2} & 0 \\ 0 & 0 & 1 \otimes \mathbf{1}_{r_6} \end{pmatrix} , \tag{4.80}$$

to obtain the quiver

$$(-\mathbf{2},-\mathbf{2},\mathbf{2})_{\mathbf{6}} \;\overset{\Phi_\alpha^1}{\Lleftarrow\!\!\leftarrow}\; (\mathbf{2},-\mathbf{1},\mathbf{1})_{\mathbf{3}} \;\overset{\Phi_\alpha^2}{\Lleftarrow\!\!\leftarrow}\; (\mathbf{6},\mathbf{0},\mathbf{0}) \tag{4.81}$$

[6] Since SU(4) has rank three, another obvious choice is to use the weights associated to the Cartan generator I_9. For the examples considered here, this yields the same quivers due to (4.45).

and the Higgs fields

$$Y_{\bar{\alpha}} \;=\; \begin{pmatrix} 0_6 & \Phi^1_\alpha & 0 \\ 0 & 0_3 & \Phi^2_\alpha \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for} \quad \alpha \;=\; 1,2,3 \qquad \text{and} \qquad Y_4 \;=\; 0 \; . \tag{4.82}$$

The Hermitian Yang-Mills equations impose the quiver relations

$$\Phi^1_\alpha \, \Phi^2_\beta - \Phi^1_\beta \, \Phi^2_\alpha \;=\; 0 \tag{4.83}$$

as well as the stability conditions

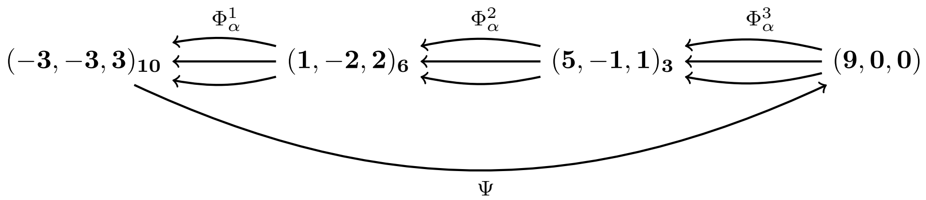
$$\sum_{\alpha=1}^{3} \Phi^1_\alpha \, \Phi^{1\,\dagger}_\alpha = -\xi_{-2} \, \mathbf{1}_6 \otimes \mathbf{1}_{r_{-2}} \; , \qquad \sum_{\alpha=1}^{3} \big(\Phi^2_\alpha \, \Phi^{2\,\dagger}_\alpha - \Phi^{1\,\dagger}_\alpha \, \Phi^1_\alpha\big) = -\xi_2 \, \mathbf{1}_3 \otimes \mathbf{1}_{r_2} \; ,$$
$$\sum_{\alpha=1}^{3} \Phi^{2\,\dagger}_\alpha \, \Phi^2_\alpha = \xi_6 \, 1 \otimes \mathbf{1}_{r_6} \; . \tag{4.84}$$

Here the arrows related to $Y_{\bar{\alpha}}$ for $\alpha = 1,2,3$ occur whenever the difference in the powers of ζ_k in (4.72) is equal to one, so that the underlying structure of the quivers shares the main features with the diagrams in [46]. On the five-sphere, the morphisms induced by the distinguished Higgs field only appear for differences of 2, while our higher-dimensional version requires differences of 3 in the powers of the root of unity. Thus, there is no arrow Y_4 in the quiver diagram for the 10-dimensional representation of SU(4). By virtue of (4.45), the results are generalized to representations $C^{l,0,0}$ straightforwardly: these representations yield A_{l+1}-quivers with three independent Higgs fields between adjacent vertices, and an arrow Y_4 connects any two vertices at a distance of 3.

Representation $\underline{\mathbf{20}}$. The above result is illustrated by the 20-dimensional representation $C^{3,0,0}$. Its decomposition

$$\underline{\mathbf{20}}\,\big|_{\mathrm{SU}(3)} \;=\; \underline{(\mathbf{-3},\mathbf{-3},\mathbf{3})_{\mathbf{10}}} \;\oplus\; \underline{(\mathbf{1},\mathbf{-2},\mathbf{2})_{\mathbf{6}}} \;\oplus\; \underline{(\mathbf{5},\mathbf{-1},\mathbf{1})_{\mathbf{3}}} \;\oplus\; \underline{(\mathbf{9},\mathbf{0},\mathbf{0})_{\mathbf{1}}} \tag{4.85}$$

and the corresponding choice of $\gamma(h)$ lead to the quiver diagram



(4.86)

Note that the allowed transitions Y_α for $\alpha = 1,2,3$ in the above examples of quiver diagrams are those which follow from the actions considered on the orbifold $S^7/\mathbb{Z}_{q+1}$ in Section 4.4.2 if only equivariance with respect to the discrete subgroup is imposed. This confirms the choice of the group action on the fibers, based on the discussion of [46], in this section.

4.7 Higher-dimensional spheres and discussion

The examples of quiver diagrams on S^7 and the resulting instanton equations studied in this chapter turned out to be the higher-dimensional analogues of those obtained for (orbifolds of) the five-sphere. Of course, dealing with a quiver gauge theory of rank 3, new types of quiver diagrams may be found for more complicated representations of SU(4), for which exactly the same construction procedure, based on the root system (B.12), applies. An exhaustive survey is left for future work.

The analogy of the results on the five- and on the seven-sphere for the representations $C^{l,0,\cdots}$ as well as the adjoint representation motivate to spend some words on higher-dimensional spheres. Due to the completely regular construction of odd-dimensional spheres as coset spaces $S^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n)$ and analogous fibrations over $\mathbb{C}P^n$ (cf. [114]), one expects the typical quiver diagrams for representations $C^{l,0,\cdots}$ and the adjoint representation to look the same for all dimensions n. It is worth pointing out that also the results for Calabi-Yau cones over generic Sasaki-Einstein manifolds [19, 34, 51] rely on such a universality of the construction.

4.7.1 Generalization to odd-dimensional spheres S^{2n+1}

By virtue of the standard embedding of $H = \mathrm{SU}(n)$ into $G = \mathrm{SU}(n+1)$,

$$\mathrm{SU}(n) \longmapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{SU}(n) \end{array}\right) \subset \mathrm{SU}(n+1), \tag{4.87}$$

and the analogues of the local section (4.1), one obtains from the canonical flat connection and the Lie algebra splitting $\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m}$ immediately

$$\mathfrak{m} = \left(\begin{array}{c|ccc} \mathrm{i}ne^{2n+1} & \Theta^1 & \dots & \Theta^n \\ \hline -\Theta^{\bar{1}} & -\mathrm{i}e^{2n+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Theta^{\bar{n}} & 0 & \dots & -\mathrm{i}e^{2n+1} \end{array}\right), \tag{4.88}$$

which yields structure equations analogous to (B.1) for the geometry of the coset space $\mathfrak{m} \cong T_e(G/H) = T_e S^{2n+1}$.

Fundamental representation. The structure of the coset space (4.88) implies the splitting $\underline{\mathbf{n+1}}|_{\mathrm{SU}(n)} = \underline{(\mathbf{n}, \dots)}_1 \oplus \underline{(-\mathbf{1}, \dots)}_n$ of the fundamental representation $\underline{\mathbf{n+1}}$ of SU(n+1) under restriction to SU(n), generalizing the decomposition (4.27). Therefore, the fundamental representation always yields the A_2-quiver with vertex loops (4.28), where the Higgs fields are given by

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & 0 \\ \phi \otimes I^{(\alpha)} & 0 \end{pmatrix} \quad \text{with} \quad I^{(\alpha)} = (\delta_{i\alpha})^T_{1\leq i\leq n} \tag{4.89}$$

as well as $X_7 = \mathrm{diag}(\psi_n, \psi_{-1} \otimes \mathbf{1}_n)$. This may serve as a model for symmetry breaking in equivariant dimensional reduction according to

$$\mathrm{U}(k_n + nk_{-1}) \to \mathrm{U}(k_n) \times \mathrm{U}(k_{-1}). \tag{4.90}$$

The discussion [34, 51] of the Hermitian Yang-Mills equations on Calabi-Yau cones over generic Sasaki-Einstein manifolds M^{2n+1} provides the equations

$$\dot{\phi}^{(\alpha)} = -\tfrac{n+1}{n}\phi^{(\alpha)} - \mathrm{i}[\phi^{(\alpha)}, X_7], \quad \dot{X}_7 = -2nX_7 - 2\mathrm{i}\sum_{\alpha=1}^{n}[\phi^{(\alpha)}, \phi^{(\alpha)\dagger}] \tag{4.91}$$

with the algebraic conditions $[\phi^{(\alpha)}, \phi^{(\beta)}] = 0$. Hence, the fundamental representation $\underline{\mathbf{n+1}}$ of $\mathrm{SU}(n+1)$ yields the instanton matrix equations

$$\begin{aligned} &\dot{\phi} = -\tfrac{n+1}{n}\phi - \mathrm{i}\phi\psi_n + \mathrm{i}\psi_{-1}\phi, \\ &\dot{\psi}_n = -2n\psi_n + 2n\mathrm{i}\phi^\dagger\phi, \qquad \dot{\psi}_{-1} = -2n\psi_{-1} - 2\mathrm{i}\phi\phi^\dagger, \end{aligned} \tag{4.92}$$

which generalizes the result (4.33). We will comment on the general structure of the equations below, but we can consider the special case of constant solutions here. Stationary points must satisfy

$$\psi_n = \mathrm{i}\phi^\dagger\phi, \quad \psi_{-1} = -\tfrac{\mathrm{i}}{n}\phi\phi^\dagger, \quad \phi(-\mathbf{1}_{k_n} + \phi^\dagger\phi) = (-\mathbf{1}_{k_{-1}} + \phi\phi^\dagger)\phi\ , \tag{4.93}$$

which holds for the trivial solution $\phi = 0$, corresponding to the canonical connection, as well as for Higgs fields such that $\phi^\dagger\phi = \mathbf{1}_{k_n}$ and $\phi\phi^\dagger = \mathbf{1}_{k_{-1}}$. The latter conditions are exactly those we have encountered for the reduction to the complex projective space $\mathbb{C}P^3$ in (4.51), as already mentioned in Section 4.5. Choosing the attached vector spaces 1-dimensional yields $\psi_n \propto \psi_{-1}$ and reproduces the scalar ansatz studied in [19]. Thus, in order to obtain new instanton solutions one has actually to consider matrix-valued Higgs fields, where the off-diagonal parts might reveal new insights.

Representations $C^{l,0,\cdots}$. Similarly, $\mathrm{SU}(n+1)$ representations consisting of l adjacent $\mathrm{SU}(n)$ representations, denoted as $C^{l,0,\cdots}$ according to their highest weights, always lead to A_{l+1}-quiver diagrams with the typical loop modification due to the $\mathrm{U}(1)$-factor of the vertical Higgs field.

Adjoint representation. One also expects the adjoint representation $\underline{(\mathbf{n+1})^{\mathbf{2}} - \mathbf{1}}$ of $\mathrm{SU}(n+1)$ to yield the square quiver diagram (4.47) for all dimensions n because the adjoint representation splits under restriction to $\mathrm{SU}(n)$ into the adjoint, one fundamental, one anti-fundamental and one trivial representation thereof,

$$\underline{\mathbf{n^2+2n}}|_{\mathrm{SU}(n)} = \underline{\mathbf{1}} \oplus \underline{\mathbf{n}} + \underline{\bar{\mathbf{n}}} \oplus \underline{\mathbf{n^2-1}}. \tag{4.94}$$

This decomposition can be easily obtained from representation theory or recalling that by the embedding (4.87) the subgroup $\mathrm{SU}(n)$ acts on a trivial part, an n-dimensional column vector, an n-dimensional row vector and an $\mathfrak{su}(n)$-matrix in (4.88), which yields the four subrepresentations in the above splitting.

4.7.2 On the matrix equations

We have seen that the instanton equations for the explicit examples of quiver diagrams on the 7-dimensional round sphere correspond to the general results derived in [34, 51] for $n = 3$. Due to these results and the regularity of the construction of the round spheres, one expects similar instanton equations in all cases.

The striking feature of *Sasakian* quiver gauge theories consists of the loop contributions caused by the U(1)-factor of the contact direction, which yields a characteristic coupling in the instanton equations: there always occur terms of the form $\phi\phi^\dagger$ or $\phi^\dagger\phi$ in the flow equation for the vertical Higgs field X_{2n+1}, which renders the system of equations intricate. Finding analytic solutions to the instanton equations therefore seems unlikely, and also the scalar ansatz in [19, Sec. 4.2] provided an analytic result only in three dimensions. Furthermore, it is noteworthy that the description of the moduli spaces solves the real equation, which is responsible for the challenging terms, only implicitly by general existence arguments on variational problems.

Therefore, the instanton matrix equations we have derived for the various representations of SU(4) in this chapter should be studied numerically, as it has been carried out for the scalar ansatz in [19]. While diagonal Higgs fields reproduce copies of the scalar ansatz, the dynamics of the off-diagonal contributions of the matrices X_μ might yield interesting new solutions. As a possible starting point, one may study small perturbations around (non-stable) stationary solutions.

5 Quiver gauge theory on the space $T^{1,1}$

The previous chapter has shown that – on the level of the examples taken into account – the quiver gauge theory on the round seven-sphere S^7 leads to results analogous to those on the sphere S^5 [46]. In five dimensions the two most prominent (and the only compact homogeneous [65]) examples of Sasaki-Einstein manifolds are S^5 as U(1)-bundle over $\mathbb{C}P^2$ and a certain U(1)-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$. The latter is known as $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$ due to Romans [115] in physics and as Stiefel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2) = \mathrm{SO}(3) \times \mathrm{SO}(3)/\mathrm{SO}(2)$ in mathematics.[1]

While string theory on $\mathrm{AdS}_5 \times S^5$ leads to $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory, the compactification $\mathrm{AdS}_5 \times T^{1,1}$ gives rise to an $\mathcal{N} = 1$ super-conformal field theory [116, 117]. The supersymmetry equations for backgrounds $\mathrm{AdS}_5 \times M_5$ in type IIB supergravity can be found in [85], and this setting also allows for exceptional Sasaki-Einstein structures [90]. The metric cone over $T^{1,1}$ is the famous *conifold* whose conical singularity provides the background for certain D-brane configurations [118–120]. The singularity can be removed either by a deformation or by a small resolution, and the transition between both approaches is known as *conifold transition* [121].

The focus of our work here will be placed on constructing $(\mathrm{SU}(2)\times\mathrm{SU}(2))$-equivariant gauge connections on $M^d \times T^{1,1}$ and on studying the conditions imposed by the instanton equation on the conifold. We will compare the results to quiver gauge theory of the underlying Kähler manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$, which has been studied in [43], and also to the case of the five-sphere [46]. In contrast to Chapter 4, the simplicity of the representation theory of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ allows us to discuss the quiver gauge theory associated to a generic representation of G without obtaining lengthy expressions.

The content and parts of the discussion here are based on the collaboration [47] with O. Lechtenfeld, A. D. Popov and R. J. Szabo. However, this chapter also provides some extensions and a comparison with quiver gauge theories on round spheres.

[1] Stiefel manifolds are defined as the set of all k-frames in $\mathbb{R}^n$, which is diffeomorphic to the quotient $\mathrm{SO}(n)/\mathrm{SO}(n-k)$, see [93] for instance. In order to obtain the above expression of $T^{1,1}$, one also employs the exceptional isomorphism $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

5.1 Geometric structure

In this section we will review the geometric structure of the homogeneous space $T^{1,1} = \mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$. A description of the five-dimensional Stiefel manifold $V_{4,2}$ as a naturally reductive space can be found in [122, Section 9.1] and in [123, Sec. 5] for example.

5.1.1 Local section

We start by describing explicit local coordinates on $\mathrm{SU}(2) \simeq S^3$ and $\mathbb{C}P^1 \simeq S^2$, based on the defining representation of the Lie group SU(2) on $\mathbb{C}^2$ and the Maurer-Cartan form. Each element of SU(2) can be locally written as [31]

$$\underbrace{\frac{1}{(1+y_l\,\bar{y}_l)^{1/2}}\begin{pmatrix} 1 & -\bar{y}_l \\ y_l & 1 \end{pmatrix}}_{=g_l\ \in\ \mathbb{C}P^1\ \subset\ \mathrm{SU}(2)}\underbrace{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_l} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi_l} \end{pmatrix}}_{\in\ \mathrm{U}(1)}\ , \tag{5.1}$$

where y_l and $\bar{y}_l$ are stereographic coordinates on S^2, defined as in [43], and the index $l = 1, 2$ refers to the two copies of S^2 which are contained in $T^{1,1}$. The Maurer-Cartan form $A_l = g_l^{-1}\mathrm{d}g_l$ on the homogeneous spaces $\mathbb{C}P^1$ is given by

$$A_l = \frac{1}{1+y_l\,\bar{y}_l}\begin{pmatrix} \frac{1}{2}\,(\bar{y}_l\,\mathrm{d}y_l - y_l\,\mathrm{d}\bar{y}_l) & -\mathrm{d}\bar{y}_l \\ \mathrm{d}y_l & \frac{1}{2}\,(y_l\,\mathrm{d}\bar{y}_l - \bar{y}_l\,\mathrm{d}y_l) \end{pmatrix} =: \begin{pmatrix} a_l & -\bar{\beta}_l \\ \beta_l & -a_l \end{pmatrix}. \tag{5.2}$$

This provides SU(2)-invariant 1-forms on $\mathbb{C}P^1 \times \mathbb{C}P^1$,

$$a_l \;=\; -\bar{a}_l \;=\; \tfrac{1}{2}\left(\bar{y}_l\,\beta_l - y_l\,\bar{\beta}_l\right)\ , \qquad \beta_l \;=\; \frac{\mathrm{d}y_l}{1+y_l\,\bar{y}_l} \tag{5.3}$$

with differentials

$$\mathrm{d}a_l \;=\; -\beta_l \wedge \bar{\beta}_l\ , \qquad \mathrm{d}\beta_l \;=\; 2a_l \wedge \beta_l\ , \qquad \mathrm{d}\bar{\beta}_l \;=\; -2a_l \wedge \bar{\beta}_l\ . \tag{5.4}$$

Since the geometry of $T^{1,1}$ involves the Hopf fibration $S^3 \to S^2$ in (5.1), it has a close relation to quantities associated with magnetic monopoles, as the appearance of the monopole forms (5.3) indicates. To deal with the two copies of SU(2) contained in $T^{1,1}$, we express an arbitrary element of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ locally as

$$\underbrace{\mathrm{diag}\,(g_1, g_2)}_{\in\ \mathbb{C}P^1\times\mathbb{C}P^1} \times \underbrace{\mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1}, \mathrm{e}^{-\mathrm{i}\varphi_1}, \mathrm{e}^{\mathrm{i}\varphi_2}, \mathrm{e}^{-\mathrm{i}\varphi_2}\right)}_{\in\ \mathrm{U}(1)\times\mathrm{U}(1)}. \tag{5.5}$$

To pass to the coset spaces $T^{p,q}$, one has to factor by the U(1) subgroup whose embedding is described by the integers p and q. We specialize to the Sasaki-Einstein case $p = q = 1$, so that the subalgebra is embedded[2] as $\mathfrak{h} = \langle I^3_{(1)} - I^3_{(2)}\rangle$, where $I^3_{(l)}$

[2] The minus sign of the embedding is a convention chosen for aesthetic reasons. Changing to the opposite convention simply inverts the complex structure on one of the two-spheres S^2 contained in $T^{1,1}$.

denote the Cartan generators of the two copies of $\mathfrak{su}(2)$. Therefore we change the $\mathrm{U}(1)$ coordinates to $\varphi := \frac{1}{2}(\varphi_1 + \varphi_2)$ and $\psi := \frac{1}{2}(\varphi_1 - \varphi_2)$, so that the $\mathrm{U}(1) \times \mathrm{U}(1)$ factor in (5.5) reads

$$\begin{aligned} &\mathrm{diag}\big(\mathrm{e}^{\mathrm{i}(\varphi+\psi)}, \mathrm{e}^{-\mathrm{i}(\varphi+\psi)}, \mathrm{e}^{\mathrm{i}(-\varphi+\psi)}, \mathrm{e}^{\mathrm{i}(\varphi-\psi)}\big) \\ &\qquad = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}\right) \times \mathrm{diag}\big(\mathrm{e}^{\mathrm{i}\psi}, \mathrm{e}^{-\mathrm{i}\psi}, \mathrm{e}^{-\mathrm{i}\psi}, \mathrm{e}^{\mathrm{i}\psi}\big) \ . \end{aligned} \tag{5.6}$$

By passing to the coset space $T^{1,1}$, the second factor is divided out and one ends up with elements of the form

$$V = \mathrm{diag}\,(g_1, g_2) \times \ \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}\right) \ . \tag{5.7}$$

Hence, our description of $T^{1,1}$ is founded on using a local section of the bundle $\mathrm{SU}(2) \times \mathrm{SU}(2) \to T^{1,1}$ with coordinates $(y_1, \bar{y}_1, y_2, \bar{y}_2, \varphi)$, and we derive a basis of $(\mathrm{SU}(2) \times \mathrm{SU}(2))$-left-invariant 1-forms on $T^{1,1}$ by considering the canonical flat connection

$$A_0 := V^{-1}\,\mathrm{d}V = \begin{pmatrix} \mathrm{i}\ \mathrm{d}\varphi + a_1 & -\mathrm{e}^{-2\mathrm{i}\varphi}\,\bar{\beta}_1 & 0 & 0 \\ \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_1 & -(\mathrm{i}\ \mathrm{d}\varphi + a_1) & 0 & 0 \\ 0 & 0 & \mathrm{i}\ \mathrm{d}\varphi + a_2 & -\mathrm{e}^{-2\mathrm{i}\varphi}\,\bar{\beta}_2 \\ 0 & 0 & \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_2 & -(\mathrm{i}\ \mathrm{d}\varphi + a_2) \end{pmatrix} . \tag{5.8}$$

After introducing real 1-forms

$$\begin{aligned} a &:= \tfrac{1}{2}(a_1 - a_2) \ , & \mathrm{i}\kappa\, e^5 &:= \mathrm{i}\ \mathrm{d}\varphi + \tfrac{1}{2}(a_1 + a_2) \ , \\ \alpha_1\Theta^1 &:= \alpha_1\left(e^1 - \mathrm{i}e^2\right) := \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_1 \ , & \alpha_2\Theta^2 &:= \alpha_2\left(e^3 - \mathrm{i}e^4\right) := \mathrm{e}^{2\mathrm{i}\varphi}\,\beta_2 \ , \end{aligned} \tag{5.9}$$

where α_1, α_2 and κ are real constants to be determined from the Sasaki-Einstein condition, one obtains the structure equations

$$\begin{aligned} \mathrm{d}e^1 &= 2\kappa e^{52} - 2\mathrm{i}a \wedge e^2, & \mathrm{d}e^2 &= -2\kappa e^{51} + 2\mathrm{i}a \wedge e^1, \\ \mathrm{d}e^3 &= 2\kappa e^{54} + 2\mathrm{i}a \wedge e^4, & \mathrm{d}e^4 &= -2\kappa e^{53} - 2\mathrm{i}a \wedge e^3, \\ \mathrm{d}e^5 &= -\tfrac{1}{\kappa}(\alpha_1^2 e^{12} + \alpha_2^2 e^{34}), \end{aligned} \tag{5.10}$$

from the flatness of the connection (5.8).

Sasaki-Einstein condition. A five-dimensional Sasaki-Einstein manifold can be described [32, 124] as a special $\mathrm{SU}(2)$ structure consisting of an orthonormal basis of 1-forms $e^1, \ldots, e^5$ (with contact form $e^5 \equiv \eta$) and 2-forms

$$\omega^1 = e^{23} + e^{24}, \quad \omega^2 = e^{31} + e^{24}, \quad \omega^3 = e^{12} + e^{34} \tag{5.11}$$

which satisfy the equations

$$\mathrm{d}\eta = 2\omega^3, \quad \mathrm{d}\omega^1 = -3\eta \wedge \omega^2, \quad \mathrm{d}\omega^2 = 3\eta \wedge \omega^1. \tag{5.12}$$

With the structure equations (5.10) one derives

$$\mathrm{d}\omega^1 = 4\kappa\eta \wedge \omega^2, \quad \mathrm{d}\omega^2 = -4\kappa\eta \wedge \omega^1, \quad \mathrm{d}\eta = -\tfrac{1}{\kappa}(\alpha_1^2 e^{12} + \alpha_2^2 e^{34}), \tag{5.13}$$

which fixes the real parameters to the values $\kappa = -\frac{3}{4}$ and $\alpha_1^2 = \alpha_2^2 = \frac{3}{2}$. Equivalently, the Sasaki-Einstein property is verified by showing that the metric cone over $T^{1,1}$ is Calabi-Yau, as carried out in Appendix C.1.

Since we study an extension of the quiver gauge theory on $\mathbb{C}P^1 \times \mathbb{C}P^1$, let us briefly comment on the implication of these fixed parameters. Our geometry consists of two copies of $\mathbb{C}P^1 \cong S^2$, and therefore the round Kähler metric [43]

$$g_{S^2\times S^2} \;=\; 4R_1^2\, \beta_1 \otimes \bar{\beta}_1 + 4R_2^2\, \beta_2 \otimes \bar{\beta}_2, \tag{5.14}$$

parameterized by two radii R_l, appears. On the other hand, the metric on $T^{1,1}$ reads

$$g \;=\; \sum_\mu e^\mu \otimes e^\mu \;=\; \tfrac{2}{3}\,\beta_1 \otimes \bar{\beta}_1 + \tfrac{2}{3}\,\beta_2 \otimes \bar{\beta}_2 + \eta \otimes \eta\ , \tag{5.15}$$

so that imposing the Sasaki-Einstein condition requires $R_1^2 = R_2^2 = \frac{1}{6}$ (see also the metric [120, Eq. (2.11)]). In particular, one cannot rescale the radii as for Kähler structures on the coset space $\mathbb{C}P^1 \times \mathbb{C}P^1$.

5.1.2 Canonical connection

The 3-form P and 4-form Q (2.15) that enter the canonical connection and the instanton equation now read

$$P = e^{125} + e^{345}, \quad Q = e^{1234}, \tag{5.16}$$

and the torsion components (2.16) of the canonical connection are given by

$$T^1 = -\tfrac{3}{2}e^{52}, \quad T^2 = \tfrac{3}{2}e^{51}, \quad T^1 = -\tfrac{3}{2}e^{54}, \quad T^4 = \tfrac{3}{2}e^{53}, \quad T^5 = 2e^{12} + 2e^{34}. \tag{5.17}$$

Plugging this torsion into the structure equations (C.5), recalling $\mathrm{d}e^\mu = -\Gamma^\mu_\nu \wedge e^\nu + T^\mu$, one identifies the canonical connection

$$\Gamma = a \otimes I_6 \qquad \text{with} \qquad I_6 := I_3^{(1)} - I_3^{(2)}. \tag{5.18}$$

Similar to the Sasaki-Einstein manifold S^7 in the previous chapter, the canonical connection (5.18) in the sense of manifolds with Killing spinors coincides with that of $T^{1,1}$ as a homogeneous space G/H. This U(1)-connection yields the curvature

$$\mathcal{F}_\Gamma = \mathrm{d}\Gamma = \tfrac{3}{2}\mathrm{i}\; I_6 \otimes (-e^{12} + e^{34}), \tag{5.19}$$

which satisfies the instanton equation (2.10) for Q given above.

5.2 Equivariance and instanton equation

We use the canonical connection associated to the Sasaki-Einstein structure of $T^{1,1}$ to apply the typical ansatz (3.10) for the gauge connection in the Hermitian vector bundle $\mathcal{E}$ over the base $M^d \times T^{1,1}$:

$$\mathcal{A} = A + \Gamma + \sum_{\mu=1}^{5} X_\mu \otimes e^\mu = A + a \otimes I_6 + \sum_{\alpha=1}^{2} (\phi^{(\alpha)} \otimes \Theta^{\bar{\alpha}} - \text{h.c.}) + X_5 \otimes e^5, \quad (5.20)$$

where the connection A on the bundle $E \to M^d$ is compatible with the isotopical decomposition (3.4) of the fibers. We have introduced the two complex Higgs fields $\phi^{(1)} := \frac{1}{2}(X_1 - \mathrm{i}X_2)$ and $\phi^{(2)} := \frac{1}{2}(X_3 - \mathrm{i}X_4)$ as well as the vertical field X_5.

5.2.1 Equivariance condition

Invariance of the gauge connection (5.20) is ensured by the equivariance condition (3.13). For the case at hand, it requires conditions with respect to the Cartan generator I_6 only:

$$\begin{aligned} &[I_6, X_1] = -2\mathrm{i}X_2, && [I_6, X_2] = 2\mathrm{i}X_1, && \\ &[I_6, X_3] = 2\mathrm{i}X_4, && [I_6, X_4] = -2\mathrm{i}X_3, && [I_6, X_5] = 0. \end{aligned} \quad (5.21)$$

Denoting the quantum numbers with respect to the two Cartan generators by (ν_5, ν_6), one therefore obtains the action of the Higgs fields according to

$$\begin{aligned} &\phi^{(1)} : (\nu_5, \nu_6) \longmapsto (*, \nu_6 + 2)\,, && \phi^{(2)} : (\nu_5, \nu_6) \longmapsto (*, \nu_6 - 2)\,, \\ &X_5 : (\nu_5, \nu_6) \longmapsto (*, \nu_6)\,. && \end{aligned} \quad (5.22)$$

Since only one of the two Cartan generators of G is involved in the equivariance condition (5.21), the action of the Higgs fields on the quantum numbers might, in principle, comprise more general contributions than the ladder operators of G, as discussed in Section 3.3. Due to the ansatz with quiver bundles induced by the G-action on the vertices in the weight diagram, it is again reasonable to restrict our attention to Higgs fields[3] that act according to (C.23).

In particular, one then finds a grading with each $\phi^{(l)}$ acting only on one of the two copies of SU(2). The resulting equivariant bundle is that constructed in [43] with a U(1)-modification due to the vertical Higgs field X_5. To make the structure of the Higgs fields more evident, one may introduce projection operators as in [43] (see Appendix C.2.1) for representations of G on $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$, where Latin indices always refer to the first factor and Greek indices to the second one. Writing $I_{i\alpha,j\beta}$ for

[3] We provide an example of Higgs fields with more general contributions in (C.16) and refer to [47] for details.

the square matrix with entry 1 at the position $(i\alpha, j\beta)$ and zero otherwise, we express the Higgs fields as

$$\phi^{(\alpha)} = \sum_{j,k=0}^{m_1} \sum_{\beta,\gamma=0}^{m_2} I_{j\beta,k\gamma}\ \phi^{(a)}_{j\beta,k\gamma} \quad \text{with} \quad \phi^{(a)}_{j\beta,k\gamma} \in \mathrm{Hom}(E_{k\gamma}, E_{j\beta}). \tag{5.23}$$

Using the representation (C.21), one obtains the commutation relation

$$[\Upsilon^{(1)} \pm \Upsilon^{(2)}, \phi^{(a)}] \;=\; \sum_{j,k=0}^{m_1} \sum_{\beta,\gamma=0}^{m_2} 2\,(k - j \pm (\gamma - \beta))\ I_{j\beta,k\gamma}\ \phi^{(a)}_{j\beta,k\gamma}, \tag{5.24}$$

and the equivariance condition (5.21) therefore restricts the Higgs fields as follows:

$$\begin{aligned} &\phi^{(1)}_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,1}\ \phi^{(1)}_{j\beta,k\gamma}\ , \quad \phi^{(2)}_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,-1}\ \phi^{(2)}_{j\beta,k\gamma}, \\ &\psi_{j\beta,k\gamma} = \delta_{k-j-\gamma+\beta,0}\ \psi_{j\beta,k\gamma}\ , \end{aligned} \tag{5.25}$$

where $\psi_{j\beta,k\gamma}$ are the components of the vertical Higgs field X_5. Indeed, these conditions are more general since only the *relative* charge $c_{i\alpha} := m_1 - m_2 - 2i + 2\alpha$ of the vertices $(i\alpha)$ enters. If also the conditions

$$[I_5, \phi^{(1)}] = 2\phi^{(1)}, \quad [I_5, \phi^{(2)}] = -2\phi^{(2)}, \quad [I_5, X_5] = 0, \tag{5.26}$$

are imposed, the fields are further restricted to the form of the ladder operators,

$$\begin{aligned} &\phi^{(1)}_{j\beta,k\gamma} = \delta_{k-j,1}\delta_{\gamma-\beta,0}\ \phi^{(1)}_{j\beta,k\gamma}\ , \quad \phi^{(2)}_{j\beta,k\gamma} \;=\; \delta_{k-j,0}\delta_{\gamma-\beta,1}\ \phi^{(2)}_{j\beta,k\gamma}, \\ &\psi_{j\beta,k\gamma} = \delta_{k-j,0}\delta_{\gamma-\beta,0}\ \psi_{j\beta,k\gamma}\ . \end{aligned} \tag{5.27}$$

That is, $\phi^{(1)}$ acts only on the first copy of SU(2), $\phi^{(2)}$ only on the second one, and X_5 leads to the typical loop contributions of Sasakian quiver gauge theories. We will see in Section 5.3.3 that this grading yields a discussion of quiver diagrams for generic representations of G similar to [43].

5.2.2 Action functional and instanton equations

Using the orthonormality of the basis $e^1, \dots, e^5$, one obtains as Yang-Mills action functional for the equivariant gauge connection (5.20) the expression

$$\begin{aligned} S_{\mathrm{YM}} = \mathrm{Vol}(T^{1,1}) \int_{M^d} \mathrm{d}^d y\ \sqrt{g}\ \frac{1}{2}\,\mathrm{Tr}\Big(&\tfrac{1}{2}\,F_{mn}\ (F^{mn})^\dagger + \sum_{m=1}^{d} \sum_{\mu=1}^{5} |D_m X_\mu|^2 \\ &+ \Big|[X_1, X_2] + 2X_5 - \tfrac{3}{2}\mathrm{i}I_6\Big|^2 + \Big|[X_3, X_4] + 2X_5 + \tfrac{3}{2}\mathrm{i}I_6\Big|^2 + |[X_1, X_3]|^2 \\ &+ |[X_1, X_4]|^2 + |[X_2, X_3]|^2 + |[X_2, X_4]|^2 + |[X_1, X_5] - \tfrac{3}{2}X_2|^2 \\ &+ |[X_2, X_5] + \tfrac{3}{2}X_1|^2 + |[X_3, X_5] - \tfrac{3}{2}X_4|^2 + |[X_4, X_5] + \tfrac{3}{2}X_3|^2\Big), \end{aligned} \tag{5.28}$$

where the covariant derivatives D_m are defined as in (4.21). We will evaluate this expression for generic representations of G later, making use of the grading of the gauge connection. The volume of the coset space $T^{1,1}$, appearing as prefactor in the above functional, is given by $\mathrm{Vol}(T^{1,1}) = \frac{16\pi^3}{27}$ [117, 125].

Instanton equation on $T^{1,1}$. The instanton equations on $T^{1,1}$ read

$$\mathcal{F}_{12} = -\mathcal{F}_{34}, \quad \mathcal{F}_{13} = \mathcal{F}_{24}, \quad \mathcal{F}_{14} = -\mathcal{F}_{23}, \quad \mathcal{F}_{a5} = 0, \quad a = 1, \dots 4, \tag{5.29}$$

in analogy to (4.22). The last condition can also be formulated as

$$[X_5, \phi^{(1)}] = -\tfrac{3}{2}\mathrm{i}\phi^{(1)}, \quad [X_5, \phi^{(2)}] = -\tfrac{3}{2}\mathrm{i}\phi^{(2)}, \tag{5.30}$$

resembling the form of equivariance conditions on $\mathbb{C}P^1 \times \mathbb{C}P^1$, which will be discussed in Section 5.4. Inserting the instanton conditions into the action functional (5.28) leads to the vanishing of the last four terms.

Instanton equation on the conifold. The evaluation of the HYM equations on the conifold yields

$$\dot{\phi}^{(\alpha)} = -\tfrac{3}{2}\phi^{(\alpha)} - \mathrm{i}[\phi^{(\alpha)}, X_5], \qquad \alpha = 1, 2, \qquad \text{with} \quad [\phi^{(1)}, \phi^{(2)}] = 0, \tag{5.31a}$$

$$\dot{X}_5 = -4X_5 - 2\mathrm{i}[\phi^{(1)}, \phi^{(1)\dagger}] - 2\mathrm{i}[\phi^{(2)}, \phi^{(2)\dagger}] \ . \tag{5.31b}$$

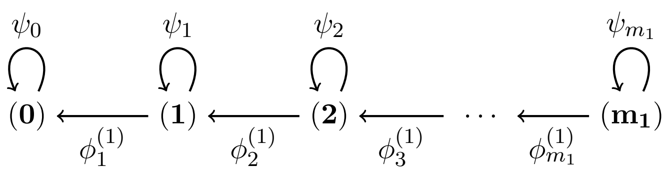
Like the instanton equations (4.26) on the round seven-sphere, they agree with the general result (4.91), derived in [34, 51], for the case $n = 2$.

5.3 Quiver diagrams

We start by studying two examples of quiver diagrams for low-dimensional representations of G and then describe the generic case, using the grading of the gauge connection. The product structure of the group G and choices for representations thereof are collected in Appendix C.

5.3.1 Representation $\underline{(\mathbf{m}, \mathbf{0})}$

Choosing as representation for the second factor SU(2) the trivial representation yields a holomorphic chain of length $m + 1$ with vertex loops as quiver diagram[4],



(5.32)

with $\phi_i^{(1)} \in \mathrm{Hom}(E_i, E_{i-1})$, $\psi_i \in \mathrm{End}(E_i)$ and $E_0, \dots, E_{m_1}$ the vector spaces attached to the vertices. Consistency required recasting this modified version of the holomorphic chain from Section 3.3.1 if the representation of one factor of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$

[4] To keep the double index notation simple, we label the vertices here, in contrast to the previous chapter, with their indices (i, α) rather than the quantum numbers $c_{i\alpha} = m_1 - m_2 - 2i + 2\alpha$.

is chosen to be trivial. Since I_5, which determines the shape of the vertical Higgs field X_5, acts with eigenvalue $\widetilde{c}_{i\alpha} = m_1 + m_2 - 2i - 2\alpha$ on the vertices, some loops $\psi_{i\alpha}$ may be set to zero, as already mentioned for the adjoint representation of SU(4) in the previous chapter.

Specializing to the case $m = 1$, one obtains a "holomorphic triple" [36,43] modified by vertex loops, and the equivariant gauge connection (5.20) takes the form

$$\mathcal{A} = \begin{pmatrix} A_0 + \psi_0 \otimes e^5 + \mathbf{1}_{k_0} \otimes a & \phi \otimes \Theta^{\bar{1}} \\ -\phi^\dagger \otimes \Theta^1 & A_1 + \psi_1 \otimes e^5 - \mathbf{1}_{k_1} \otimes a \end{pmatrix} \tag{5.33}$$

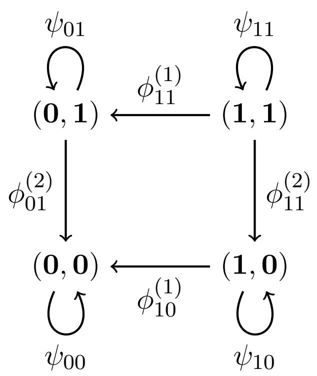
with $\phi := \phi_1^{(1)} \in \mathrm{Hom}(E_1, E_0)$, $\psi_0 \in \mathrm{End}(E_0)$ and $\psi_1 \in \mathrm{End}(E_1)$. The instanton equations on the conifold (5.31) yield the flow equations

$$\begin{aligned} \dot{\phi} &= -\tfrac{3}{2}\phi - \mathrm{i}\phi\psi_1 + \mathrm{i}\psi_0\phi, \\ \dot{\psi}_0 &= -4\psi_0 - 2\mathrm{i}\phi\phi^\dagger, \qquad\qquad \dot{\psi}_1 = -4\psi_1 - 2\mathrm{i}\phi^\dagger\phi. \end{aligned} \tag{5.34}$$

whereas the algebraic condition $[\phi^{(1)}, \phi^{(2)}] = 0$ is trivially satisfied. Arising from a 2-quiver, the instanton equations (5.34) have the same structure as those for the diagrams (4.28) and (4.35). This setup is the counterpart of the fundamental representation $C^{1,0}$ of SU(3) in [46, Sec. 4.3.1], with different dimensions of the representation spaces V_j and coefficients in the equations of the vertical field, of course.

5.3.2 Representation $\underline{(1,1)}$

Our second example is the tensor product of the defining representations for both factors SU(2), which yields as quiver diagram[5] the square



(5.35)

with $\phi_{i\alpha}^{(1)} \in \mathrm{Hom}(E_{i\alpha}, E_{i-1\,\alpha})$, $\phi_{i\alpha}^{(2)} \in \mathrm{Hom}(E_{i\alpha}, E_{i\,\alpha-1})$ and $\psi_{i\alpha} \in \mathrm{End}(E_{i\alpha})$. As above one may set $\psi_{10} = 0 = \psi_{01}$ because the U(1)-charge $\widetilde{c}_{i\alpha}$ vanishes on these

[5] We describe more general Higgs fields in (C.16), see also [47].

vertices. The quiver (5.35) encodes the gauge connection

$$\mathcal{A} = \begin{pmatrix} A_{00} + \psi_{00}\, e^5 & \phi^{(2)}_{01} \otimes \Theta^{\bar{2}} & \phi^{(1)}_{01} \otimes \Theta^{\bar{1}} & 0 \\ -\phi^{(2)\dagger}_{01} \otimes \Theta^{2} & A_{01} + \psi_{01}\, e^5 & 0 & \phi^{(1)}_{11} \otimes \Theta^{\bar{1}} \\ -\phi^{(1)\dagger}_{10} \otimes \Theta^{1} & 0 & A_{10} + \psi_{10}\, e^5 & \phi^{(2)}_{11} \otimes \Theta^{\bar{2}} \\ 0 & -\phi^{(1)\dagger}_{11} \otimes \Theta^{1} & -\phi^{(2)}_{11} \otimes \Theta^{2} & A_{11} + \psi_{11}\, e^5 \end{pmatrix} + \Gamma \tag{5.36}$$

with $\Gamma = \mathrm{diag}(0, \mathbf{1}_{k_{01}}\ 2a, -\mathbf{1}_{k_{10}}\ 2a, 0)$. In this expression one recognizes the twist of the ladder operators (C.15) in the trivial flat bundle with homomorphisms.

The instanton equations (5.31) for this case comprise

$$\begin{aligned} \dot{\phi}^{(1)}_{10} &= -\tfrac{3}{2}\phi^{(1)}_{10} - \mathrm{i}\phi^{(1)}_{10}\psi_{10} + \mathrm{i}\psi_{00}\phi^{(1)}_{10}\,, & \dot{\phi}^{(1)}_{11} &= -\tfrac{3}{2}\phi^{(1)}_{11} - \mathrm{i}\phi^{(1)}_{11}\psi_{11} + \mathrm{i}\psi_{10}\phi^{(1)}_{11}\,, \\ \dot{\phi}^{(2)}_{01} &= -\tfrac{3}{2}\phi^{(2)}_{01} - \mathrm{i}\phi^{(2)}_{01}\psi_{01} + \mathrm{i}\psi_{00}\phi^{(2)}_{01}\,, & \dot{\phi}^{(2)}_{11} &= -\tfrac{3}{2}\phi^{(2)}_{11} - \mathrm{i}\phi^{(2)}_{11}\psi_{11} + \mathrm{i}\psi_{10}\phi^{(2)}_{11}\,, \\ \dot{\psi}_{00} &= -4\psi_{00} - 2\mathrm{i}\phi^{(1)}_{10}\phi^{(1)\dagger}_{10} - 2\mathrm{i}\phi^{(2)}_{01}\phi^{(2)\dagger}_{01}\,, & \dot{\psi}_{01} &= -4\psi_{01} - 2\mathrm{i}\phi^{(1)}_{11}\phi^{(1)\dagger}_{11} + 2\mathrm{i}\phi^{(2)\dagger}_{01}\phi^{(2)}_{01}\,, \\ \dot{\psi}_{10} &= -4\psi_{10} + 2\mathrm{i}\phi^{(1)\dagger}_{10}\phi^{(1)}_{10} - 2\mathrm{i}\phi^{(2)}_{11}\phi^{(2)\dagger}_{11}\,, & \dot{\psi}_{11} &= -4\psi_{11} + 2\mathrm{i}\phi^{(1)\dagger}_{11}\phi^{(1)}_{11} + 2\mathrm{i}\phi^{(2)\dagger}_{11}\phi^{(2)}_{11}\,, \\ 0 &= \phi^{(1)}_{10}\phi^{(2)}_{11} - \phi^{(2)}_{01}\phi^{(1)}_{11}\,. \end{aligned} \tag{5.37}$$

Again the flow equations show the characteristic behavior expected for Sasakian quiver gauge theories, and the grading as $A_2 \otimes A_2$-quiver is eminent. In contrast to the first example, the instanton conditions comprise a quiver relation which imposes commutativity of the diagram (5.35) due to the last line of (5.37).

The quiver (5.35) looks like the quiver diagram (4.47) obtained on round spheres $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ for the adjoint representation, in particular the quiver diagram for the adjoint representation $C^{1,1}$ on the five-sphere in [46, Sec. 4.3.3]. Setting $\psi_{01} = 0 = \psi_{10}$ due to the vanishing of $\widetilde{c}_{i\alpha}$ on the corresponding vertices, one obtains almost exactly the same structure[6] of the equations as (4.27) and (4.28) in [46], apart from different numerical factors.

5.3.3 Generic case $\underline{(\mathbf{m_1}, \mathbf{m_2})}$

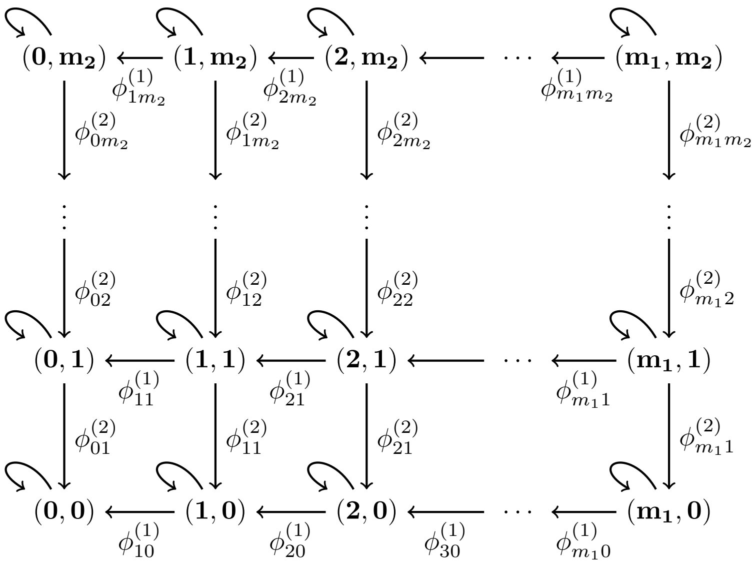
We now consider quiver gauge theories associated to a generic representation $\underline{(\mathbf{m_1}, \mathbf{m_2})}$ with generators given by (C.14). The non-vanishing parts of the invariant gauge connection read

$$\begin{aligned} \mathcal{A}^{i\alpha,i\alpha} &= A^{i\alpha} + \phi^{(3)}_{i\alpha} \otimes e^5 + c_{i\alpha}\, \mathbf{1}_{k_{i\alpha}} \otimes a \qquad \text{with} \quad c_{i\alpha} \ = \ m_1 - m_2 - 2i + 2\alpha\ , \\ \mathcal{A}^{i\,\alpha,i+1\,\alpha} &= \phi^{(1)}_{i+1\,\alpha} \otimes \Theta^{\bar{1}} \ = \ -\left(\mathcal{A}^{i+1\,\alpha,i\,\alpha}\right)^\dagger , \\ \mathcal{A}^{i\,\alpha,i\,\alpha+1} &= \phi^{(2)}_{i\,\alpha+1} \otimes \Theta^{\bar{2}} \ = \ -\left(\mathcal{A}^{i\,\alpha+1,i\,\alpha}\right)^\dagger\ , \end{aligned} \tag{5.38}$$

[6] For a comparison of the quiver diagram (5.35) with [46, Eq. (4.24)] one identifies $(\phi^{(1)}_{11}, \phi^{(2)}_{11}, \phi^{(2)}_{01}, \phi^{(1)}_{10}, \psi_{11}, \psi_{00})$ from this work with $(\phi^+_1, \phi^-_1, \phi^-_0, \phi^+_0, \psi^-, \psi^+)$ in the reference.

where we again denote the homomorphisms as $\phi^{(1)}_{i+1\alpha} := \phi^{(1)}_{i\alpha,i+1\alpha} \in \mathrm{Hom}(E_{i+1,\alpha}, E_{i\alpha})$ and $\phi^{(2)}_{i\alpha+1} := \phi^{(1)}_{i\alpha,i\alpha+1} \in \mathrm{Hom}(E_{i,\alpha+1}, E_{i\alpha})$ as well as $\psi_{i\alpha} \in \mathrm{End}(E_{i\alpha})$ to simplify the notation. As for the previous examples, the quantum number associated to I_5 is given by $\widetilde{c}_{i\alpha} := m_1 + m_2 - 2i - 2\alpha$, so that one may set some entries of X_5 to zero; we keep them for the symmetry of the exposition here and recall that they are compatible with the equivariance condition.

The quiver diagram associated to the gauge connection (5.38) is the lattice which describes quiver gauge theories on $\mathbb{C}P^1 \times \mathbb{C}P^1$ [43], modified by vertex loops:



(5.39)

Using the graded gauge connection (5.38), one derives the non-vanishing contributions of the field strength (C.24) and the action functional on $M^d \times T^{1,1}$ as

$$
\begin{aligned}
S_{\mathrm{YM}} = & \frac{16\pi^3}{27} \int_{M^d} \mathrm{d}^d x \, \sqrt{\det(g_{M^d})} \, \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} \mathrm{Tr}_{i\alpha} \Big\{ \tfrac{1}{4} F^{i\alpha\,\dagger}_{ab} F^{i\alpha\, ab} \\
& + \tfrac{1}{2} \left(D_a \psi_{i\alpha}\right) \left(D^a \psi_{i\alpha}\right)^\dagger + \left(D_a \phi^{(1)}_{i\alpha}\right)^\dagger \left(D^a \phi^{(1)}_{i\alpha}\right) + \left(D_a \phi^{(1)}_{i+1\,\alpha}\right) \left(D^a \phi^{(1)}_{i+1\,\alpha}\right)^\dagger \\
& + \left(D_a \phi^{(2)}_{i\alpha}\right)^\dagger \left(D^a \phi^{(2)}_{i\alpha}\right) + \left(D_a \phi^{(2)}_{i\,\alpha+1}\right) \left(D^a \phi^{(2)}_{i\,\alpha+1}\right)^\dagger \\
& + 2 \left| \phi^{(1)\,\dagger}_{i\alpha} \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i+1\,\alpha} \phi^{(1)}_{i+1\,\alpha}{}^\dagger + \mathrm{i}\psi_{i\alpha} + \tfrac{3}{4} c_{i\alpha} \mathbf{1}_{k_{i\alpha}} \right|^2 \\
& + 2 \left| \phi^{(2)\,\dagger}_{i\alpha} \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i\,\alpha+1} \phi^{(2)}_{i\,\alpha+1}{}^\dagger + \mathrm{i}\psi_{i\alpha} - \tfrac{3}{4} c_{i\alpha} \mathbf{1}_{k_{i\alpha}} \right|^2 \\
& + 2 \left| \phi^{(1)}_{i+1\,\alpha} \phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1} \phi^{(1)}_{i+1\,\alpha+1} \right|^2 + 2 \left| \left(\phi^{(1)}_{i\,\alpha-1} \phi^{(2)}_{i\alpha} - \phi^{(2)}_{i-1\,\alpha} \phi^{(1)}_{i\alpha} \right)^\dagger \right|^2 \\
& + 2 \left| \phi^{(2)\,\dagger}_{i\alpha} \phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha} \phi^{(2)}_{i+1\,\alpha}{}^\dagger \right|^2 + 2 \left| \left(\phi^{(2)}_{i-1\,\alpha+1}{}^\dagger \phi^{(1)}_{i\alpha} - \phi^{(1)}_{i\,\alpha+1} \phi^{(2)}_{i\,\alpha+1}{}^\dagger \right)^\dagger \right|^2 \\
& + \left| \left(\phi^{(1)}_{i\alpha} \psi_{i\alpha} - \psi_{i-1\,\alpha} \phi^{(1)}_{i\alpha} - \tfrac{3}{2} \mathrm{i} \phi^{(1)}_{i\alpha} \right)^\dagger \right|^2 + \left| \phi^{(1)}_{i+1\,\alpha} \psi_{i+1\,\alpha} - \psi_{i\alpha} \phi^{(1)}_{i+1\,\alpha} - \tfrac{3}{2} \mathrm{i} \phi^{(1)}_{i+1\,\alpha} \right|^2 \\
& + \left| \left(\phi^{(2)}_{i\alpha} \psi_{i\alpha} - \psi_{i\,\alpha-1} \phi^{(2)}_{i\alpha} - \tfrac{3}{2} \mathrm{i} \phi^{(2)}_{i\alpha} \right)^\dagger \right|^2 + \left| \phi^{(2)}_{i\,\alpha+1} \psi_{i\,\alpha+1} - \psi_{i\alpha} \phi^{(2)}_{i\,\alpha+1} - \tfrac{3}{2} \mathrm{i} \phi^{(2)}_{i\,\alpha+1} \right|^2 \Big\} ,
\end{aligned}
\tag{5.40}
$$

where we use the abbreviation $|X|^2 := X\, X^\dagger$, and the covariant derivatives now take the form

$$\begin{aligned} D\phi^{(1)}_{i+1\,\alpha} &= \mathrm{d}\phi^{(1)}_{i+1\,\alpha} + A^{i\alpha}\,\phi^{(1)}_{i+1\,\alpha} - \phi^{(1)}_{i+1\,\alpha}\,A^{i+1\,\alpha}\ , \\ D\phi^{(2)}_{i\,\alpha+1} &= \mathrm{d}\phi^{(2)}_{i\,\alpha+1} + A^{i\alpha}\,\phi^{(2)}_{i\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\,A^{i\,\alpha+1}\ , \\ D\phi^{(3)}_{i\alpha} &= \mathrm{d}\phi^{(3)}_{i\alpha} + A^{i\alpha}\,\phi^{(3)}_{i\alpha} - \phi^{(3)}_{i\alpha}\,A^{i\alpha}\ . \end{aligned} \tag{5.41}$$

Due to the quiver approach, the overall trace in the Yang-Mills action splits into the sum of the traces taken over each single vector space $E_{i\alpha}$ attached to the vertices of the quiver diagram. Plugging the field strength (C.24d) into the instanton equations (5.31) on the conifold yields the algebraic relation

$$\phi^{(1)}_{i+1\,\alpha}\phi^{(2)}_{i+1\,\alpha+1} = \phi^{(2)}_{i\,\alpha+1}\phi^{(1)}_{i+1\,\alpha+1} \tag{5.42}$$

for all $i = 0, 1, \ldots m_1 - 1$ and $\alpha = 0, 1, \ldots, m_2 - 1$. That is, the instanton condition requires all cells of the quiver diagram (5.39) to commute, analogously to the result for $\underline{(\mathbf{1}, \mathbf{1})}$. The curvature components (C.24b) and (C.24c) lead to the flow equations

$$\dot{\phi}^{(1)}_{i+1\,\alpha} = -\tfrac{3}{2}\phi^{(1)}_{i+1\,\alpha} - \mathrm{i}\phi^{(1)}_{i+1\,\alpha}\psi_{i+1\,\alpha} + \mathrm{i}\psi_{i\,\alpha}\phi^{(1)}_{i+1\,\alpha}, \tag{5.43a}$$

$$\dot{\phi}^{(2)}_{i\,\alpha+1} = -\tfrac{3}{2}\phi^{(2)}_{i\,\alpha+1} - \mathrm{i}\phi^{(2)}_{i\,\alpha+1}\psi_{i\,\alpha+1} + \mathrm{i}\psi_{i\,\alpha}\phi^{(2)}_{i\,\alpha+1}, \tag{5.43b}$$

generalizing the systems of equations obtained for the two previous examples. Finally, the equation of motion for the entries of X_5 follows from the curvature component (C.24a), which yields

$$\dot{\psi}_{i\,\alpha} = -4\psi_{i\,\alpha} - 2\mathrm{i}\phi^{(1)}_{i+1\,\alpha}\phi^{(1)\dagger}_{i+1\,\alpha} + 2\mathrm{i}\phi^{(1)\dagger}_{i\,\alpha}\phi^{(1)}_{i\,\alpha} - 2\mathrm{i}\phi^{(2)}_{i\,\alpha+1}\phi^{(2)\dagger}_{i\,\alpha+1} + 2\mathrm{i}\phi^{(2)\dagger}_{i\,\alpha}\phi^{(2)}_{i\,\alpha}. \tag{5.44}$$

Thus, the flow equation for the entry of the vertical Higgs field at a given vertex couples to all in- and out-going arrows of the horizontal Higgs fields. For vertices (i, α) with vanishing U(1)-charge $\widetilde{c}_{i\alpha}$, i.e. setting $\psi_{i\alpha} = 0$, this equation yields a further algebraic constraint.

5.4 Reduction to $\mathbb{C}P^1 \times \mathbb{C}P^1$

We now consider the reduction of the quiver gauge theory on $T^{1,1}$ to that on the Kähler base $\mathbb{C}P^1 \times \mathbb{C}P^1$, analogously to the procedure in Section 4.4.1. For this purpose, one has to fix the endomorphism X_5 by setting $X_5 = -\frac{3}{4}\mathrm{i}I_5$ (according to (5.8)), where $I_5 = I_3^{(1)} + I_3^{(2)}$ denotes the second Cartan generator of G. This yields the additional equivariance condition (5.26), which is already satisfied for the Higgs fields we have been considering so far.

The quiver diagrams are therefore given by the lattices (5.39) without vertex loops, and this reproduces the results of [35, 43]. From (5.40) one obtains the following expression for the Yang-Mills action functional on $M^d \times \mathbb{C}P^1 \times \mathbb{C}P^1$:

$$
\begin{aligned}
S_{\mathrm{r}} = \frac{16\pi^3}{27} \int_{M^d} \mathrm{d}^d x \, \sqrt{\det(g_{M^d})} \sum_{i=0}^{m_1} \sum_{\alpha=0}^{m_2} \mathrm{Tr} \Big\{ & \tfrac{1}{4} F_{ab}^{i\alpha\,\dagger} F^{i\alpha\, ab} + \big(D_a \phi_{i\alpha}^{(1)}\big)^\dagger \big(D^a \phi_{i\alpha}^{(1)}\big) \\
& + \big(D_a \phi_{i\alpha}^{(2)}\big)^\dagger \big(D^a \phi_{i\alpha}^{(2)}\big) + \big(D_a \phi_{i+1\,\alpha}^{(1)}\big) \big(D^a \phi_{i+1\,\alpha}^{(1)}\big)^\dagger + \big(D_a \phi_{i\,\alpha+1}^{(2)}\big) \big(D^a \phi_{i\,\alpha+1}^{(2)}\big)^\dagger \\
& + 2 \big| \phi_{i\alpha}^{(1)\,\dagger} \phi_{i\alpha}^{(1)} - \phi_{i+1\,\alpha}^{(1)} \phi_{i+1\,\alpha}^{(1)\,\dagger} + \tfrac{3}{2} (m_1 - 2i) \mathbf{1}_{k_{i\alpha}} \big|^2 \\
& + 2 \big| \phi_{i\alpha}^{(2)\,\dagger} \phi_{i\alpha}^{(2)} - \phi_{i+1\,\alpha}^{(2)} \phi_{i+1\,\alpha}^{(2)\,\dagger} + \tfrac{3}{2} (m_2 - 2\alpha) \mathbf{1}_{k_{i\alpha}} \big|^2 \\
& + 2 \big| \phi_{i+1\,\alpha}^{(1)} \phi_{i+1\,\alpha+1}^{(2)} - \phi_{i\,\alpha+1}^{(2)} \phi_{i+1\,\alpha+1}^{(1)} \big|^2 + 2 \big| \big(\phi_{i\,\alpha-1}^{(1)} \phi_{i\alpha}^{(2)} - \phi_{i-1\,\alpha}^{(2)} \phi_{i\alpha}^{(1)} \big)^\dagger \big|^2 \\
& + 2 \big| \phi_{i\alpha}^{(2)\,\dagger} \phi_{i+1\,\alpha-1}^{(1)} - \phi_{i+1\,\alpha}^{(1)} \phi_{i+1\,\alpha}^{(2)\,\dagger} \big|^2 + 2 \big| \big(\phi_{i-1\,\alpha+1}^{(2)\,\dagger} \phi_{i\alpha}^{(1)} - \phi_{i\,\alpha+1}^{(1)} \phi_{i\,\alpha+1}^{(2)\,\dagger} \big)^\dagger \big|^2 \Big\} \, .
\end{aligned}
\tag{5.45}
$$

For comparison with the findings of [43] one has to recall that the definitions of the Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ differ by a factor of $\sqrt{\frac{3}{2}}$, due to $\Theta^\alpha = \sqrt{\frac{2}{3}}\beta_\alpha$ in the limit $\varphi = 0$. This correctly reproduces the result and, moreover, shows once more that the Sasaki-Einstein condition has fixed the two S^2-radii as $R_1^2 = R_2^2 = \frac{1}{6}$.

In the instanton equations on the conifold, one has to replace the entries $\psi_{i\alpha}$ of X_5 by identity operators with suitable prefactors $\widetilde{c}_{i\alpha}$. The equations (5.34) for the modified holomorphic triple, for example, turn into

$$
\dot{\phi} = 0, \qquad \tfrac{2}{3}\phi\phi^\dagger = \mathbf{1}_{k_0} \qquad -\tfrac{2}{3}\phi^\dagger\phi = \mathbf{1}_{k_1}, \tag{5.46}
$$

which are part of the equations for a holomorphic triple and chain vortex equations [35, 36, 43].

For generic $\underline{(\mathbf{m_1}, \mathbf{m_2})}$, one has to set $\psi_{i\alpha} = -\frac{3}{4}\mathrm{i}I_5 = -\frac{3}{4}\mathrm{i}(m_1 + m_2 - 2i - 2\alpha)\mathbf{1}_{k_{i\alpha}}$, which reduces the expressions (5.43) and (5.44) to

$$
c_{i\alpha}\, \mathbf{1}_{k_{i\alpha}} = \tfrac{2}{3}\phi_{i+1\,\alpha}^{(1)}\phi_{i+1\,\alpha}^{(1)\dagger} - \tfrac{2}{3}\phi_{i\alpha}^{(1)\dagger}\phi_{i\alpha}^{(1)} + \tfrac{2}{3}\phi_{i\,\alpha+1}^{(2)}\phi_{i\,\alpha+1}^{(2)\dagger} - \tfrac{2}{3}\phi_{i\alpha}^{(2)\dagger}\phi_{i\alpha}^{(2)} \tag{5.47}
$$

and requires that the fields $\phi_{i+1\,\alpha}^{(1)}$ and $\phi_{i\,\alpha+1}^{(2)}$ are constant. Furthermore, the Higgs fields are still subject to the quiver relation (5.42). Taking into account the appearance of the factor of $\sqrt{\frac{2}{3}}$, one regains the BPS equations [43, Eq. (4.7)] and [35, Eq. (5.3)]. In our discussion, the metric cone is considered as $M^d \times T^{1,1}$ with $M^d = \mathbb{R}^+$ without a further external part, but also a Kähler manifold M^{2n} may be included. Then the HYM equations also comprise the Kähler form of M^{2n} and yield additional contributions. In particular, for obtaining non-trivial solutions to the BPS conditions on $M^d \times \mathbb{C}P^1 \times \mathbb{C}P^1$, one may consider a noncommutative external manifold $\mathbb{R}_\theta^{2n}$ [37, 43].

5.5 Moduli space of HYM instantons

Since the instanton equations on the conifold (5.31) take the expected form [34,51] for a Calabi-Yau cone over a five-dimensional Sasaki-Einstein manifold, we stick to [46,51] and repeat the discussion from Section 4.5. Using the rescaled matrices[7]

$$\phi^{(\alpha)} =: \mathrm{e}^{-\frac{3}{2}\tau} W_\alpha \quad \text{for} \quad \alpha = 1,2 \quad \text{and} \quad \tfrac{\mathrm{i}}{2} X_5 =: \mathrm{e}^{-4\tau} Z \ , \tag{5.48}$$

and the coordinate $s := -\frac{1}{4}\mathrm{e}^{-4\tau}$, one obtains the flow equations

$$\frac{\mathrm{d}W_\alpha}{\mathrm{d}s} = 2[W_\alpha, Z], \qquad \alpha = 1,2 \ , \tag{5.49a}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}(Z + Z^\dagger) + 2[Z, Z^\dagger] + 2\sum_{\alpha=1}^{2} (-4s)^{-\frac{5}{4}} \, [W_\alpha, W_\alpha^\dagger] =: \mu(s), \tag{5.49b}$$

and the algebraic condition $[W_1, W_2] = 0$. These Nahm-type equations are those obtained for the case $n = 2$ in [51], and due to the generality of the approach, they also occur for HYM instantons on the cone over the five-sphere S^5 [46]. However, the equivariance conditions are not universal and depend on the geometry of the explicit coset space G/H. The real equation now follows as the equation of motion of the Lagrangian

$$\mathcal{L}[g] = \int_{\mathcal{I}} \mathrm{d}s \ \mathrm{Tr} \left\{ |Z^g + Z^{g\dagger}|^2 + 2\,(-4s)^{-\frac{5}{4}} \sum_{\alpha=1}^{2} |W_\alpha^g|^2 \right\} , \tag{5.50}$$

by the same arguments as in the previous case. Hence, the moduli space of $(\mathrm{SU}(2) \times \mathrm{SU}(2))$-equivariant HYM instantons on the conifold can be described as coadjoint orbit of two commuting complex matrices (T_1, T_2), where the same boundary conditions (4.67) are imposed [46, 51]. The description of the moduli space as a Kähler quotient is also applicable.

5.6 Discussion

The construction of Sasakian quiver gauge theories on the space $T^{1,1}$ has completed the picture in five dimensions since it is the only other compact homogeneous Sasaki-Einstein space in that dimension besides the five-sphere. As expected, the instanton matrix equations on the conifold and the description of their moduli space match the general results from [51] for the case $n = 2$.

The equivariant gauge connection on bundles over $T^{1,1}$ and over the conifold inherits a grading from the product structure of the Kähler base $\mathbb{C}P^1 \times \mathbb{C}P^1$, which yields quiver diagrams of the form $A_{m_1+1} \otimes A_{m_2+1}$ with vertex loops and a description analogous to [43]. The bundle structure of the five-sphere – being a U(1)-bundle over $\mathbb{C}P^2$ – is different, but one ends up with similar quiver diagrams since in both cases Lie algebras of rank 2 are considered (see for instance the weight and quiver diagrams in [35, 40, 46]). Differences appear in the dimensions of the representation spaces attached to the vertices and in the contribution of the canonical connection: on $T^{1,1}$

[7] Again one may re-introduce the matrix X_τ.

the vertices carry monopole charges, while they carry non-abelian gauge connections on the five-sphere. Although $T^{1,1}$ and the sphere S^5 yield different gauge theories, their quiver diagrams and instanton equations require a similar treatment.

Since the construction procedure essentially yields U(1)-modifications of the underlying Kähler quiver gauge theories, the comparison between the Sasakian theories on $T^{1,1}$ and S^5 can be traced back to comparing equivariant dimensional reduction on $\mathbb{C}P^2$ [40] to that on $\mathbb{C}P^1 \times \mathbb{C}P^1$ [43]. In both cases, the U(1)-factor induced by the contact direction of the Sasaki-Einstein structure gives rise to vertex loops in the quiver diagrams and the typical structure of the differential equations for the vertical Higgs field X_5.

Part III

3-Sasakian quiver gauge theories

6 Quiver gauge theory on the squashed seven-sphere

After considering Sasakian quiver gauge theories in the two previous chapters, we now turn to quiver gauge theories on 7-dimensional manifolds endowed with 3-Sasakian structures. The archetype of such a 3-Sasakian manifold is the *squashed* seven-sphere $\mathrm{Sp}(2)/\mathrm{Sp}(1)$.

Being a particular 7-dimensional Sasaki-Einstein manifold, it appears in Freund-Rubin compactifications of the form $\mathrm{AdS}_4 \times M_7$ in M-theory [22, 103]. As anticipated in the discussion of the round seven-sphere, the squashed metric leads to $\mathcal{N} = 1$ supergravity[1] [105, 106, 126], in contrast to the effective $\mathcal{N} = 8$ supergravity on the round sphere. Tri-Sasakian reductions of M-theory solutions have been discussed in [50], where the squashed seven-sphere turned out to be the prototype of universal reductions. Generalized instantons associated to the G_2 and Spin(7) geometry of $\mathrm{Sp}(2)/\mathrm{Sp}(1)$, inherent in its 3-Sasakian structure, have been considered in [28]. By virtue of the local bundle structure as $\mathrm{SU}(2) \times S^4$, the geometry of the squashed seven-sphere comprises the setup for SU(2) gauge fields on S^4, in particular BPST instantons from the quaternionic Hopf bundle $S^3 \hookrightarrow S^7 \to S^4$ (see e.g. [127]).

The construction of 3-Sasakian quiver gauge theories on the coset space $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ resembles the procedure for the previously discussed Sasakian quiver gauge theories, but the description of the moduli space is not exhaustive since its structure is not yet fully understood. Parts of the following exposition and most results are based on a collaboration [49] with O. Lechtenfeld, A. D. Popov and R. J. Szabo. The discussion of the moduli space of hyper-Kähler instantons partially relies on joint work with M. Sperling [128].

6.1 Geometric structure

This section describes the fundamental geometric properties of the homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)$, starting from a local section and the resulting structure equations. A description of the squashed seven-sphere as coset space may also be found in [28, Sec.

[1] Recall the breaking to 3/8 maximal supersymmetry by 3-Sasakian structures and the reduction to 1/8 by a nearly parallel G_2 structure [54].

3.5] and in [22, Sec. 8]. For details on generic 3-Sasakian manifolds, beyond the review in Section 2.1, we again refer to [53, 56, 65].

6.1.1 Local section

While the Sasaki-Einstein manifolds of the previous chapters have been described as U(1)-bundles over Kähler (-Einstein) manifolds, regular 3-Sasakian structures admit a fibration as SU(2) – or SO(3) – bundles over a quaternionic Kähler manifold [65]. In this way, the symplectic group $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$ can be locally constructed as an $(\mathrm{Sp}(1)\times\mathrm{Sp}(1))$-bundle[2] over S^4,

$$\mathrm{Sp}(2) \longrightarrow S^4 \cong \mathrm{Sp}(2)\,/\,(\mathrm{Sp}(1)\times\mathrm{Sp}(1)) \ , \tag{6.1}$$

where we are following the explicit form given in [31, Sec. 4]. A local section of this bundle can be written as

$$\mathrm{Sp}(2) \ni Q \ := \ f^{-1/2}\begin{pmatrix} \mathbf{1}_2 & -x \\ x^\dagger & \mathbf{1}_2 \end{pmatrix} \qquad \text{with} \quad x \ = \ x^\mu\,\tau_\mu \ , \quad (\tau_\mu) \ = \ (-\mathrm{i}\sigma_i, \mathbf{1}_2) \tag{6.2}$$

and $f := 1 + x^\dagger\,x \ = \ 1 + \delta_{\mu\nu}\,x^\mu\,x^\nu$; here σ_i for $i = 1,2,3$ are the standard Pauli matrices. The canonical flat connection is locally given by

$$A_0 \ = \ Q^{-1}\,\mathrm{d}Q \ =: \ \begin{pmatrix} A^- & -\phi \\ \phi^\dagger & A^+ \end{pmatrix} \tag{6.3}$$

with $\phi = f^{-1}\mathrm{d}x$ and the other contributions defined as in [31], providing left-invariant 1-forms on S^4. An element of the fiber $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ can be written in local coordinates as in Section 5.1,

$$\mathrm{Sp}(1) \ni g\cdot h \ := \ \frac{1}{(1+z\,\bar z)^{1/2}}\begin{pmatrix} 1 & -\bar z \\ z & 1 \end{pmatrix}\cdot\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \ , \tag{6.4}$$

so that a section of $\mathrm{Sp}(2) \to \mathrm{Sp}(2)/\mathrm{Sp}(1)$ is obtained by

$$\mathrm{Sp}(2) \ni \widetilde{Q} \ := \ Q \times \begin{pmatrix} g\,h & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \ . \tag{6.5}$$

Consider first the flat connection on the twistor space $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))$ given by the Maurer-Cartan form of the section $\hat Q := Q\,g$:

$$\hat A_0 \ = \ \hat Q^{-1}\,\mathrm{d}\hat Q \ = \ g^{-1}\,A_0\,g + g^{-1}\,\mathrm{d}g \ =: \ \begin{pmatrix} g^{-1}\,A^-\,g + g^{-1}\,\mathrm{d}g & -g^{-1}\,\phi \\ \phi^\dagger\,g & A^+ \end{pmatrix} \ , \tag{6.6}$$

[2] It is useful, also for the relation to hyper-Kähler geometry on the cone, to recall the construction using the quaternions $\mathbb{H}$ explicitly, see [127] for example.

where the explicit form of the contributions in terms of the coordinates $x_1, \ldots, x_4$ on the 4-sphere and the $\mathbb{C}P^1$ coordinate z are given in [31]. In the final step, the matrix $\widetilde{Q} := \hat{Q}\, h$ provides a local section of the bundle $\mathrm{Sp}(2) \to \mathrm{Sp}(2)/\mathrm{Sp}(1)$, yielding

$$\widetilde{A}_0 = \begin{pmatrix} h^{-1}\, \hat{A}^-\, h + h^{-1}\, \mathrm{d}h & -h^{-1}\, \hat{\phi} \\ \hat{\phi}^\dagger\, h & A^+ \end{pmatrix} =: \begin{pmatrix} \mathrm{i}e^7 & \Theta^3 & \Theta^2 & \Theta^1 \\ -\Theta^{\bar{3}} & -\mathrm{i}e^7 & \Theta^{\bar{1}} & -\Theta^{\bar{2}} \\ -\Theta^{\bar{2}} & -\Theta^1 & -\mathrm{i}e^8 & -\Theta^{\bar{4}} \\ -\Theta^{\bar{1}} & \Theta^2 & \Theta^4 & \mathrm{i}e^8 \end{pmatrix} , \tag{6.7}$$

where $\hat{A}^- := g^{-1}\, A^-\, g + g^{-1}\, \mathrm{d}g$ and $\hat{\phi} := g^{-1}\, \phi$. For a better geometric intuition[3], it is worth emphasizing the structure of this expression as a quaternionic (2×2)-matrix:

$$\widetilde{A}_0 =: \begin{pmatrix} \mathrm{Im}(q_0) & q_1 \\ -\bar{q}_1 & \mathrm{Im}(q_2) \end{pmatrix} \tag{6.8}$$

with $q_i \in \Omega^1(\mathbb{H})$. Due to $\mathrm{Im}(\mathbb{H}) \cong \mathfrak{su}(2) \cong \mathfrak{sp}(1)$, the imaginary parts of the two quaternionic forms q_0 and q_2 represent the two commuting subalgebras $\mathfrak{sp}(1)$ that are contained in $\mathfrak{sp}(2)$. This quaternionic point of view will be useful for the generalization to higher-dimensional squashed spheres in (6.62) and the comparison with round spheres.

The flatness of $\widetilde{A}_0$ leads to the structure equations (D.1) with structure constants (D.2) for the Sp(2)-invariant complex 1-forms from (6.7). We have already defined them in such a way that they give rise to a 3-Sasakian structure on the coset space with left-invariant metric[4]

$$\mathrm{d}s^2 = \sum_{\mu=1}^{7} e^\mu \otimes e^\mu . \tag{6.9}$$

To verify the 3-Sasakian property, one considers the structure equations

$$\begin{aligned} \mathrm{d}e^1 &= e^{27} - e^{28} - e^{35} - e^{46} + e^{39} + e^{4,10}, & \mathrm{d}e^2 &= -e^{17} + e^{18} + e^{45} - e^{36} + e^{49} - e^{3,10}, \\ \mathrm{d}e^3 &= e^{47} + e^{48} + e^{15} + e^{26} - e^{19} + e^{2,10}, & \mathrm{d}e^4 &= -e^{37} - e^{38} - e^{25} + e^{16} - e^{29} - e^{1,10}, \\ \mathrm{d}e^5 &= 2e^{67} - 2e^{13} + 2e^{24}, & \mathrm{d}e^6 &= -2e^{57} - 2e^{14} - 2e^{23}, \\ \mathrm{d}e^7 &= 2e^{56} + 2e^{12} + 2e^{34}, \end{aligned} \tag{6.10a}$$

together with

$$\begin{aligned} \mathrm{d}e^8 &= -2e^{12} + 2e^{34} + 2e^{9,10}, & \mathrm{d}e^9 &= 2e^{13} + 2e^{24} - 2e^{8,10}, \\ \mathrm{d}e^{10} &= 2e^{14} - 2e^{23} + 2e^{89} \end{aligned} \tag{6.10b}$$

[3] See also the construction in terms of quaternions in [22, Sec. 8].

[4] This metric agrees with the result $||m||^2 = -B(\sigma,\sigma) - \frac{1}{2}B(m',m')$ [129, Thm. 4], where B denotes the Killing form and $m = \sigma + m' \in \mathfrak{sp}(1) \oplus \mathfrak{m}' = \mathfrak{m}$ with $\mathfrak{m}'$ the quaternionic part; the Killing form on $\mathfrak{sp}(2)$ is given in (D.19).

for the real 1-forms e^μ defined as

$$e^1 - \mathrm{i}e^2 \;:=\; \Theta^1, \quad e^3 - \mathrm{i}e^4 \;:=\; \Theta^2, \quad e^5 - \mathrm{i}e^6 \;:=\; \Theta^3, \quad e^9 - \mathrm{i}e^{10} \;:=\; \Theta^4. \tag{6.11}$$

From these structure equations, one deduces that the triple $(\eta^5, \eta^6, \eta^7) := (e^5, e^6, e^7)$ satisfies the defining relations of a 3-Sasakian structure (as employed[5] e.g. in [19])

$$\mathrm{d}\eta^\alpha \;=\; \epsilon^\alpha{}_{\beta\gamma}\,\eta^\beta \wedge \eta^\gamma + 2\,\omega^\alpha, \quad \mathrm{d}\omega^\alpha \;=\; 2\,\epsilon^\alpha{}_{\beta\gamma}\,\eta^\beta \wedge \omega^\gamma, \quad \alpha, \beta, \gamma \;=\; 5, 6, 7\;, \tag{6.12a}$$

where we identify the 2-forms

$$\omega^5 \;:=\; -e^{13} + e^{24}\;, \qquad \omega^6 \;:=\; -e^{14} - e^{23} \qquad \text{and} \qquad \omega^7 \;:=\; e^{12} + e^{34}\;. \tag{6.12b}$$

Alternatively, one checks the closure of the defining forms of hyper-Kähler geometry, as it has been done for the Kähler form (4.8) and top-degree form (4.9) on the metric cone over S^7 in Chapter 4; the relevant forms can be found in (D.4) and (D.5).

6.1.2 Canonical connection

Following the construction of the canonical connection, reviewed in Section 2.2.1, we determine the 3-form P from (2.19):

$$\begin{aligned} P &= \frac{1}{3}\,\eta^{567} + \frac{1}{3}\sum_{\alpha=5}^{7} \eta^\alpha \wedge \omega^\alpha \\ &= \tfrac{1}{3}\left(e^{567} - e^{135} + e^{245} - e^{146} - e^{236} + e^{127} + e^{347}\right)\;. \end{aligned} \tag{6.13}$$

Plugging the torsion components (2.20) into the structure equations (6.10a), we find the connection matrix

$$\mathrm{d}\begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} = \begin{pmatrix} 0 & e^8 & -e^9 & -e^{10} \\ -e^8 & 0 & e^{10} & -e^9 \\ e^9 & -e^{10} & 0 & -e^8 \\ e^{10} & e^9 & e^8 & 0 \end{pmatrix} \wedge \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} + \begin{pmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \end{pmatrix} \tag{6.14}$$

and $\mathrm{d}e^\alpha \;=\; T^\alpha$ for $\alpha = 5, 6, 7$. Recalling the adjoint representation on the tangent space, one identifies the canonical connection

$$\Gamma \;=\; I_8 \otimes e^8 + I_9 \otimes e^9 + I_{10} \otimes e^{10}\;. \tag{6.15}$$

As expected from the general theory, also for this coset space the canonical connection in the sense of [19] is identical with that of the reductive homogeneous space, defined by the torsion components $T(X,Y) = -[X,Y]_{\mathfrak{m}}$. However, the canonical connection of [19] associated to one of the Sasaki-Einstein structures would not coincide with this

[5] To obtain the same notation as in [19, 128], one has to transform our choice of indices as $(e^1, e^2, e^3, e^4, e^5, e^6, e^7) \mapsto (e^4, e^5, e^6, e^7, -e^2, -e^3, e^1)$.

connection. This observation will have some important consequences for the treatment of the moduli space in Section 6.5. The curvature of the canonical connection (6.15) reads

$$\mathcal{F}_\Gamma = 2\, I_8 \otimes \left(-\, e^{12} + e^{34} \right) + 2\, I_9 \otimes \left(e^{13} + e^{24} \right) + 2\, I_{10} \otimes \left(e^{14} - e^{23} \right) \ , \tag{6.16}$$

and solves the generalized self-duality equation (2.10) for the 4-form $Q_Z = e^{1234}$.

Instanton equations on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$**.** Written in components of the field strength, the instanton equations on the squashed seven-sphere read

$$\mathcal{F}_{12} \ = \ -\mathcal{F}_{34} \ , \qquad \mathcal{F}_{13} \ = \ \mathcal{F}_{24} \ , \qquad \mathcal{F}_{14} \ = \ -\mathcal{F}_{23} \ , \tag{6.17a}$$

$$\mathcal{F}_{a\alpha} \ = \ 0 \ = \ \mathcal{F}_{\alpha\beta} \qquad \text{for} \quad a \ = \ 1,2,3,4 \ , \ \alpha,\beta \ = \ 5,6,7 \ . \tag{6.17b}$$

The first line (6.17a) consists of the usual 4-dimensional self-duality equations, which was to be expected because the quaternionic Hopf fibration yields a BPST instanton on S^4 (see e.g. [127]). The further conditions (6.17b) require the vanishing of all vertical curvature contributions, analogously to (4.22) and (5.29), where in the 3-Sasakian case the vertical part is a group SU(2) rather than U(1) only.

6.2 Equivariance condition and instantons on the cone

Also in this setup, we employ the typical ansatz (3.10) for the gauge connection $\mathcal{A}$ in a Hermitian vector bundle $\mathcal{E}$ over $M^d \times \mathrm{Sp}(2)/\mathrm{Sp}(1)$, writing

$$\mathcal{A} = A + \Gamma + \sum_{a=1}^{7} X_\mu \otimes e^\mu \ = \ A + \sum_{j=8}^{10} I_j \otimes e^j + \sum_{a=1}^{7} X_\mu \otimes e^\mu \tag{6.18}$$

with A a connection on the vector bundle E over M^d. On the metric cone and the conformally equivalent cylinder, one applies exactly the same approach, where the endomorphisms then may depend on the radial coordinate, $X_\mu = X_\mu(\tau)$. According to the bundle structure of 3-Sasakian manifolds, the fields $X_1, \ldots, X_4$ are referred to as horizontal Higgs fields and X_5, X_6, X_7 as vertical ones.

Despite the 3-Sasakian structure inherent in all relations, it is useful to introduce the complex matrices $\phi^{(1)} = \frac{1}{2}\left(X_1 - \mathrm{i}X_2\right)$, $\phi^{(2)} = \frac{1}{2}\left(X_3 - \mathrm{i}X_4\right)$ and even $\phi^{(3)} = \frac{1}{2}\left(X_5 - \mathrm{i}X_6\right)$ (which is now a *vertical* Higgs field) with respect to the complex structure determined by Ω_7. They simplify the notation in the quiver diagrams and improve the comparison with Sasakian quiver gauge theories; since we use the generator dual to e^7 as Cartan generator, this choice is natural.

6.2.1 Equivariance

As before in the case of Sasakian quiver gauge theories, equivariance requires the vanishing of the mixed curvature terms, which now requires

$$[\widetilde{I}_8, \phi^{(1)}] = \phi^{(1)} \ , \quad [\widetilde{I}_8, \phi^{(2)}] = -\phi^{(2)} \ , \quad [\widetilde{I}_8, X_\alpha] = 0 \tag{6.19a}$$

with respect to the Cartan generator $\widetilde{I}_8$ (dual to $\mathrm{i}e^8$) and

$$\begin{aligned} &[I_4^+, \phi^{(1)}] = 0 \ , && [I_4^+, \phi^{(2)}] = \phi^{(1)} \ , && [I_4^+, X_\alpha] = 0 \ , \\ &[I_4^-, \phi^{(1)}] = -\phi^{(2)} \ , && [I_4^-, \phi^{(2)}] = 0 \ , && [I_4^-, X_\alpha] = 0 \end{aligned} \tag{6.19b}$$

with respect to the ladder operators of the subalgebra $\mathfrak{sp}(1)$. The equations (6.19a) fix the action of the Higgs fields in the weight diagram to be

$$\begin{aligned} &\phi^{(1)} : (\nu_7, \nu_8) \longmapsto (*, \nu_8 + 1) \, , && \phi^{(2)} : (\nu_7, \nu_8) \longmapsto (*, \nu_8 - 1) \, , \\ &X_\alpha : (\nu_7, \nu_8) \longmapsto (*, \nu_8) \, , \end{aligned} \tag{6.20}$$

for $\alpha = 5, 6, 7$. Again, these conditions are weaker than the action (D.20) of the ladder operators in the weight diagrams. However, as for the other examples of quiver gauge theories before, we restrict to endomorphisms that are induced by the G-action and therefore consider Higgs fields which act in the weight diagrams according to (D.20).

The scalar ansatz for G-invariant connections on the squashed seven-sphere has been applied in [28] and the resulting instanton equations with respect to the G_2 and $\mathrm{Spin}(7)$ structure have been derived there.

6.2.2 Action functional

After the implementation of the equivariance conditions (6.19), the Yang-Mills action functional on $M^d \times \mathrm{Sp}(2)/\mathrm{Sp}(1)$ reads

$$\begin{aligned} S_{\mathrm{YM}} = \mathrm{Vol}(\mathrm{Sp}(2)/\mathrm{Sp}(1)) \int_{M^d} \mathrm{d}^d y \, \sqrt{g} \, \frac{1}{2} \, \mathrm{Tr}\Big(&\tfrac{1}{2} F_{mn} \, (F^{mn})^\dagger + \sum_{m=1}^{d} \sum_{\mu=1}^{7} |D_m X_\mu|^2 \\ &+ |[X_1, X_2] + 2\, X_7 - 2\, I_8|^2 + |[X_1, X_3] - 2\, X_5 + 2\, I_9|^2 + |[X_1, X_4] - 2\, X_6 + 2\, I_{10}|^2 \\ &+ |[X_2, X_3] - 2\, X_6 - 2\, I_{10}|^2 + |[X_2, X_4] + 2\, X_5 + 2\, I_9|^2 + |[X_3, X_4] + 2\, X_7 + 2\, I_8|^2 \\ &+ |[X_1, X_5] + X_3|^2 + |[X_1, X_6] + X_4|^2 + |[X_1, X_7] - X_2|^2 + |[X_2, X_5] - X_4|^2 \\ &+ |[X_2, X_6] + X_3|^2 + |[X_2, X_7] + X_1|^2 + |[X_3, X_5] - X_1|^2 + |[X_3, X_6] - X_2|^2 \\ &+ |[X_3, X_7] - X_4|^2 + |[X_4, X_5] + X_2|^2 + |[X_4, X_6] - X_1|^2 + |[X_4, X_7] + X_3|^2 \\ &+ |[X_5, X_6] + 2\, X_7|^2 + |[X_5, X_7] - 2\, X_6|^2 + |[X_6, X_7] + 2\, X_5|^2 \Big) \, , \end{aligned} \tag{6.21}$$

where the covariant derivatives D_m are defined as in (4.21) and F_{mn} denotes the field strength of the gauge connection A in the vector bundle $E \to M^d$. The symmetry of the 3-Sasakian manifold consisting of the vertical SU(2)-triplet and the quaternionic quadruple (X_1, X_2, X_3, X_4) is manifest in the above expression.

When imposing the instanton equations (6.17) on Sp(2)/Sp(1), the terms in the last four lines of (6.21) vanish and the action functional simplifies to

$$S_{\mathrm{YM}}^{\mathrm{inst}} = \mathrm{Vol}(\mathrm{Sp}(2)/\mathrm{Sp}(1)) \int_{M^d} \mathrm{d}^d y \,\sqrt{g}\, \frac{1}{2} \mathrm{Tr}\Big(\tfrac{1}{2} F_{mn}\, (F^{mn})^\dagger + \sum_{m=1}^{d} \sum_{\mu=1}^{7} |D_m X_\mu|^2 \\ + 2\,|[X_1, X_2] + 2\,X_7 - 2\,I_8|^2 + 2\,|[X_1, X_3] - 2\,X_5 + 2\,I_9|^2 \\ + 2\,|[X_1, X_4] - 2\,X_6 + 2\,I_{10}|^2 \Big)\,. \tag{6.22}$$

The vanishing of the torsion term in the generalized Yang-Mills equation (2.11) is explicitly verified in Appendix D.1, and also the form of the Chern-Simons-type term from (2.12) can be found there.

6.2.3 Orbifold

As for the round seven-sphere, one may introduce the action of the cyclic group $\mathbb{Z}_{q+1}$ which gives rise to orbifolds of the squashed sphere. For having an action that commutes with the subgroup $H = \mathrm{Sp}(1)$ which is divided out, one considers the action given by multiplication with elements

$$h = \mathrm{diag}(\zeta_{q+1}, \zeta_{q+1}^{-1}, 1, 1), \qquad \zeta_{q+1} \coloneqq \mathrm{e}^{2\pi \mathrm{i}/(q+1)}. \tag{6.23}$$

Only the last step (6.7) in the local section has to be adapted by setting $\varphi \mapsto \frac{\varphi}{q+1}$. The action on the form part can be directly deduced from (6.7):

$$\Theta^\alpha \mapsto \zeta_{q+1}^{-1}\, \Theta^\alpha, \quad \alpha = 1, 2, \qquad \Theta^3 \mapsto \zeta_{q+1}^{-2}\, \Theta^3, \qquad e^7 \mapsto e^7. \tag{6.24}$$

As in Section 4.4.2, the additional equivariance condition (4.56) has to be included and this requires that the Higgs fields act on states in the weight diagrams in the same way as the ladder operators; thus, it does not further restrict Higgs fields that are already constructed according to (D.20).

However, note that this way of orbifolding breaks the SU(2)-symmetry of the fibers and therefore the 3-Sasakian structure, but it still preserves the Sasaki-Einstein structure with respect to I_7 [54, Ex. 43]. The 3-Sasakian lens space $L(p;q)$ is constructed [65, Ex. 2.2.6] by considering the squashed seven-sphere as the unit sphere in $\mathbb{H}^2 \ni (u_1, u_2)$ and dividing by the action $(u_1, u_2) \mapsto (\tau u_1, \tau^q u_2)$ with $\tau^p = 1$ and $(p, q) = 1$.

6.2.4 Sp(2) instanton equations on the metric cone

The metric cone over the 3-Sasakian manifold Sp(2)/Sp(1) is hyper-Kähler and therefore admits Sp(2) instantons. Such instantons can be defined in terms of the generalized self-duality equation $\star_8 \mathcal{F} = -\mathcal{F} \wedge \star_8 Q_Z$ with $r^4 Q_Z \coloneqq \frac{1}{3} \sum_{a=5}^{7} \Omega_\alpha^{1,1} \wedge \Omega_\alpha^{1,1}$, where

$\Omega^{1,1}_\alpha$ are the three induced Kähler forms on the cone. Equivalently, Sp(2) instantons[6] are described [75] as Hermitian Yang-Mills instantons with respect to all three complex structures J_α. Evaluating the holomorphicity conditions associated to the three Kähler forms $\Omega^{1,1}_\alpha$, one obtains the set of equations (D.10).

Note that the holomorphicity conditions with respect to any *two* Kähler forms imply also those of the third one. This is not surprising since two orthogonal characteristic vector fields (uniquely) imply a 3-Sasakian structure [53], and the corresponding property also holds for the metric cone. Moreover, the holomorphicity conditions already include the three stability-like conditions $\Omega^{1,1}_\alpha \lrcorner \mathcal{F} = 0$, as can be seen from the explicit expressions (D.10) and (D.11); this result has also been shown generally in [49, 128]. While in the special case of 4-dimensional hyper-Kähler structures the three instanton equations can be equally well considered either as holomorphicity or as moment map conditions (see Appendix A.2), in the higher-dimensional cases the holomorphicity conditions turn out to be more fundamental, simply because of the higher number of relations necessary to restrict the curvature.

By the choice of the canonical connection Γ as starting point in the matrix formulation of the gauge connection (6.18), the components occurring in the fields strength are completely determined by the *universal* geometry of the 3-Sasakian manifold, once the equivariance conditions have been implemented. Explicitly, the instanton equations (D.10) then yield the flow equations

$$\begin{aligned}
\dot{X}_1 &= -X_1 + [X_3, X_5] = -X_1 + [X_4, X_6] = -X_1 - [X_2, X_7], \\
\dot{X}_2 &= -X_2 - [X_4, X_5] = -X_2 + [X_3, X_6] = -X_2 + [X_1, X_7], \\
\dot{X}_3 &= -X_3 - [X_1, X_5] = -X_3 - [X_2, X_6] = -X_3 - [X_4, X_7], \\
\dot{X}_4 &= -X_4 + [X_2, X_5] = -X_4 - [X_1, X_6] = -X_4 + [X_3, X_7],
\end{aligned} \tag{6.25a}$$

and

$$\dot{X}_\alpha = -2X_\alpha - \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}[X_\beta, X_\gamma], \quad \alpha = 5, 6, 7 \tag{6.25b}$$

together with the algebraic relations

$$\begin{aligned}
&4X_5 = [X_1, X_3] - [X_2, X_4], \qquad 4X_6 = [X_1, X_4] + [X_2, X_3], \\
&4X_7 = -[X_1, X_2] - [X_3, X_4].
\end{aligned} \tag{6.25c}$$

By virtue of these algebraic relations, the differential equations (6.25b) for the vertical Higgs fields are implied by the equations (6.25a); see Appendix D.2 for details. Furthermore, the flow equations (6.25a) are cubic in the horizontal Higgs fields due to the algebraic conditions. The above flow equations agree with the general results derived in [34].

[6] A description of instantons on hyper-Kähler cones applying a completely quaternionic picture can also be found in [130].

Complex fields. For the further discussion and comparison with the Calabi-Yau setup of the previous chapters it is worth providing the complex form of these equations:

$$\dot{\phi}^{(1)} = -\phi^{(1)} - [\phi^{(2)\dagger}, \phi^{(3)}] = -\phi^{(1)} - \mathrm{i}[\phi^{(1)}, X_7] \tag{6.26a}$$

$$\dot{\phi}^{(2)} = -\phi^{(2)} + [\phi^{(1)\dagger}, \phi^{(3)}] = -\phi^{(2)} - \mathrm{i}[\phi^{(2)}, X_7] \tag{6.26b}$$

together with the algebraic relations

$$[\phi^{(1)}, \phi^{(2)}] = 2\phi^{(3)}, \quad [\phi^{(1)}, \phi^{(3)}] = 0 = [\phi^{(2)}, \phi^{(3)}], \tag{6.27a}$$

$$[\phi^{(1)}, \phi^{(1)\dagger}] + [\phi^{(2)}, \phi^{(2)\dagger}] = 2\mathrm{i}X_7 \ . \tag{6.27b}$$

HYM equations with respect to Ω_7. For a direct comparison with the Sasakian instanton equations, we also consider the subset of equations that has to be solved if the Hermitian Yang-Mills equations with respect to only *one* Kähler form are imposed; without loss of generality we consider Ω_7. The instanton condition then reads

$$\dot{\phi}^{(\alpha)} = -\phi^{(\alpha)} - \mathrm{i}[\phi^{(\alpha)}, X_7] \quad \text{for} \quad \alpha = 1, 2, \tag{6.28a}$$

$$\dot{X}_7 = -6X_7 - 2\mathrm{i}[\phi^{(1)}, \phi^{(1)\dagger}] - 2\mathrm{i}[\phi^{(2)}, \phi^{(2)\dagger}] - 2\mathrm{i}[\phi^{(3)}, \phi^{(3)\dagger}], \tag{6.28b}$$

$$[\phi^{(1)}, \phi^{(2)}] = 2\phi^{(3)}, \quad [\phi^{(1)}, \phi^{(3)}] = 0 = [\phi^{(2)}, \phi^{(3)}], \tag{6.28c}$$

and differentiating the algebraic relation gives us also $\dot{\phi}^{(3)} = -2\phi^{(3)} - \mathrm{i}[\phi^{(3)}, X_7]$. These HYM equations differ from those obtained for Sasaki-Einstein manifolds, (4.25) and (4.26), in some important aspects. Firstly, the fields $\phi^{(1)}$ and $\phi^{(2)}$ do not commute with each other (apart from setting $\phi^{(3)} \equiv 0$), so that the structure of the moduli space will be more involved (see the discussion in Section 6.5).

Secondly, while the coefficient of the linear term in the flow equation (6.28b) for X_7 matches the Sasaki-Einstein case, those of the other Higgs fields differ. In particular, the field $\phi^{(3)}$ stemming from the vertical part scales differently than the horizontal Higgs fields $\phi^{(1)}$ and $\phi^{(2)}$ in (6.28a).

The differences in the instanton equations can be attributed to the fact that the geometry is not that of a U(1)-bundle over a Kähler manifold but an SU(2)-bundle over a quaternionic manifold. Therefore, the triplet (X_5, X_6, X_7) differs from the other four matrices. Moreover, the starting point Γ in the ansatz for the gauge connection (6.18) is the canonical connection with respect to the 3-Sasakian geometry, which is *a* Hermitian Yang-Mills instanton. However, it is *not* (the lift of) the canonical connection with respect to the Sasaki-Einstein geometry, and therefore the generic discussion of [34, 51] does not apply here. This different situation has already been observed in the discussion of HYM instantons on the metric cone over the Aloff-Wallach space $X_{1,1}$ in [48].

Scalar ansatz. Despite the more complicated structure, the system of instanton equations admits – in contrast to the Sasaki-Einstein case – an analytic solution for the scalar ansatz of [19]: setting $X_a = \phi(\tau) I_a$ for $a = 1, \ldots, 4$ and $X_\alpha = \psi(\tau) I_\alpha$ for $\alpha = 5, 6, 7$ turns the instanton equations (6.25) into

$$\dot{\phi} = \phi(\psi - 1), \quad \dot{\psi} = 2\psi(\psi - 1), \quad \phi^2 = \psi, \tag{6.29}$$

which yields the analytic solution

$$\psi(\tau) = \left(1 + \mathrm{e}^{2(\tau - \tau_0)}\right)^{-1} \quad \text{and} \quad \phi(\tau) = \pm \psi(\tau)^{1/2}. \tag{6.30}$$

6.3 Examples of quiver diagrams

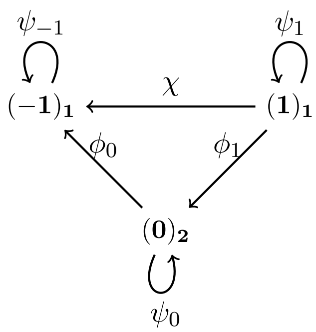
In this section we study four examples of quiver diagrams which arise from the representations $\underline{\mathbf{4}}$, $\underline{\mathbf{5}}$, $\underline{\mathbf{10}}$ and $\underline{\mathbf{14}}$, collected in Appendix D.3 (see also [108, Ch. 16]). We do not aim at a discussion of quiver diagrams for generic representations of Sp(2). As for the quiver gauge theories on $T^{1,1}$ and SU(4)/SU(3), we will consider the resulting quiver diagrams and derive the instanton matrix equations on the metric cone/cylinder.

6.3.1 Representation $\underline{\mathbf{4}}$

The simplest example stems from the fundamental representation (D.18) with the diamond (D.22) as weight diagram. It splits under restriction to the subgroup Sp(1) as

$$\underline{\mathbf{4}}|_{\mathrm{Sp}(1)} = \underline{(\mathbf{1},\mathbf{0})_{\mathbf{1}}} \oplus \underline{(-\mathbf{1},\mathbf{0})_{\mathbf{1}}} \oplus \underline{(\mathbf{0},-\mathbf{1})_{\mathbf{2}}}. \tag{6.31}$$

Collapsing the weight diagram along the subalgebra $\mathfrak{sp}(1)$ yields the following quiver:[7]



(6.32)

[7] The most general form of Higgs fields that are compatible with the equivariance conditions (6.19) alone is given in (D.24); for details, see also [49].

with the Higgs fields

$$\phi^{(1)} = \left(\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \phi_0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \phi_1 & 0 & 0 & 0 \end{array}\right) , \ \phi^{(2)} = \left(\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\phi_0 \\ \hline \phi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) , \ \phi^{(3)} = \left(\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ \hline \chi & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ,$$

$$X_7 = \mathrm{diag}(\psi_1, \psi_{-1}, \psi_0 \otimes \mathbf{1}_2) \ , \tag{6.33}$$

where the degrees of freedom are three homomorphisms $\phi_0 \in \mathrm{Hom}(E_0, E_{-1})$, $\phi_1 \in \mathrm{Hom}(E_1, E_0)$ and $\chi \in \mathrm{Hom}(E_1, E_{-1})$ as well as the endomorphisms $\psi_j \in \mathrm{End}(E_j)$ for $j \in \{0, \pm 1\}$. Since I_7 acts with eigenvalue zero on the vertex $(\mathbf{0})_\mathbf{2}$, one may set the corresponding endomorphism ψ_0 to zero, but we again keep it for the symmetry of the exposition, as done for the adjoint representation of SU(4) in (4.47) too.

In contrast to the Sasakian quiver gauge theories, one has obtained a triangular quiver with an arrow χ stemming from two of the vertical Higgs fields. In the spirit of the discussion in [48] and partially anticipating the reduction to S^4, which yields the quiver diagram (6.52), one might consider the result from a quaternionic point of view, based on the expression (6.8). Then the quiver diagram (6.32) is interpreted as a 2-quiver with a triplet $\mathbf{X} := (X_5, X_6, X_7)$ – combined into a single quaternionic quantity $\mathbf{X}$– acting *diagonally* on the two vertices: it acts trivially on the lower vertex and with fundamental action on the upper one. This approach corresponds to identifying vertices with the same quantum number ν_8, i.e. identifying along horizontal lines in the quiver diagrams.

Although it might be instructive to pursue this approach, in particular with respect to the quaternionic behavior of the instanton equations (6.25) and the generalization to higher-dimensional spheres (6.62), we will keep the normal complex weight diagrams for the description of the gauge connections.

The underlying triangular structure of the above quiver diagram (without the vertex loops) resembles the "holomorphic triangle" [42, Eq. (3.48)] in quiver gauge theory on the flag manifold Q_3 (cf. Section 7.4.2). The equivariant gauge connection (6.18) takes the form

$$\mathcal{A} = \Gamma + \left(\begin{array}{c|c|cc} A_1 + \psi_1 e^7 & -\chi^\dagger \otimes \Theta^3 & -\phi_1^\dagger \otimes \Theta^2 & -\phi_1^\dagger \otimes \Theta^1 \\ \hline \chi \otimes \Theta^{\bar 3} & A_{-1} + \psi_{-1} e^7 & \phi_0 \otimes \Theta^{\bar 1} & -\phi_0 \otimes \Theta^{\bar 2} \\ \hline \phi_1 \otimes \Theta^{\bar 2} & -\phi_0^\dagger \otimes \Theta^1 & A_0 + \psi_0 e^7 & 0 \\ \phi_1 \otimes \Theta^{\bar 1} & \phi_0^\dagger \otimes \Theta^2 & 0 & A_0 + \psi_0 e^7 \end{array}\right) . \tag{6.34}$$

As expected, this connection is an extension of the flat connection (6.7) by bundle maps, where the form of the canonical connection (6.15) follows from the generators (D.18). Equivariance requires the breaking of the structure group of the bundle $E \to M^d$ as

$$\mathrm{U}(2k_0 + k_1 + k_{-1}) \to \mathrm{U}(k_0) \times \mathrm{U}(k_1) \times \mathrm{U}(k_{-1}) \ , \tag{6.35}$$

where $k_j := \dim(E_j)$ denote the dimensions of the attached vector spaces of the quiver representation. The instanton matrix equations (6.26) and (6.27) consist of the two flow equations

$$\begin{aligned} \dot{\phi}_0 &= -\phi_0 + \chi\phi_1^\dagger = -\phi_0 - \mathrm{i}\phi_0\psi_0 + \mathrm{i}\psi_{-1}\phi_0 , \\ \dot{\phi}_1 &= -\phi_1 + \phi_0^\dagger\chi = -\phi_1 - \mathrm{i}\phi_1\psi_1 + \mathrm{i}\psi_0\phi_1 \end{aligned} \tag{6.36a}$$

and the quiver relations

$$\mathrm{i}\psi_1 = -\phi_1^\dagger\phi_1 , \quad \mathrm{i}\psi_{-1} = \phi_0\phi_0^\dagger , \quad 2\mathrm{i}\psi_0 = \phi_1\phi_1^\dagger - \phi_0^\dagger\phi_0 , \quad \chi = \phi_0\phi_1 . \tag{6.36b}$$

As a necessary consequence of the 3-Sasakian structure and in contrast to the Sasaki-Einstein case in the previous chapters, we observe the coupling to all three vertical fields in the flow equations (6.36a). The equations of motions for the vertical components can be obtained by differentiating the algebraic relations (6.36b) above, which yields $\dot{\chi} = -2\chi - \mathrm{i}\chi\psi_1 + \mathrm{i}\psi_{-1}\chi$ for instance. Due to the non-trivial algebraic relations, the entries of X_7 themselves (and not only the derivatives as on Sasaki-Einstein manifolds) couple to the horizontal fields. Similarly, the vertical field χ enters a quiver relation which requires commutativity of the triangle (6.32).

The instanton equations (6.36) admit the analytic solution (6.30) for the scalar ansatz. Stationary solutions, which might give some first insights into the moduli space of instantons and serve as starting points for numerical studies, require the homomorphisms ϕ_0 and ϕ_1 to satisfy

$$\phi_0(\mathbf{1}_{k_0} - \phi_1\phi_1^\dagger) = 0 , \qquad (\phi_0^\dagger\phi_0 - \mathbf{1}_{k_0})\phi_1 = 0 , \tag{6.37}$$

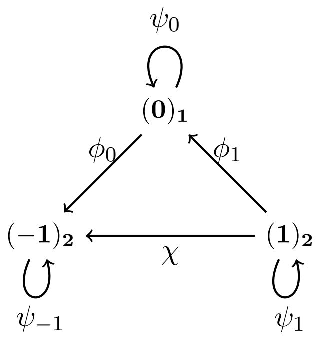
which again motivates setting $\psi_0 = 0$, due to (6.36b). Considering matrix-valued functions, one might find more general solutions than $\phi_1\phi_1^\dagger = \phi_0^\dagger\phi_0 = \mathbf{1}_{k_0}$ and the trivial solution.

6.3.2 Representation $\underline{5}$

The 5-dimensional representation of Sp(2) is given by the generators (D.28) and the weight diagram (D.27), so that the decomposition under Sp(1) yields

$$\underline{\mathbf{5}}|_{\mathrm{Sp}(1)} = \underline{(-\mathbf{1},-\mathbf{1})_{\mathbf{2}}} \oplus \underline{(\mathbf{0},\mathbf{0})_{\mathbf{1}}} \oplus \underline{(\mathbf{1},-\mathbf{1})_{\mathbf{2}}} \ . \tag{6.38}$$

The collapsing procedure of the weight diagram gives rise to the quiver



(6.39)

and the Higgs fields

$$\phi^{(1)} = \left(\begin{array}{cc|c|cc} 0 & 0 & \phi_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \phi_1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \phi^{(2)} = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_0 & 0 & 0 \\ \hline 0 & 0 & 0 & \phi_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\phi^{(3)} = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & \chi & 0 \\ 0 & 0 & 0 & 0 & -\chi \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad X_7 = \mathrm{diag}(\psi_{-1} \otimes \mathbf{1}_2, \psi_0, \psi_1 \otimes \mathbf{1}_2) \tag{6.40}$$

with $\phi_0 \in \mathrm{Hom}(E_0, E_{-1})$, $\phi_1 \in \mathrm{Hom}(E_1, E_0)$, $\chi \in \mathrm{Hom}(E_1, E_{-1})$, and $\psi_j \in \mathrm{End}(E_j)$ for $j \in \{0, \pm 1\}$, where one may set $\psi_0 = 0$ due to the action of I_7. The structure group of the equivariant bundle is broken, according to (3.5), as

$$\mathrm{U}(k_0 + 2k_1 + 2k_{-1}) \to \mathrm{U}(k_0) \times \mathrm{U}(k_1) \times \mathrm{U}(k_{-1}) \ . \tag{6.41}$$

Since the quiver diagram resembles the quiver (6.32) of the fundamental representation, one obtains instanton equations with the same structure, similar to the correspondences between quiver diagrams for different representations in Sasakian quiver gauge theory. More precisely, the instanton conditions comprise the differential equations

$$\begin{aligned} \dot{\phi}_0 &= -\phi_0 + \chi\phi_1^\dagger = -\phi_0 - \mathrm{i}\phi_0\psi_0 + \mathrm{i}\psi_{-1}\phi_0, \\ \dot{\phi}_1 &= -\phi_1 + \phi_0^\dagger\chi = -\phi_1 - \mathrm{i}\phi_1\psi_1 + \mathrm{i}\psi_0\phi_1 \end{aligned} \tag{6.42a}$$

and the quiver relations

$$2\mathrm{i}\psi_{-1} = \phi_0\phi_0^\dagger, \quad \mathrm{i}\psi_0 = \phi_1\phi_1^\dagger - \phi_0^\dagger\phi_0, \quad 2\mathrm{i}\psi_1 = -\phi_1^\dagger\phi_1, \quad 2\chi = \phi_0\phi_1. \tag{6.42b}$$

The flow equations (6.42a) coincide with (6.36a), while the algebraic conditions (6.42b) differ from the equations derived for the fundamental representation only in numerical values of the coefficients. Despite yielding the same quivers and similar instanton equations, this representation constitutes a different gauge theory since the dimensions of the internal spaces V_j in the quiver bundle and the form of the canonical connection differ from the case $\underline{\mathbf{4}}$.

Stationary points (ϕ_0, ϕ_1) of the instanton equations (6.42) have to satisfy

$$\phi_0(\mathbf{1}_{k_0} - \tfrac{1}{2}\phi_1\phi_1^\dagger) = -\phi_0 + \tfrac{1}{2}\phi_0\phi_0^\dagger\phi_0 = 0 = (\mathbf{1}_{k_0} - \tfrac{1}{2}\phi_0^\dagger\phi_0)\phi_1 = -\phi_1 + \tfrac{1}{2}\phi_1\phi_1^\dagger\phi_1, \tag{6.43}$$

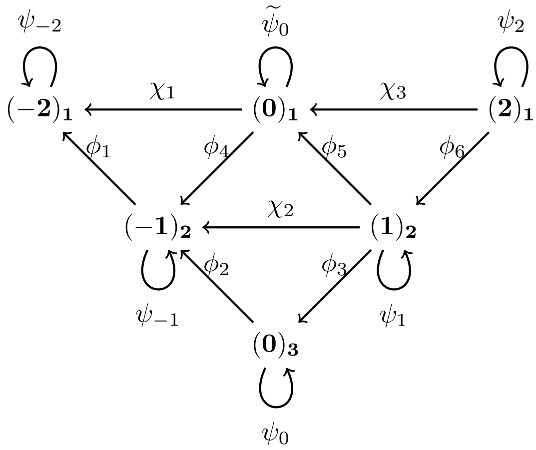
which is solved for $\frac{1}{2}\phi_1\phi_1^\dagger = \frac{1}{2}\phi_0^\dagger\phi_0 = \mathbf{1}_{k_0}$ and the trivial solution.

6.3.3 Adjoint representation $\underline{10}$

According to the weight diagram (D.30), the 10-dimensional adjoint representation (D.32) of Sp(2) decomposes into six subspaces,

$$\underline{\mathbf{10}}|_{\mathrm{Sp}(1)} = \underline{(\mathbf{-2},\mathbf{0})_{\mathbf{1}}} \oplus \underline{(\mathbf{-1},\mathbf{-1})_{\mathbf{2}}} \oplus \underline{(\mathbf{0},\mathbf{-2})_{\mathbf{3}}} \oplus \underline{(\mathbf{0},\mathbf{0})_{\mathbf{1}}} \oplus \underline{(\mathbf{1},\mathbf{-1})_{\mathbf{2}}} \oplus \underline{(\mathbf{2},\mathbf{0})_{\mathbf{1}}}. \tag{6.44}$$

Collapsing the weight diagram of $\underline{\mathbf{10}}$ along the action of I_4^- yields the quiver



(6.45)

and the Higgs fields schematically take the block matrix form

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & \phi_1 \otimes I_1^{(\alpha)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_2 \otimes I_2^{(\alpha)} & \phi_4 \otimes I_4^{(\alpha)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_3 \otimes I_3^{(\alpha)} & 0 \\ 0 & 0 & 0 & 0 & \phi_5 \otimes I_5^{(\alpha)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 \otimes I_6^{(\alpha)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.46a}$$

for $\alpha = 1, 2$, where the matrices $I_i^{(\alpha)}$ act inside the H representations according to the entries of the generators (D.32). The vertical Higgs fields are given by

$$\phi^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \chi_1 \otimes I_1^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi_2 \otimes I_2^{(3)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \chi_3 \otimes I_3^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.46b}$$

and the diagonal contribution $X_7 = \mathrm{diag}(\psi_{-2}, \psi_{-1} \otimes \mathbf{1}_2, \psi_0 \otimes \mathbf{1}_3, \widetilde{\psi}_0 . \psi_1 \otimes \mathbf{1}_2, \psi_2)$, where the same comments as before apply to ψ_0 and $\widetilde{\psi}_0$. The structure group is broken into a product of six smaller unitary groups. Evaluating the instanton equations (6.26) and (6.27) yields the following system of differential equations:

$$
\begin{aligned}
\dot\phi_1 &= -\phi_1 + \chi_1\phi_4^\dagger &&= -\phi_1 - \mathrm{i}\phi_1\psi_{-1} + \mathrm{i}\psi_{-2}\phi_1,\\
\dot\phi_2 &= -\phi_2 + \chi_2\phi_3^\dagger &&= -\phi_2 - \mathrm{i}\phi_2\psi_0 + \mathrm{i}\psi_{-1}\phi_2,\\
\dot\phi_3 &= -\phi_3 + \phi_2^\dagger\chi_2 &&= -\phi_3 - \mathrm{i}\phi_3\psi_1 + \mathrm{i}\psi_0\phi_3,\\
\dot\phi_4 &= -\phi_4 + \phi_1^\dagger\chi_1 - \chi_2\phi_5^\dagger &&= -\phi_4 - \mathrm{i}\phi_4\widetilde\psi_0 + \mathrm{i}\psi_{-1}\phi_4,\\
\dot\phi_5 &= -\phi_5 - \phi_4^\dagger\chi_2 + \chi_3\phi_6^\dagger &&= -\phi_5 - \mathrm{i}\phi_5\psi_1 + \mathrm{i}\widetilde\psi_0\phi_5,\\
\dot\phi_6 &= -\phi_6 + \phi_5^\dagger\chi_3 &&= -\phi_6 - \mathrm{i}\phi_6\psi_2 + \mathrm{i}\psi_1\phi_6.
\end{aligned} \tag{6.47a}
$$

The algebraic conditions read

$$
\begin{aligned}
&\chi_1 = \phi_1\phi_4, && \chi_3 = \phi_5\phi_6, && \phi_1\chi_2 = \chi_1\phi_5,\\
&\chi_2\phi_6 = \phi_4\chi_3, && 2\chi_2 = 3\phi_2\phi_3 - \phi_4\phi_5, &&
\end{aligned} \tag{6.47b}
$$

as well as

$$
\begin{aligned}
\mathrm{i}\psi_{-2} &= \phi_1\phi_1^\dagger, & 2\mathrm{i}\psi_{-1} &= 3\phi_2\phi_2^\dagger - \phi_1^\dagger\phi_1 + \phi_4\phi_4^\dagger,\\
\mathrm{i}\psi_0 &= \phi_3\phi_3^\dagger - \phi_2^\dagger\phi_2, & \mathrm{i}\widetilde\psi_0 &= \phi_5\phi_5^\dagger - \phi_4^\dagger\phi_4,\\
2\mathrm{i}\psi_1 &= -3\phi_3^\dagger\phi_3 + \phi_6\phi_6^\dagger - \phi_5^\dagger\phi_5, & \mathrm{i}\psi_2 &= -\phi_6^\dagger\phi_6,
\end{aligned} \tag{6.47c}
$$

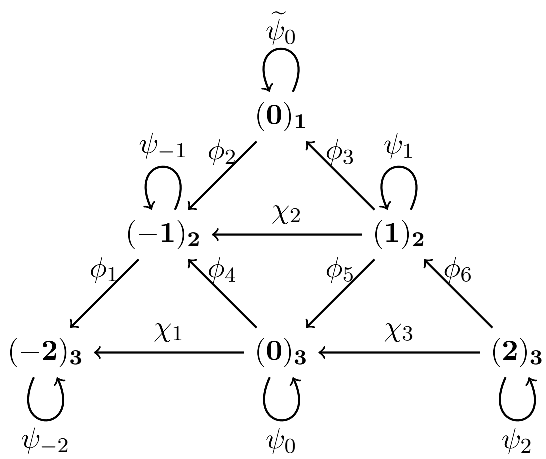
which impose quiver relations on the diagram (6.45). A priori, one recognizes commutativity of the two triangles in the upper left and upper right corner of (6.44) as well as commutativity of the two rhombi $\overline{(2,0,-1,1)}$ and $\overline{(-2,-1,1,0)}$. Including also the last condition in (6.47b) and manipulating the expressions, we find that commutativity of all cells in the quiver diagram satisfies the algebraic conditions.

6.3.4 Representation $\underline{14}$

From the weight diagram (D.33) one sees that the 14-dimensional representation (D.35) decomposes under restriction to the subgroup Sp(1) as

$$
\underline{\mathbf{14}}|_{\mathrm{Sp}(1)} = \underline{(\mathbf{-2,-2})_3} \oplus \underline{(\mathbf{-1,-1})_2} \oplus \underline{(\mathbf{0,-2})_3} \oplus \underline{(\mathbf{0,0})_1} \oplus \underline{(\mathbf{1,-2})_2} \oplus \underline{(\mathbf{2,-2})_3}. \tag{6.48}
$$

This splitting and the equivariance condition lead to the following quiver diagram:



(6.49)

It has the same shape as that of the adjoint action, and the form of the Higgs fields shows the same block structure as (6.46), so that we refrain from presenting them again. Consequently, also the matrix equations for Sp(2) instantons are very similar:

$$\begin{aligned}
\dot{\phi}_1 &= -\phi_1 - \chi_1\phi_4^\dagger &&= -\phi_1 - \mathrm{i}\phi_1\psi_{-1} + \mathrm{i}\psi_{-2}\phi_1, \\
\dot{\phi}_2 &= -\phi_2 - \chi_2\phi_3^\dagger &&= -\phi_2 - \mathrm{i}\phi_2\widetilde{\psi}_0 + \mathrm{i}\psi_{-1}\phi_2, \\
\dot{\phi}_3 &= -\phi_3 - \phi_2^\dagger\chi_2 &&= -\phi_3 - \mathrm{i}\phi_3\psi_1 + \mathrm{i}\widetilde{\psi}_0\phi_3, \\
\dot{\phi}_4 &= -\phi_4 - \phi_1^\dagger\chi_1 + \chi_2\phi_5^\dagger &&= -\phi_4 - \mathrm{i}\phi_4\psi_0 + \mathrm{i}\psi_{-1}\phi_4, \\
\dot{\phi}_5 &= -\phi_5 + \phi_4^\dagger\chi_2 - \chi_3\phi_6^\dagger &&= -\phi_5 - \mathrm{i}\phi_5\psi_1 + \mathrm{i}\psi_0\phi_5, \\
\dot{\phi}_6 &= -\phi_6 - \phi_5^\dagger\chi_3 &&= -\phi_6 - \mathrm{i}\phi_6\psi_2 + \mathrm{i}\psi_1\phi_6,
\end{aligned} \tag{6.50a}$$

together with the algebraic relations

$$\begin{aligned}
\chi_1 &= -\phi_1\phi_4, & 2\chi_2 &= 3\phi_4\phi_5 - \phi_2\phi_3, & \chi_3 &= -\phi_5\phi_6, \\
\chi_1\phi_5 &= \phi_1\chi_2, & \chi_2\phi_6 &= \phi_4\chi_3,
\end{aligned} \tag{6.50b}$$

and

$$\begin{aligned}
\mathrm{i}\psi_{-2} &= 2\phi_1\phi_1^\dagger, & 2\mathrm{i}\psi_{-1} &= 3\phi_4\phi_4^\dagger - 3\phi_1^\dagger\phi_1 + \phi_2\phi_2^\dagger, \\
\mathrm{i}\psi_0 &= \phi_5\phi_5^\dagger - \phi_4^\dagger\phi_4, & \mathrm{i}\widetilde{\psi}_0 &= \phi_3\phi_3^\dagger - \phi_2^\dagger\phi_2, \\
\mathrm{i}\psi_2 &= -\phi_6^\dagger\phi_6, & 2\mathrm{i}\psi_1 &= 3\phi_6\phi_6^\dagger - 3\phi_5^\dagger\phi_5 - \phi_3^\dagger\phi_3.
\end{aligned} \tag{6.50c}$$

The same structure of the quiver diagram compared to the adjoint representation has again induced the same flow equations with minor changes in coefficients occurring in the algebraic relations (6.50b) and (6.50c).

6.4 Reduction to S^4

The structure of the squashed seven-sphere as Sp(1)-bundle over S^4 motivates to study the reduction of the 3-Sasakian quiver gauge theory from Sp(2)/Sp(1) to S^4. Similarly to the reductions discussed in Sections 4.4.1 and 5.4, one has to set the Higgs fields proportional to the relevant generators to be divided out, which reads $\phi^{(3)} = I_3^-$ and $X_7 = I_7$ in the case at hand.[8] One obtains the additional equivariance conditions

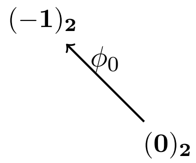
$$[\widetilde{I}_8, \phi^{(1)}] = \phi^{(1)}, \qquad [\widetilde{I}_8, \phi^{(2)}] = -\phi^{(2)}, \tag{6.51a}$$

$$[I_3^-, \phi^{(1)}] = 0 = [I_3^-, \phi^{(2)}], \qquad [I_3^-, \phi^{(1)\dagger}] = -\phi^{(2)}, \quad [I_3^-, \phi^{(2)\dagger}] = \phi^{(1)}, \tag{6.51b}$$

[8]The reduction $X_7 = I_7$ yields quiver gauge theories on the twistor space over S^4, whose quiver diagrams simply constitute the quivers for the squashed seven-sphere without the vertex loops.

which require a (further) collapsing of the weight and quiver diagrams along the action of the ladder operator I_3^-, according to the root system (D.21).

The first example we consider is the fundamental representation $\underline{\mathbf{4}}$, which yielded the quiver diagram (6.32). Imposing the additional conditions (6.51) requires $\phi_1 = -\phi_0^\dagger$, and the quiver diagram (after collapsing) is given by the chain



(6.52)

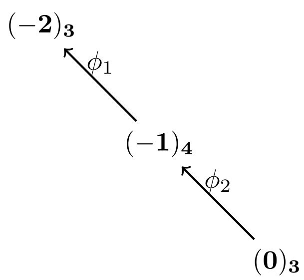
The remaining two complex Higgs fields read

$$\phi^{(1)} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\phi_0^\dagger & 0 & 0 & 0 \end{array}\right) , \qquad \phi^{(2)} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_0 \\ \hline -\phi_0^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) , \tag{6.53}$$

with the homomorphism $\phi_0 \in \mathrm{Hom}(E_{k_0}, E_{k_{-1}})$ being the only remaining degree of freedom. By repeating the quaternionic interpretation given above, the quiver (6.32) may be thought of as the "quaternionic chain" (6.52) acted on by a triplet (X_5, X_6, X_7) of vertical fields, according to the 3-Sasakian bundle structure.

The Sp(2)-instanton equations (6.36a) and (6.36b) simplify to $\phi_0\phi_0^\dagger = \mathbf{1}_{k_{-1}}$ and require a constant field ϕ_0. Thus, as expected, they satisfy the conditions for stationary solutions (6.37).

In the case of the adjoint representation $\underline{\mathbf{10}}$, the additional equivariance conditions (6.51) turn the quiver (6.45) into a chain of length 3:



(6.54)

On the level of the Higgs fields (6.46), this comprises the relations $\phi_3 = -\phi_2^\dagger$, $\phi_4 = \frac{1}{\sqrt{2}}\phi_1^\dagger$, $\phi_5 = -\frac{1}{\sqrt{2}}\phi_1$, and $\phi_6 = \phi_1^\dagger$. Hence, the instanton equations (6.47a) and (6.47b) reduce to the conditions

$$\mathbf{1}_{k_{-2}} = \tfrac{1}{2}\phi_1\phi_1^\dagger, \qquad \mathbf{1}_{k_{-1}} = \tfrac{3}{2}\phi_2\phi_2^\dagger - \tfrac{1}{4}\phi_1^\dagger\phi_1 \tag{6.55}$$

and require constant fields ϕ_1 and ϕ_2; again one encounters BPS equations of the typical form.

6.5 Moduli space of $\mathrm{Sp}(2)$ instantons

This section deals with the description of the moduli space of Sp(2) instantons, which applies not only to the hyper-Kähler cone over Sp(2)/Sp(1) [49], but also to the cone over $X_{1,1}$ (cf. Chapter 7). A detailed discussion of $\mathrm{Sp}(n)$ instantons on metric cones over generic 3-Sasakian manifolds, analogous to the study of HYM instantons on cones over generic Sasaki-Einstein manifolds [51], can be found in [128]. The result of [75] and the discussion of the Sp(2)-instanton equations in Section 6.2.4 imply the following features of the moduli space $\mathcal{M}$:

(i) The moduli space $\mathcal{M}$ of Sp(2) instantons is given by the intersection of three Hermitian Yang-Mills moduli spaces M_α with respect to the Kähler forms Ω_α, $\mathcal{M} = \mathcal{M}_5 \cap \mathcal{M}_6 \cap \mathcal{M}_7$.

(ii) Each of these single moduli spaces $\mathcal{M}_\alpha$ can be discussed similarly to the moduli spaces of instantons on Calabi-Yau cones, but the algebraic conditions differ.

(iii) The intersection of the three moduli spaces $\mathcal{M}_\alpha$ is completely characterized by the holomorphicity conditions alone.

Orbit construction. Let us first discuss the moduli space $\mathcal{M}_\alpha$ defined by a single set of HYM equations, w.l.o.g. (6.28). By a rescaling of the Higgs fields similar to (4.61) and (5.48), the instanton equations (6.28) turn into [48, 128]

$$\frac{\mathrm{d}W_\alpha}{\mathrm{d}s} = 2[W_\alpha, Z], \quad \text{for} \quad \alpha = 1,2,3 \ , \tag{6.56a}$$

$$[W_1, W_2] = 2W_3, \qquad [W_1, W_3] = 0 = [W_2, W_3], \tag{6.56b}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}(Z + Z^\dagger) + 2[Z, Z^\dagger] + 2\sum_{\alpha=1}^{2} (-6s)^{-\frac{5}{3}} [W_\alpha, W_\alpha^\dagger] + 2\,(-6s)^{-\frac{4}{3}} [W_3, W_3^\dagger], \tag{6.56c}$$

where one has defined $s := -\frac{1}{6}\mathrm{e}^{-6\tau}$ as in the Sasaki-Einstein case, but the other scaling functions differ. The gauge transformation (4.64) is still applicable, so that one obtains a local solution as orbit of a constant tuple that satisfies the algebraic conditions (6.56b). As before, the real equation can be solved as equation of motion of a suitable Lagrangian, where the different scaling of the matrices causes minor changes in the functions in front of the commutators of the Higgs fields, compared to (4.66), (5.50) and [51, Eq. (3.24)]. More precisely, the real equation now follows as the equation of motion of the Lagrangian [48, 128]

$$\mathcal{L}[g] = \int_{\mathcal{I}} \mathrm{d}s \ \mathrm{Tr}\left\{ |Z^g + Z^{g\dagger}|^2 + 2\,(-6s)^{-\frac{5}{4}} \sum_{\alpha=1}^{2} |W_\alpha^g|^2 + 2\,(-6s)^{-\frac{4}{3}}\, |W_3^g|^2 \right\}, \tag{6.57}$$

which holds for the same arguments as (4.66). Therefore, the discussion of the moduli space reduces to describing orbits of constant solutions of the complex equations under certain boundary conditions. For Sasakian quiver gauge theories these matrices could be chosen as elements of a Cartan subalgebra because the algebraic conditions required mutually commuting Higgs fields $\phi^{(\alpha)}$ (cf. (4.25) and (5.31)). In the hyper-Kähler case, however, the algebraic conditions (6.56b) constitute a (complexified) Heisenberg algebra, comprising the non-trivial relation $[\phi^{(1)}, \phi^{(2)}] = 2\phi^{(3)}$. Therefore, one cannot consider regular elements of a Cartan subalgebra as model solutions to the instanton equations only, as it was applicable in [51, 111, 112], unless one sets $\phi^{(3)} \equiv 0$. The latter subcase is, of course, not of particular interest because then the truly new features of 3-Sasakian quiver gauge theories are lost.

It is worth recalling the importance of boundary conditions for the description of the moduli space of Nahm's equations and its adaptation to instantons on Calabi-Yau cones [45]. General solutions are not only characterized by elements of a Cartan subalgebra but may include a second set of matrices which satisfy the $\mathfrak{su}(2)$ commutation relations of the original equations and which commute with the elements of the Cartan subalgebra [112, Sec. 4] (see also [44, Sec. 6]). The extreme case in which solutions are only described by this set has been studied by Kronheimer in [131] and led to nilpotent orbits. This treatment can be extended to HYM instantons on Calabi-Yau cones of any dimension [128]. Intermediate cases containing both sets of matrices have been considered by Biquard [132] and Kovalev [133] for Nahm's equations.

For the single HYM moduli spaces arising in the context of hyper-Kähler instantons it seems necessary to study possible boundary conditions and their implications in more detail, analogously to the more general cases that can occur for Nahm's equations. Such an exhaustive treatment of the equations (6.28) requires a suitable description of the (complexified) Heisenberg algebra contained in the algebraic relations (6.28c).

Structure of the intersection. While a single moduli space M_α still admits a similar description by virtue of the gauge transformation (4.64), the intersection $\mathcal{M} = \bigcap_\alpha \mathcal{M}_\alpha$ is very restrictive. Because of the SU(2)-symmetry of the equations (6.25a), all three vertical Higgs fields have to transform in the very same way. However, the non-trivial algebraic relations (6.25c) and the corresponding flow equations then only allow for constant gauge transformations (4.64). Hence, a description in terms of orbits of constant elements is not applicable for the intersection, unless a suitable gauge transformation which preserves the 3-Sasakian symmetries is found. A purely quaternionic formulation may prove useful for this purpose.

The moduli space $\mathcal{M} = \bigcap_\alpha \mathcal{M}_\alpha$ constitutes a *tri-holomorphic* or *quaternionic space* because it satisfies the holomorphicity conditions with respect to all three complex structures J_α, which are compatible with the equivariance conditions [128].[9] Using

[9] As discussed in Section 2.4, the invariant 4-form (2.22) also occurs as calibration form for quater-

the real structure constants defined by (6.10a) and the complex forms as in (D.11), one verifies the compatibility of the equivariance conditions

$$[I_8, X_1] = X_2, \quad [I_8, X_2] = -X_1, \quad [I_8, X_3] = -X_4, \quad [I_8, X_4] = X_3, \tag{6.58}$$

with the three complex structures:

$$\begin{aligned}
[I_8, J_5X_1] &= [I_8, -X_3] = X_4 = J_5X_2, \\
[I_8, J_6X_1] &= [I_8, -X_4] = -X_3 = J_6X_2, \\
[I_8, J_7X_1] &= [I_8, X_2] = -X_1 = J_7X_2.
\end{aligned} \tag{6.59}$$

The compatibility of the remaining conditions follows from the symmetries inherent in the 3-Sasakian and hyper-Kähler geometry.

Remark on quotient construction. As mentioned earlier, the case of 4-dimensional hyper-Kähler instantons is very special because then the three instanton equations may be considered as the zero locus of the three moment map conditions, without further conditions [62]. This directly leads to a description of the moduli space as a hyper-Kähler quotient.

On the 8-dimensional cone over Sp(2)/Sp(1), however, one has to impose the set of holomorphicity conditions that already imply the three moment map conditions. Therefore, the space of tri-holomorphic connections may be regarded as a quaternionic subspace of the zero locus of the moment maps, but its geometric structure is not yet fully understood. In particular, one cannot deduce that this subspace is hyper-Kähler. Even if it was hyper-Kähler, the usual quotient construction would be trivial because the moment maps, which correspond to the stability-like conditions, are identically zero on the space of tri-holomorphic connections.

6.6 Higher-dimensional spheres and discussion

We conclude this chapter by sketching some aspects of the generalization to higher-dimensional squashed spheres and summarizing the main differences compared to Sasakian quiver gauge theories. Some features of quiver gauge theories on generic squashed spheres can be deduced by simple dimension arguments.

Fundamental representation. The Lie group $\mathrm{Sp}(m)$ has dimension $2m^2+m$, and the fundamental representation acts on $\mathbb{C}^{2m}$ because of the embedding $\mathrm{Sp}(m) \hookrightarrow \mathrm{U}(2m)$

nionic submanifolds of $\mathbb{R}^8 \cong \mathbb{H}^2$ [92, Sec. 3.2], so that the characterization of $\mathcal{M}$ as a quaternionic subspace of the space of connections fits into the close relationship between both fields.

(or simply because of $\mathbb{H} = \mathbb{C}^2$). When restricting the fundamental representation $\underline{\mathbf{2(m+1)}}$ of $\mathrm{Sp}(m+1)$ to the subgroup $\mathrm{Sp}(m)$, one therefore obtains the splitting

$$\underline{\mathbf{2(m+1)}}|_{\mathrm{Sp}(m)} = \underline{\mathbf{2m}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{1}}, \tag{6.60}$$

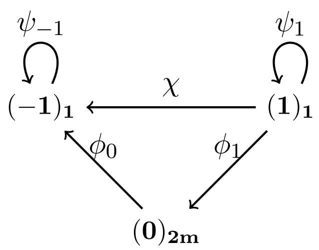
which suggests the triangular quiver diagram (6.32) for all dimensions m. More precisely, the embedding of $\mathrm{Sp}(m)$ into $\mathrm{Sp}(m+1)$ is similar to the case of the round spheres (4.87), but now one has to arrange all contributions in complex (2×2)-blocks (allowing for the quaternionic action):

$$\mathrm{Sp}(m) \longmapsto \left(\begin{array}{c|c} \mathbf{1}_2 & 0 \\ \hline 0 & \mathrm{Sp}(m) \end{array}\right) \subset \mathrm{Sp}(m+1), \tag{6.61}$$

and the local section (6.8) is generalized to

$$\mathcal{A}_0 = \left(\begin{array}{c|ccc} \mathrm{Im}(q_0) & q_1 & \dots & q_m \\ \hline -\bar{q}_1 & & & \\ \vdots & & \mathfrak{sp}(m) & \\ -\bar{q}_m & & & \end{array}\right). \tag{6.62}$$

Here the q_j denote quaternionic 1-forms (or certain complex (2×2)-matrices) which may be parametrized as the matrix x in the local section (6.2). The $\mathrm{SU}(2)$-factor giving rise to the family of Sasaki-Einstein structures is identified with the imaginary quaternions $\mathrm{Im}(q_0) \in \mathrm{Im}(\mathbb{H}) \cong \mathfrak{su}(2)$. Therefore, the quiver diagram for the fundamental representation of any squashed sphere is indeed given by



(6.63)

where the form of the Higgs fields directly follows from the choice of the generators based on (6.62). This yields a breaking of the structure group as

$$\mathrm{U}(k_1 + k_{-1} + 2mk_0) \to \mathrm{U}(k_1) \times \mathrm{U}(k_{-1}) \times \mathrm{U}(k_0). \tag{6.64}$$

Adjoint representation. For the adjoint representation $\underline{\mathbf{2m^2+5m+3}}$ of $\mathrm{Sp}(m+1)$, a simple counting argument yields the decomposition

$$\underline{\mathbf{2m^2+5m+3}}|_{\mathrm{Sp}(m)} = \underline{\mathbf{2m^2+m}} \oplus \underline{\mathbf{2m}} \oplus \underline{\overline{\mathbf{2m}}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{1}}, \tag{6.65}$$

i.e. a splitting into one adjoint, one fundamental and one anti-fundamental as well as three trivial representations of $\mathrm{Sp}(m)$. This splitting can be directly seen in (6.62) as well. It induces the generalization of the quiver diagram (6.45) with six vertices for all dimensions m.

Instanton equations. Based on the general expressions for the torsion components of the canonical connection on 3-Sasakian manifolds, one derives the instanton equations for generic hyper-Kähler cones [34, 128], which are collected in Appendix D.2.1.

These equations reveal the same properties of instantons on the metric cones over any squashed sphere $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ for $m \geq 1$. In particular, one again faces the non-trivial algebraic conditions that cause the more complicated structure of the moduli space, compared to HYM instantons on Calabi-Yau cones. On the other hand, due to the completely regular instanton equations for any m, one can easily discuss the generic case once instantons on the cone over the squashed seven-sphere are fully understood.

In other words, the description of instantons on generic Calabi-Yau cones was possible as a generalization of Nahm's equations on the cone over $\mathrm{SU}(2) \simeq \mathrm{U}(1) \times \mathbb{C}P^1$ to the case of cones over Sasaki-Einstein manifolds as U(1)-bundles over $\mathbb{C}P^n$ (or, more generally, $2n$-dimensional Kähler manifolds). For hyper-Kähler cones however, the literature approach deals only with the trivial 3-Sasakian manifold SU(2), whose bundle structure is of the form $\{\text{point}\} \times \mathrm{SU}(2)$. The squashed seven-sphere represents the first non-trivial example, being of the local form $\mathrm{QK}_4 \times \mathrm{SU}(2)$.

Therefore, the proper description of the moduli space seems to require new methods that are adapted to both the underlying quaternionic symmetry and the SU(2) triplet of vertical Higgs fields. Tools for the description of the quaternionic base might be borrowed from algebraic discussions of quaternionic instantons, such as [75, 130], but one has to include the differential equations (6.25a) with their inherent symmetries as well.

6.6.1 Comparison with Sasakian quiver gauge theories

The most fundamental difference to the case of *Sasakian* quiver gauge theory is the occurrence of the SU(2)-triplet of vertical Higgs fields in the 3-Sasakian setup, as a manifestation of the different bundle structure of 3-Sasakian manifolds compared to Sasaki-Einstein structures.

As a consequence, the examples of quiver diagrams studied here typically contain "holomorphic triangles" with vertex loops as building blocks, in contrast to the chains or rectangles we have seen for the Sasakian quiver gauge theories. This is caused by the fact that two of the vertical Higgs fields represent a further ladder operator in the weight diagrams. We sketched a quaternionic interpretation in which one combines the three vertical Higgs fields into a single quaternionic quantity. It corresponds to collapsing the quiver along the vertical ladder operator. This point of view is beneficial for the general understanding and for elaborating the analogies to the round spheres.

It might prove useful also for the instanton equations since their more complicated structure seems to require an entirely quaternionic formulation.

Since the Lie algebra $\mathfrak{sp}(2)$ is of rank 2 and symmetrically contains two subalgebras $\mathfrak{sp}(1)$, its weight diagrams [108] are typically squares or diamonds, rotated by $\pi/4$. Hence, collapsing such weight diagrams yields triangles and leads to correspondences of quiver diagrams for different representations of Sp(2). For example, both the fundamental and the 5-dimensional representation induced the same quiver diagram, and the representations $\underline{\mathbf{10}}$ and $\underline{\mathbf{14}}$ also yielded a quiver of the same shape. Like the similarities of some quivers found in the previous chapters, triangular quiver diagrams of the same shape encode the same flow equations, and only the algebraic conditions differ in some coefficients.

7 Quiver gauge theory on the Aloff-Wallach space $X_{1,1}$

Having constructed 3-Sasakian quiver gauge theories on the squashed seven-sphere in the previous chapter, we now include examples of quiver diagrams on the Aloff-Wallach space $X_{1,1}$ [134], given as coset $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$. The Sasaki-Einstein manifold $T^{1,1}$ completed the studies in dimension 5, being the only homogeneous compact Sasaki-Einstein manifold besides the sphere. Similarly, the Aloff-Wallach space $X_{1,1}$ is a natural candidate for 3-Sasakian quiver gauge theory in dimension 7 since every simply-connected compact spin manifold with regular 3-Sasakian structure is isometric either to $X_{1,1}$ or the squashed seven-sphere [53, 59]. Applications of $X_{1,1}$ in string theory arise due to its Sasaki-Einstein geometry or because of the G_2 and Spin(7) structure inherent in 7-dimensional 3-Sasakian manifolds [21, 135, 136]. Instantons with respect to the Spin(7) structure on metric cones over 3-Sasakian coset spaces, including $X_{1,1}$, have been considered in [27, 28].

The author's article [48] has studied Sasakian quiver gauge theory on $X_{1,1}$ associated to *one* of the three Sasaki-Einstein structures, which comprises examples of quiver diagrams for low-dimensional representations of SU(3). We extend this discussion to the full 3-Sasakian picture here. The results will be compared to the quiver gauge theories on Sp(2)/Sp(1) from Chapter 6 as well as to those on the underlying flag manifold Q_3, constructed in [41, 42, 98].

7.1 Geometric structure

As for the other homogeneous spaces before, the geometric structure of $X_{1,1}$ is described by starting from a local section. The choice of generators and the discussion of the 3-Sasakian geometry closely follows [27]. A detailed discussion of the spinors on $X_{1,1}$ can be found in [53], while descriptions of the related flag manifold Q_3 are given e.g. in [41, 42]. The Aloff-Wallach[1] spaces [134] $X_{k,l}$ (with coprime integers k and l) are defined as quotients

$$X_{k,l} \coloneqq \mathrm{SU}(3)/\mathrm{U}(1)_{k,l}, \tag{7.1}$$

[1]The notations $N_{k,l}$ and $N(k,l)$ for the Aloff-Wallach spaces $X_{k,l}$ appear in the literature as well, and the flag manifold Q_3 is also known as F_3 or $F(1,2)$.

where the embedding of $h \in \mathrm{U}(1)_{k,l}$ into $G = \mathrm{SU}(3)$ is given by

$$h = \mathrm{diag}(\mathrm{e}^{\mathrm{i}(k+l)\varphi}, \mathrm{e}^{-\mathrm{i}k\varphi}, \mathrm{e}^{-\mathrm{i}l\varphi}). \tag{7.2}$$

In this thesis, we are only interested in the case $k = l = 1$, which leads not only to a Sasaki-Einstein but even a 3-Sasakian manifold; the study of Spin(7) instantons in [27] includes the general case $X_{k,l}$.

7.1.1 Local section

Making use of the 3-Sasakian bundle structure of the space $X_{1,1}$, one obtains a local section starting from the underlying quaternionic manifold and then attaching the fiber SO(3), analogously to the procedure in Section 6.1. Since the twistor space over $\mathbb{C}P^2$ has already been constructed in [31,41] explicitly, we have to add the U(1)-fiber only. Similarly to (4.3), one introduces local coordinates on $\mathbb{C}P^2$ as

$$T \coloneqq (\overline{y_2}, y_1)^T \tag{7.3}$$

and obtains a section of the bundle $\mathrm{SU}(3) \to \mathbb{C}P^2$ from the matrix

$$V \coloneqq \frac{1}{\gamma}\begin{pmatrix} 1 & -T^\dagger \\ T & W \end{pmatrix}, \quad \gamma^2 \coloneqq 1 + T^\dagger T, \qquad W \coloneqq \gamma \mathbf{1}_2 - \tfrac{1}{\gamma+1} T T^\dagger. \tag{7.4}$$

Note the twist in the choice of the coordinates in (7.3), which is already adapted to our purposes. The canonical flat connection $V^{-1}\mathrm{d}V$ yields SU(3)-left-invariant forms on $\mathbb{C}P^2$ whose structure equations can be found in [31, Sec. 5]. This fibration is promoted to a local section of the twistor space $\mathcal{T}(\mathbb{C}P^2) \cong \mathbb{C}P^2 \times \mathbb{C}P^1$ by

$$\hat{V} \coloneqq V \times \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}, \quad g \coloneqq \frac{1}{(1+z\bar{z})^{1/2}}\begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \in \mathbb{C}P^1 . \tag{7.5}$$

The resulting structure equations and the choices necessary for Kähler geometry are discussed in [31]. Finally, the factor U(1) is included in the same way as in (6.4), yielding a local section $\widetilde{V} \coloneqq \hat{V} \times h$ of the bundle $\mathrm{SU}(3) \to X_{1,1}$. Based on the results of [27, 31], we already choose the basis 1-forms such that the tensors defining the 3-Sasakian geometry take their standard expressions by setting

$$\widetilde{A}_0 = \widetilde{V}^{-1}\mathrm{d}\widetilde{V} \eqqcolon \begin{pmatrix} \frac{2\mathrm{i}}{\sqrt{3}}e^8 & \sqrt{2}\Theta^2 & -\sqrt{2}\Theta^{\bar{1}} \\ -\sqrt{2}\Theta^{\bar{2}} & -\frac{\mathrm{i}}{\sqrt{3}}e^8 - \mathrm{i}e^7 & -\Theta^{\bar{3}} \\ \sqrt{2}\Theta^1 & \Theta^3 & -\frac{\mathrm{i}}{\sqrt{3}}e^8 + \mathrm{i}e^7 \end{pmatrix}. \tag{7.6}$$

This provides SU(3)-invariant 1-forms on $X_{1,1}$ described by the local coordinates $\{y_1, \overline{y_1}, y_2, \overline{y_2}, z, \bar{z}, \varphi\}$. The flatness of the connection $\widetilde{A}_0$ yields the structure equations (E.1) with structure constants (E.2). In terms of real 1-forms defined via

$e^1 - \mathrm{i}e^2 := \Theta^1$, $e^3 - \mathrm{i}e^4 := \Theta^2$ and $e^5 - \mathrm{i}e^6 := \Theta^3$, one obtains the equations

$$\begin{aligned} \mathrm{d}e^1 &= \sqrt{3}e^{82} - e^{72} - e^{35} - e^{46}, & \mathrm{d}e^2 &= -\sqrt{3}e^{81} + e^{71} - e^{36} + e^{45}, \\ \mathrm{d}e^3 &= -\sqrt{3}e^{84} - e^{74} + e^{15} + e^{26}, & \mathrm{d}e^4 &= \sqrt{3}e^{83} + e^{73} + e^{16} - e^{25}, \\ \mathrm{d}e^5 &= 2e^{67} - 2e^{13} + 2e^{24}, & \mathrm{d}e^6 &= 2e^{75} - 2e^{14} - 2e^{23}, \\ \mathrm{d}e^7 &= 2e^{12} + 2e^{34} + 2e^{56}, & \mathrm{d}e^8 &= -2\sqrt{3}e^{12} + 2\sqrt{3}e^{34}. \end{aligned} \tag{7.7}$$

The structure equations for the triplet $(\eta^5, \eta^6, \eta^7) := (e^5, e^6, e^7)$ coincide with those of the squashed seven-sphere (6.10a) and the same discussion of the geometry applies here. Therefore, the orthonormal metric $g = \sum_{\mu=1}^{7} e^\mu \otimes e^\mu$ defines a 3-Sasakian structure on $X_{1,1}$. Again, this property could be also verified by the closure of the Kähler form, the holomorphic top-degree form and the complex symplectic form on the metric cone; they take the same form as for the squashed seven-sphere, given in (D.4) and (D.5). In particular, the Kähler form which fixes our notation of holomorphicity on the metric cone reads

$$\Omega_7^{1,1} \;:=\; -\tfrac{\mathrm{i}}{2}\, r^2\, (\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}} + \Theta^{0\bar{0}}). \tag{7.8}$$

From the bundle construction above, it can be seen that $X_{1,1}$ is a U(1)-bundle over the Kähler manifold Q_3, which is, in turn, the twistor space over the self-dual manifold $\mathbb{C}P^2$ (cf. [53]). By this web of fibrations, the quiver gauge theories and instanton equations obtained on (the metric cone over) $X_{1,1}$ may be related to similar studies on these spaces, which will be considered in Section 7.4.

7.1.2 Canonical connection

Since the internal geometry described by the structure equations (7.7) is the same as that of the squashed seven-sphere, the forms P and Q again – as in (6.13) – read

$$Q = e^{1234}, \quad P = \tfrac{1}{3}e^{567} - \tfrac{1}{3}e^{135} + \tfrac{1}{3}e^{245} - \tfrac{1}{3}e^{146} - \tfrac{1}{3}e^{236} + \tfrac{1}{3}e^{127} + \tfrac{1}{3}e^{347}. \tag{7.9}$$

Using the torsion components of the canonical connection, $T^\alpha = 3P_{\alpha\mu\nu}e^{\mu\nu}$ and $T^a = \frac{3}{2}P_{a\mu\nu}e^{\mu\nu}$ (for $\alpha = 5, 6, 7$ and $a = 1, 2, 3, 4$), one derives the connection matrix

$$\mathrm{d}\begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} + \begin{pmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \end{pmatrix}, \tag{7.10}$$

and $\mathrm{d}e^\alpha = T^\alpha$. Therefore, the canonical connection of the 3-Sasakian structure on $X_{1,1}$ is the $\mathrm{U}(1)_{1,1}$-connection

$$\Gamma = I_8 \otimes e^8, \tag{7.11}$$

which is also adapted to the bundle structure of $X_{1,1}$ as coset space G/H. This abelian connection served as starting point for constructing G_2 and Spin(7) instantons in [27] since the 3-Sasakian metric is nearly parallel G_2 as well. Furthermore, it has been used for the study of HYM instantons on the metric cone over $X_{1,1}$ with respect to the Kähler form Ω_7 [48], whose discussion we are extending here. The curvature of the canonical connection (7.11),

$$\mathcal{F}_\Gamma = \mathrm{d}\Gamma = 2\sqrt{3} I_8 \otimes (-e^{12} + e^{34}), \tag{7.12}$$

satisfies the instanton equations (6.17) of a 3-Sasakian manifold.

7.2 Equivariance condition and action functional

As in the previous examples of quiver gauge theories, the ansatz (3.10) for the gauge connection on the Hermitian vector bundle $\mathcal{E} \to M^d \times X_{1,1}$ reads

$$\mathcal{A} = A + \Gamma + \sum_{\mu=1}^{7} X_\mu \otimes e^\mu = A + I_8 \otimes e^8 + \sum_{\mu=1}^{7} X_\mu \otimes e^\mu, \tag{7.13}$$

where A denotes a connection diagonal on the vertices, according to the isotopical decomposition which is induced by the decomposition of the G-representation under the subgroup $H = \mathrm{U}(1)_{1,1}$. Once more, the matrices X_μ are valued in the Lie algebra $\mathfrak{u}(k)$ and therefore skew-Hermitian, $X_\mu^\dagger = -X_\mu$.

7.2.1 Equivariance condition

Since the canonical connection is a U(1)-connection (like that on $T^{1,1}$), the equivariance condition does not involve any ladder operators of SU(3) and one has to impose the following equivariance conditions only:

$$\begin{aligned} &[I_8, X_1] = \sqrt{3} X_2, && [I_8, X_2] = -\sqrt{3} X_1, \\ &[I_8, X_3] = -\sqrt{3} X_4, && [I_8, X_4] = \sqrt{3} X_3, \\ &[I_8, X_\alpha] = 0 \quad \text{for} \quad \alpha = 5,6,7. && \end{aligned} \tag{7.14}$$

In complex notation $\phi^{(1)} := \frac{1}{2}(X_1 - \mathrm{i}X_2)$ and $\phi^{(2)} := \frac{1}{2}(X_3 - \mathrm{i}X_4)$, the Higgs fields act on vertices in the weight diagrams as

$$\begin{aligned} &\phi^{(1)} : (\nu_7, \nu_8) \longmapsto (*, \nu_8 + 3), && \phi^{(2)} : (\nu_7, \nu_8) \longmapsto (*, \nu_8 - 3), \\ &X_\alpha : (\nu_7, \nu_8) \longmapsto (*, \nu_8), && \end{aligned} \tag{7.15}$$

where the quantum numbers are defined with respect to the rescaled Cartan generators $\tilde{I}_7 = -\mathrm{i}I_7$ and $\tilde{I}_8 = -\mathrm{i}\sqrt{3}I_8$. Comparing with (6.20), one notices that the equivariance condition on $X_{1,1}$ imposes the same action with respect to the Cartan generator of the subgroup H (up to a numerical constant in the definition of the quantum number ν_8). As before, we will not only impose (7.15) but restrict the Higgs fields[2] to the action of the ladder operators (E.5).

[2]The arising of additional contributions is discussed in [48] and resembles the analogous discussion for the squashed seven-sphere.

Since the subgroup H is a torus, the subrepresentations are 1-dimensional, so that one can construct the quiver diagram directly from the weight diagram without collapsing along ladder operators and without the need of knowing the precise numerical values of the entries. On the other hand, this implies a large number of arrows even for low-dimensional representations of G, as the examples in Section 7.3 will show.

7.2.2 Action functional and instanton equations

Due to the same internal 3-Sasakian geometry of $X_{1,1}$ and Sp(2)/Sp(1), the Yang-Mills functional on $M^d \times X_{1,1}$ resembles (6.21). Explicitly, one finds

$$\begin{aligned}
S_{\text{YM}} = \text{Vol}(X_{1,1}) \int_{M^d} \mathrm{d}^d y\, \sqrt{g}\, \tfrac{1}{2}\, \text{Tr}\Big\{ \tfrac{1}{2} F_{mn} (F^{mn})^\dagger + \sum_{m=1}^{d} \sum_{\mu=1}^{7} |D_m X_\mu|^2 \\
+ \Big|[X_1, X_2] + 2X_7 - 2\sqrt{3} I_8\Big|^2 + |[X_1, X_3] - 2X_5|^2 + |[X_1, X_4] - 2X_6|^2 \\
+ |[X_2, X_3] - 2X_6|^2 + |[X_2, X_4] + 2X_5|^2 + \Big|[X_3, X_4] + 2X_7 + 2\sqrt{3} I_8\Big|^2 \\
+ |[X_1, X_5] + X_3|^2 + |[X_1, X_6] + X_4|^2 + |[X_1, X_7] - X_2|^2 + |[X_2, X_5] - X_4|^2 \\
+ |[X_2, X_6] + X_3|^2 + |[X_2, X_7] + X_1|^2 + |[X_3, X_5] - X_1|^2 + |[X_3, X_6] - X_2|^2 \\
+ |[X_3, X_7] - X_4|^2 + |[X_4, X_5] + X_2|^2 + |[X_4, X_6] - X_1|^2 + |[X_4, X_7] + X_3|^2 \\
+ |[X_5, X_6] + 2X_7|^2 + |[X_5, X_7] - 2X_6|^2 + |[X_6, X_7] + 2X_5|^2 \Big\}\,,
\end{aligned} \tag{7.16}$$

where the notation $|X|^2 \coloneqq XX^\dagger$ is employed again. The only difference occurs in the contribution of the canonical connection Γ which is a U(1)-connection in the case at hand, in contrast to the non-abelian connection (6.15) on the squashed seven-sphere. The 3-Sasakian instanton equations (6.17) yield the same set of matrix equations as on Sp(2)/Sp(1), and the action functional reduces to

$$\begin{aligned}
S_{\text{YM}}^{\text{inst}} = \text{Vol}(X_{1,1}) \int_{M^d} \mathrm{d}^d y\, \sqrt{g}\, \frac{1}{2}\, \text{Tr}\Big\{ \tfrac{1}{2} F_{mn} (F^{mn})^\dagger + \sum_{m=1}^{d} \sum_{\mu=1}^{7} |D_m X_\mu|^2 \\
+ 2\Big|[X_1, X_2] + 2X_7 - 2\sqrt{3} I_8\Big|^2 + 2\,|[X_1, X_3] - 2X_5|^2 + 2\,|[X_1, X_4] - 2X_6|^2 \Big\}
\end{aligned} \tag{7.17}$$

for matrices X_μ which are subject to the equivariance and instanton conditions. The torsion term in (2.11) vanishes by the same calculation as in Appendix D.1, and the expression for the Chern-Simons term in (2.12) follows from inserting the above curvature components into (D.8).

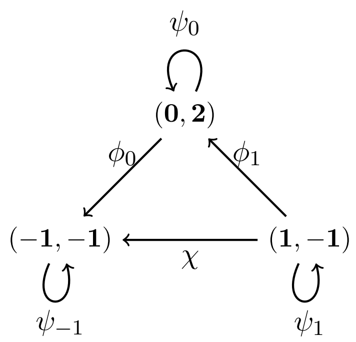
Similarly, the flow equations and algebraic relations (6.25) from the previous chapter also describe instantons on the hyper-Kähler cone over the Aloff-Wallach space $X_{1,1}$.

7.3 Examples of quiver diagrams

In this section we consider the quiver diagrams and resulting instanton matrix equations for the representations **3**, **6**, and **8** of SU(3) as explicit examples. The relevant weight diagrams are collected in Appendix E.2.

7.3.1 Fundamental representation

The simplest example follows from the fundamental representation (E.3) whose weight diagram (E.7) induces the triangular quiver



(7.18)

with the Higgs fields

$$\phi^{(1)} = \begin{pmatrix} 0 & 0 & \phi_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ \phi_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \chi \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.19}$$

as well as $X_7 = \mathrm{diag}(\psi_0, \psi_{-1}, \psi_1)$. The arrows in the diagram represent the homomorphisms $\phi_1 \in \mathrm{Hom}(E_1, E_0)$, $\phi_0 \in \mathrm{Hom}(E_0, E_{-1})$, $\chi \in \mathrm{Hom}(E_1, E_{-1})$ and the endomorphism ψ_j for $j \in \{0, \pm 1\}$. As for the other quiver gauge theories, one could set ψ_0 to zero because I_7 acts with eigenvalue zero on the corresponding vertex. Since the equivariance condition does not involve ladder operators of the subalgebra, all entries of the Higgs fields $\phi^{(\alpha)}$ are independent, in contrast to the situation on $\mathrm{SU}(4)/\mathrm{SU}(3)$ or $\mathrm{Sp}(2)/\mathrm{Sp}(1)$. The structure group of the quiver bundle is broken according to

$$\mathrm{U}(k_0 + k_1 + k_{-1}) \to \mathrm{U}(k_0) \times \mathrm{U}(k_1) \times \mathrm{U}(k_{-1}) \ , \tag{7.20}$$

where the dimensions of the vector spaces E_j are denoted as k_j. The equivariant gauge connection (7.13) takes the form

$$\mathcal{A} = A + \begin{pmatrix} \frac{2\mathrm{i}}{\sqrt{3}} \mathbf{1}_{k_0} e^8 + \psi_0 e^7 & -\phi_0^\dagger \otimes \Theta^2 & \phi_1 \otimes \Theta^{\bar{1}} \\ \phi_0 \otimes \Theta^{\bar{2}} & -\frac{\mathrm{i}}{\sqrt{3}} \mathbf{1}_{k_{-1}} e^8 + \psi_{-1} e^7 & \chi \otimes \Theta^{\bar{3}} \\ -\phi_1^\dagger \otimes \Theta^1 & -\chi^\dagger \otimes \Theta^3 & \frac{\mathrm{i}}{\sqrt{3}} \mathrm{i} \mathbf{1}_{k_1} e^8 + \psi_1 e^7 \end{pmatrix}, \tag{7.21}$$

which is again seen to be a twisted version of the flat connection (7.6). Imposing the instanton equations (6.26) on the Higgs fields yields

$$\begin{aligned} \dot{\phi}_0 &= -\phi_0 - \chi \phi_1^\dagger = -\phi_0 - \mathrm{i}\phi_0\psi_0 + \mathrm{i}\psi_{-1}\phi_0, \\ \dot{\phi}_1 &= -\phi_1 - \phi_0^\dagger \chi = -\phi_1 - \mathrm{i}\phi_1\psi_1 + \mathrm{i}\psi_0\phi_1, \end{aligned} \tag{7.22a}$$

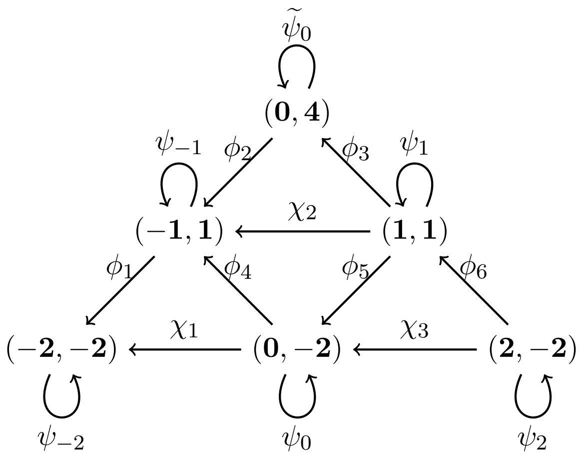
while the algebraic conditions (6.27) impose the quiver relations

$$2\chi = -\phi_0\phi_1, \qquad 2\mathrm{i}\psi_0 = \phi_1\phi_1^\dagger - \phi_0^\dagger\phi_0,$$
$$2\mathrm{i}\psi_{-1} = \phi_0\phi_0^\dagger, \qquad 2\mathrm{i}\psi_1 = -\phi_1^\dagger\phi_1. \tag{7.22b}$$

The quiver (7.18) resembles the diagrams obtained for the representations $\underline{\mathbf{4}}$ and $\underline{\mathbf{5}}$ of Sp(2) on the squashed seven-sphere. Therefore, the instanton matrix equations are similar to (6.36) and (6.42), only differing in the values of some coefficients in the algebraic relations.

7.3.2 Representation $\underline{6}$

The next example is based on the 6-dimensional representation with weight diagram (E.8), which leads to the quiver diagram



(7.23)

This diagram looks like the quivers (6.45) and (6.49), obtained for the representations $\underline{\mathbf{10}}$ and $\underline{\mathbf{14}}$ of Sp(2) on the squashed seven-sphere. The instanton conditions comprise the matrix flow equations

$$\begin{aligned}
\dot{\phi}_1 &= -\phi_1 - \chi_1\phi_4^\dagger = -\phi_1 - \mathrm{i}\phi_1\psi_{-1} + \mathrm{i}\psi_{-2}\phi_1,\\
\dot{\phi}_2 &= -\phi_2 - \chi_2\phi_3^\dagger = -\phi_2 - \mathrm{i}\phi_2\widetilde{\psi}_0 + \mathrm{i}\psi_{-1}\phi_2,\\
\dot{\phi}_3 &= -\phi_3 - \phi_2^\dagger\chi_2 = -\phi_3 - \mathrm{i}\phi_3\psi_1 + \mathrm{i}\widetilde{\psi}_0\phi_3,\\
\dot{\phi}_4 &= -\phi_4 - \phi_1^\dagger\chi_1 + \chi_2\phi_5^\dagger = -\phi_4 - \mathrm{i}\phi_4\psi_0 + \mathrm{i}\psi_{-1}\phi_4,\\
\dot{\phi}_5 &= -\phi_5 + \phi_4^\dagger\chi_2 - \chi_3\phi_6^\dagger = -\phi_5 - \mathrm{i}\phi_5\psi_1 + \mathrm{i}\psi_0\phi_5,\\
\dot{\phi}_6 &= -\phi_6 - \phi_5^\dagger\chi_3 = -\phi_6 - \mathrm{i}\phi_6\psi_2 + \mathrm{i}\psi_1\phi_6,
\end{aligned} \tag{7.24a}$$

and the quiver relations

$$2\mathrm{i}\psi_{-2} = \phi_1\phi_1^\dagger, \qquad 2\mathrm{i}\psi_{-1} = \phi_4\phi_4^\dagger + \phi_2\phi_2^\dagger - \phi_1^\dagger\phi_1,$$
$$2\mathrm{i}\psi_0 = -\phi_4^\dagger\phi_4 + \phi_5\phi_5^\dagger, \qquad 2\mathrm{i}\widetilde{\psi}_0 = \phi_3\phi_3^\dagger - \phi_2^\dagger\phi_2,$$
$$2\mathrm{i}\psi_1 = \phi_6\phi_6^\dagger - \phi_3^\dagger\phi_3 - \phi_5^\dagger\phi_5, \qquad 2\mathrm{i}\psi_2 = -\phi_6^\dagger\phi_6, \tag{7.24b}$$

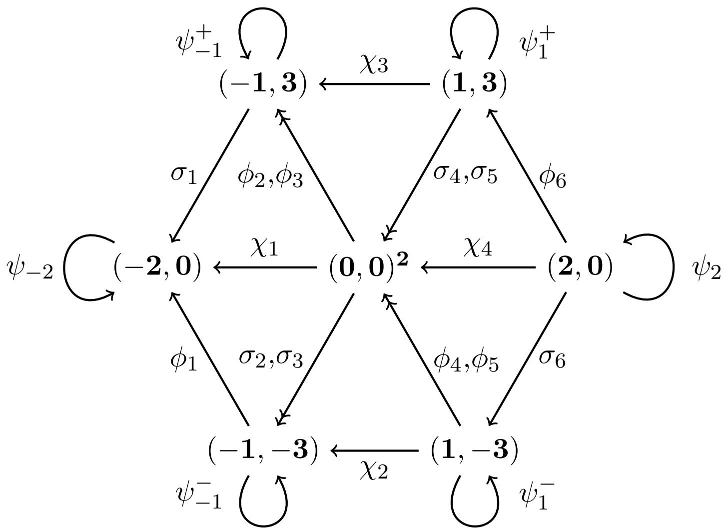
as well as

$$\phi_4\chi_3 = \chi_2\phi_6, \qquad \phi_1\chi_2 = \chi_1\phi_5, \qquad 2\chi_1 = -\phi_1\phi_4,$$
$$2\chi_3 = -\phi_5\phi_6, \qquad 2\chi_2 = \phi_4\phi_5 - \phi_2\phi_3. \tag{7.24c}$$

These equations are the counterparts of (6.47) and (6.50), and the same comments from those discussions apply here.

7.3.3 Adjoint representation

The last example of a quiver diagram is derived from the 8-dimensional adjoint representation of SU(3) with weight diagram (E.9):



(7.25)

The quiver gauge theory encoded in this diagram contains the differential equations

$$\begin{aligned}
\dot\phi_1 &= -\phi_1 + \chi_1\sigma_2^\dagger && = -\phi_1 - \mathrm{i}\phi_1\psi^-_{-1} + \mathrm{i}\psi_{-2}\phi_1, \\
\dot\phi_2 &= -\phi_2 - \sigma_1^\dagger\chi_1 + \chi_3\sigma_4^\dagger && = -\phi_2 - \mathrm{i}\phi_2\psi_0 + \mathrm{i}\psi^+_{-1}\phi_2, \\
\dot\phi_3 &= -\phi_3 + \chi_3\sigma_5^\dagger && = -\phi_3 - \mathrm{i}\phi_3\widetilde\psi_0 + \mathrm{i}\psi^+_{-1}\phi_3, \\
\dot\phi_4 &= -\phi_4 - \sigma_2^\dagger\chi_2 + \chi_4\sigma_6^\dagger && = -\phi_4 - \mathrm{i}\phi_4\psi^-_1 + \mathrm{i}\psi_0\phi_4, \\
\dot\phi_5 &= -\phi_5 - \sigma_3^\dagger\chi_2 && = -\phi_5 - \mathrm{i}\phi_5\psi^-_1 - \widetilde\psi_0\phi_5, \\
\dot\phi_6 &= -\phi_6 - \sigma_4^\dagger\chi_4 && = -\phi_6 - \mathrm{i}\phi_6\psi_2 + \mathrm{i}\psi^+_1\phi_6,
\end{aligned} \tag{7.26a}$$

and similar matrix equations for the entries of $\phi^{(2)}$:

$$\begin{aligned}
\dot\sigma_1 &= -\sigma_1 + \chi_1\phi_2^\dagger && = -\sigma_1 - \mathrm{i}\sigma_1\psi^+_{-1} + \mathrm{i}\psi_{-2}\sigma_1, \\
\dot\sigma_2 &= -\sigma_2 + \phi_1^\dagger\chi_1 - \chi_2\phi_4^\dagger && = -\sigma_2 - \mathrm{i}\sigma_2\psi_0 + \mathrm{i}\psi^-_{-1}\sigma_2, \\
\dot\sigma_3 &= -\sigma_3 - \chi_2\phi_5^\dagger && = -\sigma_3 - \mathrm{i}\sigma_3\widetilde\psi_0 + \mathrm{i}\psi^-_{-1}\sigma_3, \\
\dot\sigma_4 &= -\sigma_4 + \phi_2^\dagger\chi_3 - \chi_4\phi_6^\dagger && = -\sigma_4 - \mathrm{i}\sigma_4\psi^+_1 + \mathrm{i}\psi_0\sigma_4, \\
\dot\sigma_5 &= -\sigma_5 + \phi_3^\dagger\chi_3 && = -\sigma_5 - \mathrm{i}\sigma_5\psi^+_1 + \mathrm{i}\widetilde\psi_0\sigma_5, \\
\dot\sigma_6 &= -\sigma_6 + \phi_4^\dagger\chi_4 && = -\sigma_6 - \mathrm{i}\sigma_6\psi_2 + \mathrm{i}\psi^-_1\sigma_6.
\end{aligned} \tag{7.26b}$$

The algebraic conditions consist of the relations

$$
\begin{aligned}
2\mathrm{i}\psi_{-2} &= \phi_1\phi_1^\dagger + \sigma_1\sigma_1^\dagger, & 2\mathrm{i}\psi_2 &= -\phi_6^\dagger\phi_6 - \sigma_6^\dagger\sigma_6, \\
2\mathrm{i}\psi_{-1}^- &= -\phi_1^\dagger\phi_1 + \sigma_2\sigma_2^\dagger + \sigma_3\sigma_3^\dagger, & 2\mathrm{i}\psi_{-1}^+ &= -\sigma_1^\dagger\sigma_1 + \phi_2\phi_2^\dagger + \phi_3\phi_3^\dagger, \\
2\mathrm{i}\psi_0 &= \phi_4\phi_4^\dagger + \sigma_4\sigma_4^\dagger - \phi_2^\dagger\phi_2 - \sigma_2^\dagger\sigma_2, & 2\mathrm{i}\widetilde{\psi}_0 &= \phi_5\phi_5^\dagger + \sigma_5\sigma_5^\dagger - \phi_3^\dagger\phi_3 - \sigma_3^\dagger\sigma_3, \\
2\mathrm{i}\psi_1^- &= \sigma_6\sigma_6^\dagger - \phi_4^\dagger\phi_4 - \phi_5^\dagger\phi_5, & 2\mathrm{i}\psi_1^+ &= -\sigma_4^\dagger\sigma_4 - \sigma_5^\dagger\sigma_5 + \phi_6\phi_6^\dagger,
\end{aligned}
\tag{7.26c}
$$

as well as

$$
\begin{aligned}
\phi_1\sigma_3 &= \sigma_1\phi_3, & \phi_5\sigma_6 &= \sigma_5\phi_6, & \phi_1\chi_2 &= \chi_1\phi_4, \\
\phi_2\chi_4 &= \chi_3\phi_6, & \sigma_1\chi_3 &= \chi_1\sigma_4, & \sigma_2\chi_4 &= \chi_2\sigma_6, \\
2\chi_1 &= \phi_1\sigma_2 - \sigma_1\phi_2, & 2\chi_2 &= -\sigma_2\phi_4 - \sigma_3\phi_5, & 2\chi_3 &= \phi_2\sigma_4 + \phi_3\sigma_5, \\
2\chi_4 &= \phi_4\sigma_6 - \sigma_4\phi_6, & 0 &= \phi_4\phi_5^\dagger + \sigma_4\sigma_5^\dagger - \phi_2^\dagger\phi_3 - \sigma_2^\dagger\sigma_3. & &
\end{aligned}
\tag{7.26d}
$$

The system of equations is more complicated due to the large number of arrows and the degeneracy of the origin, but it admits the typical features already encountered in all previous examples. Once more it is instructive to visualize the paths and relations imposed by the above equations in the quiver diagram (7.25).

7.4 Reduction to related geometries

Following the previous discussions, we now include the reduction from the quiver gauge theory on the 3-Sasakian manifold $X_{1,1}$ to that on the underlying quaternionic manifold in dimension 4 and on the 6-dimensional Kähler manifold Q_3.

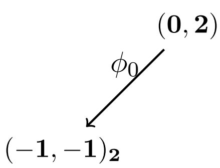
7.4.1 Reduction along the $\mathrm{SO}(3)$ fiber

When considering the gauge theory on the quaternionic base of the fibration, one has to impose, as in Section 6.4, the additional equivariance conditions

$$
[\tilde{I}_8, \phi^{(1)}] = \phi^{(1)}, \qquad [\tilde{I}_8, \phi^{(2)}] = -\phi^{(2)}, \tag{7.27a}
$$

$$
[I_{\bar{3}}^-, \phi^{(1)}] = 0 = [I_{\bar{3}}^-, \phi^{(2)}], \qquad [I_{\bar{3}}^-, \phi^{(1)\dagger}] = -\phi^{(2)}, \quad [I_{\bar{3}}^-, \phi^{(2)\dagger}] = \phi^{(1)}. \tag{7.27b}
$$

Because of the analogy to the procedure in Section 6.4 and the appearance of quiver diagrams of the same shape, the corresponding discussion applies here. Therefore, we study only the implications for the 3-dimensional representation of SU(3) explicitly. The collapsing of the quiver diagram (7.18) yields the 2-quiver



(7.28)

which resembles the quiver diagram (6.52), of course. Since the ladder operator I_3^- is involved in the additional equivariance condition (7.27), both Higgs fields now depend on the same homomorphism. More precisely, the two remaining Higgs fields read

$$\phi^{(1)} = \left(\begin{array}{c|cc} 0 & 0 & \phi_0^\dagger \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \qquad \phi^{(2)} = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline \phi_0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \tag{7.29}$$

i.e. one has to set $\phi_0 = \phi_1^\dagger$ in (7.19). The instanton equations (7.22) then reduce to the conditions

$$\mathbf{1}_{k_{-1}} = \tfrac{1}{2}\phi_0\phi_0^\dagger, \qquad \phi_0 = \text{const.}, \tag{7.30}$$

which is again the typical form of BPS equations in even-dimensional quiver gauge theories. Due to the bundle structure and the collapsing procedure, one can compare these results with quiver gauge theory on $\mathbb{C}P^2$ [40,42]. Recall that those quiver gauge theories are constructed on the *Kähler* manifold $\mathbb{C}P^2$, whereas our fibration is based on the *quaternionic* structure of $\mathbb{C}P^2$. Therefore, the quiver diagrams will be of the same shape, but the orientations of the arrows may differ.

Taking the limit to the underlying quaternionic base casts the quiver diagram (E.9) of the adjoint representation into the typical square-quiver which occurs for adjoint representations of SU(3) [40, 42].

7.4.2 Reduction to the underlying twistor space

For this reduction, one has to fix the Higgs field X_7 by setting $X_7 = I_7$, which yields the quiver diagram (7.18) without the loop contributions. The Hermitian Yang-Mills equations associated to the complex structure determined by Ω_7 then reduce to the set of conditions

$$2\chi = -\phi_0\phi_1, \quad \phi_1\phi_1^\dagger = \phi_0^\dagger\phi_0, \quad \mathbf{1}_{k_{-1}} = \tfrac{1}{3}(\phi_0\phi_0^\dagger + \chi\chi^\dagger), \quad \mathbf{1}_{k_1} = \tfrac{1}{3}(\phi_1^\dagger\phi_1 + \chi^\dagger\chi) \tag{7.31}$$

and require $\phi_0, \phi_1, \chi = \text{const}$. As expected, taking the limit to the underlying Kähler manifold $\mathcal{T}(\mathbb{C}P^2) \cong \mathbb{C}P^2 \times \mathbb{C}P^1 \cong Q_3$ yields the "holomorphic triangle" found for the fundamental representation of SU(3) [42,98]. Bearing in mind the different notation[3], one notices that the above BPS equations correctly reproduce the results (4.29) and (4.30) of [42].

In considering the reduction for this example, we have checked consistency of the constructed quiver gauge theories with known results, as it was also carried out for Sasakian quiver gauge theories on S^7 (over $\mathbb{C}P^3$) and on $T^{1,1}$ (over $\mathbb{C}P^1 \times \mathbb{C}P^1$) above.

[3] Our labeling of the vertices in the quiver diagram (7.18) differs from the conventions used for [42, Eq. (3.48)]. The comparison is established via the identifications ${}^0\phi^-_{0,-2} \equiv \phi_0^\dagger$, ${}^0\phi^+_{0,-2} \equiv \chi^\dagger$ and ${}^1\phi^0_{-1,1} \equiv \phi_1^\dagger$ (up to prefactors).

7.5 Discussion

The inclusion of the Aloff-Wallach space $X_{1,1}$ in the framework of 3-Sasakian quiver gauge theories yields examples of quiver diagrams and instanton equations similar to those on the squashed seven-sphere. More precisely, our findings for the fundamental representation of SU(3) resemble the results obtained for the representations $\underline{\mathbf{4}}$ and $\underline{\mathbf{5}}$ of Sp(2), while the 6-dimensional representation of SU(3) leads to a quiver similar to that for the representations $\underline{\mathbf{10}}$ and $\underline{\mathbf{14}}$ on the squashed seven-sphere. In contrast to the coset Sp(2)/Sp(1), where the vertices carry SU(2) gauge connections, the case at hand only involves a canonical connection which is abelian. To conclude, the comparison of the 3-Sasakian quiver gauge theories on $X_{1,1}$ with the results on Sp(2)/Sp(1) resembles the direct comparison of Sasakian quiver gauge theories on $T^{1,1}$ with those on S^5. Namely, within each class of quiver gauge theories of the same dimension, one finds diagrams and instanton equations of the same form which arise for different representations and coset spaces.

Like for the Sasakian quiver gauge theories in dimension 5, the two examples Sp(2)/Sp(1) and $X_{1,1}$ basically exhaust the list of 3-Sasakian manifolds and therefore of corresponding quiver gauge theories in dimension 7. Although we considered only a few examples of quiver diagrams, we elaborated the main features of 3-Sasakian quiver gauge theories. For the Aloff-Wallach space $X_{1,1}$ one may aim at a discussion of the quiver diagrams and instanton equations associated to generic representations of SU(3), as in [40, 42, 46] for instance. However, also the generic case will consist of holomorphic triangles with vertex loops as building blocks.

8 Conclusions and outlook

In this thesis, I have constructed new Sasakian quiver gauge theories on the homogeneous spaces $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$ and $\mathrm{SU}(4)/\mathrm{SU}(3)$ as well as 3-Sasakian quiver gauge theories on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ and $X_{1,1} = \mathrm{SU}(3)/\mathrm{U}(1)$. The studies of all four cases comprised explicit examples of quiver diagrams for low-dimensional representations of the relevant Lie groups, which have been compared with the results for similar geometric setups. I discussed the instanton equations on the Calabi-Yau and hyper-Kähler cones over the coset spaces and the description of their moduli spaces.

Results in detail

The manifold $T^{1,1}$ yields a second class of Sasakian quiver gauge theory in dimension 5 besides the existing study on S^5 [46]. The product structure of the group $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ induces a grading of the gauge connection and yields quiver diagrams of the form $A_{m_1+1} \otimes A_{m_2+1}$ with vertex loops. Therefore, a discussion of quiver gauge theory for generic representations of G, analogous to [43], is applicable, and taking the limit $\mathbb{C}P^1 \times \mathbb{C}P^1$ correctly reproduces the results for quiver gauge theory on this Kähler base. As expected, the instanton equations reveal the typical quadratic terms in the differential equations for the vertical Higgs field, as a consequence of the vertex loops.

Stemming from another rank-2 quiver gauge theory in dimension 5, our results on $T^{1,1}$ resemble the quiver diagrams and instanton equations obtained on S^5 to some extent, as we illustrated in the course of the discussion. Since we employed the ansatz starting from the canonical connection, the instanton equations match the expectations: the moduli space can be discussed following the general results for instantons on metric cones over generic Sasaki-Einstein manifolds from the literature [51].

In dimension 7, we constructed new examples of Sasakian quiver gauge theories on $\mathrm{SU}(4)/\mathrm{SU}(3)$, which enabled us to compare Sasakian quiver gauge theories of different dimensions within the class of round spheres. We studied quiver diagrams for low-dimensional representations of $\mathrm{SU}(4)$, and they turned out to be the higher-dimensional analogues of the results for the five-sphere [46], apart from the exceptional representation $\underline{\mathbf{6}}$. We therefore sketched some generalizations to arbitrary round spheres, showing that quiver diagrams of type A_l occur on all odd-dimensional spheres. Due to the higher rank of the Lie group $\mathrm{SU}(4)$, one expects also new types of quiver diagrams from more complicated weight diagrams, which can be obtained by applying the collapsing procedure according to the root system of SU(4). As a by-product, one can derive quiver gauge theories on the Kähler base $\mathbb{C}P^3$. The instanton equations on the Calabi-Yau cone over S^7 yield the expected results, so that the moduli space can be described according to the general treatment [51].

3-Sasakian quiver gauge theories have been constructed on the squashed seven-sphere and the Aloff-Wallach space $X_{1,1}$. The examples of quiver diagrams considered here show as typical feature a triangular shape, based on holomorphic triangles with vertex loops as building blocks, in constrast to the chains or rectangular quivers in Sasakian quiver gauge theory. This difference is easily attributed to the different bundle structure of 3-Sasakian manifolds whose vertical part comprises both vertex loops and a ladder operator in the weight diagrams. We also employed a purely quaternionic interpretation which (schematically) renders the quiver diagrams in the typical form known from Sasakian quiver gauge theories and emphasizes the connection to the underlying quaternionic manifold. Analogously to some similarities between quivers for $T^{1,1}$ and for S^5, several representations of Sp(2) yield quiver diagrams of the same form, and these diagrams also resemble quivers of $X_{1,1}$. The reduction of the quiver gauge theories to the underlying twistor and quaternionic manifolds has been considered as well.

According to [75] and the explicit equations here, Sp(2) instantons on the metric cones over $X_{1,1}$ and Sp(2)/Sp(1) can be described by the intersection of three HYM moduli spaces with respect to the triplet of Kähler forms. The discussion of each single moduli space may be adapted to the different algebraic conditions, comprising a complexified Heisenberg algebra rather than an abelian algebra, as it appears for instantons on Calabi-Yau cones. However, the structure of the intersection and therefore the Sp(2)-instanton moduli space itself is not yet fully understood.

Future research

After deriving various sets of instanton matrix equations, based on certain quiver diagrams, it is desirable to find explicit solutions beyond the special subclass of constant solutions arising from the underlying manifolds. Due to the larger number of degrees of freedom in quiver gauge theories, one may hope to find new solutions. In particular, the dynamics of the off-diagonal contributions in the homomorphisms might provide interesting novel insights, whereas purely diagonal fields reproduce copies of the scalar ansatz. The quadratic coupling in the flow equations for the vertical Higgs field in Sasakian quiver gauge theory seems unlikely to admit analytic solutions, so that one should apply numerical studies as in [19]. Also the implications of the possible additional arrows, which we have briefly discussed, on the level of the instanton equations should be studied in detail.

Once the approach has proved useful for the construction of new instantons, one may aim at discussing further representations of G in the above examples and may even include G_2 and Spin(7)-instantons [28] in the framework of *quiver* gauge theories. Including more examples might also clarify the similarities of quiver diagrams found among different representations and coset spaces. In addition, it might be interesting to consider also homogeneous spaces involving exceptional Lie algebras like the quotient $G_2/\mathrm{Sp}(1)$.

Since quiver bundles are intimately related to symmetry breaking in gauge theories, one may study these effects in detail for phenomenologically interesting dimensions k_j of the vector spaces E_j attached to the vertices of the quiver, as done in [38–40]. Moreover, fermionic terms can be included, in order to create a physically more realistic theory. Following the approaches in [37, 43, 102, 137], one may apply a noncommutative deformation of the external manifold M^d in order to obtain noncommutative BPS solutions, which might yield new brane interpretations.

The description of the moduli space of $\mathrm{Sp}(m)$ instantons on hyper-Kähler cones over 3-Sasakian manifolds should be completed. This requires a better understanding of the intersection, for which one has to take into account both the SU(2)-symmetry of the vertical Higgs fields and the quaternionic symmetry of the base. In particular, by virtue of the non-trivial algebraic relations, the flow equations turn into a system of cubic ODEs which only involves the fields associated to the quaternionic Kähler base. Therefore, it seems beneficial to search for a suitable adaptation of existing (algebraic) methods for the description of quaternionic instantons, such as generalized ADHM constructions, to our system with the radial dependence.

A Details on instantons

This appendix collects some standard properties of 4-dimensional instantons and provides details on their generalization to higher dimensions. Since the discussion of the moduli spaces of HYM instantons on cones over Sasaki-Einstein manifolds is based on an adaptation [51] of Donaldson's and Kronheimer's approaches [111, 112] to Nahm's equations, we review these equations as well.

A.1 Yang-Mills action and generalized instantons

In this section, we briefly report some basic features on gauge theory which can be found in standard textbooks and reviews of instantons, see [12, 138, 139] for example.

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. On the space of $\mathfrak{g}$-valued p-forms $\Omega^p(M^d, \mathfrak{g})$ an inner product can be defined via

$$(A, B) := -\int_{M^d} \mathrm{Tr}(A \wedge \star_d B) \tag{A.1}$$

for $A, B \in \Omega^p(M^d, \mathfrak{g})$, where the metric of the underlying manifold M^d enters via the Hodge star operator $\star_d$. Although we will provide final results which are independent of the choice of the Hodge star operator or use different techniques (like the HYM equations) in most cases, let us fix the convention. For an orthornomal basis $e^1, \dots, e^d$, we use the definition

$$\star_d \left(e^{p_1} \wedge \dots \wedge e^{p_k}\right) = \tfrac{1}{(d-k)!}\, \epsilon_{p_1 p_2 \dots p_k q_1 \dots q_{d-k}}\; e^{q_1} \wedge \dots \wedge e^{q_{d-k}} \tag{A.2}$$

with the completely anti-symmetric tensor $\epsilon_{12\dots d} = 1$ (for all combinations of indices raised or lowered). In Euclidean signature the Hodge star operator satisfies

$$\star_d \left(\star_d \omega\right) = (-1)^{k(d-k)} \omega \tag{A.3}$$

for any k-form ω. In particular, it follows that $\star^2 = \mathbf{1}$ when acting on 2-forms, and therefore one can decompose the curvature 2-form into a self-dual (SD) and an anti-self-dual (ASD) component in four dimensions, $\star \mathcal{F}_\pm = \pm \mathcal{F}_\pm$.

A.1.1 Yang-Mills equation

With the inner product (A.1) one defines the Yang-Mills action of a gauge connection $\mathcal{A} \in \Omega^1(M^d, \mathfrak{g})$ with curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ over a Riemannian manifold M^d as

$$S_{\mathrm{YM}} = -\tfrac{1}{4} \int_{M^d} \mathrm{Tr}(\mathcal{F} \wedge \star \mathcal{F}). \tag{A.4}$$

Varying the gauge connection $\mathcal{A} \mapsto \mathcal{A} + \delta\mathcal{A}$ and employing the definition of the field strength yields

$$\begin{aligned} \delta\mathcal{S}_{\text{YM}} &\propto \int_{M^d} \text{Tr}\left[(\text{d}(\delta\mathcal{A}) + \mathcal{A}\wedge\delta\mathcal{A} + \delta\mathcal{A}\wedge\mathcal{A})\wedge\star\mathcal{F}\right] \\ &\propto \int_{M^d} \text{Tr}\left[\delta\mathcal{A}(-\text{d}\star\mathcal{F} - \mathcal{A}\wedge\star\mathcal{F} + (-1)^{d-1}\star\mathcal{F}\wedge\mathcal{A})\right], \end{aligned} \tag{A.5}$$

where one has used integration by parts and cyclicity of the trace. Consequently, one obtains from $\delta\mathcal{S}_{\text{YM}} = 0$ the *Yang-Mills equation*

$$0 = \text{d}\star\mathcal{F} + [\mathcal{A}, \star\mathcal{F}] =: D_{\mathcal{A}}\star\mathcal{F} \tag{A.6}$$

as equation of motion of the Yang-Mills functional (A.4). The definition of the curvature immediately implies the Bianchi identity $D_{\mathcal{A}}\mathcal{F} = 0$ since

$$\begin{aligned} D_{\mathcal{A}}\mathcal{F} &:= \text{d}\mathcal{F} + \mathcal{A}\wedge\mathcal{F} - \mathcal{F}\wedge\mathcal{A} = \text{d}\mathcal{A}\wedge\mathcal{A} - \mathcal{A}\wedge\text{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{F} - \mathcal{F}\wedge\mathcal{A} \\ &= (\mathcal{F} - \mathcal{A}\wedge\mathcal{A})\wedge\mathcal{A} - \mathcal{A}\wedge(\mathcal{F} - \mathcal{A}\wedge\mathcal{A}) + \mathcal{A}\wedge\mathcal{F} - \mathcal{F}\wedge\mathcal{A} = 0. \end{aligned} \tag{A.7}$$

Thus, (anti-)self-dual connections $\star_4\mathcal{F} = \pm\mathcal{F}$ automatically satisfy the Yang-Mills equation (A.6) in four dimensions.

Bogomolny bound. Conversely, recall the *Bogomolny trick* for the 4-dimensional Yang-Mills action: Using $\mathcal{F}\wedge\mathcal{F} = \star\mathcal{F}\wedge\star\mathcal{F}$ and $\text{Tr}(\mathcal{F}\wedge\star\mathcal{F}) = \text{Tr}(\star\mathcal{F}\wedge\mathcal{F})$ one obtains a bound of the Yang-Mills functional from below by completing the square as follows (see e.g. [12, 138, 139])

$$\begin{aligned} S_{\text{YM}} &= -\frac{1}{4}\int_{M^4} \text{Tr}\left[\mathcal{F}\wedge\star\mathcal{F}\right], \\ &= -\frac{1}{8}\int_{M^4} \text{Tr}\left[(\mathcal{F}\pm\star\mathcal{F})\wedge\star(\mathcal{F}\pm\star\mathcal{F})\right] \pm \frac{1}{4}\int_{M^4}\text{Tr}\left[\mathcal{F}\wedge\mathcal{F}\right]. \end{aligned} \tag{A.8}$$

Recall that the first two *Chern classes* are given by

$$c_1 = \frac{\text{i}}{2\pi}\text{Tr}(\mathcal{F}), \quad c_2 = \frac{1}{8\pi}\left[\text{Tr}(\mathcal{F}\wedge\mathcal{F}) - \text{Tr}(\mathcal{F})\wedge\text{Tr}(\mathcal{F})\right], \tag{A.9}$$

where one has $\text{Tr}(\mathcal{F}) = 0$ in the typical setup of $\text{SU}(n)$ gauge theories. In that case, the term $\int_{M^4}\text{Tr}\left[\mathcal{F}\wedge\mathcal{F}\right]$ is a topological invariant, and since the first summand in (A.8) is non-negative and non-degenerate, the Yang-Mills action is bounded by the second Chern number with equality if and only if the connection is (anti-)self-dual.

Therefore, instantons not only satisfy the Yang-Mills equation, but actually minimize the action functional within their topological sector, which is characterized by the *instanton number* $\propto \int \text{Tr}\ (\mathcal{F}\wedge\mathcal{F})$. That this contribution is topological can be seen as follows: the integrand is the differential of the Chern-Simons term (see e.g. [138])

$$\text{CS}_3 := \frac{1}{8\pi^2}\text{Tr}\left[\text{d}\mathcal{A}\wedge\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right]. \tag{A.10}$$

Taking the differential of this expression, employing the definition of $\mathcal{F}$ and using $\mathrm{Tr}(A^4) = 0$ (due to cyclicity of the trace and anti-symmetry of the form part), yields the second Chern class c_2 in (A.9). Hence, the lower bound of the action for instantons is a topological term.

A.1.2 Generalized self-duality equation

We review some details related to the definition of higher-dimensional instantons, which are discussed in [20, 25, 28, 33, 70, 71] for instance. Given a gauge connection $\mathcal{A}$ whose curvature satisfies the generalized self-duality equation

$$0 = \star\mathcal{F} + \mathcal{F} \wedge \star Q, \tag{A.11}$$

one takes the exterior derivative to obtain

$$\begin{aligned} 0 &= \mathrm{d}\star\mathcal{F} + \mathrm{d}\mathcal{F} \wedge \star Q + \mathcal{F} \wedge \mathrm{d}\star Q \\ &= \mathrm{d}\star\mathcal{F} + (\mathrm{d}\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge \mathrm{d}\mathcal{A}) \wedge \star Q + \mathcal{F} \wedge \mathrm{d}\star Q \\ &= \mathrm{d}\star\mathcal{F} + [(\mathcal{F} - \mathcal{A} \wedge \mathcal{A}) \wedge \mathcal{A} - \mathcal{A} \wedge (\mathcal{F} - \mathcal{A} \wedge \mathcal{A})] \wedge \star Q + \mathcal{F} \wedge \mathrm{d}\star Q \\ &= \mathrm{d}\star\mathcal{F} + (-1)^d \mathcal{F} \wedge \star Q \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} \wedge \star Q + \mathcal{F} \wedge \mathrm{d}\star Q \\ &= D_{\mathcal{A}}\star\mathcal{F} + \mathcal{F} \wedge \mathrm{d}\star Q, \end{aligned} \tag{A.12}$$

where the instanton equation (A.11) has been used in the last step. The first part is the usual Yang-Mills equation (A.6), while the second contribution is a torsion term, also written as 3-form $\star H := \mathrm{d}\star Q$. It follows from the variation of a Chern-Simons term:

$$\begin{aligned} \delta \int_{M^d} \mathrm{Tr}\left[\mathcal{F} \wedge \mathcal{F} \wedge \star Q\right] &= 2 \int_{M^d} \mathrm{Tr}\left[(\mathrm{d}(\delta\mathcal{A}) \wedge \mathcal{F} + \delta\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F} + \mathcal{A} \wedge \delta\mathcal{A} \wedge \mathcal{F}) \wedge \star Q\right], \\ &= 2 \int_{M^d} \mathrm{Tr}\left[-\delta\mathcal{A} \wedge \mathrm{d}\mathcal{F} \wedge \star Q - \delta\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F} \wedge \mathrm{d}\star Q \right. \\ &\qquad \left. + \,\delta\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F} \wedge \star Q - \delta\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{A} \wedge \star Q\right] \\ &= -2 \int_{M^d} \mathrm{Tr}\left[\delta\mathcal{A} \wedge \mathcal{F} \wedge \mathrm{d}\star Q\right], \end{aligned} \tag{A.13}$$

where the Bianchi identity has been used to obtain the last line. To sum it up, the Yang-Mills equation with torsion (A.12) is the equation of motion of the action functional

$$S \propto \int_{M^d} \mathrm{Tr}\left[\mathcal{F} \wedge \star\mathcal{F} + (-1)^d \mathcal{F} \wedge \mathcal{F} \wedge \star Q\right]. \tag{A.14}$$

Since for our cases of interest the torsion term vanishes, we will not consider the second contribution of the above action functional in detail, even if we do not impose the instanton equation from the very beginning.

Remark. This thesis is only concerned with the evaluation of the BPS equation (A.11), which is an *algebraic* condition on the curvature $\mathcal{F}$ implying the differential equation (A.12). Equation (A.11) is conformally invariant for the metric cone and the cylinder (see [34, Eq. (1.5)] for instance) so that we need not distinguish and usually use both expressions interchangeably. However, due to the derivative, the Yang-Mills equation (A.6) is *not* conformally equivalent and the difference between cylinder and cone does matter, as is discussed, for example, in [33].

A.2 Instantons and Nahm's equations

This section briefly reviews Nahm's equations because their discussion by Donaldson and Kronheimer yielded inspiration for the description of instantons on Calabi-Yau cones [51]. Furthermore, the setting is closely related to the Sasaki-Einstein 3-manifolds $\mathrm{SU}(2)/\Gamma$ in [44], where the notion of *Sasakian quiver gauge theories* has been introduced. The following exposition is based on the standard references [62, 111, 112, 131].

SU(2)-**invariant connection.** One considers $\mathbb{R}^4 \setminus \{0\}$ as metric cone over $\mathrm{SU}(2) \cong S^3$ and studies the gauge connection

$$\mathcal{A} = A_0 \otimes e^0 + \sum_{i=1}^{3} A_i(r) \otimes e^i, \tag{A.15}$$

where the e^i denote the usual left-invariant 1-forms on $\mathfrak{su}(2)$ with structure equations $\mathrm{d}e^i = \epsilon^i{}_{jk} e^j \wedge e^k$ and the definition $e^0 := e^\tau := \mathrm{d}r/r$. The gauge connection (A.15) is nothing else than the typical formulation (3.10) of equivariant connections on homogeneous spaces, where the base space here may be interpreted as trivial quotient $\mathrm{SU}(2)/\{1\}$, and which has also been considered in [44, Sec. 6]. The curvature of this SU(2)-equivariant gauge connection reads

$$\begin{aligned}\mathcal{F} = &\sum_{i=1}^{3}([A_0, A_i] + \dot{A}_i) \otimes e^{0i} + ([A_1, A_2] + 2A_3) \otimes e^{12} \\ &+ ([A_2, A_3] + 2A_1) \otimes e^{23} + ([A_3, A_1] + 2A_2) \otimes e^{31},\end{aligned} \tag{A.16}$$

where we write $\dot{A}_i := \frac{\mathrm{d}}{\mathrm{d}\tau} A_i$ with respect to the rescaled cone-coordinate $\tau := \ln(r)$. Evaluating the 4-dimensional instanton equations yields

$$\begin{aligned}\dot{A}_1 &= -2A_1 - [A_0, A_1] - [A_2, A_3], \\ \dot{A}_2 &= -2A_2 - [A_0, A_2] - [A_3, A_1], \\ \dot{A}_3 &= -2A_3 - [A_0, A_3] - [A_1, A_2].\end{aligned} \tag{A.17}$$

The linear terms can be eliminated by rescaling $B_i = \mathrm{e}^{2\tau} A_i$ and writing $s(\tau) := -\frac{1}{2}\mathrm{e}^{-2\tau}$ as in [111]. Setting $B_0 = 0$, the system reduces to the famous Nahm's equations

$$\frac{\mathrm{d}B_i}{\mathrm{d}s} = -\tfrac{1}{2}\epsilon_{ijk}[B_j, B_k]. \tag{A.18}$$

They arise in the context of the *Nahm transform* for the construction of magnetic monopoles [139], and they can also be considered as the gradient flow equations [112] of

$$\Psi(B_1, B_2, B_3) \coloneqq \langle B_1, [B_2, B_3]\rangle. \tag{A.19}$$

The discussion of their moduli space is based on the gauge transformation [111]

$$B_0 \mapsto \mathrm{Ad}(g)B_0 - \tfrac{\mathrm{d}g}{\mathrm{d}s}g^{-1}, \quad B_j \mapsto \mathrm{Ad}(g)B_j, \quad j = 1, 2, 3. \tag{A.20}$$

Real and complex equation. After introducing the complex combinations $\alpha \coloneqq \frac{1}{2}(B_0 + \mathrm{i}B_1)$ and $\beta \coloneqq \frac{1}{2}(B_2 + \mathrm{i}B_3)$, the rescaled equations (A.17) take the form

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\beta + 2[\alpha, \beta] = 0, \tag{A.21a}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^\dagger) + 2([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]) = 0, \tag{A.21b}$$

where the first equation is referred to as *complex equation* and the second one as *real equation*. Based on the gauge transformation (A.20), it is shown that the moduli space (for suitable boundary conditions) can be described by coadjoint orbits of regular triples of a Cartan subalgebra [111,112], where the real equation is solved as equation of motion of the Lagrangian [111, Lem. 2.3]

$$\mathcal{L}[g] = \int_{\mathcal{I}} |\alpha + \alpha^\dagger|^2 + 2|\beta|^2. \tag{A.22}$$

The most general case is not only characterized by the regular triple, but also depends on a second set of matrices, which lead to nilpotent varieties in the extreme case studied in [131]. Both approaches can be promoted to Hermitian Yang-Mills instantons on cones over generic Sasaki-Einstein manifolds [51,128].

Hyper-Kähler quotient. For comparison with the space of Sp(2)-instantons, it is useful to recall that the flat manifold $\mathbb{R}^4/\Gamma \cong \mathbb{H}/\Gamma$ admits a hyper-Kähler structure with the three moment maps [62]

$$\begin{aligned} &\mu_1 \coloneqq [A_0, A_1] + [A_2, A_3], \quad \mu_2 \coloneqq [A_0, A_2] + [A_3, A_1], \\ &\mu_3 \coloneqq [A_0, A_3] + [A_1, A_2], \end{aligned} \tag{A.23}$$

or, in complex notation, $\mu_{\mathbb{C}} \coloneqq [\alpha, \beta]$ and $\mu_{\mathbb{R}} \coloneqq [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]$. Therefore, the 4-dimensional self-duality equations can be understood as the zero locus of a triplet of moment maps, as explained in [62]. Including non-vanishing Fayet-Iliopoulos terms Ξ_α in the three stability-like conditions of the HYM equations corresponds to deformations of the complex structures and leads to Yang-Mills theory with sources, as discussed in [113] for instance.

It is worth pointing out that the case of instantons on 4-dimensional hyper-Kähler manifolds is rather special: the set of instanton equations consists of three conditions that can be equally well interpreted either as three holomorphicity or as the three stability-like conditions, which are naturally related to moment maps. In higher dimensions, however, as the study of the squashed seven-sphere and the discussion of generic hyper-Kähler manifolds show, the three moment map conditions are already implied by the three sets of holomorphicity conditions. They are not sufficient to recover the instanton equations due to the higher dimension.

B Details of the round seven-sphere

This appendix provides details on the Sasaki-Einstein geometry of $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$ and explicit representations of the Lie group $G = \mathrm{SU}(4)$ which yield the quiver diagrams discussed in the main text.

B.1 Geometry of $\mathrm{SU}(4)/\mathrm{SU}(3)$

The definition of the 1-forms in (4.7) and the flatness of the connection lead to the structure equations

$$\begin{aligned}
\mathrm{d}\Theta^1 &= -4\mathrm{i}\mu_7 e^7 \wedge \Theta^1 + 2\mathrm{i}\mu_8 e^8 \wedge \Theta^1 + \tfrac{\zeta_2}{\zeta_1}\Theta^{2\bar{4}} + \tfrac{\zeta_3}{\zeta_1}\Theta^{3\bar{5}}, \\
\mathrm{d}\Theta^2 &= -4\mathrm{i}\mu_7 e^7 \wedge \Theta^2 - \mathrm{i}\mu_8 e^8 \wedge \Theta^2 - \mathrm{i}\mu_9 e^9 \wedge \Theta^2 - \tfrac{\zeta_1}{\zeta_2}\Theta^{14} + \tfrac{\zeta_3}{\zeta_2}\Theta^{3\bar{6}}, \\
\mathrm{d}\Theta^3 &= -4\mathrm{i}\mu_7 e^7 \wedge \Theta^3 - \mathrm{i}\mu_8 e^8 \wedge \Theta^3 + \mathrm{i}\mu_9 e^9 \wedge \Theta^3 - \tfrac{\zeta_1}{\zeta_3}\Theta^{15} - \tfrac{\zeta_2}{\zeta_3}\Theta^{26}, \\
\mathrm{d}e^7 &= -\tfrac{\mathrm{i}}{3\mu_7}(\zeta_1^2\Theta^{1\bar{1}} + \zeta_2^2\Theta^{2\bar{2}} + \zeta_3^2\Theta^{3\bar{3}}),
\end{aligned} \tag{B.1a}$$

as well as

$$\begin{aligned}
\mathrm{d}\Theta^4 &= -3\mathrm{i}\mu_8 e^8 \wedge \Theta^4 - \mathrm{i}\mu_9 e^9 \wedge \Theta^4 + \zeta_1\zeta_2\Theta^{\bar{1}2} + \Theta^{5\bar{6}}, \\
\mathrm{d}\Theta^5 &= -3\mathrm{i}\mu_8 e^8 \wedge \Theta^5 + \mathrm{i}\mu_9 e^9 \wedge \Theta^5 + \zeta_1\zeta_3\Theta^{\bar{1}3} - \Theta^{46}, \\
\mathrm{d}\Theta^6 &= 2\mathrm{i}\mu_9 e^9 \wedge \Theta^6 + \zeta_2\zeta_3\Theta^{\bar{2}3} + \Theta^{\bar{4}5}, \\
\mathrm{d}e^8 &= -\tfrac{\mathrm{i}}{6\mu_8}(-2\zeta_1^2\Theta^{1\bar{1}} + \zeta_2^2\Theta^{2\bar{2}} + \zeta_3^2\Theta^{3\bar{3}} + 3\Theta^{4\bar{4}} + 3\Theta^{5\bar{5}}), \\
\mathrm{d}e^9 &= -\tfrac{\mathrm{i}}{2\mu_9}(\zeta_2^2\Theta^{2\bar{2}} - \zeta_3^2\Theta^{3\bar{3}} + \Theta^{4\bar{4}} - \Theta^{5\bar{5}} - 2\Theta^{6\bar{6}}),
\end{aligned} \tag{B.1b}$$

where the shorthand notation $\Theta^{\alpha\beta} := \Theta^\alpha \wedge \Theta^\beta$ etc. is applied. The values of the parameters μ_7 and ζ_i in (4.7) and (B.1) have to be fixed by the Sasaki-Einstein geometry. From the closure of the Kähler form (4.8) one obtains

$$\begin{aligned}
2\mathrm{i}\ \mathrm{d}\Omega^{1,1} = r^2 \Big[&\left(\tfrac{\zeta_2}{\zeta_1} - \tfrac{\zeta_1}{\zeta_2}\right)\left(\Theta^{\bar{1}2\bar{4}} - \Theta^{1\bar{2}4}\right) + \left(\tfrac{\zeta_2}{\zeta_3} - \tfrac{\zeta_3}{\zeta_2}\right)\left(\Theta^{2\bar{3}6} - \Theta^{\bar{2}3\bar{6}}\right) \\
&+ \left(\tfrac{\zeta_1}{\zeta_3} - \tfrac{\zeta_3}{\zeta_1}\right)\left(\Theta^{1\bar{3}5} - \Theta^{\bar{1}3\bar{5}}\right) + \left(1 - \tfrac{\zeta^2}{3\mu_7}\right)\left(\Theta^0 + \Theta^{\bar{0}}\right) \wedge \left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}\right)\Big],
\end{aligned} \tag{B.2}$$

where we have already used the condition $\zeta_1 = \zeta_2 = \zeta_3 \equiv \zeta$, imposed by the first terms, to simplify the last summand. The differential of the holomorphic 4-form reads

$$\mathrm{d}\Omega^{4,0} = 4r^4(1 - 3\mu_7)\Theta^{\bar{0}} \wedge \Theta^0 \wedge \Theta^1 \wedge \Theta^2 \wedge \Theta^3, \tag{B.3}$$

which leads to $\mu_7 = \frac{1}{3}$ and therefore $\zeta^2 = 1$.

B.1.1 Vanishing of the torsion term

Let us show explicitly, without using the general theory, that upon imposing the instanton equations (4.22) the torsion term $\mathcal{F} \wedge \mathrm{d} \star Q$ in (2.11) vanishes. Because of $\star_7 Q = (e^{12} + e^{34} + e^{56}) \wedge e^7 = \omega \wedge e^7$, the differential is given by

$$\mathrm{d} \star_7 Q = \mathrm{d}(\omega \wedge e^7) = 2\omega \wedge \omega = 4(e^{1234} + e^{1256} + e^{3456}). \tag{B.4}$$

Already using $\mathcal{F}_{\mu 7} = 0$ from (4.22), we obtain

$$\mathcal{F} \wedge \mathrm{d} \star_7 Q = 4 \left[\mathcal{F}_{12} + \mathcal{F}_{34} + \mathcal{F}_{56}\right] e^{123456} = 0 \ , \tag{B.5}$$

which indeed vanishes due to the instanton equations (4.22). The Chern-Simons term in (2.12), without specializing to instanton solutions, reads

$$\begin{aligned} \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F} \wedge \star Q) = 4 \ \mathrm{Tr}(&\mathcal{F}_{12}\mathcal{F}_{34} + \mathcal{F}_{12}\mathcal{F}_{56} + \mathcal{F}_{34}\mathcal{F}_{56} - \mathcal{F}_{13}\mathcal{F}_{24} + \mathcal{F}_{14}\mathcal{F}_{23} \\ &+ \mathcal{F}_{14}\mathcal{F}_{23} - \mathcal{F}_{15}\mathcal{F}_{26} + \mathcal{F}_{16}\mathcal{F}_{25} - \mathcal{F}_{35}\mathcal{F}_{46} + \mathcal{F}_{36}\mathcal{F}_{45}) \ e^1 \wedge \ldots \wedge e^7. \end{aligned} \tag{B.6}$$

Imposing the instanton equations (4.22) turns (B.6) into

$$\begin{aligned} \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F} \wedge \star Q)|_{\mathrm{inst}} = -\mathrm{Tr}\Big[&2(\mathcal{F}_{12}^2 + \mathcal{F}_{34}^2 + \mathcal{F}_{56}^2) \\ &+ 4 \ (\mathcal{F}_{13}^2 + \mathcal{F}_{14}^2 + \mathcal{F}_{15}^2 + \mathcal{F}_{16}^2 + \mathcal{F}_{35}^2 + \mathcal{F}_{36}^2)\Big] \ e^1 \wedge \ldots \wedge e^7. \end{aligned} \tag{B.7}$$

B.2 Representations of $\mathrm{SU}(4)$

This section contains the representations of SU(4) used in Section 4.3. We will always work with the complexified version, i.e. representations of $\mathrm{SL}(4,\mathbb{C})$, whose representation theory is discussed in [108, Ch. 15] for instance.

The definition of the 1-forms in (4.7) has fixed the dual generators of the Lie algebra $\mathfrak{su}(4)$ in the fundametal representation:

$$\begin{aligned} I_1^+ &:= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_2^+ := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_3^+ := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ I_4^+ &:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_5^+ := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_6^+ := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{B.8a}$$

with the other ladder operators defined by $I_{\bar{\alpha}}^- = -\left(I_\alpha^+\right)^\dagger$ for $\alpha = 1, \ldots 6$, and the rescaled Cartan generators $\widetilde{I}_j := -\mathrm{i}\mu_j^{-1} I_j$:

$$\widetilde{I}_7 = \mathrm{diag}(3, -1, -1, -1), \quad \widetilde{I}_8 = \mathrm{diag}(0, 2, -1, -1), \quad \widetilde{I}_9 = \mathrm{diag}(0, 0, -1, 1). \tag{B.8b}$$

With respect to these generators the non-vanishing structure constants are given by

$$
\begin{aligned}
&C^4_{84}=C^5_{85}=3,\ C^4_{94}=1,\ C^5_{95}=-1,\ C^6_{96}=-2,\ C^8_{4\bar4}=C^8_{5\bar5}=-\tfrac12,\ C^9_{4\bar4}=-\tfrac12,\\
&C^9_{5\bar5}=\tfrac12,\ C^9_{6\bar6}=1,\ C^4_{5\bar6}=-1,\ C^5_{\bar46}=1,\ C^6_{\bar45}=-1,\ C^1_{71}=C^2_{72}=C^3_{73}=4,\\
&C^1_{81}=-2,\ C^2_{82}=C^3_{83}=1,\ C^2_{92}=1,\ C^3_{93}=-1,\ C^7_{1\bar1}=C^7_{2\bar2}=C^7_{3\bar3}=-\tfrac13,\ C^8_{1\bar1}=\tfrac13,\\
&C^8_{2\bar2}=-\tfrac16,\ C^8_{3\bar3}=-\tfrac16,\ C^9_{2\bar2}=-\tfrac12,\ C^9_{3\bar3}=\tfrac12,\ C^1_{2\bar4}=-1,\ C^1_{3\bar5}=-1,\ C^2_{1\bar4}=1,\\
&C^2_{3\bar6}=-1,\ \ C^3_{1\bar5}=1,\ \ C^3_{2\bar6}=1,\ \ C^4_{\bar12}=-1,\ \ C^5_{\bar13}=-1,\ \ C^6_{\bar23}=-1,
\end{aligned}
\tag{B.9}
$$

as well as the conjugated ones $C^{\alpha}_{j\alpha}=-C^{\bar\alpha}_{j\bar\alpha}$, $C^{\gamma}_{\alpha\beta}=C^{\bar\gamma}_{\bar\alpha\bar\beta}$ and $C^{\gamma}_{\alpha\bar\beta}=C^{\bar\gamma}_{\bar\alpha\beta}$ for $\alpha,\beta,\gamma=1,\dots,6$ and $j=7,8,9$. Expressing the complex forms Θ^α in (4.7) in terms of real 1-forms would provide unitary generators $I_1,\dots,I_{15}$, as chosen in [70, Sec. 7] for instance. Since we are always working in the holomorphic setup, it is convenient to keep (B.8) here. The Killing form has the only non-vanishing contributions

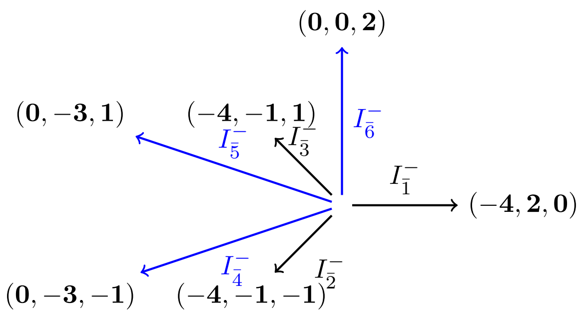
$$
\mathrm{Tr}(I^-_{\bar\alpha}\ I^+_{\alpha})=-1,\quad \mathrm{Tr}(\widetilde I_7\ \widetilde I_7)=12,\qquad \mathrm{Tr}(\widetilde I_8\ \widetilde I_8)=6,\quad \mathrm{Tr}(\widetilde I_9\ \widetilde I_9)=2,
\tag{B.10}
$$

for $\alpha=1,\dots 6$, so that the (real) generators are orthogonal and the induced metric on the coset space is indeed given by (piecewise) rescaling of the Killing form.

Denoting the quantum numbers with respect to the Cartan generators $\widetilde I_j$ by ν_j, the action of the ladder operators reads

$$
\begin{aligned}
&I^-_{\bar1}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7-4,\nu_8+2,\nu_9)\,, &&I^-_{\bar4}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7,\nu_8-3,\nu_9-1)\,,\\
&I^-_{\bar2}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7-4,\nu_8-1,\nu_9-1)\,, &&I^-_{\bar5}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7,\nu_8-3,\nu_9+1)\,,\\
&I^-_{\bar3}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7-4,\nu_8-1,\nu_9+1)\,, &&I^-_{\bar6}:(\nu_7,\nu_8,\nu_9)\longmapsto(\nu_7,\nu_8,\nu_9+2)\,.
\end{aligned}
\tag{B.11}
$$

Consequently, the root system consists of the two triangular lattices

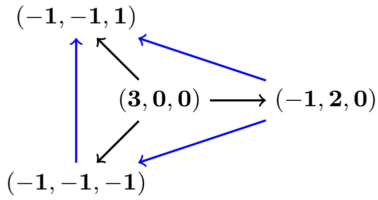


(B.12)

Since the quiver diagrams in the main text are obtained by collapsing along the ladder operators of SU(3), we stress the corresponding operators $I^-_{\bar4}$, $I^-_{\bar5}$, $I^-_{\bar6}$ by depicting them as blue arrows. Using this root system, we will construct some SU(4) representations, referring to [108] for details and a more systematic treatment.

B.2.1 Fundamental representation $\underline{4}$

The generators are those of the defining representation (B.8), and the weight diagram is the tetrahedron

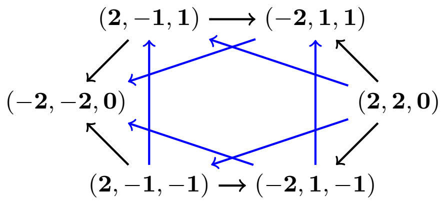


(B.13)

Under restriction to SU(3) it therefore decomposes into a trivial and a fundamental representation of the subalgebra and yields the quiver diagram (4.28) with two vertices.

B.2.2 Representation $\underline{6}$

Due to the isomorphism $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$ (or, on Lie algebra level, the coincidence of the Dynkin diagrams A_3 and D_3) one has also a six-dimensional representation of SU(4). By virtue of the root system (B.12) one can construct the weight diagram



(B.14)

For a more detailed discussion of this octahedron, we again refer to [108, Ch. 15]. The Cartan generators read $\widetilde{I}_7 = \mathrm{diag}(2,2,2,-2,-2,-2)$, $\widetilde{I}_8 = \mathrm{diag}(2,-1,-1,-2,1,1)$ and $\widetilde{I}_9 = \mathrm{diag}(0,1,-1,0,1,-1)$, while the ladder operators take the form (with $\alpha = 1,2,3$ and $\beta = 4,5,6$)

$$I_{\bar\alpha}^- = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ I^{(\alpha)} & \mathbf{0}_3 \end{pmatrix}, \quad I^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; I^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; I_{(3)} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

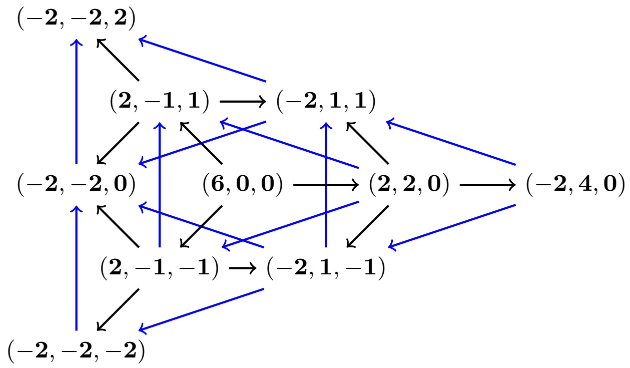
$$I_{\bar\beta}^- = \begin{pmatrix} I_1^{(\beta)} & \mathbf{0}_3 \\ \mathbf{0}_3 & I_2^{(\beta)} \end{pmatrix}, \quad I_1^{(4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; I_2^{(4)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I_1^{(5)} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$I_2^{(5)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_1^{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2^{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.15}$$

with $I_{\bar\alpha}^- = -(I_\alpha^+)^\dagger$ and $I_{\bar\beta}^- = -(I_\beta^+)^\dagger$. Under restriction to SU(3) this diagram yields the quiver (4.35) with two vertices, induced by the sum of fundamental and anti-fundamental representation of SU(3).

B.2.3 Representation 10

The weight diagram of the 10-dimensional representation is a tretrahedron of length 2, consisting of three layers of triangular SU(3) representations with fixed charges $\nu_7 \in \{-2, 2, 6\}$:



(B.16)

An explicit realization of the Cartan generators is given by

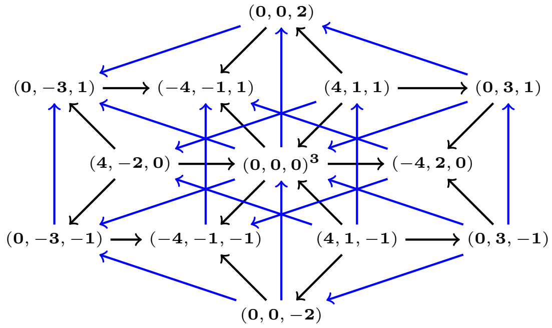
$$\begin{aligned} \widetilde{I}_7 &= \operatorname{diag}(6, 2, 2, 2, -2, -2, -2, -2, -2, -2), \\ \widetilde{I}_8 &= \operatorname{diag}(0, 2, -1, -1, 4, 1, 1, -2, -2, -2), \\ \widetilde{I}_9 &= \operatorname{diag}(0, 0, -1, 1, 0, -1, 1, 0, -2, 2), \end{aligned} \tag{B.17a}$$

and the non-vanishing entries of the ladder operators $I_{\bar{\alpha}}^- = -(I_\alpha^+)^\dagger$ may be chosen as

$$\begin{aligned}
&(I_{\bar{1}}^-)_{2,1} = \sqrt{2}, && (I_{\bar{1}}^-)_{5,2} = -\sqrt{2}, && (I_{\bar{1}}^-)_{6,3} = -1, && (I_{\bar{1}}^-)_{7,4} = -1, \\
&(I_{\bar{2}}^-)_{3,1} = \sqrt{2}, && (I_{\bar{2}}^-)_{6,2} = -1, && (I_{\bar{2}}^-)_{9,3} = -\sqrt{2}, && (I_{\bar{2}}^-)_{8,4} = -1, \\
&(I_{\bar{3}}^-)_{4,1} = \sqrt{2}, && (I_{\bar{3}}^-)_{7,2} = -1, && (I_{\bar{3}}^-)_{8,3} = -1, && (I_{\bar{3}}^-)_{10,4} = -\sqrt{2}, \\
&\left(I_{\bar{4}}^-\right)_{3,2} = -1, && \left(I_{\bar{4}}^-\right)_{8,7} = -1, && \left(I_{\bar{4}}^-\right)_{6,5} = -\sqrt{2}, && \left(I_{\bar{4}}^-\right)_{9,6} = -\sqrt{2}, \\
&(I_{\bar{5}}^-)_{4,2} = -1, && (I_{\bar{5}}^-)_{7,5} = -\sqrt{2}, && (I_{\bar{5}}^-)_{8,6} = -1, && (I_{\bar{5}}^-)_{10,7} = -\sqrt{2}, \\
&(I_{\bar{6}}^-)_{4,3} = -1, && (I_{\bar{6}}^-)_{7,6} = -1, && (I_{\bar{6}}^-)_{10,8} = -\sqrt{2}, && (I_{\bar{6}}^-)_{8,9} = -\sqrt{2}.
\end{aligned} \tag{B.17b}$$

B.2.4 Representation 15

Based on the root system (B.12), one constructs the following weight diagram for the 15-dimensional adjoint representation:



(B.18)

The degeneracy of the arrows involving the origin is not indicated for a better readibility of the diagram. This representation yields the decomposition

$$\underline{\mathbf{15}}|_{\mathrm{SU}(3)} = \underline{(\mathbf{0},\mathbf{0},\mathbf{2})_8} \oplus \underline{(\mathbf{4},\mathbf{1},\mathbf{1})_3} \oplus \underline{(-\mathbf{4},-\mathbf{1},\mathbf{1})_3} \oplus \underline{(\mathbf{0},\mathbf{0},\mathbf{0})_1}, \tag{B.19}$$

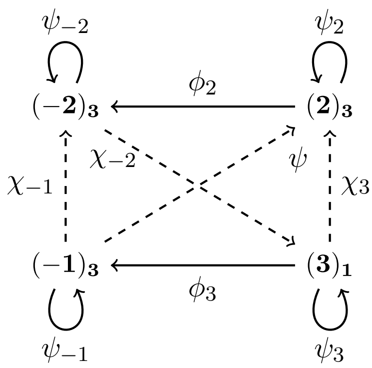
and collapsing the weight diagram (B.18) along the ladder operators of $\mathfrak{h}$ induces the quiver (4.47).

B.2.5 Higgs fields for $\underline{\mathbf{6}} \oplus \underline{\mathbf{4}}$

As discussed in Section 3.2, the construction of quiver diagrams by collapsing the generators of the complement $\mathfrak{m}$ along the action of the ladder operators of $\mathfrak{h}$ and twisting the resulting matrices by bundle maps is more restrictive than the evaluation of the equivariance condition (3.13) alone, for Sasakian coset spaces G/H. In the case at hand, the equivariance condition requires (4.16), which is not sufficient to recast the action according to (B.11) in general.

As discussed in the main text, for the adjoint representation $\underline{\mathbf{15}}$ also an arrow between fundamental and anti-fundamental representation of SU(3) in (4.47) would be compatible with (4.16). Here we illustrate the same effect by considering the reducible representation $\underline{\mathbf{6}} \oplus \underline{\mathbf{4}}$. If we stick to the construction of quiver bundles from Section 3.1, which we follow in the main text, the resulting quiver will consist of the known diagrams (4.28) and (4.35) for $\underline{\mathbf{6}}$ and $\underline{\mathbf{4}}$ *separately* since the G-action does not intertwine them.

In constrast, let us start from the explicit form of the generators of $\mathfrak{h} = \mathfrak{su}(3)$ in the chosen representation $\underline{\mathbf{6}} \oplus \underline{\mathbf{4}}$ and then forget about their origin from $\mathfrak{su}(4)$. Based on the above representations, they take the form $I_j = \bigoplus_l I_j^{(l)} \times \mathbf{1}_{k_l}$, and imposing only (4.16) on the matrices $X_\mu \in \mathfrak{u}(k)$ is compatible with the quiver



(B.20)

which connects the two quiver diagrams for $\underline{\mathbf{4}}$ and $\underline{\mathbf{6}}$ by additional arrows, indicated as dashed lines. The endomorphisms therefore take the following block shape

$$\phi^{(\alpha)} = \begin{pmatrix} 0 & 0 & \chi_3 \otimes \tilde{I}_3^{(\alpha)} & 0 \\ \phi_2 \otimes I_2^{(\alpha)} & 0 & 0 & \chi_{-1} \otimes I_{-1}^{(\alpha)} \\ 0 & \chi_{-2} \otimes I_{-2}^{(\alpha)} & 0 & 0 \\ 0 & 0 & \phi_3 \otimes I_3^{(\alpha)} & 0 \end{pmatrix}, \tag{B.21}$$

where $I_2^{(\alpha)}$ and $I_3^{(\alpha)}$ are given by the matrices $I^{(\alpha)}$ in (4.30) and (4.36). The further contributions are defined as

$$\begin{aligned} &\tilde{I}_3^{(1)} = (1,0,0)^T, && \tilde{I}_3^{(2)} = (0,0,1)^T, && \tilde{I}_3^{(3)} = (0,1,0)^T, \\ &I_{-2}^{(1)} = (1,0,0)^T, && I_{-2}^{(2)} = (0,-1,0)^T, && I_{-2}^{(3)} = (0,0,-1)^T, \\ &I_{-1}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, && I_{-1}^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, && I_{-1}^{(3)} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.22}$$

The vertical field reads

$$X_7 = \begin{pmatrix} \psi_2 \otimes \mathbf{1}_3 & 0 & 0 & \psi \otimes \widetilde{I} \\ 0 & \psi_{-2} \otimes \mathbf{1}_3 & 0 & 0 \\ 0 & 0 & \psi_3 & 0 \\ -\psi^\dagger \otimes \widetilde{I}^\dagger & 0 & 0 & \psi_{-1}\mathbf{1}_3 \end{pmatrix} \quad \text{with} \quad \widetilde{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{B.23}$$

Of course, under restriction to the action of the SU(4) generators, the additional arrows are ruled out and the quiver (B.20) decomposes into two separate diagrams since we consider, by construction, a reducible representation of G. As we have seen in the main text, the possibility of additional contributions which are compatible with the equivariance condition only appears for the sphere itself, since its orbifolds impose a condition also with respect to I_7 (cf. Section 4.4.2).

The Hermitian Yang-Mills equations for the gauge connection characterized by (B.20) comprise the differential equations

$$\begin{aligned} &\dot{\phi}_2 = -\tfrac{4}{3}\phi_2 - \mathrm{i}\phi_2\psi_2 + \mathrm{i}\psi_{-2}\phi_2 + \mathrm{i}\chi_{-1}\psi^\dagger, && \dot{\phi}_3 = -\tfrac{4}{3}\phi_3 - \mathrm{i}\phi_3\psi_{-1} + \mathrm{i}\psi_{-1}\phi_3 - \mathrm{i}\psi^\dagger\chi_3, \\ &\dot{\psi}_2 = -6\psi_2 + 4\mathrm{i}\phi_2^\dagger\phi_2 - 2\mathrm{i}\chi_3\chi_3^\dagger, && \dot{\psi}_{-1} = -6\psi_{-1} - 2\mathrm{i}\phi_3\phi_3^\dagger + 4\mathrm{i}\chi_{-1}^\dagger\chi_{-1}, \end{aligned}$$

$$\begin{aligned} \dot{\psi}_{-2} &= -6\psi_{-2} - 4\mathrm{i}\phi_2\phi_2^\dagger + 2\mathrm{i}\chi_{-2}^\dagger\chi_{-2} - 4\mathrm{i}\chi_{-1}\chi_{-1}^\dagger, \\ \dot{\psi}_3 &= -6\psi_3 + 6\mathrm{i}\phi_3^\dagger\phi_3 - 6\mathrm{i}\chi_{-2}\chi_{-2}^\dagger + 6\mathrm{i}\chi_3^\dagger\chi_3, \end{aligned} \tag{B.24a}$$

as well as

$$\begin{aligned} &\dot{\chi}_3 = -\tfrac{4}{3}\chi_3 - \mathrm{i}\chi_3\psi_3 + \mathrm{i}\psi_2\chi_3 + \mathrm{i}\psi\phi_3, && \dot{\chi}_{-1} = -\tfrac{4}{3}\chi_{-1} - \mathrm{i}\chi_{-1}\psi_{-1} + \mathrm{i}\psi_{-2}\chi_{-1} - \mathrm{i}\phi_2\psi, \\ &\dot{\chi}_{-2} = -\tfrac{4}{3}\chi_{-2} - \mathrm{i}\chi_{-2}\psi_{-2} + \mathrm{i}\psi_3\chi_{-2}, && \dot{\psi} = -6\psi - 2\mathrm{i}\chi_3\phi_3^\dagger + 4\mathrm{i}\phi_2^\dagger\chi_{-1}. \end{aligned} \tag{B.24b}$$

Furthermore, one obtains the quiver relations

$$\chi_3\chi_{-2} = 0, \quad \chi_{-2}\phi_2 = 0, \quad \phi_3\chi_{-2} = 0, \quad \chi_{-2}\chi_{-1} = 0, \quad \phi_2\chi_3 = -\chi_{-1}\phi_3. \tag{B.24c}$$

As claimed before and as it is necessary, one can consistently set the additional maps in (B.24b) to zero, and the flow equations (B.24a) then reduce to those of $\underline{\mathbf{4}}$ and $\underline{\mathbf{6}}$ separately. The algebraic relations (B.24c) motivate to set (at least) the arrow χ_{-2} to zero, on the level of instanton solutions.

C Details of the space $T^{1,1}$

This appendix provides details on the geometric structure of $T^{1,1}$, the chosen representations and some technical expressions for the field strength components of the graded gauge connection.

C.1 Geometry of $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$

By the choice of the forms in the Maurer-Cartan form (5.8) one obtains from the flatness of the connection the following complex structure equations

$$\begin{aligned} \mathrm{d}\Theta^1 &= 2\mathrm{i}\kappa e^5 \wedge \Theta^1 + 2a \wedge \Theta^1, & \mathrm{d}\Theta^2 &= 2\mathrm{i}\kappa e^5 \wedge \Theta^2 - 2a \wedge \Theta^2, \\ \mathrm{d}e^5 &= \tfrac{\mathrm{i}}{2\kappa}\alpha_1^2 \Theta^1 \wedge \Theta^{\bar{1}} + \tfrac{\mathrm{i}}{2\kappa}\alpha_2^2 \Theta^2 \wedge \Theta^{\bar{2}}, & \mathrm{d}a &= -\tfrac{1}{2}\alpha_1^2 \Theta^1 \wedge \Theta^{\bar{1}} + \tfrac{1}{2}\alpha_2^2 \Theta^2 \wedge \Theta^{\bar{2}}. \end{aligned} \tag{C.1}$$

After introducing the third holomorphic form $\Theta^0 := \mathrm{d}\tau - \mathrm{i}e^5$ on the metric cone, the Kähler form and top-degree form read

$$\Omega^{1,1} := -\tfrac{\mathrm{i}}{2} r^2 (\Theta^{0\bar{0}} + \Theta^{1\bar{1}} + \Theta^{2\bar{2}}), \qquad \Omega^{3,0} := r^3 \Theta^0 \wedge \Theta^1 \wedge \Theta^2. \tag{C.2}$$

The vanishing of the differential of the Kähler form yields the condition

$$0 = (\Theta^0 + \Theta^{\bar{0}}) \wedge \left[\left(1 + \frac{\alpha_1^2}{2\kappa}\right) \Theta^{1\bar{1}} + \left(1 + \frac{\alpha_2^2}{2\kappa}\right) \Theta^{2\bar{2}} \right], \tag{C.3}$$

and the closure of $\Omega^{3,0}$ requires

$$0 = \left(\mathrm{d}\tau + \tfrac{4}{3}\kappa \mathrm{i} e^5\right) \wedge \Theta^0 \wedge \Theta^1 \wedge \Theta^2, \tag{C.4}$$

so that one obtains $\kappa = -\frac{3}{4}$ and $\alpha_1^2 = \alpha_2^2 = \frac{3}{2}$. Then the structure equations in terms of the real 1-forms are given by

$$\begin{aligned} \mathrm{d}e^1 &= -\tfrac{3}{2} e^{52} - 2\mathrm{i}a \wedge e^2, & \mathrm{d}e^2 &= \tfrac{3}{2} e^{51} + 2\mathrm{i}a \wedge e^1, \\ \mathrm{d}e^3 &= -\tfrac{3}{2} e^{54} + 2\mathrm{i}a \wedge e^4, & \mathrm{d}e^4 &= \tfrac{3}{2} e^{53} - 2\mathrm{i}a \wedge e^3, \\ \mathrm{d}e^5 &= 2e^{12} + 2e^{34}, & \mathrm{d}a &= -\tfrac{3}{2}\mathrm{i}e^{12} + \tfrac{3}{2}\mathrm{i}e^{34}. \end{aligned} \tag{C.5}$$

Curvature of the Levi-Civita connection. While the canonical connection (5.18) was obtained by using the torsion components (5.17), the structure equations (C.5) yield for the Levi-Civita connection, i.e. $T^\mu \equiv 0$, the connection matrix with the only non-vanishing contributions

$$\begin{aligned} \Gamma^1{}_2 &= -\tfrac{1}{2}e^5 - 2\mathrm{i}a, & \Gamma^1{}_5 &= e^2, & \Gamma^2{}_5 &= -e^1, \\ \Gamma^3{}_4 &= -\tfrac{1}{2}e^5 + 2\mathrm{i}a, & \Gamma^3{}_5 &= e^4, & \Gamma^4{}_5 &= -e^3. \end{aligned} \tag{C.6}$$

They lead to the curvature components

$$\begin{aligned} &R^1{}_2 = 3e^{12} - 2e^{34}\ , && R^1{}_3 = -e^{24}\ , && R^1{}_4 = e^{23}\ , && R^1{}_5 = e^{15} && R^2{}_3 = e^{14}\ , \\ &R^3{}_4 = -2e^{12} + 3e^{34} && R^2{}_4 = -e^{13}\ , && R^2{}_5 = e^{25}\ , && R^3{}_5 = e^{35}\ , && R^4{}_5 = e^{45}\ . \end{aligned} \tag{C.7}$$

Contraction over the indices gives us the Ricci-tensor $\mathrm{Ric} = 4\,g$ and the scalar curvature $s = 20$, as expected from the general result $s = 2n(2n+1)$ on a Sasaki-Einstein manifold of dimension $2n+1$. One also recognizes the known property that the curvature (C.7) spans the entire Lie algebra $\mathfrak{so}(5)$, i.e. has generic holonomy [54].

Vanishing of the torsion term. Similarly to Section B.1.1, the vanishing of the torsion term in the generalized Yang-Mills equation (A.12) is easily verified. On $T^{1,1}$ one obtains $\star_5 Q = \star_5 e^{1234} = e^5$ and therefore $\mathrm{d} \star_5 Q = 2\omega = 2(e^{12} + e^{34})$. For connections subject to the instanton equation (5.29), the torsion term is given by

$$\mathrm{d} \star_5 Q \wedge \mathcal{F} = 2(e^{12} + e^{34}) \wedge \mathcal{F} = 4(\mathcal{F}_{12} + \mathcal{F}_{34}) \wedge e^{1234} = 0\ , \tag{C.8}$$

analogously to (B.5). The Chern-Simons term in (2.12) on a five-dimensional Sasaki-Einstein manifold reads

$$\mathrm{Tr}\ (\mathcal{F} \wedge \mathcal{F} \wedge \star Q) = 2\ \mathrm{Tr}\ (\mathcal{F}_{12}\mathcal{F}_{34} - \mathcal{F}_{13}\mathcal{F}_{24} + \mathcal{F}_{14}\mathcal{F}_{23})\ e^{12345}, \tag{C.9}$$

and imposing the instanton equations (5.29) turns it into

$$\mathrm{Tr}\ (\mathcal{F} \wedge \mathcal{F} \wedge \star Q) = -2\ \mathrm{Tr}\ (\mathcal{F}_{12}\mathcal{F}_{12} + \mathcal{F}_{13}\mathcal{F}_{13} + \mathcal{F}_{14}\mathcal{F}_{14})\ e^{12345}. \tag{C.10}$$

C.2 Representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$

We construct representations of the Lie algebra of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ by using tensor products of single $\mathrm{SU}(2)$ representations. For the fundamental representation of $\mathrm{SU}(2)$ we choose the Cartan generator and the two ladder operators

$$I^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad I^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad I^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.11}$$

with their usual commutation relations

$$[I_3, I_\pm] = \pm 2 I_\pm, \qquad [I_+, I_-] = -I_3. \tag{C.12}$$

The above representation can be generalized to an irreducible representation of SU(2) on $\mathbb{C}^{m+1}$ for any positive m by using the generators

$$I^+_{(m)} := \begin{pmatrix} 0 & \gamma_0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{m-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} , \quad I^-_{(m)} := -\left(I^+_{(m)}\right)^\dagger \tag{C.13}$$

and $I^3_{(m)} := \text{diag}(m, m-2, \dots, -m+2, -m)$ with the definition $\gamma_i^2 := (i+1)(m-i)$ for $i = 0, 1, \dots, m-1$. For any pair of integers (m_1, m_2) a representation of SU(2)×SU(2) on $\mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1}$ is obtained by taking the tensor products

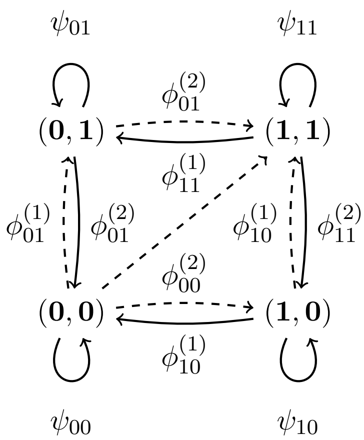
$$\begin{aligned} I^{(1)}_\pm &:= I^\pm_{(m_1)} \otimes \mathbf{1}_{m_2+1}, & I^{(1)}_3 &:= I^3_{(m_1)} \otimes \mathbf{1}_{m_2+1}, \\ I^{(2)}_\pm &:= \mathbf{1}_{m_1+1} \otimes I^\pm_{(m_2)}, & I^{(2)}_3 &:= \mathbf{1}_{m_1+1} \otimes I^3_{(m_2)}. \end{aligned} \tag{C.14}$$

For instance, in the fundamental representation of both SU(2) factors, the generators read

$$\begin{aligned} I^{(1)}_+ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\left(I^{(1)}_-\right)^\dagger, & I^{(2)}_+ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\left(I^{(2)}_-\right)^\dagger, \\ I^{(1)}_3 &= \text{diag}(1,1,-1,-1), & I^{(2)}_3 &= \text{diag}(1,-1,1,-1). \end{aligned} \tag{C.15}$$

These generators lead to the Higgs fields and equivariant connection (5.36) encoded in the quiver diagram (5.36).

General Higgs fields. According to the discussion in the main text, without specializing to the G-action when considering the induced equivariant quiver bundle, the condition (5.22) alone is compatible with more general arrows. For the representation $\underline{(\mathbf{1},\mathbf{1})}$ one obtains



(C.16)

as the most general quiver diagram, where arrows not belonging to generators of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ are denoted as dashed lines. In particular, this quiver diagram admits two different arrows between adjacent vertices and even a new transition between $(0,0)$ and $(1,1)$ since only the relative quantum number $c_{i\alpha}$ determines the relations. Effectively, in terms of the relevant quantum number, one might even think of a modified holomorphic chain with double arrows which is obtained by identifying vertices along the diagonal lines. For details on more general Higgs fields see the discussion in [47], which also includes the explicit example $\underline{(\mathbf{2},\mathbf{1})}$.

C.2.1 Projection operators

The discussion of the equivariance condition benefits from expressing the generator $I_6 = I_3^{(1)} - I_3^{(2)}$ of $\mathfrak{h}$ on $\mathbb{C}^k$ by natural projection operators [43, 47] on $\mathbb{C}^{m_1+1}$ and $\mathbb{C}^{m_2+1}$, respectively,

$$\begin{aligned} \Pi_i : \mathbb{C}^{m_1+1} &\longrightarrow \mathbb{C} \ , \quad \Pi_i \ = \ (\delta_{i,j}\delta_{i,k})_{1\le j,k\le m_1+1} \ , \qquad i \ = \ 0,1,\dots,m_1, \\ \Pi_\alpha : \mathbb{C}^{m_2+1} &\longrightarrow \mathbb{C} \ , \quad \Pi_\alpha \ = \ (\delta_{\alpha,\beta}\delta_{\alpha,\gamma})_{1\le\alpha,\gamma\le m_2+1} \ , \quad \alpha \ = \ 0,1,\dots,m_2, \end{aligned} \tag{C.17}$$

where Latin indices always refer to the first copy of SU(2) and Greek indices to the second copy. The projection from the tensor product $\mathbb{C}^{m_1+1}\otimes\mathbb{C}^{m_2+1}$ to the component with indices i and α is given by the operator

$$\Pi_{i\alpha} : \mathbb{C}^{m_1+1} \otimes \mathbb{C}^{m_2+1} \longrightarrow \mathbb{C} \ , \qquad \Pi_{i\alpha} \ \coloneqq \ \Pi_i \otimes \Pi_\alpha, \tag{C.18}$$

and thus by the diagonal square matrix

$$\Pi_{i\alpha} \ = \ \left(\delta_{ij}\,\delta_{\alpha\beta}\,\delta_{ik}\,\delta_{\alpha\gamma}\right)^{j,k=0,1,\dots,m_1}_{\beta,\gamma=0,1,\dots,m_2} \tag{C.19}$$

of size $[(m_1+1)\,(m_2+1)]^2$. Furthermore, one introduces the operators

$$\begin{aligned} \Pi_i^{(1)} &\coloneqq \sum_{\alpha=0}^{m_2} \Pi_{i\alpha} = \Pi_i \otimes \sum_{\alpha=0}^{m_1} \Pi_\alpha = \Pi_i \otimes \mathbf{1}_{m_2+1} \ , \\ \Pi_\alpha^{(2)} &\coloneqq \sum_{i=0}^{m_1} \Pi_{i\alpha} = \sum_{i=0}^{m_1} \Pi_i \otimes \Pi_\alpha = \mathbf{1}_{m_1+1} \otimes \Pi_\alpha \ , \end{aligned} \tag{C.20}$$

which project on all components with a fixed value of the first or second index, respectively. This yields a representation of the generators of the maximal torus of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ by the diagonal matrices

$$\begin{aligned} \Upsilon^{(1)} &\coloneqq \sum_{i=0}^{m_1} (m_1 - 2i)\,\Pi_i^{(1)} = I^3_{(m_1)} \otimes \mathbf{1}_{m_2+1} \ , \\ \Upsilon^{(2)} &\coloneqq \sum_{\alpha=0}^{m_2} (m_2 - 2\alpha)\,\Pi_\alpha^{(2)} = \mathbf{1}_{m_1+1} \otimes I^3_{(m_2)} \ . \end{aligned} \tag{C.21}$$

In particular, the Lie algebra $\mathfrak{h}$ is generated by

$$I_6 \ = \ \Upsilon^{(1)} - \Upsilon^{(2)} \ = \ \sum_{i=0}^{m_1}\sum_{\alpha=0}^{m_2} (m_1 - m_2 - 2i + 2\alpha)\,\Pi_{i\alpha} \ , \tag{C.22}$$

and we define the corresponding U(1)-charge $c_{i\alpha} := m_1 - m_2 - 2i + 2\alpha$. The action of the ladder operators on the quantum numbers associated to I_6 and $I_5 := I_3^{(1)} + I_3^{(2)}$ reads

$$I_+^{(1)} : \ (\nu_5, \nu_6) \longmapsto (\nu_5 + 2, \nu_5 + 2), \qquad I_+^{(2)} : \ (\nu_5, \nu_6) \longmapsto (\nu_5 + 2, \nu_6 - 2). \tag{C.23}$$

C.3 Details graded connection

This section provides technical details for the discussion of quiver diagrams and instantons for generic representations $\underline{(\mathbf{m_1}, \mathbf{m_2})}$ which encode the graded gauge connection (5.38). The non-vanishing parts of the field strength are given by

$$\begin{aligned}
\mathcal{F}^{i\alpha,i\alpha} &= \mathrm{d}\mathcal{A}^{i\alpha,i\alpha} + \mathcal{A}^{i\alpha,i\alpha} \wedge \mathcal{A}^{i\alpha,i\alpha} + \mathcal{A}^{i\,\alpha,i+1\,\alpha} \wedge \mathcal{A}^{i+1\,\alpha,i\,\alpha} \\
&\quad + \mathcal{A}^{i\,\alpha,i-1\,\alpha} \wedge \mathcal{A}^{i-1\,\alpha,i\,\alpha} + \mathcal{A}^{i\,\alpha,i\,\alpha+1} \wedge \mathcal{A}^{i\,\alpha+1,i\,\alpha} + \mathcal{A}^{i\,\alpha,i\,\alpha-1} \wedge \mathcal{A}^{i\,\alpha-1,i\,\alpha} \\
&= F^{i\alpha} + D\psi_{i\alpha} \wedge e^5 \\
&\quad + \left(\phi^{(1)}_{i+1\,\alpha}\, \phi^{(1)}_{i+1\,\alpha}{}^{\dagger} - \phi^{(1)\,\dagger}_{i\alpha}\, \phi^{(1)}_{i\alpha} - \mathrm{i}\psi_{i\alpha} - \tfrac{3}{4}\, c_{i\alpha}\, \mathbf{1}_{k_{i\alpha}}\right) \Theta^1 \wedge \Theta^{\bar{1}} \\
&\quad + \left(\phi^{(2)}_{i\,\alpha+1}\, \phi^{(2)}_{i\,\alpha+1}{}^{\dagger} - \phi^{(2)\,\dagger}_{i\alpha}\, \phi^{(2)}_{i\alpha} - \mathrm{i}\psi_{i\alpha} + \tfrac{3}{4}\, c_{i\alpha}\, \mathbf{1}_{k_{i\alpha}}\right) \Theta^2 \wedge \Theta^{\bar{2}} \ ,
\end{aligned} \tag{C.24a}$$

$$\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha} &= \mathrm{d}\mathcal{A}^{i\,\alpha,i+1\,\alpha} + \mathcal{A}^{i\alpha,i\alpha} \wedge \mathcal{A}^{i\,\alpha,i+1\,\alpha} + \mathcal{A}^{i\,\alpha,i+1\,\alpha} \wedge \mathcal{A}^{i+1\,\alpha,i+1\,\alpha} \\
&= D\phi^{(1)}_{i+1\,\alpha} \wedge \Theta^{\bar{1}} + \left(\phi^{(1)}_{i+1\,\alpha}\, \psi_{i+1\,\alpha} - \psi_{i\,\alpha}\phi^{(1)}_{i+1\,\alpha} - \tfrac{3\mathrm{i}}{2}\, \phi^{(1)}_{i+1\,\alpha}\right) \Theta^{\bar{1}} \wedge e^5, \\
&= -\left(\mathcal{F}^{i+1\,\alpha,i\,\alpha}\right)^{\dagger} \ ,
\end{aligned} \tag{C.24b}$$

$$\begin{aligned}
\mathcal{F}^{i\,\alpha,i\,\alpha+1} &= \mathrm{d}\mathcal{A}^{i\,\alpha,i\,\alpha+1} + \mathcal{A}^{i\alpha,i\alpha} \wedge \mathcal{A}^{i\,\alpha,i\,\alpha+1} + \mathcal{A}^{i\,\alpha,i\,\alpha+1} \wedge \mathcal{A}^{i\,\alpha+1,i\,\alpha+1} \\
&= D\phi^{(2)}_{i\,\alpha+1} \wedge \Theta^{\bar{2}} + \left(\phi^{(2)}_{i\,\alpha+1}\, \psi_{i\,\alpha+1} - \psi_{i\alpha}\, \phi^{(2)}_{i\,\alpha+1} - \tfrac{3\mathrm{i}}{2}\, \phi^{(2)}_{i\,\alpha+1}\right) \Theta^{\bar{2}} \wedge e^5 \\
&= -\left(\mathcal{F}^{i\,\alpha+1,i\,\alpha}\right)^{\dagger},
\end{aligned} \tag{C.24c}$$

$$\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha+1} &= \mathcal{A}^{i\,\alpha,i+1\,\alpha} \wedge \mathcal{A}^{i+1\,\alpha,i+1\,\alpha+1} + \mathcal{A}^{i\,\alpha,i\,\alpha+1} \wedge \mathcal{A}^{i\,\alpha+1,i+1\,\alpha+1} \\
&= \left(\phi^{(1)}_{i+1\,\alpha}\, \phi^{(2)}_{i+1\,\alpha+1} - \phi^{(2)}_{i\,\alpha+1}\, \phi^{(1)}_{i+1\,\alpha+1}\right) \Theta^{\bar{1}\bar{2}} \ = \ -\left(\mathcal{F}^{i+1\,\alpha+1,i\,\alpha}\right)^{\dagger} \ ,
\end{aligned} \tag{C.24d}$$

$$\begin{aligned}
\mathcal{F}^{i\,\alpha,i+1\,\alpha-1} &= \mathcal{A}^{i\,\alpha,i+1\,\alpha} \wedge \mathcal{A}^{i+1\,\alpha,i+1\,\alpha-1} + \mathcal{A}^{i\,\alpha,i\,\alpha-1} \wedge \mathcal{A}^{i\,\alpha-1,i+1\,\alpha-1} \\
&= \left(\phi^{(2)\dagger}_{i\alpha}\, \phi^{(1)}_{i+1\,\alpha-1} - \phi^{(1)}_{i+1\,\alpha}\, \phi^{(2)}_{i+1\,\alpha}\right) \Theta^{\bar{1}2} \ = \ -\left(\mathcal{F}^{i+1\,\alpha-1,i\,\alpha}\right)^{\dagger} \ .
\end{aligned} \tag{C.24e}$$

In order to express the Lagrangian $\mathcal{F}_{MN}\, \mathcal{F}^{MN}$ in terms of the homomorphisms, one recalls $g = \delta_{\mu\nu}\, e^\mu \otimes e^\nu = \Theta^1 \otimes \Theta^{\bar{1}} + \Theta^2 \otimes \Theta^{\bar{2}} + e^5 \otimes e^5$ and obtains

$$\begin{aligned}
\mathcal{F}_{MN}\, \mathcal{F}^{MN} &= \mathcal{F}_{ab}\, \mathcal{F}^{ab} + 4g^{ab} \left(\mathcal{F}_{a1}\, \mathcal{F}_{b\bar{1}} + \mathcal{F}_{a\bar{1}}\, \mathcal{F}_{b1} + \mathcal{F}_{a2}\, \mathcal{F}_{b\bar{2}} + \mathcal{F}_{a\bar{2}}\, \mathcal{F}_{b2}\right) + 2g^{ab}\, \mathcal{F}_{a5}\, \mathcal{F}_{b5} \\
&\quad + 8 \left(\mathcal{F}_{1\bar{1}}\, \mathcal{F}_{\bar{1}1} + \mathcal{F}_{12}\, \mathcal{F}_{\bar{1}\bar{2}} + \mathcal{F}_{1\bar{2}}\, \mathcal{F}_{\bar{1}2} + \mathcal{F}_{\bar{1}2}\, \mathcal{F}_{1\bar{2}} + \mathcal{F}_{\bar{1}\bar{2}}\, \mathcal{F}_{12} + \mathcal{F}_{2\bar{2}}\, \mathcal{F}_{\bar{2}2}\right) \\
&\quad + 4 \left(\mathcal{F}_{15}\, \mathcal{F}_{\bar{1}5} + \mathcal{F}_{\bar{1}5}\, \mathcal{F}_{15} + \mathcal{F}_{25}\, \mathcal{F}_{\bar{2}5} + \mathcal{F}_{\bar{2}5}\, \mathcal{F}_{25}\right) \ .
\end{aligned} \tag{C.25}$$

Inserting the above curvature components (C.24) leads to the action functional (5.40). Moreover, imposing the instanton equations (5.31) on the explicit components of the field strength yields the flow equations and quiver relations discussed in Section 5.3.3.

D Details of the squashed seven-sphere

This appendix collects technical aspects related to the discussion of 3-Sasakian quiver gauge theories on the squashed seven-sphere. In particular, we provide some details on the Sp(2)-instanton equations and the representations of Sp(2) used for the quiver diagrams in Section 6.3.

D.1 Geometry of $\mathrm{Sp}(2)/\mathrm{Sp}(1)$

In terms of complex forms, the structure equations for the Lie algebra of Sp(2), induced by the flatness of the connection (6.7), read

$$\begin{aligned}
&\mathrm{d}\Theta^1 = -\mathrm{i}e^7\wedge\Theta^1 + \mathrm{i}e^8\wedge\Theta^1 - \Theta^{\bar{2}3} + \Theta^{2\bar{4}}, && \mathrm{d}\Theta^2 = -\mathrm{i}e^7\wedge\Theta^2 - \mathrm{i}e^8\wedge\Theta^2 + \Theta^{\bar{1}3} - \Theta^{14},\\
&\mathrm{d}\Theta^3 = -2\mathrm{i}e^7\wedge\Theta^3 - 2\,\Theta^{12}, && \mathrm{d}\Theta^4 = -2\mathrm{i}e^8\wedge\Theta^4 + 2\,\Theta^{\bar{1}2},\\
&\mathrm{d}e^7 = -\mathrm{i}(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}), && \mathrm{d}e^8 = \mathrm{i}(\Theta^{1\bar{1}} - \Theta^{2\bar{2}} - \Theta^{4\bar{4}})\ .
\end{aligned} \tag{D.1}$$

They yield the non-vanishing structure constants

$$\begin{aligned}
&C^1_{71} = -C^{\bar{1}}_{7\bar{1}} = 1\ , && C^2_{72} = -C^{\bar{2}}_{7\bar{2}} = 1\ , && C^3_{73} = -C^{\bar{3}}_{7\bar{3}} = 2\ , && C^1_{81} = -C^{\bar{1}}_{8\bar{1}} = -1\ ,\\
&C^2_{82} = -C^{\bar{2}}_{8\bar{2}} = 1\ , && C^4_{84} = -C^{\bar{4}}_{8\bar{4}} = 2\ , && C^{\bar{1}}_{2\bar{3}} = C^1_{\bar{2}3} = 1\ , && C^1_{2\bar{4}} = C^{\bar{1}}_{\bar{2}4} = -1\ ,\\
&C^3_{12} = C^{\bar{3}}_{\bar{1}\bar{2}} = 2\ , && C^2_{14} = C^{\bar{2}}_{\bar{1}\bar{4}} = 1\ , && C^2_{\bar{1}3} = C^{\bar{2}}_{1\bar{3}} = -1\ , && C^4_{\bar{1}2} = C^{\bar{4}}_{1\bar{2}} = -2\ ,\\
&C^7_{1\bar{1}} = C^7_{2\bar{2}} = C^7_{3\bar{3}} = -1\ , && C^8_{1\bar{1}} = -C^8_{2\bar{2}} = 1\ , && C^8_{4\bar{4}} = -1
\end{aligned} \tag{D.2}$$

with respect to the Cartan generators rescaled by a factor of $-\mathrm{i}$.

3-Sasakian property. To show that the coset space Sp(2)/Sp(1) is 3-Sasakian, one can use, equivalently to the definition applied in Section 6.1, that its metric cone must be hyper-Kähler. Introducing again a fourth holomorphic form

$$\Theta^0 \ :=\ \frac{\mathrm{d}r}{r} - \mathrm{i}e^7\ , \tag{D.3}$$

the metric cone is Calabi-Yau due to the closure of the forms

$$\Omega^{1,1} \ :=\ -\tfrac{\mathrm{i}}{2}\,r^2\left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}} + \Theta^{0\bar{0}}\right) \qquad \text{and} \qquad \Omega^{4,0} \ :=\ r^4\,\Theta^{1230}\ . \tag{D.4}$$

For the holonomy to be further reduced from SU(4) to Sp(2), one additionally requires the closure of the *complex symplectic form* [4, 27]

$$\Omega^{2,0} \ :=\ r^2\left(\Theta^{12} + \Theta^{30}\right)\ , \tag{D.5}$$

which follows from the structure equations (D.1) as well.

Vanishing of the torsion term. Also for the 3-Sasakian structure of Sp(2)/Sp(1) the vanishing of the torsion term $\mathcal{F} \wedge \mathrm{d} \star_7 Q$ in (2.11) for instanton solutions can be verified easily. Using the result $Q = e^{1234}$ and the structure equations (6.10a), we obtain

$$\begin{aligned} \mathrm{d} \star_7 Q = \mathrm{d}(e^{567}) &= 2\omega^5 \wedge e^{67} - 2\omega^6 \wedge e^{57} + 2\omega^7 \wedge e^{56} \\ &= 2(-e^{13} + e^{24}) \wedge e^{67} + 2(e^{14} + e^{23}) \wedge e^{57} + 2(e^{12} + e^{34}) \wedge e^{56} . \end{aligned} \tag{D.6}$$

Therefore, the torsion term for instanton solutions, obeying (6.17), indeed vanishes:

$$(\mathrm{d} \star_7 Q) \wedge \mathcal{F} = 2e^{1234} \wedge \Big[(\mathcal{F}_{24} - \mathcal{F}_{13})e^{67} + (\mathcal{F}_{23} + \mathcal{F}_{14})e^{57} + (\mathcal{F}_{12} + \mathcal{F}_{34})e^{56}\Big] = 0. \tag{D.7}$$

The second contribution in the action functional (2.12) is given by

$$\mathrm{Tr}\ (\mathcal{F} \wedge \mathcal{F} \wedge \star Q) = 2\ \mathrm{Tr}\ (\mathcal{F}_{12}\mathcal{F}_{34} - \mathcal{F}_{13}\mathcal{F}_{24} + \mathcal{F}_{14}\mathcal{F}_{23})\ e^1 \wedge \ldots \wedge e^7 , \tag{D.8}$$

and imposing the instanton equations (6.17) yields

$$\mathrm{Tr}\ (\mathcal{F} \wedge \mathcal{F} \wedge \star Q)|_{\mathrm{inst}} = -2\ \mathrm{Tr}\ (\mathcal{F}_{12}\mathcal{F}_{12} + \mathcal{F}_{13}\mathcal{F}_{13} + \mathcal{F}_{14}\mathcal{F}_{14})\ e^1 \wedge \ldots \wedge e^7 . \tag{D.9}$$

D.2 Details of Sp(2) instantons

This section provides some lengthy expressions for Sp(2) instantons on the hyper-Kähler cone over the squashed seven-sphere. The evaluation of the three sets of holomorphicity conditions, $\mathcal{F}^{2,0} = 0$ with respect to J_α for $\alpha = 5, 6, 7$, yields

$$\begin{aligned} \Omega_5 :\quad & \mathcal{F}_{14} = -\mathcal{F}_{23}, && \mathcal{F}_{12} = -\mathcal{F}_{34}, && \mathcal{F}_{17} = \mathcal{F}_{36}, && \mathcal{F}_{16} = -\mathcal{F}_{37}, \\ & \mathcal{F}_{26} = \mathcal{F}_{47}, && \mathcal{F}_{27} = -\mathcal{F}_{46}, && \mathcal{F}_{1\tau} = -\mathcal{F}_{35}, && \mathcal{F}_{2\tau} = \mathcal{F}_{45}, \\ & \mathcal{F}_{3\tau} = \mathcal{F}_{15}, && \mathcal{F}_{4\tau} = -\mathcal{F}_{25}, && \mathcal{F}_{6\tau} = -\mathcal{F}_{57}, && \mathcal{F}_{7\tau} = \mathcal{F}_{56}, \end{aligned} \tag{D.10a}$$

$$\begin{aligned} \Omega_6 :\quad & \mathcal{F}_{12} = -\mathcal{F}_{34}, && \mathcal{F}_{13} = \mathcal{F}_{24}, && \mathcal{F}_{15} = \mathcal{F}_{47}, && \mathcal{F}_{17} = -\mathcal{F}_{45}, \\ & \mathcal{F}_{25} = \mathcal{F}_{37}, && \mathcal{F}_{27} = -\mathcal{F}_{35}, && \mathcal{F}_{1\tau} = -\mathcal{F}_{46}, && \mathcal{F}_{2\tau} = -\mathcal{F}_{36}, \\ & \mathcal{F}_{3\tau} = \mathcal{F}_{26}, && \mathcal{F}_{4\tau} = \mathcal{F}_{16}, && \mathcal{F}_{5\tau} = \mathcal{F}_{67}, && \mathcal{F}_{7\tau} = \mathcal{F}_{56}, \end{aligned} \tag{D.10b}$$

$$\begin{aligned} \Omega_7 :\quad & \mathcal{F}_{13} = \mathcal{F}_{24}, && \mathcal{F}_{14} = -\mathcal{F}_{23}, && \mathcal{F}_{15} = \mathcal{F}_{26}, && \mathcal{F}_{16} = -\mathcal{F}_{25}, \\ & \mathcal{F}_{35} = \mathcal{F}_{46}, && \mathcal{F}_{36} = -\mathcal{F}_{45}, && \mathcal{F}_{1\tau} = \mathcal{F}_{27}, && \mathcal{F}_{2\tau} = -\mathcal{F}_{17}, \\ & \mathcal{F}_{3\tau} = \mathcal{F}_{47}, && \mathcal{F}_{4\tau} = -\mathcal{F}_{37}, && \mathcal{F}_{5\tau} = \mathcal{F}_{67}, && \mathcal{F}_{6\tau} = -\mathcal{F}_{57}. \end{aligned} \tag{D.10c}$$

These expressions show that the holomorphicity conditions with respect to any two Kähler forms already imply the third set and, moreover, also the three stability-like conditions,

$$\Omega_5 \lrcorner \mathcal{F} = -\mathcal{F}_{13} + \mathcal{F}_{24} + \mathcal{F}_{67} + \mathcal{F}_{\tau 5} = 0 , \tag{D.11a}$$

$$\Omega_6 \lrcorner \mathcal{F} = -\mathcal{F}_{14} - \mathcal{F}_{23} - \mathcal{F}_{57} + \mathcal{F}_{\tau 6} = 0 , \tag{D.11b}$$

$$\Omega_7 \lrcorner \mathcal{F} = \mathcal{F}_{12} + \mathcal{F}_{34} + \mathcal{F}_{56} + \mathcal{F}_{\tau 7} = 0 . \tag{D.11c}$$

For our choice of the fundamental forms (6.12b) defining the 3-Sasakian geometry, the 4-form Q_Z (2.21) which enters the generalized self-duality equation (2.10) on the cylinder reads

$$Q_Z = e^{1234} + \mathrm{e}^{\tau 567} + \tfrac{1}{3}(e^{67} + \mathrm{e}^{\tau 5}) \wedge (-e^{13} + e^{24}) + \tfrac{1}{3}(e^{75} + \mathrm{e}^{\tau 6}) \wedge (-e^{14} - e^{23}) \\ + \tfrac{1}{3}(e^{56} + \mathrm{e}^{\tau 7}) \wedge (e^{12} + e^{34}). \tag{D.12}$$

The explicit evaluation of $\mathcal{F} = -\mathcal{F} \wedge \star_8 Q_Z$ actually yields the same conditions as the three systems of holomorphicity conditions (D.10), which explicitly proves the equivalence of both approaches claimed in Section 6.2.4.

We now demonstrate sufficiency of taking into account only the differential equations (6.25a) for the *horizontal* Higgs fields and the algebraic conditions (6.25c). Considering X_5, without loss of generality, one obtains from the algebraic conditions and the flow equations (6.25a)

$$\begin{aligned} 4\dot{X}_5 &= [\dot{X}_1, X_3] + [X_1, \dot{X}_3] - [\dot{X}_2, X_4] - [X_2, \dot{X}_4] \\ &= [-X_1, X_3] + [-[X_2, X_7], X_3] + [X_1, -X_3] + [X_1, -[X_4, X_7]] \\ &\quad - [-X_2, X_4] - [[X_1, X_7], X_4] - [X_2, -X_4] - [X_2, [X_3, X_7]] \\ &= -2([X_1, X_3] - [X_2, X_4]) + [[X_3, X_2], X_7] - [[X_1, X_4], X_7] \\ &= -8X_5 - 4[X_6, X_7], \end{aligned} \tag{D.13}$$

where we have used the Jacobi identity. It reproduces indeed the flow equation (6.25b). Therefore, it is sufficient to study the system (6.25a), interpreting (6.25c) as *definition* of the three vertical Higgs fields.

We note that the Sp(2)-instanton equations (D.10) on the metric cone contain the usual four-dimensional self-duality conditions (6.17a) and those of the 3-Sasakian base Sp(2)/Sp(1) as well. The canonical connection (6.16) of the 3-Sasakian manifold lifts to an instanton on the metric cone.

D.2.1 Instanton equations on generic hyper-Kähler cones

Because of the general formulation of the torsion components of the canonical connection on 3-Sasakian manifolds in [19,20], one can easily derive the instanton equations on the cones over generic 3-Sasakian manifolds M^{4m+3} [34,128]. With the conventions of [19] for the Kähler forms, i.e.

$$\Omega_1 = r^2 \sum_{i=0}^{m} \left(e^{4i} \wedge e^{4i+1} + e^{4i+2} \wedge e^{4i+3} \right), \quad \Omega_2 = r^2 \sum_{i=0}^{m} \left(e^{4i} \wedge e^{4i+2} - e^{4i+1} \wedge e^{4i+3} \right), \\ \Omega_3 = r^2 \sum_{i=0}^{m} \left(e^{4i} \wedge e^{4i+3} + e^{4i+1} \wedge e^{4i+2} \right), \tag{D.14}$$

the flow equations read [34, 128]

$$\begin{aligned}
\dot{X}_{4j} &= -X_{4j} + [X_1, X_{4j+1}] &&= -X_{4j} + [X_2, X_{4j+2}] &&= -X_{4j} + [X_3, X_{4j+3}],\\
\dot{X}_{4j+1} &= -X_{4j+1} - [X_1, X_{4j}] &&= -X_{4j+1} - [X_2, X_{4j+3}] &&= -X_{4j+1} + [X_3, X_{4j+2}],\\
\dot{X}_{4j+2} &= -X_{4j+2} + [X_1, X_{4j+3}] &&= -X_{4j+2} - [X_2, X_{4j}] &&= -X_{4j+2} - [X_3, X_{4j+1}],\\
\dot{X}_{4j+3} &= -X_{4j+3} - [X_1, X_{4j+2}] &&= -X_{4j+3} + [X_2, X_{4j+1}] &&= -X_{4j+3} - [X_3, X_{4j}],
\end{aligned} \tag{D.15}$$

for $j = 1, \ldots, m$. In addition, one obtains the algebraic conditions

$$\begin{aligned}
4\delta_{ij} X_1 &= -[X_{4i}, X_{4j+1}] - [X_{4i+2}, X_{4j+3}] = -[X_{4i}, X_{4j+1}] - [X_{4j+2}, X_{4i+3}],\\
4\delta_{ij} X_2 &= -[X_{4i}, X_{4j+2}] + [X_{4i+1}, X_{4j+3}] = -[X_{4i}, X_{4j+2}] + [X_{4j+1}, X_{4i+3}],\\
4\delta_{ij} X_3 &= -[X_{4i}, X_{4j+3}] - [X_{4i+1}, X_{4j+2}] = -[X_{4i}, X_{4j+3}] - [X_{4j+1}, X_{4i+2}],
\end{aligned} \tag{D.16a}$$

and

$$\begin{aligned}
[X_{4i}, X_{4j+1}] &= [X_{4j}, X_{4i+1}], \qquad & [X_{4i+2}, X_{4j+3}] &= [X_{4j+2}, X_{4i+3}],\\
[X_{4i}, X_{4j+2}] &= [X_{4j}, X_{4i+2}], \qquad & [X_{4i+1}, X_{4j+3}] &= [X_{4j+1}, X_{4i+3}],\\
[X_{4i}, X_{4j+3}] &= [X_{4j}, X_{4i+3}], \qquad & [X_{4i+1}, X_{4j+2}] &= [X_{4j+1}, X_{4i+2}],
\end{aligned} \tag{D.16b}$$

as well as

$$[X_{4i}, X_{4j}] = [X_{4i+1}, X_{4j+1}] = [X_{4i+2}, X_{4j+2}] = [X_{4i+3}, X_{4j+3}] \tag{D.16c}$$

for $i, j = 1, \ldots, m$. Again, differentiating the algebraic conditions (D.16a) yields the correct flow equations for the vertical Higgs fields,

$$\dot{X}_\alpha = -2X_\alpha - \tfrac{1}{2}\epsilon_\alpha{}^{\beta\gamma}[X_\beta, X_\gamma]. \tag{D.17}$$

In the above equations, the SU(2)-symmetry of the vertical Higgs fields and the quaternionic symmetry of the $4m$-dimensional base are manifest. The equations show that the case of $m = 0$, i.e. 4-dimensional hyper-Kähler instantons, is rather special because then only the conditions (D.17) have to be solved and the challenges due to the non-trivial algebraic conditions do not occur. As already pointed out in Appendix A.2, the three instanton equations can be equivalently interpreted either as holomorphicity or as moment map conditions [62] in that case. One should bear in mind that in four dimensions the definitions of being Calabi-Yau and hyper-Kähler coincide because of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

D.3 Representations of $\mathrm{Sp}(2)$

This section collects the representations of Sp(2) which have been employed in the main text. By the choice of the basis 1-forms in (6.7), the corresponding generators read

$$I_1^+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_2^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_3^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_4^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{I}_7 = \mathrm{diag}(1,-1,0,0), \qquad \tilde{I}_8 = \mathrm{diag}(0,0,-1,1), \tag{D.18}$$

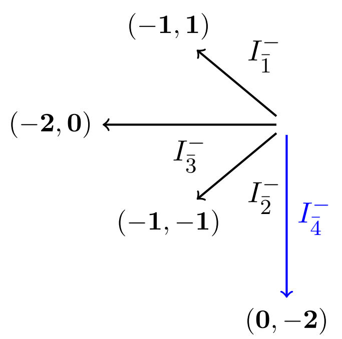
with $I_{\bar{\alpha}}^- := -(I_\alpha^+)^\dagger$ for $\alpha = 1, \dots 4$ and $\widetilde{I}_j := -\mathrm{i} I_j$ for $j = 7, 8$. The Killing form $B(X,Y) \propto \mathrm{Tr}(XY)$ yields for these generators the only non-vanishing combinations

$$\begin{aligned} &\mathrm{Tr}\,(I_{\bar{1}}^-\; I_1^+) = \mathrm{Tr}\,(I_{\bar{2}}^-\; I_2^+) = -2, \quad \mathrm{Tr}\,(I_{\bar{3}}^-\; I_3^+) = \mathrm{Tr}\,(I_{\bar{4}}^-\; I_4^+) = -1, \\ &\mathrm{Tr}\,(I_7\; I_7) = \mathrm{Tr}\,(I_8\; I_8) = -2, \end{aligned} \tag{D.19}$$

and the induced metric $g = \sum_{\mu=1}^{7} e^\mu \otimes e^\mu$ on the coset space is therefore given by the (rescaled) Killing form, according to [129, Thm. 4]. The generators act on states in the weight diagram, labeled by their quantum numbers (ν_7, ν_8) with respect to $\widetilde{I}_7$ and $\widetilde{I}_8$, as follows

$$\begin{aligned} &I_{\bar{1}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 + 1)\,, && I_{\bar{2}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 - 1)\,, \\ &I_{\bar{3}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 2, \nu_8)\,, && I_{\bar{4}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7, \nu_8 - 2)\,, \end{aligned} \tag{D.20}$$

which is encoded in the root system

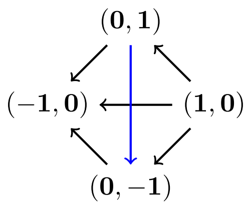


(D.21)

In the following, we will collect some low-dimensional representations of the Lie algebra of Sp(2) constructed from the root system (D.21). For more details and a systematic description, we again refer to [108].

D.3.1 Representation $\underline{4}$

The generators of Sp(2) in the fundamental representation $\underline{\mathbf{4}}$ (denoted as $\Gamma_{1,0}$ in [108]) are given by (D.18) and yield the diamond



(D.22)

as weight diagram. Consequently, the representation decomposes under restriction to the subgroup Sp(1) as

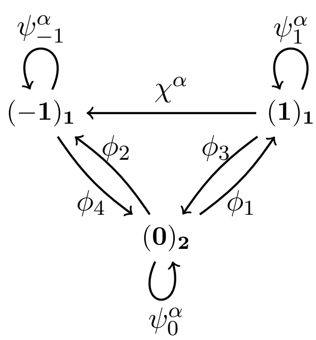
$$\underline{\mathbf{4}}|_{\mathrm{Sp}(1)} = \underline{(\mathbf{-1},\mathbf{0})_1} \oplus \underline{(\mathbf{0},\mathbf{-1})_2} \oplus \underline{(\mathbf{1},\mathbf{0})_1} \tag{D.23}$$

and induces the quiver diagram (6.32).

General Higgs fields. Analogously to (C.16) and (B.20), we also comment on the most general form of the matrices $X_\mu \in \mathfrak{u}(k)$ which are compatible with the equivariance conditions (6.19) alone, i.e. without the restriction to generators of G. Starting from the H-representation on the right-hand side of (D.23), forgetting about its origin as restriction of a G-representation, we obtain the most general Higgs fields

$$\phi^{(1)} = \left(\begin{array}{c|c|cc} 0 & 0 & \phi_2 & 0 \\ \hline 0 & 0 & \phi_0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \phi_1 & \phi_3 & 0 & 0 \end{array}\right), \quad \phi^{(2)} = \left(\begin{array}{c|c|cc} 0 & 0 & 0 & -\phi_2 \\ \hline 0 & 0 & 0 & -\phi_0 \\ \hline \phi_1 & \phi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad X^\alpha = \left(\begin{array}{c|c|cc} \psi_1^\alpha & -\chi^{\alpha\dagger} & 0 & 0 \\ \hline \chi^\alpha & \psi_{-1}^\alpha & 0 & 0 \\ \hline 0 & 0 & \psi_0^\alpha & 0 \\ 0 & 0 & 0 & \psi_0^\alpha \end{array}\right)$$

for $\alpha = 5, 6, 7$. The quiver diagram is given by



(D.24)

containing more contributions than the corresponding ladder operators of Sp(2) and the quiver (6.32). Plugging these Higgs fields into the instanton equations on the hyper-Kähler cone yields

$$
\begin{aligned}
\dot{\phi}_2 &= -\phi_2 - (\phi_1^\dagger\psi_0^5 - \psi_1^5 + \chi_5^\dagger\phi_3^\dagger) = -\phi_2 + \mathrm{i}(\phi_1^\dagger\psi_0^6 - \psi_1^6 + \chi_6^\dagger\phi_3^\dagger), \\
&= -\phi_2 - \mathrm{i}(\phi_2\psi_0^7 - \psi_1^7\phi_2 + \chi_7^\dagger\phi_0), \\
\dot{\phi}_0 &= -\phi_0 - (\phi_3^\dagger\psi_0^5 - \chi_5\phi_1^\dagger - \psi_{-1}^5\phi_3^\dagger) = -\phi_0 + \mathrm{i}(\phi_3^\dagger\psi_0^6 - \chi_6\phi_1^\dagger - \psi_{-1}^6\phi_3^\dagger), \\
&= -\phi_0 - \mathrm{i}(\phi_0\psi_0^7 - \chi_7\phi_0 - \psi_{-1}^7\phi_0), \\
\dot{\phi}_1 &= -\phi_1 - (-\phi_2^\dagger\psi_1^5 - \phi_0^\dagger\chi_5 + \psi_0^5\phi_2^\dagger) = -\phi_1 + \mathrm{i}(-\phi_2^\dagger\psi_1^6 - \phi_0^\dagger\chi_6 + \psi_0^6\phi_2^\dagger), \\
&= -\phi_1 - \mathrm{i}(\phi_1\psi_1^7 + \phi_3\chi_7 - \psi_0^7\phi_1), \\
\dot{\phi}_3 &= -\phi_3 - (\phi_2^\dagger\chi_5^\dagger - \phi_0^\dagger\psi_{-1}^5 + \psi_0^5\phi_0) = -\phi_3 + \mathrm{i}(\phi_2^\dagger\chi_5^\dagger - \phi_0^\dagger\psi_{-1}^5 + \psi_0^5\phi_0), \\
&= -\phi_3 - \mathrm{i}(-\phi_1\chi_7^\dagger + \phi_3\psi_{-1}^7 - \psi_0^7\phi_3)\ ,
\end{aligned}
\tag{D.25}
$$

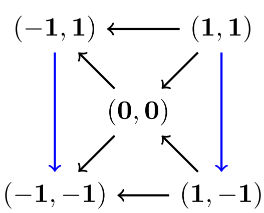
where we kept the real form of the vertical Higgs fields X_α to emphasize their SU(2)-symmetry. The algebraic conditions for this case read

$$
\begin{aligned}
\mathrm{i}\psi_1^7 &= \phi_2\phi_2^\dagger - \phi_1^\dagger\phi_1, & \mathrm{i}\psi_{-1}^7 &= \phi_0\phi_0^\dagger - \phi_3^\dagger\phi_3, \quad \chi_7 = 0, \\
2\mathrm{i}\psi_0^7 &= \phi_1\phi_1^\dagger + \phi_3\phi_3^\dagger - \phi_2^\dagger\phi_2 - \phi_0^\dagger\phi_0, & \psi_0^5 - \mathrm{i}\psi_0^6 &= -\phi_1\phi_2 - \phi_3\phi_0, \\
\psi_1^5 - \mathrm{i}\psi_1^6 &= 2\phi_2\phi_1, & \psi_{-1}^5 - \mathrm{i}\psi_{-1}^6 &= 2\phi_0\phi_3, \\
\chi_5 - \mathrm{i}\chi_6 &= 2\phi_0\phi_1, & \chi_5^\dagger - \mathrm{i}\chi_6^\dagger &= -2\phi_2\phi_3
\end{aligned}
\tag{D.26}
$$

and they again imply the flow equations for the vertical fields. Recall that the orbifold case requires also an equivariance condition with respect to I_7, which uniquely leads to the quiver (6.32) and the resulting instanton equations in the main text.

D.3.2 Representation $\underline{5}$

Due to the isomorphism $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, the Lie algebra of Sp(2) admits also a five-dimensional representation (denoted as $\Gamma_{0,1}$ in [108]) with the square



(D.27)

as weight diagram. A concrete choice for the representation of the generators is given by

$$I_{\bar{1}}^{-}=\begin{pmatrix}0&0&\sqrt{2}&0&0\\0&0&0&0&0\\0&0&0&0&-\sqrt{2}\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad I_{\bar{2}}^{-}=\begin{pmatrix}0&0&0&0&0\\0&0&\sqrt{2}&0&0\\0&0&0&-\sqrt{2}&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad I_{\bar{3}}^{-}=\begin{pmatrix}0&0&0&-1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$
$$I_{\bar{4}}^{-}=\begin{pmatrix}0&0&0&0&0\\-1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},\qquad \widetilde{I}_7=\mathrm{diag}\left(-1,-1,0,1,1\right),\qquad \widetilde{I}_8=\mathrm{diag}\left(1,-1,0,1,-1\right).\tag{D.28}$$

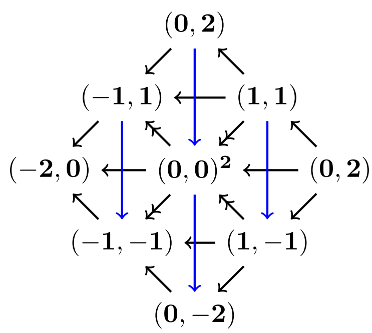
This representation yields the splitting

$$\underline{\mathbf{5}}|_{\mathrm{Sp}(1)}=\underline{(\mathbf{-1},\mathbf{-1})_{\mathbf{2}}}\oplus\underline{(\mathbf{0},\mathbf{0})_{\mathbf{1}}}\oplus\underline{(\mathbf{1},\mathbf{-1})_{\mathbf{2}}},\tag{D.29}$$

which gives rise to the quiver diagram (6.39).

D.3.3 Representation 10

The 10-dimensional adjoint representation (also referred to as $\Gamma_{2,0}$ in [108]) of Sp(2) is described by the weight diagram



(D.30)

where the double-headed arrows represent two arrows involving the degenerate origin. This representation admits the decomposition

$$\underline{\mathbf{10}}|_{\mathrm{Sp}(1)}=\underline{(\mathbf{-2},\mathbf{0})_1}\oplus\underline{(\mathbf{-1},\mathbf{-1})_2}\oplus\underline{(\mathbf{0},\mathbf{-2})_3}\oplus\underline{(\mathbf{0},\mathbf{0})_1}\oplus\underline{(\mathbf{1},\mathbf{-1})_2}\oplus\underline{(\mathbf{2},\mathbf{0})_1}.\tag{D.31}$$

The explicit realization of the generators is either obtained by the structure constants with respect to the real 1-forms $e^1,\dots,e^{10}$ (with a suitable rescaling such that the structure constants become completely antisymmetric) or can be constructed from scratch using the weight diagram (D.30). The latter approach yields, for instance, the choice

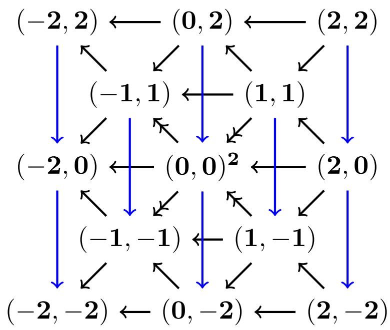
$$
\begin{aligned}
&(I_{\bar{1}}^-)_{1,2} = -\sqrt{2}, && (I_{\bar{1}}^-)_{2,4} = -\sqrt{2}, && (I_{\bar{1}}^-)_{3,5} = -1, && (I_{\bar{1}}^-)_{3,7} = -1,\\
&(I_{\bar{1}}^-)_{5,8} = 1, && (I_{\bar{1}}^-)_{6,9} = -\sqrt{2}, && (I_{\bar{1}}^-)_{7,8} = 1, && (I_{\bar{1}}^-)_{9,10} = -\sqrt{2},\\
&(I_{\bar{2}}^-)_{1,3} = \sqrt{2}, && (I_{\bar{2}}^-)_{2,5} = 1, && (I_{\bar{2}}^-)_{2,7} = -1, && (I_{\bar{2}}^-)_{3,6} = -\sqrt{2},\\
&(I_{\bar{2}}^-)_{4,8} = \sqrt{2}, && (I_{\bar{2}}^-)_{5,9} = 1, && (I_{\bar{2}}^-)_{7,9} = -1, && (I_{\bar{2}}^-)_{8,10} = -\sqrt{2},\\
&(I_{\bar{3}}^-)_{1,7} = \sqrt{2}, && (I_{\bar{3}}^-)_{1,8} = -1, && (I_{\bar{3}}^-)_{3,9} = -1, && (I_{\bar{3}}^-)_{7,10} = -\sqrt{2},\\
&(I_{\bar{4}}^-)_{2,3} = -1, && (I_{\bar{4}}^-)_{4,5} = -\sqrt{2}, && (I_{\bar{4}}^-)_{5,6} = \sqrt{2}, && (I_{\bar{4}}^-)_{8,9} = -1,
\end{aligned} \tag{D.32a}
$$

with $I_\alpha^+ = -(I_{\bar{\alpha}}^-)^\dagger$ for $\alpha = 1, \dots, 4$ and the Cartan generators

$$
\begin{aligned}
\widetilde{I}_7 &= \mathrm{diag}(-2, -1, -1, 0, 0, 0, 0, 1, 1, 2),\\
\widetilde{I}_8 &= \mathrm{diag}(0, -1, 1, -2, 0, 2, 0, -1, 1, 0).
\end{aligned} \tag{D.32b}
$$

D.3.4 Representation $\underline{14}$

The 14-dimensional representation (also known as $\Gamma_{0,2}$) of Sp(2) is described by the weight diagram



(D.33)

and the decomposition under restriction to the subgroup Sp(1) reads

$$
\underline{\mathbf{14}}|_{\mathrm{Sp}(1)} = \underline{(\mathbf{-2,-2})_3} \oplus \underline{(\mathbf{-1,-1})_2} \oplus \underline{(\mathbf{0,-2})_3} \oplus \underline{(\mathbf{0,0})_1} \oplus \underline{(\mathbf{1,-2})_2} \oplus \underline{(\mathbf{2,-2})_3}. \tag{D.34}
$$

The non-vanishing components of an explicit choice for the generators read

$$
\begin{aligned}
&(I_{\bar{1}}^-)_{2,4} = -\sqrt{2}, && (I_{\bar{1}}^-)_{3,5} = -2, && (I_{\bar{1}}^-)_{4,6} = -\sqrt{2}, && (I_{\bar{1}}^-)_{5,7} = -1, && (I_{\bar{1}}^-)_{5,9} = -\sqrt{5},\\
&(I_{\bar{1}}^-)_{7,10} = 1, && (I_{\bar{1}}^-)_{8,11} = -\sqrt{2}, && (I_{\bar{1}}^-)_{9,10} = \sqrt{5}, && (I_{\bar{1}}^-)_{11,12} = -2, && (I_{\bar{1}}^-)_{11,13} = -\sqrt{2},\\
&(I_{\bar{2}}^-)_{1,4} = -2, && (I_{\bar{2}}^-)_{2,5} = -\sqrt{2}, && (I_{\bar{2}}^-)_{4,7} = 1, && (I_{\bar{2}}^-)_{4,9} = -\sqrt{5}, && (I_{\bar{2}}^-)_{5,8} = -\sqrt{2},\\
&(I_{\bar{2}}^-)_{6,10} = \sqrt{2}, && (I_{\bar{2}}^-)_{7,11} = 1, && (I_{\bar{2}}^-)_{9,11} = -\sqrt{5}, && (I_{\bar{2}}^-)_{10,13} = \sqrt{2}, && (I_{\bar{2}}^-)_{11,14} = -2,\\
&(I_{\bar{3}}^-)_{1,6} = -\sqrt{2}, && (I_{\bar{3}}^-)_{2,7} = -\sqrt{2}, && (I_{\bar{3}}^-)_{3,8} = \sqrt{2}, && (I_{\bar{3}}^-)_{4,10} = 1,\\
&(I_{\bar{3}}^-)_{5,11} = 1, && (I_{\bar{3}}^-)_{6,12} = \sqrt{2}, && (I_{\bar{3}}^-)_{7,13} = \sqrt{2}, && (I_{\bar{3}}^-)_{8,14} = \sqrt{2},\\
&(I_{\bar{4}}^-)_{1,2} = -\sqrt{2}, && (I_{\bar{4}}^-)_{2,3} = -\sqrt{2}, && (I_{\bar{4}}^-)_{4,5} = -1, && (I_{\bar{4}}^-)_{6,7} = -\sqrt{2},\\
&(I_{\bar{4}}^-)_{7,8} = \sqrt{2}, && (I_{\bar{4}}^-)_{10,11} = -1, && (I_{\bar{4}}^-)_{12,13} = -\sqrt{2}, && (I_{\bar{4}}^-)_{13,14} = \sqrt{2},
\end{aligned} \tag{D.35a}
$$

with $I^{+}_{\alpha} = -(I^{-}_{\bar{\alpha}})^{\dagger}$ and the two Cartan generators

$$\begin{aligned}\widetilde{I}_7 &= \mathrm{diag}(-2,-2,-2,-1,-1,0,0,0,0,1,1,2,2,2)\ ,\\ \widetilde{I}_8 &= \mathrm{diag}(-2,0,2,-1,1,-2,0,2,0,-1,1,-2,0,2)\ .\end{aligned} \tag{D.35b}$$

The decomposition under the subgroup H is given by (6.48) and leads to the quiver diagram (6.49).

E Details of the Aloff-Wallach space $X_{1,1}$

This appendix contains details related to the 3-Sasakian quiver gauge theory on $X_{1,1}$. Since the instanton equations coincide with those on the squashed seven-sphere and only the weight diagrams of the well-known Lie group SU(3) are necessary, this part is significantly shorter than Appendix D discussing Sp(2)/Sp(1).

E.1 Geometry of $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$

The choice of the 1-forms in (7.6) yields the structure equations

$$\begin{aligned} &\mathrm{d}\Theta^1 = -\mathrm{i}e^7 \wedge \Theta^1 + \sqrt{3}\mathrm{i}e^8 \wedge \Theta^1 - \Theta^{\bar{2}3}, \quad &&\mathrm{d}\Theta^2 = -\mathrm{i}e^7 \wedge \Theta^2 - \sqrt{3}\mathrm{i}e^8 \wedge \Theta^2 + \Theta^{\bar{1}3}, \\ &\mathrm{d}\Theta^3 = -2\mathrm{i}e^7 \wedge \Theta^3 - 2\Theta^{12}, &&\mathrm{d}e^7 = -\mathrm{i}(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}), \\ &\mathrm{d}e^8 = \sqrt{3}\mathrm{i}(\Theta^{1\bar{1}} - \Theta^{2\bar{2}}), \end{aligned} \tag{E.1}$$

and the non-vanishing structure constants read [27]

$$\begin{aligned} &C^1_{3\bar{2}} = -C^2_{3\bar{1}} = -1 = -C^{\bar{1}}_{\bar{2}\bar{3}} = C^{\bar{2}}_{1\bar{3}}, && C^3_{12} = 2 = C^{\bar{3}}_{\bar{1}\bar{2}}, \\ &C^1_{71} = C^2_{72} = 1 = -C^{\bar{1}}_{7\bar{1}} = -C^{\bar{2}}_{7\bar{2}}, && C^3_{73} = 2 = -C^{\bar{3}}_{7\bar{3}}, \\ &C^1_{81} = -C^2_{82} = -\sqrt{3} = -C^{\bar{1}}_{8\bar{1}} = C^{\bar{2}}_{8\bar{2}}, && C^3_{83} = 0 = C^{\bar{3}}_{8\bar{3}}, \\ &C^7_{1\bar{1}} = C^7_{2\bar{2}} = C^7_{3\bar{3}} = -1, && C^8_{1\bar{1}} = -C^8_{2\bar{2}} = \sqrt{3}, \end{aligned} \tag{E.2}$$

with respect to the Cartan generators I_7 and I_8 rescaled by a factor of $-\mathrm{i}$.

The conventions here follow [27], but for a detailed geometric description, including the Killing spinors, it is worth consulting [53, Ch. 4.4] as well.

E.2 Representations of SU(3)

In this section we briefly describe the SU(3) representations studied in the main text. Since the subgroup $H = \mathrm{U}(1)$ is just a circle, there is no collapsing of the weight diagrams, and arrows compatible with the G-action can occur for any non-zero entry in the ladder operators. Therefore, it is sufficient to identify the non-vanishing entries from the weight diagrams, without the need of determining the numerical coefficients, in contrast to the procedure for SU(4) or Sp(2).

By the definition of the 1-forms in (7.6), the ladder generators in the defining representation read

$$I_1^+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad I_2^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_3^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{E.3a}$$

with $I_{\bar{\alpha}}^- = -(I_\alpha^+)^\dagger$ for $\alpha = 1, 2, 3$, and the two Cartan generators are given by

$$\tilde{I}_7 \coloneqq -\mathrm{i} I_7 = \mathrm{diag}(0, -1, 1), \qquad \tilde{I}_8 \coloneqq -\mathrm{i}\sqrt{3} I_8 = \mathrm{diag}(2, -1, -1). \tag{E.3b}$$

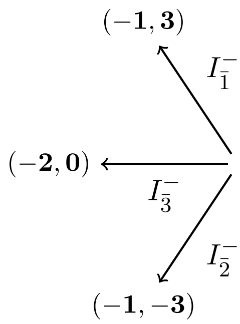
The Killing form admits the only non-vanishing contributions

$$\mathrm{Tr}(I_1^+ I_{\bar{1}}^-) = \mathrm{Tr}(I_2^+ I_{\bar{2}}^-) = \mathrm{Tr}(I_7 I_7) = \mathrm{Tr}(I_8 I_8) = -2, \quad \mathrm{Tr}(I_3^+ I_{\bar{3}}^-) = -1, \tag{E.4}$$

which corresponds to (D.19). In particular, this confirms the 3-Sasakian metric used in the main text, by virtue of [129, Thm. 4]. The generators act in the weight diagrams as

$$\begin{aligned} &I_{\bar{1}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 + 3), \qquad && I_{\bar{2}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 - 3), \\ &I_{\bar{3}}^- : (\nu_7, \nu_8) \longmapsto (\nu_7 - 2, \nu_8), \end{aligned} \tag{E.5}$$

which is depicted in the root system

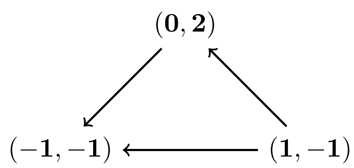


(E.6)

The notation of the ladder operators relevant for the arrows in the quiver diagrams is the same as for Sp(2) in (D.21), so that the quiver diagrams will have the same shape.

E.2.1 Fundamental Representation

The fundamental representation, defined by the generators (E.3), yields the weight diagram

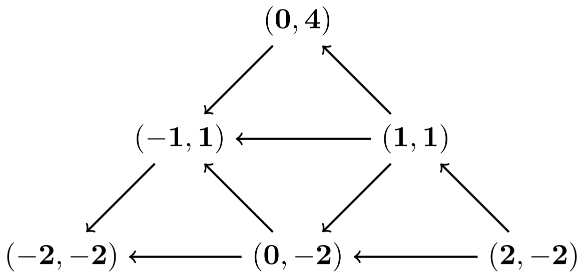


(E.7)

and gives rise to the quiver (7.18).

E.2.2 Representation $\underline{6}$

The 6-dimensional representation of SU(3) determines the weight diagram

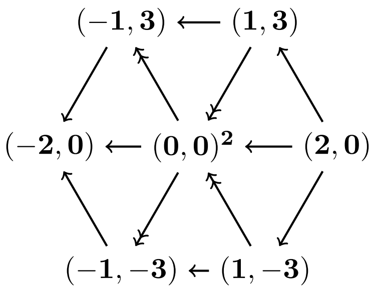


(E.8)

which induces the quiver (7.23).

E.2.3 Adjoint representation $\underline{8}$

The adjoint representation is described by the weight diagram



(E.9)

Since SU(3) has rank two, the origin of this diagram is twice degenerate, and we use double-headed arrows as shorthand notation. This representation yields the quiver diagram (7.25).

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