Adiabatic Approach for Noncommutative Multi-Solitons in 2+1 Dimensions

– Diplomarbeit –

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Abstract

One possibility to analyse slow scattering processes is the adiabatic approach, especially for non-integrable models. We test the validity of this approach in noncommutative field theory. Since the WZW-modified $U(1)$ sigma model in 2+1 dimensions allows exact multi-soliton solutions we calculate the moduli space dynamics for this model. We are able to identify the static sigma model with an unconstrained scalar field model for a special potential. The goal is to compare the geodesic motion in multi-soliton moduli space with the scattering behaviour of exact solutions. In order to determine the moduli space dynamics we calculate the metric on this space and find that it is Kähler. Due to the fact that the WZW-like term does not contribute to the moduli space dynamics the metric is the same for a whole family of models, including the standard $U(1)$ sigma model and the Ward model. Surprisingly the approximate two-soliton scattering does not fit the exact field configurations , questioning the adiabatic approach for noncommutative solitons. Finally we take first steps to calculating the moduli space metric for simple cases of $U(2)$ solutions.

Zusammenfassung

Eine Möglichkeit um langsame Streuprozesse, besonders bei nicht integrierbaren Modellen, zu analysieren ist der adiabatische Ansatz. Wir testen die Gültigkeit dieser Herangehensweise in nichtkommutativer Feldtheorie. Da das WZW-modifizierte $U(1)$ Sigma-Modell in 2+1 Dimensionen exakte Multi-Solitonen-Lösungen erlaubt, berechnen wir die Modulraum-Dynamik für dieses Modell. Wir sind in der Lage statische Sigma-Modell-Lösungen mit einem Skalarfeld-Modell ohne Zwangsbedingungen und mit einem speziellen Potential zu identifizieren. Dabei ist es das Ziel die geodätische Bewegung im Multi-Solitonen-Modulraum mit dem Streuverhalten exakter Lösungen zu vergleichen. Um die Modulraum-Dynamik zu bestimmen berechnen wir die Metrik in diesem Raum und finden heraus, dass diese kählersch ist. Aufgrund der Tatsache, dass der WZW-artige Term nicht zur Modulraum-Dynamik beiträgt bekommt man für eine ganze Familie von Modellen, zu der auch das gewöhnliche $U(1)$ Sigma-Modell und das Ward-Modell gehören die gleiche Metrik heraus. Überraschenderweise entsprechen die genäherte Zwei-Solitonen-Streuung nicht den exakten Feldkonfigurationen. Schließlich führen wir erste Schritte zur Berechnung der Modulraum-Metrik in einfachen Fällen von $U(2)$-Solitonen durch.

Contents

Introduction

Classically [1] *solitons* are special solutions of non-linear wave equations (e.g. Sine-Gordon-equation), where the non-linear terms balance the distortion in shape of the wave packets, so that the phase-velocity is equal to the group-velocity of the wave.
Additionally these wave packets should not change their shape or velocity after scattering for being solitons, else they are called *solitary waves*.
If a field theory model admits static localized finite energy solutions, solitons can be constructed by boosting these solutions with constant velocity.

Field theories, which are not integrable can actually be solved numerically only. In order to understand scattering processes of soliton solutions in these models qualitatively without numerics, one can make an *adiabatic approach*. In this approach, proposed by Manton [2, 3], one analyses the *moduli space* dynamics in the case of small velocities. Therefore static finite energy solutions (k-lumps) will be constructed and the moduli α_i parametrising these k-solitons are endowed with a time-dependence. Discussing the dynamics in the moduli space $\mathcal{M}_k$, one considers the action on this space, which is obtained by evaluating the original action on the soliton configurations, with $\alpha_i \to \alpha_i(t)$. These dynamics are then interpreted as motion of a point particle in $\mathcal{M}_k$, and describe this motion by moduli space metric $h(\alpha)$ and magnetic vector potential $A(\alpha)$. The geodesic motion possibly with magnetic forcing should then approximate the motion of k scattering solitons.

One goal of this thesis is to test the quality of the adiabatic approach. Therefore we need an integrable model admitting exact solutions to compare them with the results of our approximation.
In 2+1 dimensions there is no Lorentz-invariant model admitting soliton solutions [4]. Since the common $U(n)$ sigma model is Lorentz-invariant but not integrable, we must modify it to gain integrability. This modification is achieved by adding a Wess-Zumino-Witten-like (WZW-like) term [5, 6] to the sigma model action, leading to the equations of motion of the so called *Ward-model* [4]. This model is not Lorentz invariant anymore but integrable and the exact solutions are known. The adiabatic approach is already tested for this model in the case of $SU(2)$-valued solutions [3].

We want to discuss the decribed ideas for *noncommutative spaces*. The mathematical ideas of noncommutativity were already introduced by Snyder in 1947 [7]. In string theory it appears in 1999, when N. Seiberg and E. Witten found that some string moduli spaces are described by noncommutative gauge theory [8]. Since this discovery physicists extended all kinds of field theories to noncommutative geometry [9], which is described in detail in [10, 11, 12]. In string theory more specifically the Moyal deformation is used.
Solitons in noncommutative theories can be interpreted as D-branes with a constant magnetic field background [11, 13]. The influence of noncommutativity in d dimensions can be parametrised by constant antisymmetric matrix θ_{mn}.
Analysing noncommutative *scalar* field theories one possibility to get results is to expand the action around the $\theta \to \infty$ limit [9, 14, 15].
Generally noncommutative theories have more solutions than commutative ones, because

there exist beyond the smoothly deformed solutions derived from the commutative case additional solutions which would be singular in commutative coordinates. The *abelian*[1] sigma model treated in this thesis is such a model. While in the commutative case there exits no $U(1)$-solutions, in noncommutative geometry the model has simple static multi-soliton solutions discussed in detail in [16]. These solutions are described by k complex parameters, which can be interpreted as the locations of the multi-soliton-lumps in the noncommutative plane.

Because of this and the fact that the Ward extension of this $U(1)$ sigma model is integrable again, it is a good candidate to test the adiabatic approach in the noncommutative case. Therefore we need the exact solution for this model derived in [17] for comparison. Additionally there can be found exact solutions for the noncommutative $U(2)$ sigma model. In this *nonabelian* case the static multi-soliton solutions are not so simple but they have a commutative analogue [18, 3] and we also try to realise the adiabatic approach.

The thesis is organised as follows. In chapter 1 we introduce the noncommutative realm described by the Moyal star product or by Fock space operators and consider an isomorphism, the Moyal-Weyl map between these formalisms. Then we derive the Ward model from 2+2 dimensional selfdual Yang Mills theory in chapter 2. For the adiabatic approach we need static finite energy solutions, which are discussed in chapter 3. The adiabatic approach is described in chapter 4. Additionally we ask in chapter 4, how the WZW-like term contribute to the moduli space dynamics. After these introductive chapters we calculate the moduli space metric for generic $U(1)$ solutions in chapter 5. We obtain the same moduli space metric as [19, 14, 15] for noncommutative scalar field theory. Furthermore we discuss properties of this metric and analyse the scattering process of two solitons described by a two-lump multi-soliton. Now we are able to compare this scattering with the scattering process for exact solutions with slow relative velocity. In chapter 6 we generalise these results by permitting degeneracies. In chapter 7 we emded the $U(1)$ ward model in a generic scalar field model by adding the sigma model constraint with a Lagrangian multiplier In chapter 8 we finally consider simple examples in the $U(2)$ case and compute the moduli space metric as far as we could. In the appendices A and B lengthy calculations taken out of chapter 3 can be found. The last three appendices collect formulas and properties of coherent states, operator ideals and Kähler metrics.

[1] Solutions are called *abelian* if they are $U(1)$-valued, although they are actually not commutative, due to their dependence on noncommutative coordinates.

1. Noncommutativity

1.1. Star product formalism

One possibility to make a field theory noncommutative is by replacing the commutative pointwise multiplication of the fields with a noncommutative $\star$-product [11, 17].
Our fields are smooth functions on $\mathbb{R}^{2+1}$ with coordinates $x^i = (t, x, y)$ and Minkowskian metric

$$(\eta_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

So this algebra of smooth functions denoted by $\big(C^\infty(\mathbb{R}^{2+1}), \cdot\big)$ maps to $\big(C^\infty(\mathbb{R}^{2+1}), \star\big)$.
As associative noncommutative star product we choose the so called *Moyal-product* [1] which is for two arbitrary functions $f, g \in C^\infty(\mathbb{R}^{2+1})$ defined by

$$(f \star g)(\vec{x}) := f(\vec{x}) \exp\{\tfrac{\mathrm{i}}{2} \overleftarrow{\partial_i}\, \theta^{ij}\, \overrightarrow{\partial_j}\}\, g(\vec{x}), \tag{1.2}$$

where we suppose a constant real antisymmetric noncommutativity tensor (θ_{ij}).
Additionally we assume that (θ_{ij}) is skew diagonalized with respect to space coordinates, while the time coordinate remains commutative and the corresponding entries of (θ_{ij}) vanish. So we get

$$\theta_{xy} = -\theta_{yx} =: \theta, \tag{1.3}$$

with a real positive deformation parameter θ.
With the definiton of spatial complex coordinates and derivatives

$$z := x + \mathrm{i}y \qquad , \qquad \bar{z} := x - \mathrm{i}y \tag{1.4}$$

and derivatives

$$\partial_z := \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y) \qquad , \qquad \partial_{\bar{z}} := \tfrac{1}{2}(\partial_x + \mathrm{i}\partial_y) \tag{1.5}$$

(1.3) becomes

$$\theta^{z\bar{z}} = -\theta^{\bar{z}z} = -2\mathrm{i}\theta. \tag{1.6}$$

From the definition (1.2) we calculate some examples to get more familiar with this new product. Evaluating (1.2) up to the order θ^2

$$f \star g = fg + \tfrac{\mathrm{i}}{2}\theta\,(\partial_x f \partial_y g - \partial_y f \partial_x g) - \tfrac{1}{8}\theta^2\,\big(\partial_x^2 f \partial_y^2 g - 2\partial_x \partial_y f \partial_x \partial_y g + \partial_y^2 f \partial_x^2 g\big) + \mathcal{O}(\theta^3) \tag{1.7}$$

[1] This $\star$-product is called Moyal-product [20], although it appears in a work of Groenewold [21] for the first time [11].

we can identifiy the first order term as the Poisson bracket, and we see that terms of even order are symmetric, while terms of odd order are antisymmetric under exchange of f and g

Generally it is valid that the star product of functions depending only on one coordinate is not different from the ordinary product

$$(f \star g)(x) = (fg)(x) \qquad \text{and} \qquad (f \star g)(y) = (fg)(y). \tag{1.8}$$

For the coordinate functions $f(\vec{x}) = x$ and $f(\vec{x}) = y$ we find

$$x \star y = xy + \tfrac{\mathrm{i}}{2}\theta\{x, y\} = xy + \tfrac{\mathrm{i}}{2}\theta \tag{1.9}$$
$$y \star x = yx + \tfrac{\mathrm{i}}{2}\theta\{y, x\} = yx - \tfrac{\mathrm{i}}{2}\theta, \tag{1.10}$$

and analogously for the complex coordinates

$$z \star \bar{z} = \bar{z}z + \theta \tag{1.11}$$
$$\bar{z} \star z = z\bar{z} - \theta. \tag{1.12}$$

Thus we obtain commutation relations

$$[x, y]_\star := x \star y - y \star x = \mathrm{i}\theta \tag{1.13}$$
$$[z, \bar{z}]_\star := z \star \bar{z} - \bar{z} \star z = 2\theta \tag{1.14}$$

1.2. Operator formalism

With the star-product we have a noncommutative field theory indeed, but since the product is defined as an infinite sum, calculations could be very tedious. Therefore we remind that noncommutativity is also an essential part of quantum mechanics due to the fact that quantum mechanics is described by operators, which are in general not commutative with respect to composition. So we are looking for an algebra isomorphism connecting functions with noncommutative star-multiplication $(C^\infty(\mathbb{R}^{2+1}), \star)$ and linear continuous operators on a Hilbert space with the ordinary composition $(\mathrm{L}(\mathcal{H}), \circ)$. If there exists such a connection, we can describe our theory with operators as well and have for every calculation the choice to work either in star product formalism or in operator formalism. Indeed there exists such an isomorphism [11, 17], which we will describe in the following. At first we identify our coordinate functions x and y with operator-valued coordinates $\hat{x}$ and $\hat{y}$, while t as commutative coordinate remains a scalar. Since the coordinate functions satisfy (1.13) we should suppose

$$[\hat{x}, \hat{y}] = \mathrm{i}\theta \quad , \quad [\hat{x}, t] = 0 = [\hat{y}, t] \tag{1.15}$$

and equivalently

$$\left[\hat{z}, \hat{\bar{z}}\right] = 2\theta \quad , \quad [\hat{z}, t] = 0 = \left[\hat{\bar{z}}, t\right]. \tag{1.16}$$

Hence we see that the commutation relations of the spatial coordinates are quite similar to the Heisenberg algebra well known from quantum mechanics. Thus we rescale our coordinates as

$$\frac{1}{\sqrt{2\theta}}\hat{z} =: a \qquad \text{and} \qquad \frac{1}{\sqrt{2\theta}}\hat{\bar{z}} =: a^\dagger, \tag{1.17}$$

so that our new variables are the creation and annihilation operators a and $a^\dagger$ satisfying

$$[a, a^\dagger] = 1. \tag{1.18}$$

As we know from quantum mechanics these operators act in a simple way on harmonic oscillator states $|n\rangle$, which form an orthonomal basis of an infinite dimensional separable Hilbert space, the Fock space $\mathcal{H}$. All infinite dimensional separable Hilbert spaces are equivalent to l^2, so a acts like a left-shift and $a^\dagger$ like a right-shift

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad \text{and} \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \text{for} \qquad n \in \mathbb{N}_0. \tag{1.19}$$

Furthermore we define the number operator N by

$$N := a^\dagger a \qquad \text{with} \qquad N|n\rangle = n|n\rangle. \tag{1.20}$$

1.3. The Moyal-Weyl map

In the following we specify the relation between the coordinate functions z, $\bar{z}$ and the operators a, $a^\dagger$. There exists a map between functions $f(t, z, \bar{z})$ and operator-valued functions $\hat{f}(t, a, a^\dagger)$ called the *Moyal-Weyl map* [17, 11]

$$\begin{aligned} f(t,z,\bar{z}) \quad \longrightarrow \quad \hat{f}(t,a,a^\dagger) &= -\int \frac{\mathrm{d}p\,\mathrm{d}\bar{p}}{(2\pi)^2}\,\mathrm{d}z\,\mathrm{d}\bar{z}\; f(t,z,\bar{z})\; \mathrm{e}^{-\mathrm{i}[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]} \\ &= \text{Weyl-ordered } f(t, \sqrt{2\theta}a, \sqrt{2\theta}a^\dagger), \end{aligned} \tag{1.21}$$

and inverse

$$\begin{aligned} \hat{f}(t,a,a^\dagger) \quad \longrightarrow \quad f(t,z,\bar{z}) &= 2\pi\theta \int \frac{2\mathrm{i}\mathrm{d}p\,\mathrm{d}\bar{p}}{(2\pi)^2}\; \mathrm{tr}_{\mathcal{H}} \left\{ \hat{f}(t,a,a^\dagger)\; \mathrm{e}^{\mathrm{i}[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]} \right\} \\ &= \hat{f}_\star\left(t, \frac{z}{\sqrt{2\theta}}, \frac{\bar{z}}{\sqrt{2\theta}}\right), \end{aligned} \tag{1.22}$$

where *Weyl-ordered* means a symmetrical order taking all possibilities of ordering a and $a^\dagger$ into account, and $\hat{f}_\star$ means that products between z and $\bar{z}$ in $\hat{f}(t, z, \bar{z})$ become $\star$-products. As an example we want to map the function $f(t, z, \bar{z}) = z^2\bar{z}$ via (1.21) to $\hat{f}(t, a, a^\dagger)$ and then via (1.22) back to $f(t, z, \bar{z})$.
Transformed under the Moyal-Weyl map (1.21) f becomes operator-valued

$$\begin{aligned} \hat{f}(t,a,a^\dagger) &= \text{Weyl-ordered } f(t, \sqrt{2\theta}a, \sqrt{2\theta}a^\dagger) \\ &= (2\theta)^{\frac{3}{2}}\, \tfrac{1}{3}(a^2a^\dagger + aa^\dagger a + a{a^\dagger}^2) \end{aligned} \tag{1.23}$$

The inverse transformation (1.22) yields

$$\begin{aligned} f(t,z,\bar{z}) &= \hat{f}_\star\left(t, \frac{z}{\sqrt{2\theta}}, \frac{\bar{z}}{\sqrt{2\theta}}\right) \\ &= \tfrac{1}{3}\left((z \star z) \star \bar{z} + (z \star \bar{z}) \star z + \bar{z} \star (z \star z)\right) \\ &= \tfrac{1}{3}\left(z^2\bar{z} + 2\theta z + z\bar{z}z + \theta z - \theta z + \bar{z}z^2 - 2\theta z\right) \\ &= z^2\bar{z}. \end{aligned} \tag{1.24}$$

More simple examples are given by

$$\begin{array}{lcccccc} f: & 1 & z & \bar{z} & z\bar{z}-\theta & z\bar{z} & z\bar{z}+\theta \\ \hat{f}: & 1 & \sqrt{2\theta}a & \sqrt{2\theta}a^\dagger & 2\theta N & 2\theta(N+\frac{1}{2}) & 2\theta(N+1) \end{array} \tag{1.25}$$

The Moyal-Weyl map leads to the required isomorphism between $(\mathrm{L}(\mathcal{H}), \circ)$ and $(C^\infty(\mathbb{R}^{2+1}), \star)$

$$f \star g \quad \longrightarrow \quad \hat{f}\hat{g}. \tag{1.26}$$

Additionally we want to consider how integrals and spatial derivatives are defined in the operator formalism [17, 11].
For the integral over the plane we find

$$\int \mathrm{d}x\mathrm{d}y f \quad \longrightarrow \quad 2\pi\theta \,\mathrm{tr}_{\mathcal{H}}\, \hat{f} = 2\pi\theta \sum_{n=0}^{\infty} \langle n|\hat{f}|n\rangle \tag{1.27}$$

and the spatial derivatives become

$$\partial_x f \quad \longrightarrow \quad \hat{\partial}_x \hat{f} := \frac{\mathrm{i}}{\theta}[\hat{y}, \hat{f}] \qquad \text{and} \qquad \partial_y f \quad \longrightarrow \quad \hat{\partial}_y \hat{f} := -\frac{\mathrm{i}}{\theta}[\hat{x}, \hat{f}], \tag{1.28}$$

which implies for the complex coordinates

$$\partial_z f \quad \longrightarrow \quad \hat{\partial}_z \hat{f} = \frac{-1}{\sqrt{2\theta}}[a^\dagger, \hat{f}] \qquad \text{and} \qquad \partial_{\bar{z}} f \quad \longrightarrow \quad \hat{\partial}_{\bar{z}} \hat{f} = \frac{1}{\sqrt{2\theta}}[a, \hat{f}]. \tag{1.29}$$

As shown above the Moyal-Weyl map gives a relation between functions and operators. Due to this relation special types of functions could be seen as analogues of special classes of operators [11]. So the bounded functions $C_b(\mathbb{R}^{2+1})$ correspond to the bounded operators $L(\mathcal{H}, \mathcal{H})$, while the compact operators $\mathcal{K}(\mathcal{H}, \mathcal{H})$ can be identified with the functions $C_0(\mathbb{R}^{2+1})$ vanishing at infinity.

2. The modified sigma model in 2+1 dimensions

We start with an integrable and Lorentz-invariant model, the self-dual Yang-Mills-Higgs equations (sdYMH) in 2+2 dimensions, reduce the dimension and get the Bogomolny equations in 2+1 dimensions. By a special gauge, breaking the Lorentz-invariance, we obtain modified sigma model equations, which are integrable and admit soliton solutions. Then we can specify an action leading to these equations.

2.1. Dimensional reduction

In 2+2 dimensional sdYMH theory [2] the fields are the algebra-valued gauge potential $A_\mu \in u(n)$ and the adjoint Higgs field $\phi \in u(n)$. The covariant derivative of the Higgs field is

$$D_\mu\phi := \partial_\mu + [\, A_\mu, \phi\,]_\star \tag{2.1}$$

and the field tensor for this gauge potential is

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [\, A_\mu, A_\nu\,]_\star, \tag{2.2}$$

where Greek indices $\mu, \nu, \ldots = 1,2,3,4$, while Roman indices denote only $i, j, \ldots = 1,2,3$. For selfdual fields it is valid that

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \tag{2.3}$$

with the totally antisymmetric tensor $\varepsilon_{\mu\nu\sigma\tau}$ ($\varepsilon_{1234} = -1$). This is equivalent to the equation

$$F_{4i} = \tfrac{1}{2}\varepsilon_{ijk}F^{jk}. \tag{2.4}$$

Now we suppose that A_μ is independent of the time coordinate x^4 (dimensional reduction) and (2.4) becomes

$$-\partial_i A_4 - [\, A_i, A_4\,]_\star = \tfrac{1}{2}\varepsilon_{ijk}F^{jk} \tag{2.5}$$

In three dimensions A_4 is a scalar function. So we identify $-A_4$ with the Higgs field ϕ and get the Bogolmolny equations in 2+1 dimensions

$$D_i\phi = \tfrac{1}{2}\varepsilon_{ijk}F^{jk} \tag{2.6}$$

2.2. The equations of motion

Since we cannot specify an action leading to this equations we must reduce them to a more simple form.

(2.6) consists of three equations ($\varepsilon_{123} = \varepsilon_{txy} = -1$)

$$\partial_t\phi + [\,A_t, \phi\,]_\star = -F_{xy} = -\partial_x A_y + \partial_y A_x - [\,A_x, A_y\,]_\star \tag{2.7}$$
$$\partial_x\phi + [\,A_x, \phi\,]_\star = \quad F_{yt} = \quad \partial_y A_t - \partial_t A_y + [\,A_y, A_t\,]_\star \tag{2.8}$$
$$\partial_y\phi + [\,A_y, \phi\,]_\star = \quad F_{tx} = \quad \partial_t A_x - \partial_x A_t + [\,A_t, A_x\,]_\star. \tag{2.9}$$

With the special gauge

$$A_x = -\phi \qquad \text{and} \qquad A_y = A_t, \tag{2.10}$$

we get rid of one equation, because (2.7) and (2.9) becomes equal

$$(\partial_y + \partial_t)A_x - \partial_x A_y = 2\,[\,A_x, A_y\,]_\star \tag{2.11}$$
$$(\partial_y - \partial_t)A_y = -\partial_x A_x. \tag{2.12}$$

In order to solve (2.11) we introduce the group-valued prepotential $\Phi(t, x, y) \in U(n)$

$$2A_x =: \Phi^{-1} \star \partial_x\Phi \qquad \text{and} \qquad A_y + A_t =: \Phi^{-1} \star (\partial_y + \partial_t)\Phi \tag{2.13}$$

Inserting this we see that (2.11) is satisfied and for (2.12) we get

$$-\partial_x(\Phi^{-1} \star \partial_x\Phi) - \partial_y(\Phi^{-1} \star \partial_y\Phi) - \partial_y(\Phi^{-1} \star \partial_t\Phi) + \partial_t(\Phi^{-1} \star \partial_y\Phi) + \partial_t(\Phi^{-1} \star \partial_t\Phi) = 0. \tag{2.14}$$

If we write this in another form, we obtain the equations of motion of the modified $U(n)$ sigma model, the *Ward chiral model.* [1]

$$(\eta^{ij} + v_k\epsilon^{kij})\partial_i(\Phi^{-1} \star \partial_j\Phi) = 0, \tag{2.15}$$

where $\vec{v} = (0, 1, 0)$ The fields Φ are unitary $n \times n$-matrices satisfying the constraint $\Phi^\dagger = \Phi^{-1}$, where Φ^{-1} denotes the $\star$-inverse, i.e. $\Phi^{-1}\star\ \Phi = \mathbb{1}$
These equations of motion are not Lorentz-invariant, because the choice of $\vec{v}$ fixes one direction.

2.3. Action and energy functional

We can now specify an action [17] leading to the modified sigma model equations (2.15)

$$\begin{aligned} S = &-\frac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\, \eta^{ij}\, \mathrm{tr}\, \left(\partial_i\Phi^{-1} \star \partial_j\Phi\right) \\ &-\frac{1}{3}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \tilde{v}_\lambda\, \epsilon^{\lambda\mu\nu\sigma}\, \mathrm{tr}\, \left(\tilde{\Phi}^{-1} \star \partial_\mu\tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\nu\tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\sigma\tilde{\Phi}\right), \end{aligned} \tag{2.16}$$

where $\tilde{\Phi}$ is an extended version of $\Phi(t, x, y)$ with $\tilde{\Phi}(t, x, y, 0) = \text{const.}$ and $\tilde{\Phi}(t, x, y, 1) = \Phi(t, x, y)$ and $\tilde{v}_\lambda$ is the extended vector $(\vec{v}, 0)$, which is in our case $\vec{\tilde{v}} = (0, 1, 0, 0)$.
The proof that the variation of this action leads to (2.15) is shown in appendix A.
This action consists of two very different terms $S = S_2 + S_3$, where S_2 is a quadratic term which represents the kinetic part of the action, while the cubic totally antisymmetric term S_3 is a Wess-Zumino-Witten (WZW) like term [5, 6].
In the following these two terms will be discussed seperately.

[1] Ward proposed the same equation for the $SU(2)$ chiral model in the commutative case [4].

While explicit time dependent solutions of this model were constructed in [17], we want to discuss static solutions in chapter 3.
Additionally we want to calculate the energy functional

$$E = E_2[\Phi] + E_3[\tilde{\Phi}], \tag{2.17}$$

where E_2 and E_3 are defined as the Legendre tranformations of L_2 and L_3, which can be read off from

$$S_2 = \int \mathrm{d}t L_2 \qquad \text{and} \qquad S_3 = \int \mathrm{d}t L_3. \tag{2.18}$$

These Legendre transformations are determined in appendix B.
The result for the energy is (B.12)

$$\begin{aligned} E = \; & \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \ \mathrm{tr} \ \left(\partial_t \Phi^{-1} \star \partial_t \Phi + \partial_x \Phi^{-1} \star \partial_x \Phi + \partial_y \Phi^{-1} \star \partial_y \Phi \right) \\ & + \frac{1}{3} \int \mathrm{d}x \mathrm{d}y \int_0^1 \mathrm{d}\rho \ \mathrm{tr} \ \tilde{v}_t \, \epsilon^{t\mu\nu\sigma} \left(\tilde{\Phi}^{-1} \star \partial_\mu \tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\nu \tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\sigma \tilde{\Phi} \right). \end{aligned} \tag{2.19}$$

Hence we see that our gauge (2.10) makes a lot of sense, because it leads to a spacelike $\vec{v}$ (i.e. $v_t = 0$).
This implies $E_3 = 0$ and we get the energy functional from the unmodified $U(n)$ sigma model

$$E = \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \ \mathrm{tr} \ \left(\partial_t \Phi^{-1} \star \partial_t \Phi + \partial_x \Phi^{-1} \star \partial_x \Phi + \partial_y \Phi^{-1} \star \partial_y \Phi \right), \tag{2.20}$$

which is conserved because the divergence vanishes.
But if we would have made another choice in (2.10) we could get a timelike $\vec{v}$ (i.e. $v_t \neq 0$) and the energy functional would not be conserved [4].

3. Static Solutions and BPS Bound

3.1. Construction and BPS bound

In this chapter we will construct static solutions Φ with finite energy, especially soliton and antisoliton solutions as in [17] and [16].
Since we dicuss a $U(n)$ sigma model it is valid that $\Phi^\dagger = \Phi^{-1}$. Additionally we assume that our fields are real, i.e. $\Phi^\dagger = \Phi$, which implies

$$\Phi \star \Phi = \mathbb{1}. \tag{3.1}$$

This restricts the $U(n)$ to a Grassmanian sigma model. Thus each solution Φ can be described by a hermitian $\star$-projector P with $P \star P = P$

$$\Phi = \mathbb{1} - 2P. \tag{3.2}$$

$P^\perp$ is also some P
The energy functional for static real solutions simplifies to

$$E = \frac{1}{2} \int \mathrm{d}x\mathrm{d}y \ \mathrm{tr}\ \left(\partial_x \Phi \star \partial_x \Phi + \partial_y \Phi \star \partial_y \Phi\right). \tag{3.3}$$

In the following we want to perform all calculations in the operator formalism and in complex coordinates $z, \bar{z}$.
Thus the energy functional transforms to

$$E = 4\pi\theta \ \mathrm{Tr} \left(\hat{\partial}_z \hat{\Phi} \ \hat{\partial}_{\bar{z}} \hat{\Phi}\right), \tag{3.4}$$

where the trace is taken over Fock space $\mathcal{H}$ (1.27) and group space $U(n)$, i.e.

$$\mathrm{Tr} := \mathrm{tr}_{U(n)} \, \mathrm{tr}_{\mathcal{H}} \,. \tag{3.5}$$

With (1.29) we get

$$E = 2\pi \mathrm{Tr} \left([a^\dagger, \Phi] \ [\Phi, a]\right). \tag{3.6}$$

Inserting (3.2) we find for the energy

$$\begin{aligned} E &= 8\pi \mathrm{Tr} \left([a^\dagger, P] \ [P, a]\right) \\ &= 8\pi \mathrm{Tr} \left(a^\dagger P a - P a^\dagger P a - a^\dagger P a P + P a^\dagger a P\right) \\ &= 8\pi \mathrm{Tr} \left(a^\dagger P a - P a^\dagger P a - a^\dagger P a P + P a^\dagger P a P - P a^\dagger P a P + P a^\dagger a P\right) \\ &= 8\pi \mathrm{Tr} \left([P, a^\dagger] P [a, P] + P a^\dagger (\mathbb{1} - P) a P\right). \end{aligned} \tag{3.7}$$

Now we want to use the cyclic property of the trace. Supposing finite energy E the expression $\mathrm{Tr}\left([a^\dagger, P]\ [P, a]\right)$ is finite and we see from (D.6) that the operator $[a^\dagger, P]$ is Hilbert-Schmidt. So it is compact and due to the facts that P is bounded and the compact operators form an ideal in $L(\mathcal{H},\mathcal{H})$ the operator $[P,a^\dagger]P$ is compact as well. Using the cyclic property for these operators (D.10) we obtain

$$\begin{aligned} E &= 8\pi \mathrm{Tr}\left([a,P]\,[P,a^\dagger]P + Pa^\dagger(\mathbb{1}-P)aP\right) \\ &= 8\pi \mathrm{Tr}\left(Paa^\dagger P - PaPa^\dagger P + Pa^\dagger(\mathbb{1}-P)aP\right) \\ &= 8\pi \mathrm{Tr}\left(Pa(\mathbb{1}-P)a^\dagger P + Pa^\dagger(\mathbb{1}-P)aP\right). \end{aligned} \tag{3.8}$$

Defining a topological charge Q as in [16]

$$Q := \mathrm{Tr}\left(Pa(\mathbb{1}-P)a^\dagger P - Pa^\dagger(\mathbb{1}-P)aP\right), \tag{3.9}$$

we can rewrite the energy

$$\begin{aligned} E &= 8\pi\left(Q + \mathrm{Tr}\left(2Pa^\dagger(\mathbb{1}-P)aP\right)\right) \\ &= 8\pi\left(Q + 2|[a,P]P|^2\right) \\ &\geq 8\pi Q. \end{aligned} \tag{3.10}$$

The solutions leading to minimal energy are called soliton solutions. They satisfy the condition $(\mathbb{1}-P)aP = 0$.
As well as describing Φ by P we can describe it by its orthogonal complement $P^\perp = \mathbb{1}-P$. Writing $\Phi = -\mathbb{1} + 2P^\perp$ we gain after an analogous calculation

$$\begin{aligned} E &= 8\pi\left(-Q + \mathrm{Tr}\left(2Pa(\mathbb{1}-P)a^\dagger P\right)\right) \\ &= 8\pi\left(Q + 2|P[a,P]|^2\right) \\ &\geq -8\pi Q, \end{aligned} \tag{3.11}$$

where the condition for the minimal energy is now $(\mathbb{1}-P)a^\dagger P = 0$, and the solutions satisfying this condition are called antisolitons. Combining (3.10) and (3.11) we see that the energy obeys the BPS bound

$$E \geq 8\pi|Q|. \tag{3.12}$$

If the solutions are either solitons or antisolitons with

$$(\mathbb{1}-P)aP = 0 \qquad \text{soliton} \tag{3.13}$$

$$(\mathbb{1}-P)a^\dagger P = 0 \qquad \text{antisoliton} \tag{3.14}$$

the energy bound is saturated. Solitons and antisolitons satisfy the equations of motion (2.14), which reduce in the static case to

$$\partial_z\left(\Phi^\dagger \star \partial_{\bar{z}}\Phi\right) + \partial_{\bar{z}}\left(\Phi^\dagger \star \partial_z\Phi\right) = 0, \tag{3.15}$$

or, equivalently, to

$$[[a,[a^\dagger,P]],P] = 0. \tag{3.16}$$

Additionally we want to derive an expression for the energy for projectors of finite rank

$$\mathrm{rank}(P) = \dim(\mathrm{im}P) =: r < \infty, \tag{3.17}$$

where the projectors should fulfil (3.13) again.
In order to derive this simple expression, we start with the second line of (3.7)

$$\begin{aligned} E &= 8\pi \mathrm{Tr}\left(a^\dagger P a - 2Pa^\dagger P a + P a^\dagger a P\right) \\ &= 8\pi \mathrm{Tr}\left(a^\dagger P a - P a^\dagger P a + 2a^\dagger (\mathbb{1} - P) a P\right) \\ &= 8\pi \mathrm{Tr}\left(\left(a^\dagger a - a a^\dagger\right) P\right), \end{aligned} \tag{3.18}$$

and the energy becomes

$$E = 8\pi \mathrm{Tr}\,(P)\,. \tag{3.19}$$

This implies for projectors P of finite rank r

$$Q = \mathrm{Tr}\,(P) = r. \tag{3.20}$$

If we want to analyse the solutions more precisely we must consider the properties of the projector P. Each projector acting on $\mathbb{C}^n \otimes \mathcal{H}$ can be parametrized by an $n \times r$ matrix $|T\rangle$ of kets as.

$$P = |T\rangle\langle T|T\rangle^{-1}\langle T|. \tag{3.21}$$

Note that $\langle T|T\rangle$ is a $r \times r$ matrix. The properties of a general $U(n)$ solution $|T\rangle$ are discussed in detail in [16], while we are interested only in $U(1)$ and $U(2)$ solutions.

3.2. Abelian solutions

Any $U(1)$ solution is described by a finite rank projector.
In the generic case the rank r solution $|T\rangle$, a so called *multi-soliton* consisting of r solitons is given by a row vector with r entries [16]

$$|T\rangle = (|\alpha_1\rangle, \ldots, |\alpha_r\rangle), \tag{3.22}$$

where the $|\alpha_i\rangle$ are coherent states. These entries $|\alpha_i\rangle$ can be interpreted as the location of fixed lumps in the noncommutative complex plane. A lump sitting at the origin can be related to the ground state $|0\rangle$ of the Fock space $\mathcal{H}$. Shifting this lump with the unitary operator $D(\alpha_i)$ defined in (C.1) we get one at the position α_i,

$$D(\alpha_i)|0\rangle = |\alpha_i\rangle. \tag{3.23}$$

The properties of coherent states and the shift operator $D(\alpha)$ are described in appendix C. In this generic the solitons are centered at different positions in the noncommutative complex plane. But in general there could be several solitons at the same position so that we have a degenerated situation. In order to take this into account we remind of the fact that the lumps are nothing but shifted ground states. By computing the space that is spanned by $|\alpha_1\rangle, \ldots, |\alpha_r\rangle$ in the limit $\alpha_i \to 0$ we can derive an expression for a degenerated lump of r solitons on top of each other at the origin. Such a lump is descibed by exited states $\frac{a^{\dagger i}}{i!}|0\rangle$, $i = 0, \ldots, r-1$. Now we can shift any number r_k of these excited states by $D(\alpha_k)$ to the position α_k and get the state $\frac{a^{\dagger i}}{i!}|\alpha_k\rangle$ with $i = 0, \ldots, r_k-1$. This implies that the general rank r solution is a mulitsoliton, consisting of q lumps at $\alpha_1, \ldots, \alpha_q$. The k-th lump has rank r_k for $k = 1, \ldots, q$, where $\sum_{k=1}^{q} r_k = r$. We will write this as a vector

$$|T\rangle = (|T^{(1)}\rangle, \ldots, |T^{(q)}\rangle) \qquad \text{with} \qquad |T^{(k)}\rangle = (|\alpha_k\rangle, a^\dagger|\alpha_k\rangle, \ldots, \frac{a^{\dagger r_k - 1}}{(r_k-1)!}|\alpha_k\rangle). \tag{3.24}$$

Now we want to check if solutions parametrised by (3.24) really are soliton solutions of our model. For this they must fulfil condition (3.13). Inserting (3.21) in (3.13) it simplifies to

$$(\mathbb{1} - P)a|T\rangle = 0. \tag{3.25}$$

In section 6.2 it will be calculated that a acting on $|T\rangle$ gives

$$a|T\rangle = |T\rangle\gamma, \tag{3.26}$$

so that condition (3.25) is satisfied. The detailed form of the matrix γ will also be calculated in section 6.2. Thus (3.25) is satisfied and we have a static soliton solution for the noncommutative $U(1)$ sigma model parametrized by the coherent states $|\alpha_k\rangle$.

3.3. Nonabelian solutions

In the $U(2)$ case we will not use bra-ket-notation, because not all moduli parameters can be interpreted as soliton positions so that some of them cannot be described as coherent states. Furthermore, the $U(2)$ projectors have infinite rank in $\mathbb{C}^2 \otimes \mathcal{H}$. Therefore, it is more convenient to span im$P$ using operators rather than states and write

$$P = T\left(T^\dagger T\right)^{-1} T^\dagger, \tag{3.27}$$

where T is a 2 dimensional vector with operator-valued entries. For solitons the entries are independent of $a^\dagger$ and contain polynomials in a of degree at most r. As in the commutative case [3, 18] and in [22] we consider the rank 1 solution built from the vectors

$$T = \begin{pmatrix} \lambda a + \beta \\ a + \alpha \end{pmatrix} \tag{3.28}$$

and the rank 2 solution deduced from

$$T = \begin{pmatrix} \lambda a^2 + \beta a + \gamma \\ a^2 + \delta a + \alpha \end{pmatrix}. \tag{3.29}$$

4. Adiabatic approach

4.1. General method

In the last two chapters we have derived an action for a 2+1 dimensional model (2.16) and static solutions (3.2) with finite energy.
In order to solve the time-dependent 2+1 model via an adiabatic approach we endow the parameters representing all degrees of freedom γ_l of the static solutions with a time-dependence. Reinserting them into the action, the goal is to bring the Lagrangian into the nice form

$$L = \frac{1}{2} g_{lk} \dot{\gamma}_l \dot{\gamma}_k + A_l \dot{\gamma}_l, \tag{4.1}$$

well known from mechanics [3] for real parameters γ, which is in complex coordinates

$$L = \frac{1}{2}(h_{ij}\dot{\alpha}_i\dot{\alpha}_j + h_{\bar{i}\bar{j}}\dot{\bar{\alpha}}_i\dot{\bar{\alpha}}_j) + h_{\bar{i}j}\dot{\bar{\alpha}}_i\dot{\alpha}_j + A_i\dot{\alpha}_i + A_{\bar{i}}\dot{\bar{\alpha}}_i. \tag{4.2}$$

The parameters $\alpha_i \in \mathbb{C}$ can be identified with coordinates in a parameter space, the *moduli space* $\mathcal{M}_k$. In the mechanical analogue these coordinates correspond to positions of particles. By comparison with mechanics we see that h must be the metric of this moduli space, while A is a magnetic vector potential.
Treating the moduli space coordinates as particle positions we can derive equations of motion. In this case they describe a geodesic motion with a magnetic force and we can discuss mechanical problems like scattering.
So the dynamics in the moduli space is determined by the metric h and the magnetic vector potential A. Thus the problem is solved if the Lagrangian has the required form (4.2) and we are able to read off h and A. The geodesic motion approximates then time dependent solutions representing the motion of lumps in the complex plane described by our noncommutative coordinates z and $\bar{z}$ for small velocities.
This adiabatic approximation was used in conjunction with monopole scattering in [23] and [24]. Further information about this approach sometimes called Manton approach and moduli space dynamics can be found in [2] and [3].

4.2. The kinetical term

As described above we reinsert the static solution (3.2) with time-dependent parameters

$$\Phi = \mathbb{1} - 2P(t) \tag{4.3}$$

into the kinetic part S_2 of the action (2.16)

$$S_2 = -\frac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \ \mathrm{tr}\ (\partial_t\Phi \star \partial_t\Phi + \partial_x\Phi \star \partial_x\Phi + \partial_y\Phi \star \partial_y\Phi)\,. \tag{4.4}$$

The aim is now to bring the Lagrangian to the required form (4.2) to obtain the moduli space metric $h_{\bar{i}j}$. In the operator formalism the action becomes

$$S_2 = -\frac{1}{2}\int \mathrm{d}t\ 2\pi\theta\ \mathrm{Tr}\ \left((\hat{\partial}_t\hat{\Phi})^2 + (\hat{\partial}_x\hat{\Phi})^2 + (\hat{\partial}_y\hat{\Phi})^2\right). \tag{4.5}$$

In all following calculations we will use the operator formalism. So there is no problem omitting the hats on operators. In complex coordinates (1.5) we have

$$S_2 = -\frac{1}{2}\int \mathrm{d}t\; 2\pi\theta\; \mathrm{Tr}\; \left((\partial_t\Phi)^2 + 4(\partial_z\Phi)(\partial_{\bar{z}}\Phi)\right), \tag{4.6}$$

where we recover the finite energy E (3.4)

$$S_2 = -\int \mathrm{d}t\; \left(\pi\theta\; \mathrm{Tr}\; (\partial_t\Phi)^2 \;+\; E\right). \tag{4.7}$$

As a constant in the Lagrangian E does not contribute to the equations of motion desribing the moduli space dynamics.
Inserting (4.3) S_2 simplifies to

$$S_2 = 4\pi\theta \int \mathrm{d}t\; \mathrm{Tr}\; (\partial_t P)^2. \tag{4.8}$$

Using the parametrization (3.21) we calculate the time derivative $\partial_t P$

$$\partial_t P = \partial_t \left(|T\rangle\langle T|T\rangle^{-1}\langle T|\right). \tag{4.9}$$

We define for shorter notation

$$\langle T|T\rangle =: M, \tag{4.10}$$

and calculate

$$\partial_t M^{-1} = -M^{-1}\,\partial_t M\; M^{-1} \qquad \text{and} \qquad \partial_t M = \left(\partial_t\langle T|\right)|T\rangle + \langle T|\left(\partial_t|T\rangle\right). \tag{4.11}$$

Hence

$$\partial_t P = \partial_t|T\rangle\; M^{-1}\langle T| + |T\rangle M^{-1}\,\partial_t\langle T| - |T\rangle M^{-1}\,\partial_t M\; M^{-1}\langle T|. \tag{4.12}$$

Thus we get the action

$$\begin{aligned} S_2 = 4\pi\theta \int \mathrm{d}t\; \mathrm{Tr}\; \Big(& |T\rangle M^{-1}\,\partial_t\langle T|\,(\mathbb{1}-P)\,\partial_t\,|T\rangle M^{-1}\langle T| \\ & + (\mathbb{1}-P)\,\partial_t\,|T\rangle M^{-1}\,\partial_t\langle T|\,(\mathbb{1}-P)\Big). \end{aligned} \tag{4.13}$$

Since the energy should be finite, $\mathrm{Tr}(\partial_t P)^2$ must be finite, too. Thus the trace is well defined and the operator $\partial_t P$ is Hilbert-Schmidt and hence compact. From (4.12) we see that $\partial_t|T\rangle\, M^{-1}\langle T|$ and $|T\rangle M^{-1}\,\partial_t\langle T|$ must be compact as well. Since $(\mathbb{1}-P)$ is bounded, the operator $(\mathbb{1}-P)\partial_t\,|T\rangle M^{-1}\langle T|$ is also compact.
So both expressions $|T\rangle M^{-1}\,\partial_t\langle T|$ and $(\mathbb{1}-P)\partial_t\,|T\rangle M^{-1}$ the terms of (4.13) are composed of are compact.
Using the cyclicity of the trace (D.10) we obtain

$$S_2 = 8\pi\theta \int \mathrm{d}t\; \mathrm{Tr}\,(\mathbb{1}-P)\,\partial_t|T\rangle\; M^{-1}\,\partial_t\langle T|. \tag{4.14}$$

Further simplification of the action S_2 is not possible in the general $U(n)$ case. We are able to calculate the moduli space metric only in special cases of $U(1)$ or $U(2)$ solutions. This will be done in the following chapters.
Now we consider the WZW-like part S_3.

4.3. The WZW-like term

To get the magnetic vector potential A we proceed in the same way with the WZW-like part S_3 of the action (2.16), which reduces with the cyclic property of the trace to

$$S_3 = -\frac{1}{3}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \tilde{v}_\lambda\, \epsilon^{\lambda\mu\nu\sigma}\, \mathrm{tr}\, \left(\tilde{\Phi}^{-1}\star\partial_\mu\tilde{\Phi}\star\tilde{\Phi}^{-1}\star\partial_\nu\tilde{\Phi}\star\tilde{\Phi}^{-1}\star\partial_\sigma\tilde{\Phi}\right), \tag{4.15}$$

which reduces with $\vec{\tilde{v}} = (0,1,0,0)$ and the cyclic property of the trace to

$$S_3 = -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\;\mathrm{tr}\left(\tilde{\Phi}^{-1}\star\partial_t\tilde{\Phi}\star\left(\tilde{\Phi}^{-1}\star\partial_\rho\tilde{\Phi}\star\tilde{\Phi}^{-1}\star\partial_y\tilde{\Phi} - \tilde{\Phi}^{-1}\star\partial_y\tilde{\Phi}\star\tilde{\Phi}^{-1}\star\partial_\rho\tilde{\Phi}\right)\right). \tag{4.16}$$

From now on we will use the operator formalism and omit the hats on operators.
Recall the fact that $\tilde{\Phi}$ interpolates between

$$\tilde{\Phi}(t,x,y,0) = \text{const.} \qquad \text{and} \qquad \tilde{\Phi}(t,x,y,1) = \Phi(t,x,y) \tag{4.17}$$

for time dependent solutions Φ as above in (4.3).
This condition is satisfied, if we set

$$\tilde{\Phi} := \mathbb{1} - \chi P \qquad \text{for} \qquad \chi := 1 - \mathrm{e}^{\mathrm{i}\pi\rho}, \tag{4.18}$$

which implies

$$\tilde{\Phi}^{-1} = \tilde{\Phi}^\dagger = \mathbb{1} - \bar{\chi}P. \tag{4.19}$$

In operator formalism S_3 becomes

$$S_3 = -\int \mathrm{d}t\, 2\pi\theta\, \mathrm{tr}_{\mathcal{H}} \int_0^1 \mathrm{d}\rho\;\mathrm{tr}\left(\tilde{\Phi}^\dagger\partial_t\tilde{\Phi}\left(\tilde{\Phi}^\dagger\partial_\rho\tilde{\Phi}\,\tilde{\Phi}^\dagger\partial_y\tilde{\Phi} - \tilde{\Phi}^\dagger\partial_y\tilde{\Phi}\,\tilde{\Phi}^\dagger\partial_\rho\tilde{\Phi}\right)\right). \tag{4.20}$$

With (4.18) we can calculate the elements

$$\begin{aligned}
\tilde{\Phi}^\dagger\partial_t\tilde{\Phi} &= -\,\chi\partial_t P + |\chi|^2 P\partial_t P \\
\tilde{\Phi}^\dagger\partial_y\tilde{\Phi} &= -\,\chi\partial_y P + |\chi|^2 P\partial_y P \\
\tilde{\Phi}^\dagger\partial_\rho\tilde{\Phi} &= (\mathbb{1} - (1-\mathrm{e}^{-\mathrm{i}\pi\rho}P)\,(\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\pi\rho})P) = \mathrm{i}\pi P.
\end{aligned} \tag{4.21}$$

Inserting these expressions we find

$$\begin{aligned}
S_3 = -2\mathrm{i}\pi^2\theta \int \mathrm{d}t \int_0^1 \mathrm{d}\rho\;\mathrm{Tr}\Big(&\left(-\chi\partial_t P + |\chi|^2 P\partial_t P\right)\,\big(-\chi P\partial_y P + |\chi|^2 P\partial_y P \\
&+\chi\partial_y PP - |\chi|^2 P\partial_y PP\big)\Big).
\end{aligned} \tag{4.22}$$

With (1.28) the last term vanishes

$$P\partial_y PP = -\frac{\mathrm{i}}{\theta}\left(P[x,P]P\right) = -\frac{\mathrm{i}}{\theta}\left(PxP - PxP\right) = 0. \tag{4.23}$$

Thus we get

$$\begin{aligned}
S_3 = -2\mathrm{i}\pi^2\theta \int \mathrm{d}t \int_0^1 \mathrm{d}\rho\;\mathrm{Tr}\Big(&|\chi|^2\partial_t PP\partial_y P - \chi|\chi|^2\partial_t PP\partial_y P \\
&- |\chi|^2\partial_t P\partial_y P + \chi|\chi|^2 P\partial_t PP\partial_y P \\
&- |\chi|^2 P\partial_t PP\partial_y P + \chi|\chi|^2 P\partial_t P\partial_y P\Big)
\end{aligned} \tag{4.24}$$

In order to show that the fourth and the fifth term vanishes, we need the cyclic property of the trace (D.10). Above we have already used that $\partial_t P$ and $[x,P]$ are Hilbert-Schmidt operators. So $P\partial_t P$ and $P[x,P]$ are compact and with same arguments we are allowed to use (D.10) again.

$$P\partial_t PP[x,P] = P[x,P]P\partial_t P = 0 \tag{4.25}$$

It remains with (1.28)

$$S_3 = -2\mathrm{i}\pi^2\theta \int_0^1 \mathrm{d}\rho \left(|\chi|^2 - \chi|\chi|^2\right) \int \mathrm{d}t \; \mathrm{Tr}\left(\partial_t PP[x,P] - P\partial_t P[x,P]\right) \tag{4.26}$$

After a short calculation and writing x in terms of a and $a^\dagger$ by (1.4) and (1.17) S_3 reduces to

$$S_3 \sim \int \mathrm{d}t \, \partial_t \mathrm{Tr}\Big((a + a^\dagger)P)\Big) \tag{4.27}$$

$$= \int \mathrm{d}t \left(\left\{ \partial_{\alpha_i} \mathrm{Tr}\Big((a+a^\dagger)P)\Big) \right\} \dot{\alpha}_i + \left\{ \partial_{\bar{\alpha}_i} \mathrm{Tr}\Big((a+a^\dagger)P)\Big) \right\} \dot{\bar{\alpha}}_i \right) \tag{4.28}$$

We see that the magnetic potential is exact $A_i = \partial_{\alpha_i} f$ (and $\bar{A}_i = \partial_{\bar{\alpha}_i} f$), hence there is no magnetic forcing ($B = \mathrm{rot}\ \mathrm{grad} f = 0$) and the WZW-like term S_3 does not contribute to the moduli space dynamics.

5. Generic $U(1)$ case

5.1. Calculation of the generic metric

In order to get a Lagrangian of the form (4.2) to determine the moduli space metric we simplify the relevant term of the action in the special case of generic $U(1)$ solutions (3.22) of finite rank r. Since the WZW-like term S_3 does not contribute, the relevant part of the action is (4.14)

$$S = S_2 = 8\pi\theta \int \mathrm{d}t \ \mathrm{tr}_{\mathcal{H}} \left(\mathbb{1} - P\right) \partial_t |T\rangle \, M^{-1} \, \partial_t \langle T|. \tag{5.1}$$

Taking the trace over $\mathcal{H}$ we have only finite sums because all operators are finite so that we get for the Lagrangian

$$L = 8\pi\theta \ \mathrm{tr}_r \left(\partial_t \langle T| \, \partial_t |T\rangle M^{-1} - \partial_t \langle T| \, |T\rangle M^{-1} \langle T| \, \partial_t |T\rangle M^{-1} \right), \tag{5.2}$$

where tr_r denotes the trace over $\mathrm{im}P$.
We should calculate the time derivatives $\partial_t|T\rangle$ and $\partial_t\langle T|$. For this remember that the generic solution (3.22) has time-dependent parameters $|\alpha(t)\rangle$

$$|T\rangle = (|\alpha_1(t)\rangle, \ldots, |\alpha_r(t)\rangle). \tag{5.3}$$

With (3.23) we find for the time derivative

$$\partial_t|T\rangle = (\dot{D}(\alpha_1(t))|0\rangle, \ldots, \dot{D}(\alpha_r(t))|0\rangle). \tag{5.4}$$

Inserting the definition of the shift operator (C.3) we get

$$\partial_t|T\rangle = \left((-\tfrac{1}{2}\partial_t|\alpha_1|^2 + a^\dagger \dot{\alpha}_1)|\alpha_1(t)\rangle, \ldots, (-\tfrac{1}{2}\partial_t|\alpha_r|^2 + a^\dagger \dot{\alpha}_r)|\alpha_r(t)\rangle \right), \tag{5.5}$$

which is in matrix notation

$$\partial_t|T\rangle = a^\dagger|T\rangle \underbrace{\begin{pmatrix} \dot{\alpha}_1 & & \\ & \ddots & \\ & & \dot{\alpha}_r \end{pmatrix}}_{:=\dot{A}} + |T\rangle \underbrace{\begin{pmatrix} -\frac{1}{2}\partial_t|\alpha_1|^2 & & \\ & \ddots & \\ & & -\frac{1}{2}\partial_t|\alpha_r|^2 \end{pmatrix}}_{:=\dot{B}}. \tag{5.6}$$

We thus obtain

$$\partial_t|T\rangle = a^\dagger|T\rangle\dot{A} + |T\rangle\dot{B} \tag{5.7}$$

and analogously

$$\partial_t\langle T| = \dot{\bar{A}}\langle T|a + \dot{B}\langle T|. \tag{5.8}$$

Now we insert these results into the Lagrangian (5.2)

$$
\begin{aligned}
L = 8\pi\theta\ \mathrm{tr}_r \Big(& \dot{\bar{A}}\,\langle T|aa^\dagger|T\rangle\,\dot{A}M^{-1} + \dot{B}\,\langle T|a^\dagger|T\rangle\,\dot{A}M^{-1} \\
& + \dot{\bar{A}}\,\langle T|a|T\rangle\,\dot{B}M^{-1} + \dot{B}M\dot{B}M^{-1} \\
& - \left(\dot{\bar{A}}\,\langle T|a|T\rangle + \dot{B}\langle T|T\rangle\right) M^{-1}\left(\langle T|a^\dagger|T\rangle\,\dot{A} + \langle T|T\rangle\dot{B}\right) M^{-1}\Big)
\end{aligned}
\tag{5.9}
$$

By expansion this reduces to

$$
L = 8\pi\theta\ \mathrm{tr}_r \left(\dot{\bar{A}}\left(\langle T|aa^\dagger|T\rangle - \langle T|a|T\rangle\,M^{-1}\,\langle T|a^\dagger|T\rangle\right)\dot{A}M^{-1}\right), \tag{5.10}
$$

where the matrix $\dot{B}$ drops out.
In the generic case the solitons are at different positions, i.e. $\alpha_i \neq \alpha_j$ for $i \neq j$. Hence it follows with (C.8)

$$
a|T\rangle = (\alpha_1|\alpha_1\rangle, \ldots, \alpha_r|\alpha_r\rangle) = |T\rangle \underbrace{\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_r \end{pmatrix}}_{=A}. \tag{5.11}
$$

With this we gain the Lagrangian

$$
L = 8\pi\theta\ \mathrm{tr}_r \left(\dot{\bar{A}}\Big(\bar{A}MA + M - MAM^{-1}\bar{A}M\Big)\dot{A}M^{-1}\right). \tag{5.12}
$$

In the following we will calculate the three terms in (5.12) seperately by matrix multiplication.
For the first term we get

$$
\mathrm{tr}_r\left(\dot{\bar{A}}\bar{A}MA\dot{A}M^{-1}\right) = \dot{\bar{A}}_{ij}\,\bar{A}_{jk}\,M_{kl}\,A_{lm}\,\dot{A}_{mn}\,{M^{-1}}_{ni}, \tag{5.13}
$$

where the sum is taken over indices appearing twice.
Since $A, \bar{A}, \dot{A}, \dot{\bar{A}}$ are diagonal this reduces to

$$
\begin{aligned}
\mathrm{tr}_r\left(\dot{\bar{A}}\bar{A}MA\dot{A}M^{-1}\right) &= \dot{\bar{A}}_{ii}\,\bar{A}_{ii}\,M_{il}\,A_{ll}\,\dot{A}_{ll}\,{M^{-1}}_{li} \\
&= \sum_{i,l}^{r} \dot{\bar{\alpha}}_i\dot{\alpha}_l\left(\bar{\alpha}_i\alpha_l\,M_{il}\,{M^{-1}}_{li}\right)
\end{aligned}
\tag{5.14}
$$

The second term in (5.12) becomes

$$
\mathrm{tr}_r\left(\dot{\bar{A}}M\dot{A}M^{-1}\right) = \sum_{i,l}^{r}\dot{\bar{\alpha}}_i\dot{\alpha}_l\,M_{il}\,{M^{-1}}_{li}, \tag{5.15}
$$

and the third term is

$$
\begin{aligned}
-\,\mathrm{tr}_r\left(\dot{\bar{A}}MAM^{-1}\bar{A}M\dot{A}M^{-1}\right) &= -\,\dot{\bar{A}}_{ii}\,M_{ij}\,A_{jj}\,{M^{-1}}_{jk}\,\bar{A}_{kk}\,M_{kl}\,\dot{A}_{ll}\,{M^{-1}}_{li} \\
&= -\sum_{i,l}^{r}\dot{\bar{\alpha}}_i\dot{\alpha}_l\left(\sum_{j,k}^{r}\bar{\alpha}_k\alpha_j\,{M^{-1}}_{li}\,M_{ij}\,{M^{-1}}_{jk}\,M_{kl}\right).
\end{aligned}
\tag{5.16}
$$

We rename some indices and by summation obtain the Lagrangian

$$L = 8\pi\theta \sum_{i,j}^{r} \dot{\bar{\alpha}}_i \dot{\alpha}_j \left(\bar{\alpha}_i \alpha_j \, M_{ij} \, {M^{-1}}_{ji} + M_{ij} \, {M^{-1}}_{ji} - \sum_{k,l}^{r} \bar{\alpha}_k \alpha_l \, {M^{-1}}_{ji} \, M_{il} \, {M^{-1}}_{lk} \, M_{kj} \right). \tag{5.17}$$

If we compare this result with the Lagrangian of the required form (4.2) we find the hermitian moduli space metric

$$\begin{aligned} h_{ij} &= 0 \\ h_{\bar{i}j} &= 8\pi\theta \left(\bar{\alpha}_i \alpha_j \, M_{ij} \, {M^{-1}}_{ji} + M_{ij} \, {M^{-1}}_{ji} - \sum_{k,l}^{r} \bar{\alpha}_k \alpha_l \, {M^{-1}}_{ji} \, M_{il} \, {M^{-1}}_{lk} \, M_{kj} \right) \\ h_{\bar{i}\bar{j}} &= 0. \end{aligned} \tag{5.18}$$

5.2. Determination of the generic Kähler potential

In this section we will show that the moduli space metrik $h_{\bar{i}j}$ is Kähler and specify a Kähler potential as defined in (E.4)

$$h_{\bar{i}j} = \frac{\partial}{\partial \bar{\alpha}_i} \frac{\partial}{\partial \alpha_j} \, \mathrm{K}. \tag{5.19}$$

In order to proof that $h_{\bar{i}j}$ is a Kähler metric we could show that the conditions (E.3) are satisfied.
But if we are able to find a Kähler potential K, the metric $h_{\bar{i}j}$ will naturally be Kähler.
Going this way we guess that K is something like the expression $\ln \det M$. Thus we calculate the complex derivatives of this expression

$$\frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \bar{\alpha}_j} (\ln \det M) = \frac{\partial}{\partial \alpha_i} \left(\frac{1}{\det M} \frac{\partial}{\partial \bar{\alpha}_j} \det M \right). \tag{5.20}$$

Using the derivative of the determinant as found in [25]

$$\frac{\partial}{\partial \bar{\alpha}_j} \det M = \det M \, {M^{-1}}_{kl} \, \frac{\partial M_{lk}}{\partial \bar{\alpha}_j} \tag{5.21}$$

we get

$$\begin{aligned} \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \bar{\alpha}_j} (\ln \det M) =& \frac{\partial}{\partial \alpha_i} \left({M^{-1}}_{kl} \right) \frac{\partial M_{lk}}{\partial \bar{\alpha}_j} + {M^{-1}}_{kl} \, \frac{\partial^2}{\partial \alpha_i \partial \bar{\alpha}_j} M_{lk} \\ =& - {M^{-1}}_{km} \, {M^{-1}}_{nl} \, \frac{\partial M_{mn}}{\partial \alpha_i} \, \frac{\partial M_{lk}}{\partial \bar{\alpha}_j} + {M^{-1}}_{kl} \, \frac{\partial^2}{\partial \alpha_i \partial \bar{\alpha}_j} M_{lk}. \end{aligned} \tag{5.22}$$

In the following we compute the derivatives

$$\begin{aligned} \frac{\partial M_{mn}}{\partial \alpha_i} =& \frac{\partial}{\partial \alpha_i} \left(\mathrm{e}^{-\frac{1}{2}(|\alpha_m|^2 + |\alpha_n|^2) + \bar{\alpha}_m \alpha_n} \right) \\ =& \left(\delta_{im}(-\tfrac{1}{2}\bar{\alpha}_m) + \delta_{in}(-\tfrac{1}{2}\bar{\alpha}_n + \bar{\alpha}_m) \right) M_{mn} \end{aligned} \tag{5.23}$$

and in the same way

$$\frac{\partial M_{lk}}{\partial \bar{\alpha}_j} = \left(\delta_{jk}(-\tfrac{1}{2}\alpha_k) + \delta_{jl}(-\tfrac{1}{2}\alpha_l + \alpha_k) \right) M_{lk}. \tag{5.24}$$

This implies the second derivative

$$
\begin{aligned}
\frac{\partial^2}{\partial\alpha_i\partial\bar{\alpha}_j}M_{lk} =& \frac{\partial}{\partial\alpha_i}\left((\delta_{jk}(-\tfrac{1}{2}\alpha_k)+\delta_{jl}(-\tfrac{1}{2}\alpha_l+\alpha_k)M_{lk})\right)\\
=& \left(-\tfrac{1}{2}\delta_{jl}\delta_{li}+\delta_{il}\delta_{ki}-\tfrac{1}{2}\delta_{jk}\delta_{ik}\right)M_{lk}\\
&+\left(\delta_{jk}(-\tfrac{1}{2}\alpha_k)+\delta_{jl}(-\tfrac{1}{2}\alpha_l+\alpha_k)\right)\left(\delta_{il}(-\tfrac{1}{2}\bar{\alpha}_l)+\delta_{ik}(-\tfrac{1}{2}\bar{\alpha}_k+\bar{\alpha}_l)\right)M_{lk}\\
=&\Big(-\delta_{ij}+\delta_{jl}\delta_{ki}+\delta_{jl}\delta_{il}(-\tfrac{1}{2}\alpha_l+\alpha_k)(-\tfrac{1}{2}\bar{\alpha}_l)\\
&+\delta_{jl}\delta_{ki}(-\tfrac{1}{2}\alpha_l+\alpha_k)(-\tfrac{1}{2}\bar{\alpha}_k+\bar{\alpha}_l)\\
&+\delta_{jk}(-\tfrac{1}{2}\alpha_k)\delta_{il}(-\tfrac{1}{2}\bar{\alpha}_l)+\delta_{jk}\delta_{ik}(-\tfrac{1}{2}\alpha_k)(-\tfrac{1}{2}\bar{\alpha}_k+\bar{\alpha}_l)\Big)M_{lk}
\end{aligned}
\tag{5.25}
$$

and the product

$$
\begin{aligned}
\frac{\partial M_{mn}}{\partial\alpha_i}\,\frac{\partial M_{lk}}{\partial\bar{\alpha}_j} =&\Big(\delta_{im}\delta_{jl}(-\tfrac{1}{2}\alpha_l+\alpha_k)(-\tfrac{1}{2}\bar{\alpha}_m)\\
&+\delta_{im}\delta_{jk}(-\tfrac{1}{2}\bar{\alpha}_m)(-\tfrac{1}{2}\alpha_k)+\delta_{jl}\delta_{in}(-\tfrac{1}{2}\bar{\alpha}_n+\bar{\alpha}_m)(-\tfrac{1}{2}\alpha_l+\alpha_k)\\
&+\delta_{jk}\delta_{in}(-\tfrac{1}{2}\bar{\alpha}_n+\bar{\alpha}_m)(-\tfrac{1}{2}\alpha_k)\Big)M_{mn}\,M_{lk}.
\end{aligned}
\tag{5.26}
$$

We insert the results of (5.25) and (5.26) in the two parts of (5.22) seperatly. After a short calculation for the first term it yields

$$
\begin{aligned}
-{M^{-1}}_{km}\,{M^{-1}}_{nl}\,\frac{\partial M_{mn}}{\partial\alpha_i}\frac{\partial M_{lk}}{\partial\bar{\alpha}_j} =& -\tfrac{1}{2}\alpha_i\bar{\alpha}_i\,\delta_{ij}-{M^{-1}}_{ij}\,M_{ji}(\tfrac{1}{2}\bar{\alpha}_i\alpha_j-\tfrac{1}{2}\bar{\alpha}_j\alpha_j-\tfrac{1}{2}\bar{\alpha}_i\alpha_i)\\
&-{M^{-1}}_{kl}\,M_{li}\,{M^{-1}}_{ij}\,M_{jk}\,\bar{\alpha}_l\alpha_k+{M^{-1}}_{jl}\,M_{lj}\,\delta_{ij}(\tfrac{1}{2}\bar{\alpha}_l\alpha_j)\\
&+{M^{-1}}_{jm}\,M_{mj}\,\delta_{ij}(\tfrac{1}{2}\bar{\alpha}_i\alpha_k),
\end{aligned}
\tag{5.27}
$$

and the second term becomes

$$
\begin{aligned}
{M^{-1}}_{kl}\,\frac{\partial^2}{\partial\alpha_i\partial\bar{\alpha}_j}\,M_{lk} =& -\delta_{ij}+{M^{-1}}_{ij}\,M_{ji}+\alpha_i\bar{\alpha}_j\,{M^{-1}}_{ij}\,M_{ji}+\tfrac{1}{2}\alpha_i\bar{\alpha}_i\,\delta_{ij}\\
&+{M^{-1}}_{ij}\,M_{ji}\,(\tfrac{1}{2}\bar{\alpha}_i\alpha_j-\tfrac{1}{2}\bar{\alpha}_j\alpha_j-\tfrac{1}{2}\bar{\alpha}_i\alpha_i)\\
&-\tfrac{1}{2}\bar{\alpha}_l\alpha_j\,\delta_{ij}\,{M^{-1}}_{jl}\,M_{lj}-\tfrac{1}{2}\bar{\alpha}_i\alpha_k\,\delta_{ij}\,{M^{-1}}_{ki}\,M_{ik}.
\end{aligned}
\tag{5.28}
$$

This implies for (5.22)

$$
\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\bar{\alpha}_j}(\ln\det M) = -\delta_{ij}+{M^{-1}}_{ij}\,M_{ji}(1+\alpha_i\bar{\alpha}_j)-{M^{-1}}_{kl}\,M_{li}\,{M^{-1}}_{ij}\,M_{jk}\,\bar{\alpha}_l\alpha_k, \tag{5.29}
$$

which is nothing but the moduli space metric with interchanged indices $\frac{h_{\bar{j}\bar{i}}}{8\pi\theta}$ minus δ_{ij}.

$$
\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\bar{\alpha}_j}(\ln\det M) = \frac{h_{\bar{j}i}}{8\pi\theta}-\delta_{ij} \tag{5.30}
$$

Thus the metric is

$$
h_{\bar{i}j} = 8\pi\theta\,\frac{\partial}{\partial\bar{\alpha}_i}\frac{\partial}{\partial\alpha_j}\left(\sum_{k=1}^{r}\bar{\alpha}_k\alpha_k+\ln\det M\right), \tag{5.31}
$$

and we can read off the Kähler potential

$$K = 8\pi\theta \sum_{k=1}^{r} \bar{\alpha}_k \alpha_k + \ln\det M. \tag{5.32}$$

Inserting (5.31) into the Lagrangian (5.17) it simplifies to

$$L = 8\pi\theta \sum_{i,j}^{r} \dot{\bar{\alpha}}_i \dot{\alpha}_j \left(\delta_{ij} + \partial_{\bar{i}}\partial_j \ln\det M\right). \tag{5.33}$$

These results for metric and Kähler potential appear in [14] and [15] for the noncommutative scalar field model.
In order to find another simple expression for the Kähler potential K (5.32), we use the definition of the determinant

$$\det M = \sum_{\pi\in S_r} \operatorname{sign}(\pi)\, \mathrm{e}^{\sum_{k=1}^{r}\left(\bar{\alpha}_k\alpha_{\pi(k)} - \frac{1}{2}|\alpha_k|^2 - \frac{1}{2}|\alpha_{\pi(k)}|^2\right)}, \tag{5.34}$$

where π denotes a permutation and S_r the symmetrical group with r elements. This simplifies to

$$\begin{aligned} \det M =& \sum_{\pi\in S_r} \operatorname{sign}(\pi)\, \mathrm{e}^{-\sum_{k=1}^{r}|\alpha_k|^2 + \sum_{k=1}^{r}\bar{\alpha}_k\alpha_{\pi(k)}} \\ =& \mathrm{e}^{-\sum_{k=1}^{r}|\alpha_k|^2}\, \det \underbrace{\left(\mathrm{e}^{\bar{\alpha}_i\alpha_j}\right)}_{=:R} \end{aligned} \tag{5.35}$$

Hence we obtain for the Kähler potential

$$K = 8\pi\theta\, \partial_{\bar{i}}\partial_j \ln\det R \tag{5.36}$$

and for the Lagrangian

$$L = 8\pi\theta \sum_{i,j}^{r} \dot{\bar{\alpha}}_i \dot{\alpha}_j \ \ln\det R. \tag{5.37}$$

5.3. Properties of the generic metric

In this section we discuss the dependence of $h_{\bar{i}j}$ on the moduli space coordinates $\{\alpha_i\} =: \alpha$. We will show that the metric is translationally invariant and depends only on coordinate differences. This implies then the decoupling of the center of mass motion.
At first we ask how the metric changes under translations. Therefore we consider a shift of the coordinates by a common constant $\beta \in \mathbb{C}$

$$|T\rangle = (|\alpha_1\rangle, \dots, |\alpha_r\rangle) \quad \longrightarrow \quad (|\alpha_1+\beta\rangle, \dots, |\alpha_r+\beta\rangle) \tag{5.38}$$

So we must calculate how the shift acts on the coherent states

$$|\alpha_i + \beta\rangle = D(\alpha_i+\beta)|0\rangle, \tag{5.39}$$

where D is the shift operator (C.1). A short calculation yields

$$|\alpha_i + \beta\rangle = D(\beta)\,|\alpha_i\rangle\, \mathrm{e}^{\frac{1}{2}(\alpha_i\bar{\beta} - \bar{\alpha}_i\beta)}, \tag{5.40}$$

and the shifted $|T\rangle$ becomes

$$(|\alpha_1+\beta\rangle,\dots,|\alpha_r+\beta\rangle) = D(\beta)\,|T\rangle\,U \qquad \text{with} \qquad U := \mathrm{diag}\{\mathrm{e}^{\frac{1}{2}(\alpha_i\bar{\beta}-\bar{\alpha}_i\beta)}\}. \tag{5.41}$$

Hence the determinant of $M = \langle T|T\rangle$ is invariant under these transformations so that the Kähler potential (5.32) is changed only by functions holomorphic and antiholomorphic in α,

$$K(\alpha+\beta,\bar{\alpha}+\bar{\beta}) = 8\pi\theta\sum_{i=1}^{r}(\alpha_i\bar{\alpha}_i+\beta\bar{\beta}+\alpha_i\bar{\beta}+\bar{\alpha}_i\beta)+\ln\det\langle T|T\rangle. \tag{5.42}$$

Due to the gauge invariance of the Kähler metric (E.5),

$$h_{\bar{i}j}(\alpha+\beta,\bar{\alpha}+\bar{\beta}) = h_{\bar{i}j}(\alpha,\bar{\alpha}). \tag{5.43}$$

Thus the moduli space metric $h_{\bar{i}j}$ is invariant under constant translations.
We can show now that $h_{\bar{i}j}$ depends only on coordinate differences. It follows from (5.43) that

$$0 = \partial_\beta\, h_{\bar{i}j}(\alpha,\bar{\alpha}) = \partial_\beta\, h_{\bar{i}j}(\alpha+\beta,\bar{\alpha}+\bar{\beta}) = \sum_{k=1}^{r}\frac{\partial h_{\bar{i}j}}{\partial\alpha_k} \tag{5.44}$$

We change to new coordinates

$$\begin{array}{rcl} u_1 &=& \alpha_1-\alpha_2 \\ u_2 &=& \alpha_2-\alpha_3 \\ \vdots && \vdots \\ u_{r-1} &=& \alpha_{r-1}-\alpha_r \\ u_r &=& \alpha_1+\alpha_r \end{array} \tag{5.45}$$

and express ∂_{α_k} in these coordinates

$$\begin{array}{rcl} \partial_{\alpha_1} &=& \partial_{u_r}+\partial_{u_1} \\ \partial_{\alpha_2} &=& \partial_{u_2}-\partial_{u_1} \\ \partial_{\alpha_3} &=& \partial_{u_3}-\partial_{u_2} \\ \vdots && \vdots \\ \partial_{\alpha_{r-1}} &=& \partial_{u_{r-1}}-\partial_{u_{r-2}} \\ \partial_{\alpha_r} &=& \partial_{u_r}-\partial_{u_{r-1}}. \end{array} \tag{5.46}$$

So it yields with (5.44) that

$$0 = \sum_{k=1}^{r}\frac{\partial h_{\bar{i}j}}{\partial\alpha_k} = 2\,\frac{\partial h_{\bar{i}j}}{\partial u_r}, \tag{5.47}$$

which implies that $h_{\bar{i}j}$ is independent of u_r and does consequently only depend on the coordinate differences $u_1\dots u_{r-1}$.
We introduce a center of mass coordinate

$$s := \frac{1}{r}\sum_{i=1}^{r}\alpha_i \tag{5.48}$$

and some relative coordinates $\{w_i\} = w$ defined by

$$\alpha_i =: s + w_i \qquad \text{with} \qquad \sum_{i=1}^{r} w_i = 0. \tag{5.49}$$

From (5.42) it is obvious that the Kähler potential in these coordinates is

$$K(w, s, \bar{w}, \bar{s}) = 8\pi\theta\,(rs\bar{s} + \sum_{i=1}^{r} w_i\bar{w}_i + \ln\det M(w, \bar{w})), \tag{5.50}$$

and since the metric depends only on coordinate differences the Lagrangian becomes

$$L = 8\pi\theta \left(r\dot{\bar{s}}\dot{s} + \sum_{i,j}^{r} \dot{\bar{w}}_i\dot{w}_j\,(\delta_{ij} + \partial_{\bar{i}}\partial_j \ln\det M(w, \bar{w})) \right). \tag{5.51}$$

Hence the center of mass motion is completely decoupled from the relative coordinates.

5.4. Asymptotic cases

In the generic case we have interpreted our moduli space coordinates as different positions of single solitons. Now we consider what happens if a soliton goes to infinity ($\alpha_r \to \infty$) on the one hand , and if all solitons are converging to one position ($\alpha_i \to \alpha_r, \forall i \in \{1 \dots r\}$) on the other hand.
So without loss of generality the r-th soliton should be far away, i.e. $\alpha_r \to \infty$. Then we find for the interesting term in the Kähler potential (5.32)

$$\lim_{\alpha_r\to\infty} \ln\det M = \ln\det \left(\lim_{\alpha_r\to\infty} M_{ij} \right)_{(i,j)\in\{1\dots r\}}, \tag{5.52}$$

where the limit is taken by components

$$\lim_{\alpha_r\to\infty} M_{ir} = \lim_{\alpha_r\to\infty} \mathrm{e}^{\bar{\alpha}_i\alpha_r - \frac{1}{2}(|\alpha_r|^2+|\alpha_i|^2)} = 0 \tag{5.53}$$

$$\lim_{\alpha_r\to\infty} M_{rj} = 0 \tag{5.54}$$

$$\lim_{\alpha_r\to\infty} M_{rr} = 1. \tag{5.55}$$

Hence we get the matrix

$$\lim_{\alpha_r\to\infty} M = \left(\begin{array}{ccc|c} & & & 0 \\ & M^{(r-1)} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) \tag{5.56}$$

and for the determinant

$$\det \lim_{\alpha_r\to\infty} M = \det M^{(r-1)}, \tag{5.57}$$

where $M^{(r-1)}$ denotes a solution of lower rank $r-1$. Then the Kähler potential is

$$\lim_{\alpha_r\to\infty} K = K^{(r-1)} + 8\pi\theta\,\bar{\alpha}_r\alpha_r \tag{5.58}$$

and we obtain the metric

$$\lim_{\alpha_r\to\infty}\left(h_{\bar{i}j}\right)_{(i,j)\in\{1\ldots r\}} = \left(\begin{array}{ccc|c} & & & 0 \\ & \left(h_{\bar{i}j}\right)_{(i,j)\in\{1,\ldots,(r-1)\}} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 8\pi\theta \end{array}\right), \tag{5.59}$$

which is equal to the moduli space metric for a solution of lower rank $(r-1)$. So one can say that solitons, which are far away like our r-th soliton do not contribute to the moduli space metric of the remaining $(r-1)$ solitons.
Now we discuss the other case, where all solitons go to one point α_r, choosen without loss of generality $(\alpha_i \to \alpha_r, \forall i \in \{1\ldots r\})$. [1] Again we calculate the limit of the matrix M by components

$$\lim_{\alpha_i,\alpha_j\to\alpha_r} M_{ij} = \lim_{\alpha_i,\alpha_j\to\alpha_r} \mathrm{e}^{\bar{\alpha}_i\alpha_j-\frac{1}{2}(|\alpha_j|^2+|\alpha_i|^2)} = 1. \tag{5.60}$$

Since all entries have the value 1, the determinant of M vanishes and we get the Kähler potential

$$\lim_{\alpha_i\to\alpha_r,\forall i} K = 8\pi\theta r\bar{\alpha}_r\alpha_r + K' \tag{5.61}$$

where the last term is a singular constant, which does not contribute to the metric

$$\lim_{\alpha_i\to\alpha_r,\forall i}\left(h_{\bar{i}j}\right)_{(i,j)\in\{1\ldots r\}} = 8\pi\theta r\,\mathbb{1}. \tag{5.62}$$

This metric represents asymptotically the moduli space of a single soliton solution with "mass" r, where no scattering is possible. So it is not surprising that this metric is flat.
In next chapter considering degenerated solutions we are able to compare solutions of this asymptotic case with totally degenerated solutions, where all solitons are sitting at the point α_r on top of each other.
But before that we will calculate the moduli space metric of a simple example and analyse scattering of two solitons.

5.5. $U(1)$ solutions of rank 1 and rank 2

The simplest case of a rank 1 solution is rather trivial. There exists only one moduli parameter α and the matrix is just a number $\langle T|T\rangle = 1$. This implies the simple Kähler potential (5.32) $K = 8\pi\theta\,\bar{\alpha}\alpha$, which leads to a flat metric and we get the Lagrangian (5.33) of a free particle

$$L = 8\pi\theta\,\dot{\bar{\alpha}}\dot{\alpha}. \tag{5.63}$$

Thus we want to discuss the next simple case and compute the moduli space metric $h_{\bar{i}j}$ of a rank 2 solution. The generic rank 2 solution has two parameters α and β representing positions of solitons in moduli space

$$|T\rangle = (|\alpha\rangle, |\beta\rangle). \tag{5.64}$$

[1] Intermediate cases where only some of the α_i merge are found in the appendix of [15].

Inserting this into P (3.21) and that into Φ (3.2) we get the static solution of rank 2

$$\Phi = \mathbb{1} - \frac{2}{1-|\sigma|^2}\,(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| - \sigma|\alpha\rangle\langle\beta| - \bar{\sigma}|\beta\rangle\langle\alpha|), \tag{5.65}$$

where σ defined by $\sigma := \langle\alpha|\beta\rangle$ becomes with (C.10)

$$\sigma = \mathrm{e}^{-\frac{1}{2}\alpha\bar{\alpha}-\frac{1}{2}\beta\bar{\beta}+\bar{\alpha}\beta}, \tag{5.66}$$

which implies $|\sigma|^2 = \mathrm{e}^{-\alpha\bar{\alpha}-\beta\bar{\beta}+\bar{\alpha}\beta+\bar{\beta}\alpha} = \mathrm{e}^{-|\alpha-\beta|^2}$.
Recall that in the adiabatic approach the parameters are time-dependent. In order to get the moduli space dynamics we have reinserted the time-dependent $\Phi(t)$ into the action. We have seen that there is no contribution from the WZW-like term to the dynamics meaning no magnetic forcing in general. We have calulated an expression for the moduli space metric $h_{\bar{i}j}$ (5.31) and thus for the Lagrangian L (5.33) in the generic case.
To determine the dynamics in the example of a generic rank 2 solution we need not reinsert our special solution (5.65) into the action again, but we can use our formula (5.33) for the Lagrangian

$$L = h_{\bar{i}j}\dot{\bar{\alpha}}_i\dot{\alpha}_j = 8\pi\theta\sum_{i,j}^{r}\dot{\bar{\alpha}}_i\dot{\alpha}_j\,(\delta_{ij} + \partial_{\bar{i}}\partial_j \ln\det M)\,. \tag{5.67}$$

In this example (5.64) we find for the matrix M

$$M = \langle T|T\rangle = \begin{pmatrix} 1 & \sigma \\ \bar{\sigma} & 1 \end{pmatrix}, \tag{5.68}$$

and calculate the metric elements (5.31)

$$\begin{aligned}
h_{\bar{\alpha}\alpha} &= 8\pi\theta\,(1 + \frac{\partial}{\partial\bar{\alpha}}\frac{\partial}{\partial\alpha}\ln(1-\bar{\sigma}\sigma)) = 8\pi\theta\,(1+\frac{\partial}{\partial\bar{\alpha}}\,\frac{(\bar{\alpha}-\bar{\beta})\bar{\sigma}\sigma}{1-\bar{\sigma}\sigma}) \\
&= \frac{8\pi\theta}{(1-\bar{\sigma}\sigma)^2}\left(1-\bar{\sigma}\sigma(1+|\alpha-\beta|^2)\right), && (5.69)\\
h_{\bar{\beta}\beta} &= 8\pi\theta\,(1 + \frac{\partial}{\partial\bar{\beta}}\frac{\partial}{\partial\beta}\ln(1-\bar{\sigma}\sigma)) \\
&= \frac{8\pi\theta}{(1-\bar{\sigma}\sigma)^2}\left(1-\bar{\sigma}\sigma(1+|\alpha-\beta|^2)\right), && (5.70)\\
h_{\bar{\alpha}\beta} &= 8\pi\theta\,(\frac{\partial}{\partial\bar{\alpha}}\frac{\partial}{\partial\beta}\ln(1-\bar{\sigma}\sigma)) = 8\pi\theta\,(\frac{\partial}{\partial\bar{\alpha}}\,\frac{-(\bar{\alpha}-\bar{\beta})\bar{\sigma}\sigma}{1-\bar{\sigma}\sigma}) \\
&= \frac{8\pi\theta}{(1-\bar{\sigma}\sigma)^2}\left(\bar{\sigma}\sigma\,(-1+\bar{\sigma}\sigma+|\alpha-\beta|^2)\right), && (5.71)\\
h_{\bar{\beta}\alpha} &= 8\pi\theta\,(\frac{\partial}{\partial\bar{\beta}}\frac{\partial}{\partial\alpha}\ln(1-\bar{\sigma}\sigma)) \\
&= \frac{8\pi\theta}{(1-\bar{\sigma}\sigma)^2}\left(\bar{\sigma}\sigma\,(-1+\bar{\sigma}\sigma+|\alpha-\beta|^2)\right). && (5.72)
\end{aligned}$$

Inserting these results into the Lagrangian (5.67) we obtain

$$\begin{aligned}
L = \frac{8\pi\theta}{1-|\sigma|^2}\Big(&(\dot{\bar{\alpha}}\dot{\alpha}+\dot{\bar{\beta}}\dot{\beta})(1-\bar{\sigma}\sigma(1+|\alpha-\beta|^2)) \\
&(\dot{\bar{\alpha}}\dot{\beta}+\dot{\bar{\beta}}\dot{\alpha})(\bar{\sigma}\sigma\,(-1+\bar{\sigma}\sigma+|\alpha-\beta|^2))\Big),
\end{aligned} \tag{5.73}$$

which can be rewritten in center of mass s and relative coordinate w

$$s := \frac{1}{2}(\alpha + \beta) \qquad \text{and} \qquad w := \alpha - \beta \tag{5.74}$$

as

$$L = 4\pi\theta\, (4\dot{\bar{s}}\dot{s} + \dot{\bar{w}}\dot{w}\, \Omega(\bar{w}w)), \tag{5.75}$$

with

$$\Omega := \frac{1 - \mathrm{e}^{-2\bar{w}w} - 2\bar{w}w\, \mathrm{e}^{-\bar{w}w}}{(1 - \mathrm{e}^{-\bar{w}w})^2}. \tag{5.76}$$

The center of mass motion is decoupled so that we discuss in the following the relative motion seperately.
It is determined by the metric

$$h_{\mathrm{rel}} = h_{\bar{w}w} = 4\pi\theta\, \Omega(\bar{w}w), \tag{5.77}$$

which leads to the Kähler potential

$$K_{\mathrm{rel}} = 4\pi\theta\, (\bar{w}w + 2\ln(1 - \mathrm{e}^{-\bar{w}w})). \tag{5.78}$$

The metric depends only on the absolute value of the complex coordinate w. Thus we rewrite w in polar coordinates $w = r\mathrm{e}^{i\varphi}$ and the Lagrangian describing the relative motion L_{rel} becomes

$$L_{\mathrm{rel}} = 4\pi\theta\, \dot{\bar{w}}\dot{w}\, \Omega(\bar{w}w) = 4\pi\theta\, (\dot{r}^2 + r^2\dot{\varphi}^2)\, \Omega(r), \tag{5.79}$$

where $\Omega(r)$ simplifies to

$$\Omega(r) = \frac{\sinh(r^2) - r^2}{\cosh(r^2) - 1}. \tag{5.80}$$

The factor (5.80) tends to 1 as $r \gg 1$ and the metric becomes flat as expected from the last section. It is monotonously increasing with r and goes to 0 as $r \to 0$.

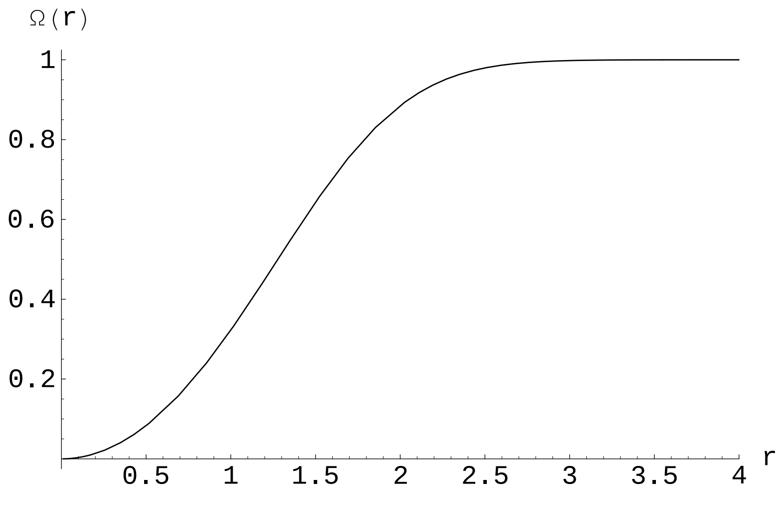

Figure 5.1.: Ω as function of relative distance r

To investigate its behaviour near 0 it is convenient to expand the hyperbolic functions in terms of r^2, which gives

$$\Omega(r) \approx \frac{1}{3} r^2 - \frac{1}{90} r^6 + \mathcal{O}(r^{10}). \tag{5.81}$$

Introducing a new coordinate $\rho = r^2$ and writing the metric as a two form we find for small r

$$h = \mathrm{d}s^2 \approx r^2 \left(\mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2\right) = \frac{1}{4} \left(\mathrm{d}\rho^2 + 4\rho^2 \mathrm{d}\varphi^2\right) = \frac{1}{4} \left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\tilde{\varphi}^2\right), \tag{5.82}$$

with $\tilde{\varphi} = 2\varphi$.
Since the angle $\tilde{\varphi}$ runs from 0 to 4π there is a conical singularity at the origin of the moduli space. Actually the moduli space is not the complex plane but $\mathcal{M}_2 = \mathbb{C}/S_2$, because we identify (α, β) with (β, α). Hence, $\varphi \in [0, \pi]$ and $\tilde{\varphi} \in [0, 2\pi]$ and the singulariy disappears in $(\rho, \tilde{\varphi})$. Further discussion about this metric is found in [15, 19]. Such a metric and a Kähler potential of our form occurs also in statistical physics [26, 27] (see also (E.6)).
In order to interpret the dynamics described by this Lagrangian we consult the mechanical analogue of a two particle scattering problem.

5.6. Two soliton scattering

As discussed above the generic rank 2 case describes two solitons with complex relative coordinate $w = r\mathrm{e}^{i\varphi}$ and can therefore be regarded as a scattering problem in moduli space. Recall that the dynamic obeys the Lagrangian L_{rel} (5.79).
The scattering problem can be discussed as an analogue to the Kepler problem. Thus we are looking for conserved quantities at first.
Due to the rotational invariance of L_{rel} we find from the equations of motion in polar coordinates the conserved angular momentum

$$l = \Omega(r)\, r^2 \dot{\varphi}. \tag{5.83}$$

Additionally we calculate the equations of motion for the coordinates w and $\bar{w}$. After a short calculation and introducing $u = \bar{w}w$ they get the form

$$\Omega\, \ddot{\bar{w}} + \frac{\partial \Omega}{\partial u}\, \dot{\bar{w}}\dot{\bar{w}} = 0 \tag{5.84}$$

$$\Omega\, \ddot{w} + \frac{\partial \Omega}{\partial u}\, \dot{w}\dot{w} = 0, \tag{5.85}$$

Calculating $\dot{w}(5.84) + \dot{\bar{w}}(5.85)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\Omega(\bar{w}, w)\, \dot{\bar{w}}\dot{w}\right) = 0, \tag{5.86}$$

and we gain the conserved kinetic-energy-like quantity

$$e = \Omega(r)\, (\dot{r}^2 + r^2 \dot{\varphi}^2). \tag{5.87}$$

Now we are able to determine the maximal deflection angle taken at minimal distance $r_{\min}$ called the scattering angle Θ.
Combining the expressions for the conserved quantities e (5.87) and l (5.83) we get

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{1}{r} \left(\sqrt{\tfrac{1}{b^2} r^2\, \Omega(r) - 1} \right)^{-1}. \tag{5.88}$$

With the constraint for minimal distance $\dot{r} = 0$ it follows

$$r_{min}^2 \, \Omega(r_{\text{min}}) = \frac{l^2}{e} =: b^2, \tag{5.89}$$

where b is called the impact parameter.
This implies the scattering angle as a function of the impact parameter [2]

$$\Theta(b) = \pi - 2 \int_{r_{\text{min}}}^{\infty} \mathrm{d}r \, \left(r \sqrt{\tfrac{1}{b^2} r^2 \, \Omega(r) - 1} \right)^{-1} \tag{5.90}$$

For large values of b we have no scattering $\Theta \approx 0$, because $b \gg 1$ implies $r_{\text{min}} \gg 1$ so $\Omega(r) \approx 1$ as in flat space. For small values of b we have a large scattering angle. From $b \ll 1$ follows $r_{\text{min}} \ll 1$, which implies $\Omega(r) \approx \frac{1}{3} r^2$ as discussed in last section. The scattering angle in this asymptotic case is

$$\Theta(b) \approx \pi - 2 \int_{r_{\text{min}}}^{\infty} \mathrm{d}r \frac{1}{r \sqrt{\frac{1}{b^2} r^4 - 1}} \qquad \text{for} \qquad b \approx r_{\text{min}}^2. \tag{5.91}$$

Integrating yields

$$\Theta \approx \pi + \arctan \frac{1}{\sqrt{\frac{r^4}{r_{\text{min}}^4} - 1}} \Bigg|_{r_{\text{min}}}^{\infty} \approx \frac{\pi}{2}. \tag{5.92}$$

This result of maximal scattering correspond to the fact that the moduli space has a conical sigularity at $r = 0$ [19].
Numerical evaluation of the scattering angle yields the following diagramm

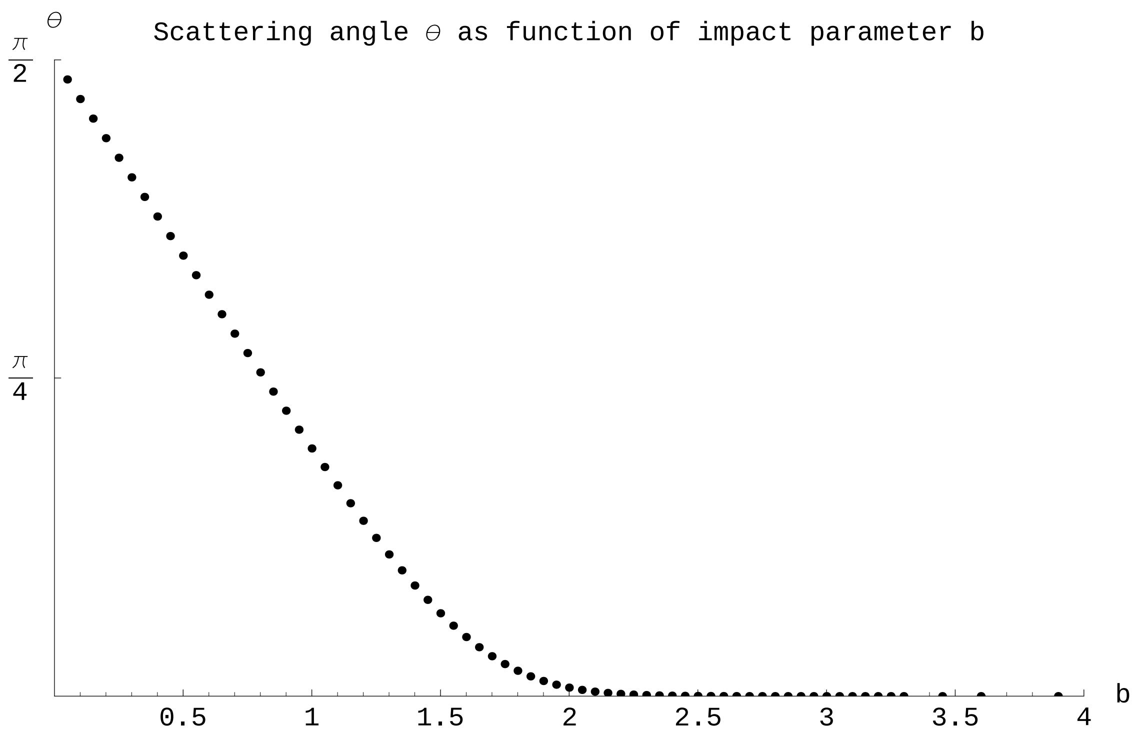

Figure 5.2.: Scattering angle Θ as function of impact parameter b.

[2]The factor two is necessary, because the soliton comes in from $\theta = 0$

5.7. Comparison to exact solutions

In this simple rank two case we are able to compare our results with the exact time-dependent solution of the 2+1-dimensional Ward model. There is no scattering in the exact solution and the motion is equivalent to motion in flat space [17]. So the scattering angle is a good measure for the quality of our approximation.
For comparison we must set same initial values. Thus we set our initial point with polar coordinates $r(t_0) =: r_0$ and $\varphi(t_0) =: \varphi_0$ in an area where the moduli space is flat ($r_0 \gg 1$) so that $\Omega(r_0) \approx 1$.
We take the initial velocity from the exact case [17]

$$\vec{v}_0 \equiv (v_{0x}, v_{0y}) = -\left(\frac{\cos\varphi}{\cosh\eta}, \frac{\sinh\eta}{\cosh\eta}\right) \quad \text{for} \quad \mu =: e^{\eta - \mathrm{i}\varphi} \tag{5.93}$$

and $\mu = -\mathrm{i}$ describes static solutions.
From (5.90) it is clear that the scattering angle depends on e and l only in the combination

$$b = \frac{l}{\sqrt{e}} = \frac{|\vec{r}_0 \times \vec{v}_0|}{|\vec{v}_0|}. \tag{5.94}$$

So we can get any trajectory (any value of Θ) for arbitrarily small choices of $\vec{v}_0$ (provided we adjust $\vec{r}_0$ so that the wanted b-value obtain). Especially we have scattering for small initial velocities as well.
In comparison with the exact known solutions [17, 28] of the modified $U(1)$ sigma model, where no scattering happens [17, 28], our approximation is not very good. Thus it should be discussed, which solutions of which models are approximated by the adiabatic approach really. This discussion will be resumed in the conclusion.

6. General $U(1)$ case

6.1. Completely degenerate case

Unlike last chapter we allow degeneracies now. Instead of (3.22) we consider a solution $|T\rangle$ with r solitons on top of each other at position α. From (3.24) we get

$$|T\rangle = (|\alpha\rangle, a^\dagger|\alpha\rangle, \ldots, \frac{{a^\dagger}^{r-1}}{(r-1)!}|\alpha\rangle). \tag{6.1}$$

In this case we do not get the same expressions for the moduli space metric and the Kähler potential as derived in generic case, because equation (5.11) is not valid anymore.
Now it is the aim to investigate the moduli space metric and Kähler potential for this completely degenerate case and compare them with the asymptotic case described in section 5.4, where all solitons of a generic solution cross at the same position.
We start with the Lagrangian for finite solutions (5.2) again

$$L = 8\pi\theta \ \mathrm{tr}_r \left(\partial_t\langle T|\, \partial_t|T\rangle M^{-1} - \partial_t\langle T|\, |T\rangle M^{-1}\langle T|\, \partial_t|T\rangle M^{-1}\right). \tag{6.2}$$

Inserting the time derivative of (6.1)

$$\partial_t|T\rangle = a^\dagger|T\rangle \underbrace{\begin{pmatrix} \dot\alpha & & \\ & \ddots & \\ & & \dot\alpha \end{pmatrix}}_{=:\dot A} + |T\rangle \underbrace{\begin{pmatrix} -\frac{1}{2}|\dot\alpha|^2 & & \\ & \ddots & \\ & & -\frac{1}{2}|\dot\alpha|^2 \end{pmatrix}}_{=:\dot B} \tag{6.3}$$

we obtain after an analogous calculation as in section 5.1

$$L = 8\pi\theta \ \mathrm{tr}_r \left(\dot{\bar A}(\langle T|aa^\dagger|T\rangle - \langle T|a|T\rangle\, M^{-1}\, \langle T|a^\dagger|T\rangle)\dot A M^{-1}\right). \tag{6.4}$$

But instead of equation (5.11) we find in the degenerate case

$$\begin{aligned} a|T\rangle &= |T\rangle\, \alpha\mathbb{1} + (0, |\alpha\rangle, a^\dagger|\alpha\rangle, \ldots, \frac{{a^\dagger}^{r-2}}{(r-2)!}|\alpha\rangle)) \\ &= |T\rangle\, \alpha\mathbb{1} + |T\rangle \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}}_{:=Z} \end{aligned} \tag{6.5}$$

Inserting in (6.4) yields

$$L = 8\pi\theta\, r\, \dot{\bar\alpha}\dot\alpha, \tag{6.6}$$

where we read off the flat metric $h_{\bar i j} = 8\pi\theta r \mathbb{1}$ and the Kähler potential $K = 8\pi\theta\, r\, \bar\alpha\alpha$.
These results correspond to the asymptotic case in 5.4. The metrics agree and the Kähler potentials differ in the infinite constant appearing in 5.4.

6.2. Calculation of the general metric

In this section we investigate general solutions $|T\rangle$ as described in (3.24)

$$|T\rangle = (|T^{(1)}\rangle, \ldots, |T^{(q)}\rangle) \qquad \text{with} \qquad |T^{(k)}\rangle = (|\alpha_k\rangle, a^\dagger|\alpha_k\rangle, \ldots, \frac{a^{\dagger r_k - 1}}{(r_k-1)!}|\alpha_k\rangle). \tag{6.7}$$

Combining the time derivatives (5.6) and (6.3) we get the time derivative of (6.7)

$$\partial_t |T\rangle = a^\dagger |T\rangle\, \dot{A} + |T\rangle\, \dot{B} \tag{6.8}$$

wherein

$$\dot{A} := \mathrm{diag}\{\underbrace{\dot{\alpha}_1, \ldots, \dot{\alpha}_1}_{r_1-\text{times}}, \ldots, \underbrace{\dot{\alpha}_q, \ldots, \dot{\alpha}_q}_{r_q-\text{times}}\} \tag{6.9}$$

$$\dot{B} := \mathrm{diag}\{\underbrace{-\tfrac{1}{2}|\dot{\alpha}_1|^2, \ldots, -\tfrac{1}{2}|\dot{\alpha}_1|^2}_{r_1-\text{times}}, \ldots, \underbrace{-\tfrac{1}{2}|\dot{\alpha}_q|^2, \ldots, -\tfrac{1}{2}|\dot{\alpha}_q|^2}_{r_q-\text{times}}\}. \tag{6.10}$$

Since (6.8) is of form like (5.6) and (6.3) we start the calculation of the moduli space metric with the same Lagrangian

$$L = 8\pi\theta\ \mathrm{tr}_r \left(\dot{\bar{A}} (\langle T|aa^\dagger|T\rangle - \langle T|a|T\rangle\, M^{-1}\, \langle T|a^\dagger|T\rangle) \dot{A} M^{-1} \right). \tag{6.11}$$

We calculate how a acts on $|T\rangle$ in the general case by combining (5.11) and (6.5)

$$a|T\rangle = |T\rangle(A+Z) \tag{6.12}$$

with

$$A := \mathrm{diag}\{\underbrace{\alpha_1, \ldots, \alpha_1}_{r_1-\text{times}}, \ldots, \underbrace{\alpha_q, \ldots, \alpha_q}_{r_q-\text{times}}\} \tag{6.13}$$

and

$$Z = \begin{pmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(q)} \end{pmatrix} \qquad \text{for} \qquad Z^{(s)} = \underbrace{\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}}_{r_s-\text{times}} \tag{6.14}$$

Inserting in (6.11) we gain

$$L = 8\pi\theta\ \mathrm{tr}_r \left(\dot{\bar{A}} \,(M + (\bar{A} + Z^\dagger) M (A+Z) - M(A+Z)M^{-1}(\bar{A}+Z^\dagger)M)\, \dot{A} M^{-1} \right) \tag{6.15}$$

The metric elements $h_{\bar{i}j}$ are nothing but the coefficients of $\dot{\bar{\alpha}}_i \dot{\alpha}_j$. We get it from the Lagrangian by differentiation with respect to $\frac{\partial}{\partial \dot{\bar{\alpha}}_i}$ and $\frac{\partial}{\partial \dot{\alpha}_j}$

$$h_{\bar{i}j} = 8\pi\theta\ \mathrm{tr}_r \left((\frac{\partial}{\partial \dot{\bar{\alpha}}_i} \dot{\bar{A}})(M + \ldots)(\frac{\partial}{\partial \dot{\alpha}_j} \dot{A}) M^{-1} \right). \tag{6.16}$$

We replace

$$\frac{\partial}{\partial \dot{\bar{\alpha}}_i} \dot{\bar{A}} = \frac{\partial}{\partial \bar{\alpha}_i} \bar{A} \qquad \text{and} \qquad \frac{\partial}{\partial \dot{\alpha}_j} \dot{A} = \frac{\partial}{\partial \alpha_j} A, \tag{6.17}$$

to obtain the metric

$$h_{\bar{i}j} = 8\pi\theta\ \mathrm{tr}_r \left((\partial_{\bar{i}} \bar{A})\, (M + (\bar{A} + Z^\dagger) M (A+Z) - M(A+Z)M^{-1}(\bar{A}+Z^\dagger)M)\, (\partial_j A) M^{-1} \right). \tag{6.18}$$

6.3. Determination of the general Kähler potential

We guess that there is a Kähler potential of the form $K = 8\pi\theta(\mathrm{tr}_r\,\bar{A}A + \ln\det M)$ satisfying $h_{\bar{i}j} = \partial_{\bar{i}}\partial_j K$ again. In order to prove this guess we make the Ansatz

$$M = \mathrm{e}^{-\frac{1}{2}\bar{A}A} R \mathrm{e}^{-\frac{1}{2}\bar{A}A} \qquad \text{i.e.} \qquad R := \mathrm{e}^{\frac{1}{2}\bar{A}A} M \mathrm{e}^{\frac{1}{2}\bar{A}A} \tag{6.19}$$

and write metric and Kähler potential in terms of R instead of M. Inserting (6.19) in (6.16) we need

$$[\mathrm{e}^{\frac{1}{2}\bar{A}A}, Z] = 0, \tag{6.20}$$

to show that $h_{\bar{i}j}$ is invariant under the transformation $|T\rangle \to |T\rangle \mathrm{e}^{\frac{1}{2}\bar{A}A}$, hence

$$h_{\bar{i}j}(M) = h_{\bar{i}j}(R). \tag{6.21}$$

The Kähler potential becomes in terms of R

$$K = 8\pi\theta(\mathrm{tr}_r\,\bar{A}A + \ln\det \mathrm{e}^{-\frac{1}{2}\bar{A}A} R \mathrm{e}^{-\frac{1}{2}\bar{A}A}) = 8\pi\theta\ \ln\det R. \tag{6.22}$$

Hoping to find the correct expression for the metric (6.16) we calculate

$$\partial_{\bar{i}}\partial_j K = 8\pi\theta\ \partial_{\bar{i}}\partial_j\ \ln\det R \tag{6.23}$$

as in (5.22) and write it in matrix notation

$$\partial_{\bar{i}}\partial_j K = 8\pi\theta\ \mathrm{tr}_r\left(R^{-1}\partial_{\bar{i}}\partial_j R - R^{-1}\partial_j R\, R^{-1}\partial_{\bar{i}} R\right). \tag{6.24}$$

In the general case

$$\langle T|T\rangle_{kl} = \left(\langle\alpha_k|\,\frac{a^{m-1}}{(m-1)!}\frac{a^{\dagger\,n-1}}{(r-1)!}\,|\alpha_l\rangle\right)_{n\in\{1,\dots,r_l\},m\in\{1,\dots,r_k\}}, \tag{6.25}$$

where the indices $k,l \in \{1,\dots,q\}$ This matrix element describes the overlap between the m-th excitation of the k-th lump with the n-th excitation of the l-th lump. Using the shift operator (C.3) we find

$$\begin{aligned}\langle T|T\rangle_{kl}^{mn} &= \langle 0|D^\dagger(\alpha_k)\,\frac{a^{m-1}}{(m-1)!}\frac{a^{\dagger\,n-1}}{(r-1)!}\,D(\alpha_l)|0\rangle \\ &= \langle 0|\mathrm{e}^{\bar{\alpha}_k a}\mathrm{e}^{-\frac{1}{2}|\alpha_k|^2}\,\frac{a^{m-1}}{(m-1)!}\frac{a^{\dagger\,n-1}}{(r-1)!}\,\mathrm{e}^{-\frac{1}{2}|\alpha_l|^2}\mathrm{e}^{\alpha_l a^\dagger}|0\rangle\end{aligned} \tag{6.26}$$

Hence we get for R in matrix notation

$$R = \left(\langle 0|\mathrm{e}^{\bar{\alpha}_k a}\,\frac{a^{m-1}}{(m-1)!}\frac{a^{\dagger\,n-1}}{(r-1)!}\,\mathrm{e}^{\alpha_l a^\dagger}|0\rangle\right) \tag{6.27}$$

and the derivative is

$$\begin{aligned}\partial_j R &= \partial_j\left(\langle 0|\mathrm{e}^{\bar{\alpha}_k a}\,\frac{a^{m-1}}{(m-1)!}\frac{a^{\dagger\,n-1}}{(r-1)!}\,\mathrm{e}^{\alpha_l a^\dagger}|0\rangle\right) \\ &= \left(\langle 0|\mathrm{e}^{\bar{\alpha}_k a}\,\frac{a^{m-1}}{(m-1)!}a^\dagger\frac{a^{\dagger\,n-1}}{(r-1)!}\,\mathrm{e}^{\alpha_l a^\dagger}|0\rangle\partial_j\alpha_l\right) \\ &= \mathrm{e}^{\frac{1}{2}\bar{A}A}\,\langle T|a^\dagger|T\rangle\,\mathrm{e}^{\frac{1}{2}\bar{A}A}\,\partial_j A \\ &= (\bar{A} + Z^\dagger)R\ (\partial_j A)\,.\end{aligned} \tag{6.28}$$

In an analogous way we gain

$$\partial_{\bar{i}} R = (\partial_{\bar{i}} A)\; R\,(A+Z)\,, \tag{6.29}$$

hence the second derivative becomes

$$\partial_{\bar{i}}\partial_j R = \Big((\partial_{\bar{i}} A) R + (\bar{A}+Z^{\dagger})(\partial_{\bar{i}} A) R (A+Z)\Big)\,(\partial_j A)\,. \tag{6.30}$$

Inserting (6.28),(6.29) and (6.30) in (6.24) we obtain the required metric

$$\begin{aligned}\partial_{\bar{i}}\partial_j K &= 8\pi\theta\ \mathrm{tr}_r \Big((\partial_{\bar{i}}\bar{A})\,(M + (\bar{A}+Z^{\dagger})R(A+Z) - R(A+Z)R^{-1}(\bar{A}+Z^{\dagger})R)\,(\partial_j A) R^{-1}\Big)\\ &= h_{\bar{i}j}.\end{aligned} \tag{6.31}$$

So it is proven that the Kähler potential in the general case is

$$K = 8\pi\theta\ \ln\det R = 8\pi\theta(\mathrm{tr}_r\, \bar{A}A + \ln\det M). \tag{6.32}$$

6.4. Scattering on a degenerated soliton

We discuss two soliton scattering similar to section 5.6 with the difference that one soliton is degenerated now.
At first we consider the simplest case, where a rank 1 soliton at the position β is scattered on a degenerated rank 2 soliton sitting at α. This process is described by a general rank 3 solution of the form (6.7)

$$|T\rangle = (|\alpha\rangle, a^{\dagger}|\alpha\rangle, |\beta\rangle). \tag{6.33}$$

We calculate the moduli space metric with the formulas derived in 6.2 and 6.3. Therefore we need the matrix M

$$M = \langle T|T\rangle = \begin{pmatrix} 1 & \bar{\alpha} & \sigma \\ \alpha & 1+\bar{\alpha}\alpha & \beta\sigma \\ \bar{\sigma} & \bar{\beta}\bar{\sigma} & 1 \end{pmatrix} \tag{6.34}$$

with $\sigma := \langle\alpha|\beta\rangle = \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\bar{\alpha}\beta}$. This implies

$$\det M = 1 - \bar{\sigma}\sigma + \bar{\sigma}\sigma(\bar{\alpha}\beta + \bar{\beta}\alpha - \bar{\alpha}\alpha - \bar{\beta}\beta). \tag{6.35}$$

Introducing center of mass s and relative coordinate w

$$s = \frac{1}{3}(2\alpha+\beta) \qquad \text{and} \qquad w = \alpha - \beta \tag{6.36}$$

(6.35) reduces to

$$\det M = 1 - (\bar{w}w+1)\,\mathrm{e}^{-\bar{w}w}. \tag{6.37}$$

Inserting in (6.32) yields the Kähler potential

$$K = 8\pi\theta\left(3\bar{s}s + \frac{2}{3}\bar{w}w + \ln\big(1-(\bar{w}w+1)\,\mathrm{e}^{-\bar{w}w}\big)\right), \tag{6.38}$$

which is in the $(r \to 0)$-expansion

$$K \approx 8\pi\theta \left(3\bar{s}s - \ln 2 + \ln(\bar{w}w)^2 + \frac{1}{36}(\bar{w}w)^2) + \frac{1}{810}(\bar{w}w)^2)^3 + \mathcal{O}(\bar{w}w)^4)\right). \tag{6.39}$$

The metric becomes by differentiation

$$h_{\bar{s}s} = 3 \tag{6.40}$$

$$h_{\bar{w}w} = \frac{2}{3} - \frac{\bar{w}^3 w^3 \mathrm{e}^{-2\bar{w}w}}{(1-(\bar{w}w+1)\,\mathrm{e}^{-\bar{w}w})^2} + \frac{(2\bar{w}w - \bar{w}^2 w^2)\mathrm{e}^{-\bar{w}w}}{1-(\bar{w}w+1)\,\mathrm{e}^{-\bar{w}w}} \tag{6.41}$$

Thus the center of mass motion is completely decoupled again and the relevant part of the Lagrangian describing the relative motion is of the same form as in (5.79)

$$L_{\mathrm{rel}} = 8\pi\theta\,(\dot{\bar{w}}\dot{w}\Omega(\bar{w},w)) = 8\pi\theta\,(\dot{r}^2 + r^2\dot{\varphi}^2)\Omega(r), \tag{6.42}$$

where Ω is only dependent of the absolute value $\bar{w}w = r^2$

$$\Omega(r) = \frac{2}{3} - \frac{r^6 \mathrm{e}^{-2r^2}}{\left(1-(r^2+1)\,\mathrm{e}^{-r^2}\right)^2} + \frac{(2r^2 - r^4)\mathrm{e}^{-r^2}}{1-(r^2+1)\,\mathrm{e}^{-r^2}}. \tag{6.43}$$

Due to the absence of the $\bar{w}w$-term in (6.39) the conformal factor Ω is in the $(r \to 0)$-limit proportional to r^2 as (5.80) in the generic case

$$\Omega(r) \approx \frac{2}{9}r^2 + \frac{2}{90}r^4 + \mathcal{O}(r^6) \tag{6.44}$$

so that we find the same conical singularity.

We can treat the scattering as in the generic case and obtain after an analogous calculation the scattering angle

$$\Theta(b) = \pi - 2\int_{r_{\mathrm{min}}}^{\infty} \mathrm{d}r \,\left(r\sqrt{\tfrac{1}{b^2}r^2\,\Omega(r) - 1}\right)^{-1}, \tag{6.45}$$

with impact parameter $b = \frac{l}{\sqrt{e}}$ and r_{min} computed from $r_{min}^2\,\Omega(r_{min}) = b^2$.

Numerical evaluation yields the figure

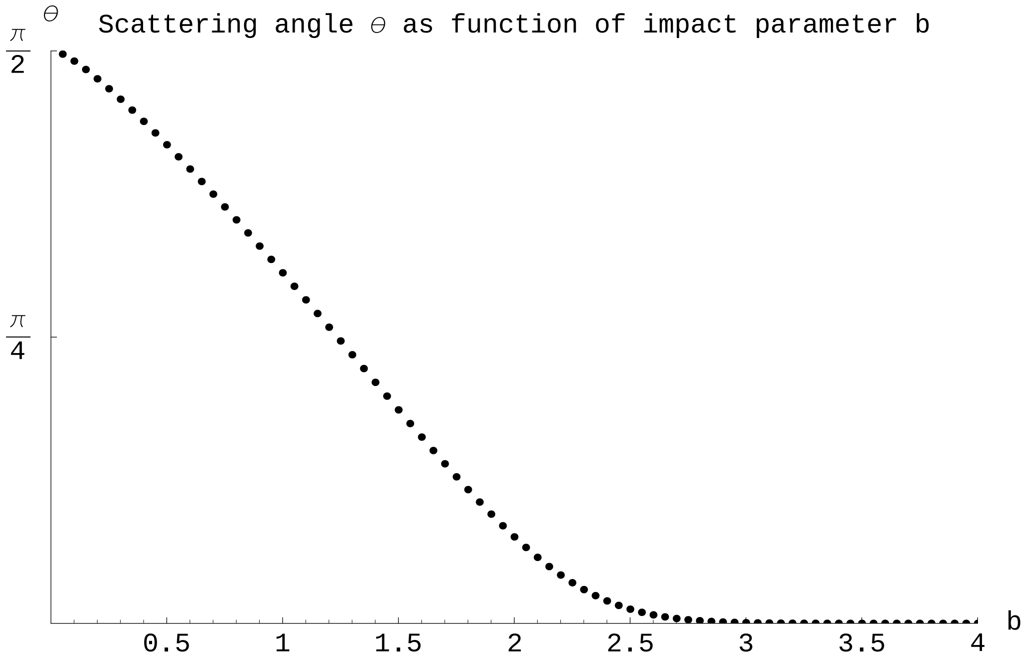

Figure 6.1.: Scattering angle Θ as function of impact parameter b,

where we see the same asymptotic behavior as in (5.90).
More complex cases, where a rank 1 soliton scatters with a higher rank one do not yield qualitatively new results. The asymptotic behavior is the same in every case.

7. Comparison with the noncommutative scalar field model

Since the results of the generic case are the same as in [19],[14] and [15] discussing a scalar field model, we want to establish a relationship between these models.
Therefore we describe first the noncommutative scalar field model as analysed in [19],[14] and [15]. Then we specialise with respect to the potential in order that we are able to embed the $U(1)$ sigma model.
We consider the action in star product formalism with the real field Φ

$$S = \int \mathrm{d}t\, \mathrm{d}\bar{z}\mathrm{d}z \left(\frac{1}{2}(\partial_t \Phi)^2 - \partial_{\bar{z}}\partial_z \Phi - m^2 V(\Phi)_\star \right). \tag{7.1}$$

The factor of m^2 is added, because the perturbation paramter appearing later should be dimensionless. The energy for static configurations is written in operators (hats omitted)

$$E[\Phi, \theta m^2] = 2\pi\ \mathrm{tr}_{\mathcal{H}} \left(\frac{1}{2}[a,\Phi][\Phi,a^\dagger] + \theta m^2 V(\Phi) \right), \tag{7.2}$$

where we assume that the potential $V(\Phi) = \sum_{k=0}^{\infty} a_k \Phi^k$ has

$$V'(0) = 0, \quad V(0) = 0 \qquad \text{and} \qquad V'(\lambda) = 0. \tag{7.3}$$

We can solve this problem in the $\theta \to \infty$ limit. Rescaling the energy yields

$$E_{-1} = \lim_{\theta m^2 \to \infty} \frac{1}{\theta m^2} E[\Phi, \theta m^2] = 2\pi\ \mathrm{tr}_{\mathcal{H}}\ V(\Phi). \tag{7.4}$$

Varying the energy we get the equations of motion

$$\frac{\partial V}{\partial \Phi} = \sum_{k=1}^{\infty} a_k\, k\, \Phi^{k-1} \stackrel{!}{=} 0 \tag{7.5}$$

One solution for this equation is found as

$$\Phi_0 = \lambda P, \tag{7.6}$$

where P is a projector of finite rank r. Doing the adiabatic approximation in this $\theta \to \infty$ limit the solitons should move slowly so that the potential energy is at its minimum and hence constant [19].
Thus the Lagrangian is proportional to $\dot{P}^2$ and we must get out a moduli space metric of same form as in our model.
If we study the more general case of finite θ we are not allowed to neglect the potential. So we consider static solutions Φ and their energy E of this finite θ model using perturbation theory. We make a perturbation expansion in $\frac{1}{\theta m^2}$

$$\Phi = \Phi_0 + \frac{1}{\theta m^2}\Phi_1 + O\left(\left(\frac{1}{\theta m^2} \right)^2 \right) \tag{7.7}$$

$$E = \theta m^2 E_{-1} + E_0 + \frac{1}{\theta m^2} E_1 + O\left(\left(\frac{1}{\theta m^2} \right)^2 \right) \tag{7.8}$$

Reinserting in (7.2) and using Taylor's theorem to evaluate

$$V(\Phi_0 + \frac{1}{\theta m^2}\Phi_1 + \dots) = V(\Phi_0) + \frac{1}{\theta m^2}\frac{V''(\Phi_0)}{2}\Phi_1^2 + \dots \tag{7.9}$$

yields up to second order

$$E = 2\pi \,\mathrm{tr}_{\mathcal{H}} \left(\theta m^2 \underbrace{V(\Phi_0)}_{E_{-1}} + \underbrace{\frac{1}{2}[a,\Phi_0][\Phi_0,a^\dagger]}_{E_0} + \frac{1}{\theta m^2} \underbrace{\left([a,\Phi_1][\Phi_0,a^\dagger] + \frac{V''(\Phi_0)}{2}\Phi_1^2 \right)}_{E_1} \right). \tag{7.10}$$

Recall $\Phi_0 = \lambda P$ with the projector P of finite rank r, which implies

$$E_0 = \pi\lambda^2 \,\mathrm{tr}_{\mathcal{H}}\, [a,P][P,a^\dagger]. \tag{7.11}$$

In order to get the equation of motion we consider the variation with respect to Φ_0

$$\frac{\partial E_0}{\partial \Phi_0} = [\Phi_0, E_0] = [P,[a^\dagger,[a,P]] \stackrel{!}{=} 0. \tag{7.12}$$

This equation is equivalent to (3.16) and is satisfied especially by the BPS solutions

$$(\mathbb{1} - P)aP = 0 \tag{7.13}$$

Using (7.13) we recover for the energy E_0 after a calculation as in (3.7) and (3.18)

$$E_0 = \pi\lambda^2 \,\mathrm{tr}_{\mathcal{H}}\ P. \tag{7.14}$$

To determine the next energy correction E_1 we must vary E_1 with respect to Φ_1

$$\frac{\partial E_1}{\partial \Phi_1} = 2\pi \,\mathrm{tr}_{\mathcal{H}} \Big([a^\dagger,[a,\Phi_0]] + \Phi_1 V''(\Phi_0) \Big) \stackrel{!}{=} 0. \tag{7.15}$$

It follows

$$\Phi_1 = -\lambda\, [a^\dagger,[a,\Phi_0]]\, \frac{1}{V''(\Phi_0)} \tag{7.16}$$

from which we could calculate E_2.
Now we want to compare these finite θ results with the results obtained in 3. The $U(1)$ sigma model corresponds to a scalar field model with an infinitely steep double well potential

$$V(\Phi) = \lim_{\beta\to\infty} \beta\,(\mathbb{1} - \Phi^2)^2 \tag{7.17}$$

representing the sigma model constraint $\Phi^2 = \mathbb{1}$ with a Lagrange-multiplier β. For the $U(1)$ model we must generalize to complex Φ. But since we restrict to a Grassmanian model a real Φ is sufficient. With respect to comparison the only modification owed to this extension is a factor of two in the energy functional.
We slightly vary the ansatz made above and write

$$V(\Phi) \geq 0 \quad , \quad V(\lambda_0) = 0 \quad \text{and} \quad V'(\Phi) = v\prod_i (\Phi - \lambda_i). \tag{7.18}$$

Instead of (7.6) in the $\theta \to \infty$ limit we obtain the general solution (up to unitary transformations)

$$\Phi_0 = \sum_i \lambda_i P_i, \tag{7.19}$$

where P_i are orthogonal projectors with

$$P_i P_j = \delta_{ij} P_i \qquad \text{and} \qquad \sum_i P_i = \mathbb{1}. \tag{7.20}$$

For our special potential (7.17) the total energy (7.2) is

$$E = 2\pi \ \mathrm{tr}_{\mathcal{H}} \left([a, \Phi][\Phi, a^\dagger] + \theta m^2 \lim_{\beta \to \infty} \beta \, (1 - \Phi^2)^2 \right) + O\left(\left(\frac{1}{\theta m^2} \right)^2 \right), \tag{7.21}$$

where the factor of two mentioned above is added. This potential (7.17) has two minima at $\lambda_{(0)} = -1$ and $\lambda_{(2)} = 1$ and one maximum at $\lambda_{(1)} = 0$. We associate with the maximum at zero the zero projector $P_{(1)} = 0$, hence the solution Φ_0 is a linear combination of two orthogonal projectors $P_{(0)} = P$ and $P_{(2)} = \mathbb{1} - P$

$$\Phi_0 = \sum_i \lambda_{(i)} P_{(i)} = \mathbb{1} - 2P. \tag{7.22}$$

Reinserting this solution we are able to calculate E_{-1} and E_0

$$E_{-1} = 2\pi \, \mathrm{tr}_{\mathcal{H}} \, P \, V(\lambda) = 0 \tag{7.23}$$

$$E_0 = 8\pi \, \mathrm{tr}_{\mathcal{H}} \, [a, P][P, a^\dagger] = 8\pi \, \mathrm{tr}_{\mathcal{H}} \ P. \tag{7.24}$$

With (7.22) the second derivative of the potential becomes

$$V''(\Phi_0) = \lim_{\beta \to \infty} \beta(12\Phi_0^2 - 4) = \lim_{\beta \to \infty} 8\beta \tag{7.25}$$

and hence

$$\Phi_1 = \lim_{\beta \to \infty} \frac{1}{2\beta} \, [a^\dagger, [a, P]] = 0. \tag{7.26}$$

This implies that the first energy correction also vanishes

$$E_1 = -\frac{2\pi}{\beta} \, \mathrm{tr}_{\mathcal{H}} \, \Big([a^\dagger, [a, P]] \Big)^2 = 0 \tag{7.27}$$

and all higher order corrections of Φ and E vanish as well.
So for this special potential the perturbative ansatz yields exact static solutions and an exact expression for their energy

$$\Phi = \mathbb{1} - 2P \qquad \text{and} \qquad E = 8\pi \, \mathrm{tr}_{\mathcal{H}} \ P \tag{7.28}$$

which are the same results as in our Ward model (3.2) and (3.19). So it is clear that the adiabatic approach in both models leads to same moduli space dynamics.

8. Simple examples in the $U(2)$ case

8.1. One-soliton metric

To determine the moduli space metric we again carry through the adiabatic approach described in 4. In this simple example of a one soliton solution we expect a flat metric. Following [3] and [18] we set in the solution (3.28) $\lambda = 0$ and fix $\beta = \mu$ so that $\alpha = \gamma(t)$ is the only time-dependent parameter

$$T = \begin{pmatrix} \mu \\ a + \gamma(t) \end{pmatrix}. \tag{8.1}$$

Using shift operators (C.4) the solution reduces to

$$T = \begin{pmatrix} \mu \\ a + \gamma(t) \end{pmatrix} = D^\dagger(\gamma) \begin{pmatrix} \mu \\ a \end{pmatrix} D(\gamma). \tag{8.2}$$

Determinating the moduli space metric we start with the Lagrangian read off from (4.14)

$$L = 8\pi\theta \operatorname{Tr} (\mathbb{1} - P)\, \partial_t T\, (T^\dagger T)^{-1}\, \partial_t T^\dagger. \tag{8.3}$$

Therefore we need the time derivative $\partial_t T$, which is in terms of T

$$\partial_t T = [\dot{D} D^\dagger, T] = \begin{pmatrix} 0 \\ \dot\gamma \end{pmatrix} \tag{8.4}$$

and analogously

$$\partial_t T^\dagger = \begin{pmatrix} 0 \\ \dot{\bar\gamma} \end{pmatrix} \tag{8.5}$$

Inserting in (8.3) we gain

$$L = 8\pi\theta \operatorname{Tr} (\mathbb{1} - P) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(D^\dagger (\bar\mu\mu + a^\dagger a)^{-1} D \right) \dot\gamma \dot{\bar\gamma} \tag{8.6}$$

This implies that only one matrix element of $(\mathbb{1} - P)$

$$(\mathbb{1} - P)_{22} = \mathbb{1} - D^\dagger a \frac{1}{\bar\mu\mu + a^\dagger a} a^\dagger D \tag{8.7}$$

contributes to the Lagrangian

$$L = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left(\frac{1}{\bar\mu\mu + a^\dagger a} - a \frac{1}{\bar\mu\mu + a^\dagger a} a^\dagger \frac{1}{\bar\mu\mu + a^\dagger a} \right) \dot\gamma \dot{\bar\gamma}. \tag{8.8}$$

Evaluating the trace yields

$$L = 8\pi\theta\, \dot\gamma \dot{\bar\gamma} \left(\sum_{n=0}^{\infty} \frac{\bar\mu\mu}{n + \bar\mu\mu + 1} \right). \tag{8.9}$$

The sum converges to one so that we obtain the Lagrangian

$$L = 8\pi\theta\, \dot{\bar\gamma} \dot\gamma, \tag{8.10}$$

where we read off the expected flat metric. Note that if we switch from a to $z = \sqrt{2\theta} a$ and rescale the moduli $\gamma = \frac{\alpha}{\sqrt{2\theta}}$ accordingly, the metric is independent from the noncommutativity parameter θ.

8.2. Two-soliton metric

We perform the adiabatic approach similar to the one-soliton case. Again we fix some parameters in the solution (3.29) as described in [3] and [18] for the commutative case

$$\lambda = 0 \quad , \quad \beta = 0 \quad \text{and} \quad \delta = 0 \tag{8.11}$$

and take $\gamma = \mu(t)$ and $\alpha = \varepsilon(t)$, i.e.

$$T = \begin{pmatrix} \mu(t) \\ a^2 + \varepsilon(t) \end{pmatrix} \tag{8.12}$$

To get the moduli space metric we start again with the Lagrangian (8.3)

$$L = 8\pi\theta \operatorname{Tr} (\mathbb{1} - P)\, \partial_t T\, (T^\dagger T)^{-1}\, \partial_t T^\dagger. \tag{8.13}$$

In this two-soliton case we find

$$(T^\dagger T)^{-1} = \Big(|\mu|^2 + (a^{\dagger 2} + \bar{\varepsilon})(a^2 + \varepsilon) \Big)^{-1} =: M^{-1} \tag{8.14}$$

and for the time derivatives

$$\partial_t T = \begin{pmatrix} \dot{\mu} \\ \dot{\varepsilon} \end{pmatrix} \qquad \text{and} \qquad \partial_t T^\dagger = \begin{pmatrix} \dot{\bar{\mu}} \\ \dot{\bar{\varepsilon}} \end{pmatrix} \tag{8.15}$$

Unfortunately we are not able to recover a simple expression for the time derivatives in terms of T as in (8.5) and in (8.6) respectively. The projector P becomes

$$P = \begin{pmatrix} \frac{|\mu|^2}{M} & \frac{\mu}{M} a^{\dagger 2} + \frac{\mu \bar{\varepsilon}}{M} \\ a^2 \frac{\bar{\mu}}{M} + \frac{\bar{\mu}\varepsilon}{M} & a^2 \frac{1}{M} a^{\dagger 2} + a^2 \frac{\bar{\varepsilon}}{M} + \frac{\varepsilon}{M} a^{\dagger 2} + \frac{|\varepsilon|^2}{M} \end{pmatrix}. \tag{8.16}$$

Inserting (8.14), (8.15) and (8.16) in the Lagrangian (8.13) we obtain

$$L = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left\{ \left((\mathbb{1} - P)_{11}\, \dot{\bar{\mu}}\dot{\mu} + (\mathbb{1} - P)_{12}\, \dot{\bar{\mu}}\dot{\varepsilon} + (\mathbb{1} - P)_{21}\, \dot{\bar{\varepsilon}}\dot{\mu} + (\mathbb{1} - P)_{22}\, \dot{\bar{\varepsilon}}\dot{\varepsilon} \right) M^{-1} \right\}, \tag{8.17}$$

where we can read off the components of the moduli space metric[1]

$$h_{\bar{\mu}\mu} = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left(\frac{|\mu|^2}{M^2} \right) \tag{8.18}$$

$$h_{\bar{\mu}\varepsilon} = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left(a^2 \frac{\bar{\mu}}{M^2} + \frac{\bar{\mu}\varepsilon}{M^2} \right) \tag{8.19}$$

$$h_{\bar{\varepsilon}\mu} = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left(\frac{\mu}{M} a^{\dagger 2} \frac{1}{M} + \frac{\mu\bar{\varepsilon}}{M} \right) \tag{8.20}$$

$$h_{\bar{\varepsilon}\varepsilon} = 8\pi\theta \operatorname{tr}_{\mathcal{H}} \left(a^2 \frac{1}{M} a^{\dagger 2} \frac{1}{M} + a^2 \frac{1}{M^2} \bar{\varepsilon} + \frac{\varepsilon}{M} a^{\dagger 2} \frac{1}{M} + \frac{|\varepsilon|^2}{M^2} \right). \tag{8.21}$$

Since we have not succeeded in inverting the operator M (8.14) we are not able to get further simplifications of the metric elements, e.g. by evaluating the trace over $\mathcal{H}$.[2]

[1] Same results are found in [22].

[2] One possibility we tried was to write the operator M (8.14) in the basis of coherent states. But then we got a rather complicated differential operator, where the inversion was also not obvious.

Conclusion and open questions

Considering a modified $U(n)$ sigma model with a WZW-like term in noncommutative setting yields some remarkable results. We have used Manton's approach to analyse soliton scattering. One important result of this thesis is the fact that the WZW-like term S_3 does not contribute to the moduli space dynamics so that magnetic forcing is absent. Thus the metric alone calculated in this thesis determines the dynamic in moduli space. It is valid for a whole family of models $(S_2 + \gamma S_3)$ parametrised by $\gamma \in [0, 1]$, where $\gamma = 1$ describes the Ward model and $\gamma = 0$ the unmodified sigma model. But for the exact results in the Ward model the WZW-like term plays an important role, because without it the model is not integrable. So it is not amazing that the moduli space trajectories as found in chapter 5 are quite different from the exact solutions as regards their scattering behaviour. We should ask the question, which solutions of which model are well approximated by the adiabatic approach really.
Part of the problem is that the exact $U(1)$ solutions do not scatter (bosonic), while for the adiabatic approximated ones scattering happens (fermionic).
One possibility to solve this problem could be finding additional solutions for the $U(1)$ Ward model having the required scattering behaviour, which are not known yet. Such additional solutions exist in the commutative $SU(2)$ Ward model, where the adiabatic approach is already successfully tested.
Another possibility is that the adiabatic approach yields good results only for the $(\gamma = 0)$ unmodified sigma model. But for this model there exist no exact solutions. So one could compare our results with numerical calculations only.
Another problem, which could not be solved, is the case of noncommutative $U(2)$ solutions, because the expression for the moduli space metric in chapter 8 could not be simplified enough to analyse scattering.
The most important results of this thesis can be found in [29].

A. Variation of the action

We will prove that the action (2.16) leads to the equations of motion (2.15). Therefore we have to calculate the variation of the action δS, treating the two parts δS_2 and δS_3 seperately again and begining with varying S_2

$$\delta S_2 = -\frac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\, \eta^{ij}\, \mathrm{tr}\ \delta\left(\partial_i\Phi^{-1}\star\partial_j\Phi\right). \tag{A.1}$$

With the identity $\partial_a\Phi^{-1} = -\Phi^{-1}\star\partial_a\Phi\star\Phi^{-1}$ this becomes

$$\delta S_2 = \frac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\, \eta^{ij}\, \mathrm{tr}\ \delta\left((\Phi^{-1}\star\partial_i\Phi)\star(\Phi^{-1}\star\partial_j\Phi)\right). \tag{A.2}$$

Using the cyclic property of the trace we get

$$\begin{aligned}\delta S_2 &= \int \mathrm{d}t\mathrm{d}x\mathrm{d}y\, \eta^{ij}\, \mathrm{tr}\ \delta\left(\Phi^{-1}\star\partial_i\Phi\right)\star\left(\Phi^{-1}\star\partial_j\Phi\right)\\ &= \int \mathrm{d}t\mathrm{d}x\mathrm{d}y\, \eta^{ij}\, \mathrm{tr}\ \left(\Phi^{-1}\star\delta\Phi\star\Phi^{-1}\star\partial_i\Phi + \Phi^{-1}\star\partial_i\delta\Phi\right)\star\left(\Phi^{-1}\star\partial_j\Phi\right).\end{aligned} \tag{A.3}$$

Partial integration implies

$$\delta S_2 = -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\ \mathrm{tr}\ \Phi^{-1}\star\delta\Phi\star\left(\eta^{ij}\partial_i(\Phi^{-1}\star\partial_j\Phi)\right). \tag{A.4}$$

For varying S_3 we define the shorter notation

$$J_m := \tilde{\Phi}^{-1}\star\partial_m\tilde{\Phi}. \tag{A.5}$$

In our special gauge ($v = (0,1,0,0)$) the WZW-like term is

$$\delta S_3 = -\frac{1}{3}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \epsilon^{rsq}\, \mathrm{tr}\ \left(J_r\star J_s\star J_q\right), \tag{A.6}$$

where $r,s,q\in\{t,y,\rho\} =$. With this notation and due to the cyclic property of the trace the variation is

$$\begin{aligned}\delta S_3 &= -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \epsilon^{rsq}\, \mathrm{tr}\ \left(\delta J_r\star J_s\star J_q\right)\\ &= \int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \epsilon^{rsq}\, \mathrm{tr}\ \left(\tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star(J_r\star J_s\star J_q) - \tilde{\Phi}^{-1}\star\partial_r\delta\tilde{\Phi}\star J_s\star J_q\right).\end{aligned} \tag{A.7}$$

Partial integration yields

$$\begin{aligned}\delta S_3 &= \int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \epsilon^{rsq}\, \mathrm{tr}\ \Big(\tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star(J_r\star J_s\star J_q) - \partial_r(\tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star J_s\star J_q)\\ &\quad - \tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star(J_r\star J_s\star J_q) - \tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star(J_r\star J_s\star J_q + J_s\star J_r\star J_q)\Big)\\ &= -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \epsilon^{rsq}\, \mathrm{tr}\ \left(\partial_r(\tilde{\Phi}^{-1}\star\delta\tilde{\Phi}\star J_s\star J_q)\right).\end{aligned} \tag{A.8}$$

Only the terms for $r = \rho$ contributes

$$\begin{aligned}\delta S_3 &= -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \int_0^1 \mathrm{d}\rho\, \partial_\rho \ \mathrm{tr}\ \left(\tilde{\Phi}^{-1} \star \delta\tilde{\Phi} \star \epsilon^{\rho s q} J_s \star J_q\right) \\ &= -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \ \mathrm{tr}\ \Phi^{-1} \star \delta\Phi \star v_c \epsilon^{cij} \partial_i (\Phi^{-1} \star \partial_j \Phi),\end{aligned} \tag{A.9}$$

with $\vec{v} = (0, 1, 0)$.
By summation we obtain

$$\delta S = \delta S_2 + \delta S_3 = -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y \ \mathrm{tr}\ \Phi^{-1} \star \delta\Phi \star \left((\eta^{ij} + v_c \epsilon^{cij}) \partial_i (\Phi^{-1} \star \partial_j \Phi)\right), \tag{A.10}$$

which should be zero ($\delta S \overset{!}{=} 0$).
This is satisfied if

$$(\eta^{ij} + v_c \epsilon^{cij}) \partial_i (\Phi^{-1} \star \partial_j \Phi) = 0, \tag{A.11}$$

and we have derived the equation of motion (2.15).

B. Legendre transformation

In this appendix we carry out the Legendre transformations of L_2 and L_3 as required in 2.3.
The calculation of E_2 in the star-formalism is very difficult, because the relevant part of the Lagrangian has the form

$$L_2 = \mathrm{tr}(\Phi^{-1} \star \dot{\Phi} \star \Phi^{-1} \star \dot{\Phi}). \tag{B.1}$$

Due to the definition of the star-product (1.2) L_2 is additionally dependent on mixed derivatives of Φ. Thus

$$L_2 = L_2(\Phi, \nabla\Phi, ..\nabla^n\Phi, .., \dot{\Phi}, \nabla\dot{\Phi}, .., \nabla^n\dot{\Phi}, ...) \tag{B.2}$$

and for each time derivative we can compute a canonical momentum. So we must compute an infinite sum of canonical momenta to obtain the energy. This is the reason, why we change to the operator formalism for the Legendre transformation of L_2, where it reduces to

$$L_2 = 2\pi\theta \ \mathrm{Tr}\left(-\tfrac{1}{2}\eta^{ij}\partial_i\hat{\Phi}^{-1}\partial_j\hat{\Phi}\right), \tag{B.3}$$

with trace taken over Fock space $\mathcal{H}$ (1.27) and over group space $U(n)$, i.e.

$$\mathrm{Tr} := \mathrm{tr}\,\mathrm{tr}_{\mathcal{H}}\,. \tag{B.4}$$

The part E_2 of the energy functional is defined as

$$E_2 := \sum \hat{\pi}^{\alpha\beta}_{ij}\dot{\hat{\Phi}}^{\alpha\beta}_{ij} - L_2, \tag{B.5}$$

where the indices i,j are necessary because $\hat{\Phi}$ is group-valued ($\hat{\Phi} \in U(n)$) and the indices α, β represent the fact that the entries of $\hat{\Phi}_{ij}$ are functions of noncommutative coordinates, which are operator valued. So $\hat{\Phi}_{ij}$ is operator valued too and we consider the entry $\hat{\Phi}^{\alpha\beta}_{ij}$ of $\hat{\Phi}$. In the following calculation we will drop the hats on the operators. At first we will calculate the entries of the canonical momentum

$$\pi^{\alpha\beta}_{ij} := \frac{\partial L_2}{\partial(\partial_t\Phi^{\alpha\beta}_{ij})} = \frac{\partial}{\partial(\partial_t\Phi^{\alpha\beta}_{ij})}\left(2\pi\theta \ \mathrm{Tr}\ \left(\tfrac{1}{2}\partial_t(\Phi^{-1})^{\gamma\delta}_{kl}\ \partial_t\Phi^{\delta\varepsilon}_{lm}\right)\right) \tag{B.6}$$

Using the identity

$$-\partial_t(\Phi^{-1})^{\gamma\delta}_{kl} = (\Phi^{-1})^{\gamma\lambda}_{ko}\ \dot{\Phi}^{\lambda\tau}_{op}\ \partial_t(\Phi^{-1})^{\tau\delta}_{pl} \tag{B.7}$$

we get

$$\begin{aligned}\pi^{\alpha\beta}_{ij} &= \frac{\partial}{\partial\dot{\Phi}^{\alpha\beta}_{ij}}\left(2\pi\theta \ \mathrm{Tr}\ \left(-\tfrac{1}{2}(\Phi^{-1})^{\gamma\lambda}_{ko}\ \dot{\Phi}^{\lambda\tau}_{op}\ (\Phi^{-1})^{\tau\delta}_{pl}\ \dot{\Phi}^{\delta\varepsilon}_{lm}\right)\right) \\ &= -\tfrac{1}{2}(2\pi\theta)\left((\Phi^{-1})^{\gamma\alpha}_{ki}\ (\Phi^{-1})^{\beta\delta}_{jl}\ \dot{\Phi}^{\delta\gamma}_{lk}\ + (\Phi^{-1})^{\beta\lambda}_{jo}\ \dot{\Phi}^{\lambda\tau}_{op}\ (\Phi^{-1})^{\tau\alpha}_{pi}\ \right),\end{aligned} \tag{B.8}$$

Inserting this, the energy written with matrices becomes

$$
\begin{aligned}
E_2 &= -\tfrac{1}{2}\left(2\pi\theta \,\mathrm{Tr}\,\left(-2\,\partial_t\Phi^{-1}\,\partial_t\Phi\right) - 2\pi\theta\,\mathrm{Tr}\,\left(-\,\partial_t\Phi^{-1}\,\partial_t\Phi \;+\; \partial_x\Phi^{-1}\,\partial_x\Phi \;+\; \partial_y\Phi^{-1}\,\partial_y\Phi\right)\right)\\
&= \tfrac{1}{2}2\pi\theta\,\mathrm{Tr}\,\left(\partial_t\hat\Phi^{-1}\,\partial_t\hat\Phi \;+\; \partial_x\hat\Phi^{-1}\,\partial_x\hat\Phi \;+\; \partial_y\hat\Phi^{-1}\,\partial_y\hat\Phi\right),
\end{aligned}
\tag{B.9}
$$

and in star-formalism

$$
E_2 = \tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\left(\partial_t\Phi^{-1}\star\partial_t\Phi + \partial_x\Phi^{-1}\star\partial_x\Phi + \partial_y\Phi^{-1}\star\partial_y\Phi\right). \tag{B.10}
$$

In the WZW-like part L_3 there are only terms linear in $\dot{\tilde\Phi}$. Thus it is not necessary to be as careful as in the Legendre transformation above, and we stay in the star formalism. So the energy E_3 is

$$
\begin{aligned}
E_3 =& -\tfrac{1}{3}\int \mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\;\mathrm{tr}\left[\frac{\partial}{\partial(\partial_t\tilde\Phi)}\left(\tilde v_\lambda\,\epsilon^{\lambda\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\right)\,\partial_t\tilde\Phi\right.\\
&\left.-\left(\tilde v_\lambda\,\epsilon^{\lambda\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\right)\right]\\
=& -\tfrac{1}{3}\int \mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\;\mathrm{tr}\left[\frac{\partial}{\partial(\partial_t\tilde\Phi)}\left(\tilde v_t\,\epsilon^{t\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\right.\right.\\
&\left.+\tilde v_\lambda\,\epsilon^{\lambda\mu\nu t}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_t\tilde\Phi\right)\right)\,\partial_t\tilde\Phi\\
&\left.-\left(\tilde v_\lambda\,\epsilon^{\lambda\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\right)\right]\\
=& -\tfrac{1}{3}\int \mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\;\mathrm{tr}\left[\tilde v_\lambda\,\epsilon^{\lambda\mu\nu t}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_t\tilde\Phi\right)\right.\\
&-\tilde v_t\,\epsilon^{t\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\\
&\left.-\tilde v_\lambda\,\epsilon^{\lambda\mu\nu t}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_t\tilde\Phi\right)\right]\\
=& \tfrac{1}{3}\int \mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\;\mathrm{tr}\,\left(\tilde v_t\,\epsilon^{t\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right)\right),
\end{aligned}
\tag{B.11}
$$

By summation we obtain the required energy

$$
\begin{aligned}
E =&\; \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\left(\partial_t\Phi^{-1}\star\partial_t\Phi + \partial_x\Phi^{-1}\star\partial_x\Phi + \partial_y\Phi^{-1}\star\partial_y\Phi\right)\\
&+\frac{1}{3}\int \mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\;\mathrm{tr}\;\tilde v_t\,\epsilon^{t\mu\nu\sigma}\left(\tilde\Phi^{-1}\star\partial_\mu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\nu\tilde\Phi\star\tilde\Phi^{-1}\star\partial_\sigma\tilde\Phi\right).
\end{aligned}
\tag{B.12}
$$

C. Coherent states

Since our general $U(1)$ solutions Φ are parametrized by coherent states $|\alpha\rangle$, it seems to be necessary to know some properties and relations of coherent states. They can be found in the literature espacially in books about quantum optics like [30], because they play an important role there.
At first we will define unitary transformations conserving the properties of the creation and annihalation operators, which describe the noncommutative coordinates (1.17).
Translations in the complex plane are defined by the unitary shift operator

$$D(\alpha) := \mathrm{e}^{\alpha a^\dagger - \bar{\alpha} a}, \tag{C.1}$$

with $\alpha \in \mathbb{C}$.
From the Baker-Campbell-Hausdorff formula we know the identity for vanishing double commutators

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B} \mathrm{e}^{\frac{1}{2}[A,B]}. \tag{C.2}$$

Using this (C.1) becomes

$$D(\alpha) = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \mathrm{e}^{\alpha a^\dagger} \mathrm{e}^{-\bar{\alpha} a}. \tag{C.3}$$

Now we can easily calculate the effect of $D(\alpha)$ on a or $a^\dagger$

$$D(\alpha) a D^\dagger(\alpha) = a - \alpha \qquad \text{and} \qquad D(\alpha) a^\dagger D^\dagger(\alpha) = a^\dagger - \bar{\alpha}. \tag{C.4}$$

The unitary operator describing *rotations* is defined by

$$R(\theta) := \mathrm{e}^{-\mathrm{i}\theta a^\dagger a} \tag{C.5}$$

with $\theta \in \mathbb{R}$, which acts on creation and annihalation operators as

$$R^\dagger(\theta) a R(\theta) = a \mathrm{e}^{-\mathrm{i}\theta} \qquad \text{and} \qquad R^\dagger(\theta) a^\dagger R(\theta) = a^\dagger \mathrm{e}^{\mathrm{i}\theta}. \tag{C.6}$$

In this context one could additionally define unitary operators for Bogolyubov transformations or squeezing operators [30], but this is not necessary for our purposes.
Having these unitary transformations we are able to construct *coherent states* with respect to a and $a^\dagger$ as shifted ground states

$$|\alpha\rangle := D(\alpha)|0\rangle. \tag{C.7}$$

The coherent state $|\alpha\rangle$ is an eigenstate to eigenvalue α of the annihalation operator a

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{C.8}$$

because

$$(a - \alpha)|\alpha\rangle = D(\alpha) a D^\dagger(\alpha) D(\alpha)|0\rangle = D(\alpha) a|0\rangle = 0. \tag{C.9}$$

Considering the scalar product we see that two different coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal

$$\langle\beta|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}\alpha\bar{\alpha}-\frac{1}{2}\beta\bar{\beta}+\bar{\beta}\alpha}, \tag{C.10}$$

which implies

$$|\langle\beta|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}. \tag{C.11}$$

The set $\{|\alpha\rangle\}$ is over-complete in the Hilbert space $\mathcal{H}$ and the coherent states are not linearly independent (for proof see [30]).
In order to derive a dependence between the states $|n\rangle$ of the Fock space $\mathcal{H}$ and the coherent states $|\alpha\rangle$ we write (C.7) in another form on the one hand

$$\begin{aligned} |\alpha\rangle &= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha a^\dagger}|0\rangle \\ &= \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle, \end{aligned} \tag{C.12}$$

and analyse, how $a^\dagger$ acts on $|\alpha\rangle$ on the other hand

$$a^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial\alpha}+\frac{\bar{\alpha}}{2}\right)|\alpha\rangle. \tag{C.13}$$

From this it follows

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{\partial}{\partial\alpha}+\frac{\bar{\alpha}}{2}\right)^n|\alpha\rangle. \tag{C.14}$$

If we want to evaluate the trace of an operator A via coherent states we can use

$$\mathrm{tr}_{\mathcal{H}}\, A = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, \langle\alpha|A|\alpha\rangle\,. \tag{C.15}$$

D. Operator ideals

In this appendix we want to introduce some types of operators. This classification is useful for an identification with different types of functions via the Moyal-Weyl map (1.21) (1.22), and necessary for answering questions like "What operators in Hilbert spaces have a well defined trace?" or "In which case are we allowed to use the cyclic property of the trace?". The definitions and theorems in this appendix are essentially from [31].
For our purposes we are only interested in operators from the seperable Hilbert space $\mathcal{H}$ into $\mathcal{H}$. We suppose that all these operators are linear. They are bounded with respect to the operator norm

$$\|A\|_{op} = \sup\{\|Ax\| \mid x \in \mathcal{H} : \|x\| \leq 1\}, \tag{D.1}$$

if for every operator A there is a number k such that

$$\|Ax\| \leq k\|x\| \tag{D.2}$$

for all $x \in \mathcal{H}$. The set of all these operators with this norm forms a Banach algebra denoted by $L(\mathcal{H}, \mathcal{H})$.
The smallest operator ideal in this algebra is the set of the finite dimensional operators $\mathcal{F}(\mathcal{H}, \mathcal{H})$. An operator $A \in L(\mathcal{H}, \mathcal{H})$ is called *finite dimensional*, if

$$A \in \mathcal{F}(\mathcal{H}, \mathcal{H}) \Leftrightarrow \dim \operatorname{Im}(A) < \infty. \tag{D.3}$$

Another important ideal is the set of all compact operators $\mathcal{K}(\mathcal{H}, \mathcal{H})$. An operator $A \in L(\mathcal{H}, \mathcal{H})$ is called *compact*, if Ax for x from the closed unit ball $\tilde{K}_1^{\|\,\|}$ is relativly compact in the norm topology

$$A \in \mathcal{K}(\mathcal{H}, \mathcal{H}) \Leftrightarrow \overline{A[\tilde{K}_1^{\|\,\|}]}^{\|\,\|} \text{compact.} \tag{D.4}$$

Since $\mathcal{K}(\mathcal{H}, \mathcal{H})$ is an ideal in $L(\mathcal{H}, \mathcal{H})$ the compositions KB and BK with a bounded operator $B \in L(\mathcal{H}, \mathcal{H})$ and a compact one $K \in \mathcal{K}(\mathcal{H}, \mathcal{H})$ are compact.
An operator A is called *trace-class* (or *nuclear*), if it has well defined trace

$$\operatorname{tr}_{\mathcal{H}} A < \infty \tag{D.5}$$

and *Hilbert-Schmidt*, if

$$\operatorname{tr}_{\mathcal{H}} A^\dagger A < \infty. \tag{D.6}$$

The sets of trace-class operators and Hilbert-Schmidt operators are ideals in $L(\mathcal{H}, \mathcal{H})$ too. Now we want to give some relations between these operator ideals. [1]
It is easy to show that

$$\mathcal{F}(\mathcal{H}, \mathcal{H}) \subseteq \mathcal{K}(\mathcal{H}, \mathcal{H}), \tag{D.7}$$

Each trace-class or Hilbert-Schmidt operator is compact as well.

[1] The proofs are found in [31].

In Hilbert spaces each compact operator can be approximated by finite dimensional operators

$$\overline{\mathcal{F}(\mathcal{H},\mathcal{H})}^{\|\,\|_{op}} = \mathcal{K}(\mathcal{H},\mathcal{H}). \tag{D.8}$$

What does this mean for the trace?
Let A and B be two compact operators and the product AB should be a trace-class operator. As compact operators A and B can be written as limits of sequences of finite dimensional operators $A_1, A_2, \cdots \in \mathcal{F}(\mathcal{H},\mathcal{H})$ and $B_1, B_2, \cdots \in \mathcal{F}(\mathcal{H},\mathcal{H})$

$$\lim_{n\to\infty} A_n = A \qquad \text{and} \qquad \lim_{n\to\infty} B_n = B. \tag{D.9}$$

So $\mathrm{tr}_{\mathcal{H}}(AB)$ is welldefined and it follows the cyclic property for infinite dimensional compact operators

$$\mathrm{tr}_{\mathcal{H}}(AB) := \lim_{n\to\infty} \mathrm{tr}_{\mathcal{H}}(A_n B_n) = \lim_{n\to\infty} \mathrm{tr}_{\mathcal{H}}(B_n A_n) = \mathrm{tr}_{\mathcal{H}}(BA). \tag{D.10}$$

E. Kähler metric and Kähler potential

The basic definitions and theorems about Kähler forms, Kähler manifolds and Kähler geometry are found in [32].
On a complex manifold M with a hermitian metric g a *Kähler form* Ω can be written as

$$\begin{aligned}\Omega &= \mathrm{i}g_{\mu\bar{\nu}}\,\mathrm{d}z^{\mu}\otimes\mathrm{d}\bar{z}^{\nu}-\mathrm{i}g_{\bar{\nu}\mu}\,\mathrm{d}z^{\nu}\otimes\mathrm{d}\bar{z}^{\mu}\\ &= \mathrm{i}g_{\mu\bar{\nu}}\,\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}.\end{aligned} \tag{E.1}$$

A *Kähler manifold* is a Hermitian manifold (M, g) whose Kähler form Ω is closed, i.e. $\mathrm{d}\Omega = 0$. The metric g is called the *Kähler metric* of M. [1]
Let g be a Kähler metric. Then the Kähler form Ω must be closed and it yields that

$$\begin{aligned}\mathrm{d}\Omega =&(\partial+\bar{\partial})\mathrm{i}g_{\mu\bar{\nu}}\,\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}\\ =&\mathrm{i}\partial_{\lambda}\,g_{\mu\bar{\nu}}\,\mathrm{d}z^{\lambda}\wedge\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}+\mathrm{i}\partial_{\bar{\lambda}}\,g_{\mu\bar{\nu}}\,\mathrm{d}\bar{z}^{\lambda}\wedge\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}\\ =&\tfrac{1}{2}\mathrm{i}(\partial_{\lambda}\,g_{\mu\bar{\nu}}-\partial_{\mu}\,g_{\lambda\bar{\nu}})\,\mathrm{d}z^{\lambda}\wedge\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}\\ &+\tfrac{1}{2}\mathrm{i}(\partial_{\bar{\lambda}}\,g_{\mu\bar{\nu}}-\partial_{\bar{\nu}}\,g_{\mu\bar{\lambda}})\,\mathrm{d}z^{\lambda}\wedge\mathrm{d}z^{\mu}\wedge\mathrm{d}\bar{z}^{\nu}=0,\end{aligned} \tag{E.2}$$

which leads to the conditions

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\lambda}}=\frac{\partial g_{\lambda\bar{\nu}}}{\partial z^{\mu}} \qquad \text{and} \qquad \frac{\partial g_{\mu\bar{\nu}}}{\partial\bar{z}^{\lambda}}=\frac{\partial g_{\mu\bar{\lambda}}}{\partial\bar{z}^{\nu}}. \tag{E.3}$$

So if we want to know, whether we have a Kähler metric ,we must check if these conditions are fulfilled.
Each Kähler metric is locally (on a chart U) expressed as

$$g_{\mu\bar{\nu}}=\frac{\partial}{\partial z_{\mu}}\frac{\partial}{\partial\bar{z}_{\nu}}\,\mathrm{K}, \tag{E.4}$$

where K is a scalar function on U named *Kähler potential.*
Hence the Kähler form on U is $\Omega = \mathrm{i}\partial\bar{\partial}K$.
Similar to a gauge invariance the Kähler potential K is determined only up to holomorphic $\Phi(z)$ (antiholomorphic $\Psi(\bar{z})$) functions, because another Kähler potential $\tilde{K}$ leads to the same metric $g_{\mu\bar{\nu}}$

$$\frac{\partial}{\partial z_{\mu}}\frac{\partial}{\partial\bar{z}_{\nu}}\,\tilde{\mathrm{K}}=\frac{\partial}{\partial z_{\mu}}\frac{\partial}{\partial\bar{z}_{\nu}}\,\left(K(z,\bar{z})+\Phi(z)+\Psi(\bar{z})\right)=\frac{\partial}{\partial z_{\mu}}\frac{\partial}{\partial\bar{z}_{\nu}}\,K=g_{\mu\nu} \tag{E.5}$$

For a physical interpretation we look in which contexts Kähler potentials occur in physics. We find in [26] and [27] that in the correspondence between the quantum statistics of identical particles and classical mechanics the Kähler potential K occurs as a normalisation factor $\mathcal{N}$

$$K=\hbar\,\ln|\mathcal{N}|^{-2}. \tag{E.6}$$

[1] All complex manifolds admit a hermitian metric, but not all also admit a Kähler metric [32].

Bibliography

[1] R. Rajaraman. *Solitons and Instantons.* North-Holland, 1982.

[2] N. Manton and P. Sutcliffe. *Topological Solitons.* Cambridge University Press, 2004.

[3] M. Dunajski and N.S. Manton. Reduced dynamics of ward solitons. *Nonlinearity*, **18** (2005) 1677-1689.

[4] R.S. Ward. Soliton solutions in an integrable chiral model in 2+1 dimensions. *J. Math. Phys.*, **29** (1988) 386-389.

[5] J.Wess and B. Zumino. Consequences of anomalous ward identities. *Phys. Lett. B*, **37** (1971) 95-97.

[6] E.Witten. Nonabelian bosonisation in two dimensions. *Commun. Math. Phys.*, **92** (1984) 455-472.

[7] H: S. Snyder. Quantized space-time. *Phys. Rev.*, **71** (1947) 38-41.

[8] N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, **9909** (1999) 032.

[9] R. Gopakumar, S. Minwalla, and A. Strominger. Noncommutative solitons. *JHEP*, **0005** (2000) 020.

[10] M. R. Douglas and N. A. Nekrasov. Noncommutative field theory. *Reviews of Modern Physics*, **73** (2001) 977.

[11] J.A. Harvey. Komaba lectures on noncommutative solitons and d-branes. `hep-th/0102076`.

[12] M. Wohlgenannt. Non-commutative geometry and physics. `hep-th/0602105`.

[13] R. J. Szabo. Magnetic backgrounds and noncommutative field theory. *International Journal of Modern Physics A*, **19** (2004) 1837.

[14] L. Hadasz, U. Lindström, M. Roček, and R. von Unge. Noncommutative multisolitons: moduli spaces, quantization, finite θ effects and stability. *JHEP*, **0106** (2001) 040.

[15] R. Gopakumar, M. Headrick, and M. Spradlin. On noncommutative multi-solitons. *Commun.Math.Phys.*, **233** (2003) 355-381.

[16] A. Domrin, O. Lechtenfeld, and S. Petersen. Sigma-model solitons in the noncommutative plane: construction and stability analysis. *JHEP*, **0503** (2005) 045.

[17] O. Lechtenfeld and A.D. Popov. Noncommutative multi-solitons in 2+1 dimensions. *JHEP*, **0111** (2001) 40.

[18] R.S. Ward. Slowly-moving lumps in the $\mathbb{CP}^1$ model in (2+1) dimensions. *Phys. Lett. B*, **158** (1985) 424-428.

[19] U. Lindström, M. Roček, and R. von Unge. Non-commutative soliton scattering. *JHEP*, **0012** (2000) 004.

[20] J. Moyal. *Proc. Camb. Phil. Soc.*, **45** (1949) 99.

[21] H. Groenewold. *Physica*, **12** (1946) 405.

[22] K. Furuta, T. Inami, H. Nakajima, and M. Yamamoto. Low-energy dynamics of noncommutative $\mathbb{CP}^1$ solitons in 2+1 dimensions. *Phys.Lett. B*, **537** (2002) 165-172.

[23] N.S. Manton. A remark on the scattering of bps monopoles. *Phys. Lett. B*, **110** (1982) 54.

[24] M.F. Atiyah and N.J. Hitchin. Low energy scattering of nonabelian monopoles. *Phys. Lett. A*, **107** (1985) 21.

[25] N. Dragon. *Geometrie der Relativitätstheorie*. `http://www.itp.uni-hannover.de/~dragon/Group.html`, february 2006.

[26] T.H. Hansson, S.B. Isakov, J.M. Leinaas, and U. Lindström. Exclusion statistics in classical mechanics. `quant-ph/0004108`.

[27] T.H. Hansson, S.B. Isakov, J.M. Leinaas, and U. Lindström. Classical phase space and statistical mechanics of identical particals. `quant-ph/0003121`.

[28] O. Lechtenfeld and A.D. Popov. Scattering of noncommutative solitons in 2+1 dimensions. *Phys.Lett. B*, **523** (2001) 178-184.

[29] M. Klawunn, O. Lechtenfeld, and S.Petersen. Moduli-space dynamics of noncommutative abelian sigma-model solitons. *JHEP*, **06** (2006) 032.

[30] H. Römer. *Theoretical Optics*. WILEY-VCH, 2005.

[31] A. Pietsch. *Operator Ideals*. North-Holland Mathematical Library 20, 1980.

[32] M. Nakahara. *Geometry, Topology and Physics*. Graduate Studies in Physics, 1996.

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Michael Klawunn, Hannover, den 28.04.2006