Aspects of Logarithmic Conformal Field Theory

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Abstract

In this thesis we consider various aspects of logarithmic conformal field theories (LCFTs). After recalling some important definitions and relations of (logarithmic) conformal field theories we study possible extensions of conformal ghost systems to LCFTs and the generic structure of the four-point function.

In the first part of this thesis we analyze the b-c ghost system of central charge $c = -2$ and compare it with the θ-$\bar{\theta}$ system, which is known to be a LCFT. For that purpose we study two different ways of constructing logarithmic conformal field theories: via symplectic fermions and by deforming the energy-momentum tensor. While the two approaches lead to the same results for $c = -2$, they are not equivalent for $c = -26$. In the latter case the construction by symplectic fermions leads to unsurmountable problems, but a variation of the deformation method yields operators satisfying the commutation relations of the Virasoro algebra. However, the operators do not act in a consistent way on the Hilbert space of states which forces us to study the theory on nontrivial Riemann surfaces. This leads to a consistent LCFT for $c = -26$ which in particular differs from known LCFTs by the structure of the Virasoro modes. The mode L_0 turns out to be diagonal and it is even possible to choose the deformation parameters appearing in the result in such a way that the global conformal group is non-logarithmic.

In the second part we investigate the generic structure of 4-point correlation functions of fields residing in indecomposable representations of arbitrary rank. To fix the generic structure we need to solve a generalization of the global conformal Ward identities (GCWI). The solutions are recursively determined by using an algorithm which we present in this part of this thesis. We also compute all results for a Jordan-rank $r = 2$ and $r = 3$ theory, making use of permutation symmetry and introduce a graphical representation. By doing so we obtain much shorter results and get a better understanding of the structure of the solutions. In case all four fields are logarithmic partner fields additional constants in the solution may appear. We discuss the origin of these additional degrees of freedom and investigate the influence of the discrete symmetry on these terms. Finally we explicitly determine the four-point correlator for arbitrary Jordan-rank for the case where up two fields are of logarithmic type and suggest the form for the case of three logarithmic fields.

Keywords: logarithmic conformal field theory, ghost system, correlation function

Zusammenfassung

In dieser Arbeit werden verschiedene Aspekte logarithmischer konformer Feldtheorien (LKFTn) betrachtet. Nach einer Wiederholung der wichtigsten Definitionen und Relationen von (logarithmischen) konformen Feldtheorien untersuchen wir, inwieweit konforme Geist-Systeme zu LKFTn erweitert werden können und bestimmen die generische Struktur von Vier-Punkts-Funktionen.

Im ersten Teil dieser Arbeit wird das b-c Geist-System mit der zentralen Ladung $c = -2$ analysiert und mit dem θ-$\bar{\theta}$ System, bei dem es sich um eine LKFT handelt, verglichen. Zu diesem Zweck betrachten wir zwei verschiedene Wege, mit denen sich LKFTn erzeugen lassen: Zum einen kann dies mittels symplektischer Fermionen geschehen, zum anderen durch eine Deformation des Energie-Impuls-Tensors. Die beiden Ansätze liefern dasselbe Resultat für $c = -2$, sind aber für $c = -26$ nicht äquivalent. Die Konstruktion mittels symplektischer Fermionen führt zu unüberwindlichen Problemen, wohingegen die Deformations-Methode Operatoren erzeugt, die die Kommutator-Relationen der Virasoro-Algebra erfüllen. Da die Operatoren jedoch auf den Hilbertraum der Zustände nicht konsistent wirken, muss die Theorie auf nicht-trivialen Riemann-Flächen untersucht werden. Dies führt zu einer konsistenten LKFT für $c = -26$, die sich insbesondere durch die Struktur der Virasoro-Moden von anderen LKFTn abhebt. Es stellt sich heraus, dass der L_0 Mode diagonal ist und durch geeignete Wahl der Deformationsparameter sogar die globale konforme Gruppe nicht-logarithmisch gewählt werden kann.

Im zweiten Teil der Arbeit untersuchen wir die generische Struktur von Vier-Punkts-Korrelationsfunktionen von Feldern, die sich in unzerlegbaren Darstellungen von beliebigem Jordan-Rang befinden. Um die generische Struktur festzulegen muss eine verallgemeinerte Version der globalen konformen Ward Identitäten gelöst werden. Mittels eines neu entwickelten Algorithmus bestimmen wir sämtliche Lösungen für eine Jordan-Rang $r = 2$ und $r = 3$ Theorie und stellen diese graphisch dar, wobei wir zusätzlich Permutationssymmetrien ausnutzen. Dadurch erhalten wir erheblich kürzere Resultate und bekommen ein besseres Verständnis für die Struktur der Lösungen. Sind alle vier Felder im Korrelator logarithmische Partner-Felder, so können zusätzliche Konstanten in der Lösung auftreten. Wir diskutieren den Grund für das Auftreten dieser zusätzlichen Freiheitsgrade und untersuchen den Einfluss der diskreten Symmetrien auf diese Terme. Zum Abschluss bestimmen wir explizit den Vier-Punkts-Korrelator für beliebigen Jordan-Rang für den Fall, dass bis zu zwei Felder logarithmisch sind und schlagen sogar eine Verallgemeinerung für drei logarithmische Felder vor.

Schlagworte: Logarithmische Konforme Feldtheorie, Geist-System, Korrelationsfunktion

„[. . .] damit die Arbeit der vergangenen Jahrhunderte
nicht nutzlos für die kommenden Jahrhunderte gewesen sei,
damit unsere Enkel nicht nur gebildeter,
sondern gleichzeitig auch tugendhafter und glücklicher werden,
und damit wir nicht sterben,
ohne uns um die Menschheit verdient gemacht zu haben."

— Denis Diderot (1751)

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Introduction

„Wir müssen wissen.
Wir werden wissen."
— David Hilbert (1930)

In 1984 Belavin, Polyakov and Zamolodchikov published their famous paper about "infinite conformal symmetry in two-dimensional quantum field theory", cf. Belavin *et al.* (1984). In these times non-trivial quantum field theories (QFTs) could almost only be studied in the perturbative regime. The work of Belavin *et al.* on two-dimensional conformal field theories (CFTs) on the other hand allows a fully non-perturbative approach, which is a consequence of the infinitely large symmetry algebra of the theory. The impact of CFT on theoretical and mathematical physics was enormous, and CFT was found to be useful in wide areas of condensed-matters physics as well as in string theory.

Only three years after BPZ's ground-breaking work Knizhnik discovered that twist fields in ghost systems lead to logarithmic divergences (as opposed to poles in ordinary CFT), cf. Knizhnik (1987). In the following years various other authors discovered aspects of what we now call "logarithmic conformal field theories" (LCFTs), e. g., Rozansky and Saleur (1993); Saleur (1992a,b). In 1993, Gurarie introduced in his paper logarithmic operators which helped to quickly establish LCFTs as a new and interesting field of studies. Since then an enormous amount of work was done to understand LCFTs and to link LCFTs to other fields in physics. Useful applications have been found in many areas, reaching from sandpile models to applications in string theory to name only two. The number of topics LCFT might play a role in is still expanding: there are suggestions about an AdS/LCFT correspondence, and recently links to Stochastic Löwner evolutions (SLE) were proposed.

In order to determine the full field content of a conformal field theory one has to understand its representation theory. Representations of LCFTs are characterized by their Jordan-cell structure. In this sense CFTs form a subset of LCFTs: CFTs are LCFTs of Jordan-rank one. Conformal field theories have been studied for more than twenty years, and in this time a powerful machinery of tools, algorithms and definitions was developed, which nowadays is indispensable for analyzing conformal field theories. These definitions and techniques include characters, null vectors, operator product expansions, correlation functions, partition function, fusion rules, to name only some of the most important. Since LCFTs were established about ten years ago, there was a great effort to port the tested and very successful machinery of CFT to LCFT. Among many others the works of Flohr, Gaberdiel, Kausch, Kogan and Rouhani contributed significantly to this endeavor. Nevertheless, the foundations of LCFT are still not settled, and there are many areas which are not particularly well understood. This includes the understanding of twist fields, in particular their geometrical meaning, modular properties of characters and partition functions and extensions of conformal systems to logarithmic ones. This thesis is a contribution to the effort to establish the foundations of LCFT and to shed some light on its dark corners.

The structure of this dissertation is as follows. In the first chapter we present the basics of conformal field theory and LCFT. This introduction should enable the reader to understand the succeeding chapters.

In the second chapter we pick up the question whether conformal systems can be extended to logarithmic conformal field theories. We compare in detail a b-c conformal ghost system of central charge $c = -2$ with the θ-$\bar{\theta}$ system of the same central charge. The latter is known to be a LCFT and has several similarities with the former, which is a CFT. We examine two different methods for a transition from the b-c to the θ-$\bar{\theta}$ system: one is via symplectic fermions, the other approach is by a deformation of the energy-momentum tensor. Both methods succeed in the sense that the θ-$\bar{\theta}$ system is recovered. In the next step we apply the method to a b-c ghost system with central charge -26. For such a system a LCFT was not known up to now. The symplectic fermion approach fails because the energy-momentum tensor cannot be constructed, but a generalization of the method of deformation is possible and yields a consistent representation of the Virasoro algebra. While the constructed operators obey the Virasoro algebra, they do not act consistently on the Hilbert space of states of the theory. Considering the origin of logarithmic operators leads to an investigation of the theory on nontrivial Riemann surfaces. Indeed, this yields a consistent theory for $c = -26$. In difference to other known LCFTs the Virasoro mode L_0 turns out to be diagonal, and a certain choice of deformation parameters even results in a non-logarithmic global conformal group.

The last part of this thesis deals with the question what the generic form of four-point correlation functions in LCFT is. This question has been answered in case of a CFT for a long time, and its solution is given in the first part of this thesis. In the logarithmic case a generalization of the global conformal Ward identities (GCWIs) needs to be solved. Finding the generic structure of a correlator is much harder because the generalized GCWIs include the predecessors of the correlator, i. e., correlators of a lower logarithmic level. The latter requires to solve the equations recursively and also leads to much longer equations. Fortunately it is possible to break the long equations into smaller ones and to solve these with the help of an algorithm which is described in this thesis. By using this algorithm, we compute all results for a Jordan-rank $r = 2$ and $r = 3$ theory which can be rewritten in a much more appealing form by exploiting permutation symmetry and introducing a graphical representation. Each solution contains several functions which only depend on the globally conformal invariant crossing ratio. The number of these functions grows heavily with the Jordan-rank r and the total level of the logarithmic partner fields. The complexity of the solutions also increases because additional degrees of freedom appear in correlators which contain logarithmic fields only. We explain these additional degrees of freedom and give an almost complete classification of the form of the additional terms which can appear. In the last subsection of this part of the thesis we explicitly compute the four-point correlator for arbitrary Jordan-rank for the case where up to two fields are of logarithmic type. Furthermore, we suggest the form of a four-point function containing up to three logarithmic fields.

Chapter 1

Foundations

"If I have been able to see further, it was only because I stood on the shoulders of giants."
— Sir Isaac Newton (February 5, 1675)

This thesis is about different aspects of logarithmic conformal field theory (LCFT). In order to provide a better understanding of the following chapters this chapter lays the foundations, gives an overview of the basic framework and contains all conventions and matters of notation. For brevity we omit almost all proofs. As a compensation, we recommend introductory literature in the beginning of each section. With the help of the given references the interested reader should be in the position to close all remaining gaps.

LCFT actually is a generalization of conformal field theory (CFT), or to put it more precisely: a CFT is a LCFT of Jordan-rank $r = 1$. Nevertheless we will proceed in the usual way and give first an overview of conformal field theory and afterwards deal with the LCFT. On the one hand this is for historical reasons, on the other hand the structures showing up in CFT are in general of a simpler form and thus easier to understand.

1.1 Conformal Field Theory

This section gives an elementary introduction to the conformal field theory. As noted before we omit most of the proofs. We refer to di Francesco *et al.* (1997); Gaberdiel (2000); Ginsparg (1988); Ketov (1995) and Schellekens (1996) for more details. These references give a more complete introduction and should leave almost no question open. For a mathematical more stringent approach towards CFT see Schottenloher (1997). For an introduction motivated by string theory the reader should consult Polchinski (1998a,b). It may also be of interest to study Gaberdiel and Goddard (2000); Schreiber (2004) and the appendix of Wald (1984).

In the next four subsections we introduce a number of terms that are needed in order to define what a conformal field theory is and that help explaining its properties. These topics can be outlined in a fairly general setting, that is on curved space in arbitrary dimension d. Shortly after this we will consider two dimensional systems only, which also has the benefit that the space in question is conformally flat, cf. subsection 1.1.4.

The much richer symmetry algebra in $d = 2$ and an algebraic structure called Virasoro algebra which has no equivalent in $d > 2$ is discussed in the following subsections. A nice property of a conformal field theory in two dimensions is that it allows for a calculation of n-point functions without having to resort to perturbation theory in a given coupling constant.

1.1.1 Conformal transformations, conformal group

On a pseudo-Riemannian manifold $(\mathcal{M}, g)$ of dimension d the line element ds and thereby the meaning of distance on an infinitesimal scale is defined by the metric tensor g as

$$ds^2 := g_{\mu\nu}(x) dx^\mu dx^\nu \ . \tag{1.1}$$

The diffeomorphisms $f : \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ form the group $\mathrm{Diff}(\mathcal{M})$. If $f \in \mathrm{Diff}(\mathcal{M})$ additionally fulfills

$$f^* g_{f(p)} = e^{2\sigma(p)} g_p \quad \forall p \in \mathcal{M} \tag{1.2}$$

with $\sigma \in \mathcal{F}(\mathcal{M}) := \{h : \mathcal{M} \to \mathbb{R} \,|\, h \text{ diffeomorphic}\}$ and f^* being the pullback, then f is called a *conformal transformation.* The set of conformal transformations on $\mathcal{M}$ forms a group, the *conformal group* $\mathrm{Conf}(\mathcal{M})$.[1] An *isometry* is defined by the same equation as above, but for fixed $\sigma = 0$, and thus the group of isometries $\mathrm{Iso}(\mathcal{M})$ is a subgroup of $\mathrm{Conf}(\mathcal{M})$. We point out that $\mathrm{Iso}(\mathcal{M})$ and $\mathrm{Conf}(\mathcal{M})$ are properties of the space only and do not depend on a physical action. From

$$g_{f(p)}(f_* X, f_* Y) = e^{2\sigma(p)} g_p(X, Y) \quad \forall X, Y \in T_p\mathcal{M} \tag{1.3}$$

[1] In the literature it is also common to refer to the the conformal group as the component that contains the identity.

with f_* being the push-forward it is clear that conformal transformations f preserve the metric up to a scale. On a Riemannian manifold we can define the angle ϕ between two vectors $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu \in T_p\mathcal{M}$ the tangent space by $\cos\phi := g_p(X,Y)/\sqrt{g_p(X,X)g_p(Y,Y)}$ and immediately derive from (1.2) that the angle is invariant under conformal transformations. Thus conformal transformations may locally change the scale, but not the shape of a manifold.[2] Expressed in components (1.2) becomes:

$$\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = e^{2\sigma(p)} g_{\mu\nu}(p) \ , \tag{1.4}$$

where x denotes the "old" coordinates in a neighborhood of p and x' the coordinates in a vicinity of $f(p)$.

If we restrict ourselves to $\mathbb{R}^{p,q}$ with the flat metric tensor $\eta_{\mu\nu}$ then the group of isometries $\mathrm{Iso}(\mathcal{M})$ is the Poincaré group which is the semi-direct product of $O(p,q)$ and the group of translations $E^{p,q}$. In particular this means that all rigid translations and Lorentz rotations also are conformal transformations. The converse is not true, the conformal group for $\mathbb{R}^{p,q}$ endowed with the flat metric is isomorphic to $O(p+1,q+1)/\mathbb{Z}_2$.[3]

1.1.2 (Conformal) Killing vector fields

A vector field $X \in \mathfrak{X}(\mathcal{M})$ induces an infinitesimal displacement ϵX on points $p \in \mathcal{M}$ with coordinates x^μ. Hereby $\mathfrak{X}(\mathcal{M})$ denotes the set of all vector fields on $\mathcal{M}$. Inserting the change of the coordinates by an infinitesimal displacement $x^\mu \to x^\mu + \epsilon X^\mu(x)$ in (1.4) (with $\sigma \equiv 0$) leads to

$$\mathfrak{L}_X g_{\mu\nu} := X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu X^\lambda + g_{\mu\lambda}\partial_\nu X^\lambda = 0 \ . \tag{1.5}$$

which is called the *Killing equation.* The operator $\mathfrak{L}_X$ denotes the Lie-derivative. A vector field X is called a *Killing vector field*, if its infinitesimal displacement generates an isometry, that means it fulfills equation (1.5). In a manner of speaking a Killing vector field represents a symmetry of a manifold.

Before we go on with the generalization of the above definitions for arbitrary σ_p, we note that the d-dimensional ($d \geq 2$) pseudo-Riemannian manifold $\mathbb{R}^{p,1}$

[2] We remark that the given definition of angle implies that the dimension d is greater than 1. In $d = 1$ every smooth map automatically is a conformal transformation.

[3] It should be added that the connected component of the identity of the conformal group is isomorphic to $SO^+(p+1,q+1)$. Here $SO^+(p+1,q+1)$ denotes the connected component of the identity in $O(p+1,q+1)$.

($d = p + 1$) endowed with the Minkowski metric tensor $\eta_{\mu\nu}$ is a *maximally symmetric space*. By definition this means that it possesses $\frac{1}{2}d(d+1)$ Killing vector fields. Of these d generate translations, $d-1$ boosts and $\frac{1}{2}(d-1)(d-2)$ generate space rotations.

A slightly more general concept is the following one: X is called a *conformal Killing vector field* if the infinitesimal displacement generates a conformal transformation, which means that

$$\mathfrak{L}_X g_{\mu\nu} = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu X^\lambda + g_{\mu\lambda}\partial_\nu X^\lambda = \psi g_{\mu\nu} \quad , \psi \in \mathcal{F}(M) \; . \tag{1.6}$$

This is known as the *conformal Killing equation* and $\sigma = \frac{1}{2}\epsilon\psi$.

1.1.3 Conformal group and algebra of $(\mathbb{R}^{p,q}, \eta_{\mu\nu})$ for $p + q > 2$

For $\mathbb{R}^{p,q}, p + q = d > 2$ endowed with the flat metric tensor $\eta_{\mu\nu}$ the conformal Killing equation can be brought to the form

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial_\lambda \varepsilon^\lambda)\,\eta_{\mu\nu} \; . \tag{1.7}$$

In this equation we have set $\varepsilon^\mu(x) := \epsilon X^\mu(x)$. One easily finds that ε_μ is at most quadratic in the coordinates x^ν which allows for solving (1.7) by using the ansatz $\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}\,x^\nu x^\rho$.

As noted before the Poincaré group forms a subgroup of the conformal group. Thus the conformal group contains the translations and rotations. Additionally, the dilations $x'^\mu = \alpha x^\mu$ and so-called *special conformal transformations*[4] (SCT) which can be written for $\boldsymbol{x}^2 \neq 0$ in the form

$$\frac{x'^\mu}{\boldsymbol{x}'^2} = \frac{x^\mu}{\boldsymbol{x}^2} - b^\mu \tag{1.8}$$

belong to the conformal group. Altogether we have d translations, $\frac{d}{2}(d-1)$ rotations, one scale transformation and d special conformal transformations. Thus we find that the group has $\frac{1}{2}(d+1)(d+2)$ generators and, as we noted before, the component containing the identity of the conformal group is isomorphic to $SO^+(p+1, q+1)$. We will find that the conformal group in two

[4]These transformations can be understood as a translation by b^μ, preceeded and followed by an inversion $x^\mu \to x^\mu/\mathbf{x}^2$ for $\boldsymbol{x}^2 \neq 0$.

dimensions has an analogous structure, but that the conformal algebra is completely different. The generators of the conformal group in $d > 2$ dimensions are

$$\begin{aligned} D &= x \cdot \partial && \text{(dilation)}, \\ P_\mu &= \partial_\mu && \text{(translation)}, \\ M_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu && \text{(rotation)}, \\ K_\mu &= 2x_\mu x \cdot \partial - \boldsymbol{x}^2 \partial_\mu && \text{(SCT)}, \end{aligned} \tag{1.9}$$

and with that the commutation rules can be easily worked out.

So far we have been dealing with conformal properties of the manifold on which physical theories will be defined. In such a physical theory the operators 1.9 will act on fields on the manifold. In the following we use the above operators respectively their commutation relations in order to show that mass terms break conformal invariance.

The commutator we need to evaluate in order to show this is $[D, P_\mu] = -P_\mu$. Then we can simply determine that

$$[D, P^2] = P_\mu [D, P^\mu] + [D, P_\mu] P^\mu = -2P_\mu P^\mu = -2m^2 \tag{1.10}$$

holds and for a massive state $|p\rangle$ with $m^2 = p^2$ we find

$$-2m^2 = \langle p| \, [D, P^2] \, |p\rangle = \langle p| \, (DP^2 - P^2 D) \, |p\rangle = 0 \; , \tag{1.11}$$

which means that the mass m has to vanish.

1.1.4 Weyl rescaling, conformal field theory, conformal flatness

In subsection 1.1.1 we discussed the effect of coordinate transformations on the metric tensor $g_{\mu\nu}$ and introduced the notion of a conformal transformation. A related concept is the following: suppose we are given two different metrics g and g' on a manifold $\mathcal{M}$. Then we say that g is *conformally related* to g' if

$$g'_p = e^{2\sigma(p)} g_p \qquad \forall p \in \mathcal{M} \; . \tag{1.12}$$

We refer to the transformation $g_p \to e^{2\sigma(p)} g_p$ as *Weyl rescaling*. "Conformally related" is an equivalence relation for the metric tensors on $\mathcal{M}$, and the equivalence class of metric tensors is called the *conformal structure*. It should be pointed out that for a Weyl rescaling of the metric tensor no coordinate transformation is involved. Weyl invariance is *not* a property of the pseudo-Riemannian manifold, but of the physical action. Now we are finally in the position to define what we understand under the term *conformal field theory*: a conformal field theory is a field theory which is invariant under Weyl rescalings at every point of spacetime.

A pseudo-Riemannian manifold $(\mathcal{M}, g)$ is called[5] *conformally flat* iff for every $p \in \mathcal{M}$ there exists a neighborhood U such that the metric tensor within is conformally equivalent to the metric of a flat space $\eta_{\mu\nu}$. From (1.2) it is obvious that conformally flat manifolds possess the same local conformal group as the flat Minkowski spacetime, though the global topology may be different from the topology of a flat space. Further it can be shown that a two-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$ always is conformally flat.[6]

Let us consider an example for Weyl invariance. The following action describes a free massless boson ϕ in d dimensions on a curved background. The parameter α denotes some constant we are not interested in for the moment,

$$S = \alpha \int d^d x \sqrt{|g|} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \ . \tag{1.13}$$

For $d = 2$ the action is known as *Polyakov action.*[7] As every physical action or mathematical meaningful object the action (1.13) is invariant under transformations $f \in \mathrm{Diff}(\mathcal{M})$. What is remarkable is that the Polyakov action is invariant under Weyl rescalings iff the dimension $d = 2$. This can easily be seen as follows: under $g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu}$, the inverse metric tensor transforms as $g^{\mu\nu} \mapsto e^{-2\sigma} g^{\mu\nu}$ and $g := |\det(g_{\mu\nu})| \mapsto e^{2d\sigma} g$. Since no coordinate transformation takes place, $\partial_\mu \phi$ and $d^d x$ are invariant under Weyl rescalings, such that the rescaled action S' becomes

$$S' = \alpha \int d^d x \sqrt{|g|} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi)\, e^{(d-2)\sigma} \ . \tag{1.14}$$

Obviously the Polyakov action is on a classical level invariant under Weyl rescalings iff $d = 2$. Of course the "critical dimension" depends on the form of the action. For the Yang-Mills action the critical dimension is four, i. e. $d = 4$.

[5] An equivalent definition is that the Weyl tensor vanishes, cf. Nakahara (1990).

[6] A simple derivation based on counting degrees of freedom is given in Schreiber (2004) a more rigorous proof can be found in Nakahara (1990).

[7] Though the name "Polyakov action" is well established in the literature it was in fact Brink, Di Vecchia and Howe (1976) and Deser and Zumino (1976) who first had the idea to study this action. Polyakov though was the first to stress its importance for quantizing the bosonic string.

1.1.5 Energy-momentum tensor, conserved currents

The *energy-momentum tensor* can be defined by varying the action with respect to the metric tensor

$$T_{\mu\nu} := \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} \ . \tag{1.15}$$

The so defined energy-momentum tensor is always symmetric.[8] Translational isometries imply that the tensor is locally conserved, meaning that

$$\nabla_\mu T^{\mu\nu} = 0 \ , \tag{1.16}$$

where ∇_μ denotes the appropriate covariant derivative. Furthermore, one can easily show that the Weyl invariance of the action S also guarantees that the stress-energy tensor $T^{\mu\nu}$ is traceless, $T^\mu{}_\mu = 0$. With the help of the energy-momentum tensor we can now define a *conformal current*

$$j_\mu(\varepsilon) := T_{\mu\nu}\,\varepsilon^\nu \ , \tag{1.17}$$

that is associated with an infinitesimal conformal transformation ε^μ, where ε^μ has to satisfy equation (1.7). The short calculation $\partial^\mu j_\mu = \frac{1}{2} T^\mu{}_\mu (\partial \cdot \varepsilon) = 0$ immediately shows us that the current has vanishing divergence, too.

1.1.6 Conformal field theory in two dimensions

The structure we have investigated so far was mostly a pseudo-Riemannian manifold $(\mathcal{M}, g)$ in arbitrary dimension d. There is a number of reasons why two-dimensional systems have a beauty on their own and why it is an appealing endeavor to solely focus on such systems. For the implications of conformal invariance for dimensions $d > 2$ we refer the interested reader to Erdmenger (1996).

First we noticed that the Polyakov action, which is of importance for string theory, is invariant under Weyl rescalings if and only if the system is two-dimensional. This by the way is why strings are favored by physicists compared to extended objects like membranes. We also have noted that a two-dimensional pseudo-Riemannian manifold always is conformally flat which means that there exists a parameterization such that the coordinates of the

[8] In a theory that is invariant under the rotation group it is at least possible to make the energy-momentum tensor $T_{\mu\nu}$ symmetric.

metric locally have the form $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$. Using this feature together with the Weyl rescaling invariance of the theory makes further calculations much easier because we can use the flat metric tensor $\eta_{\mu\nu}$ on the whole worldsheet.

Another important feature of two-dimensional conformal field theory, which we have not discussed up to now, is that the conformal Lie algebra is infinite-dimensional.[9] Upon quantization, this leads to a powerful algebraic structure, the so-called Virasoro algebra, which will show up only in two-dimensional conformal field theory.

For the above reasons we will investigate the two-dimensional manifold $\mathbb{R}^{1,1}$ with the flat metric tensor $\eta_{\mu\nu}$ from now on only. By virtue of a *Wick rotation* we can even replace the Minkowski metric with a flat Euclidean metric $\delta_{\mu\nu}$.[10] This turns out to be advantageous if we go to complex coordinates in the following, as (1.7) for $d = 2$ and $g_{\mu\nu} = \delta_{\mu\nu}$ becomes

$$\partial_1\varepsilon_1 = \partial_2\varepsilon_2 \ , \qquad \partial_1\varepsilon_2 = -\partial_2\varepsilon_1 \ . \tag{1.18}$$

which are the well-known Cauchy-Riemann differential equations.

Complexification and the conformal algebra in two dimensions

In the beginning we started with a time coordinate x^0 and a space coordinate x^1. Upon Wick rotating, the time coordinate became imaginary: $x^0 \mapsto -ix^2$. We now choose $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$ as coordinates on the complex plane. Also we introduce $\varepsilon(z,\bar{z}) = \varepsilon_1 + i\varepsilon_2$ and $\bar{\varepsilon}(z,\bar{z}) = \varepsilon_1 - i\varepsilon_2$ such that the infinitesimal coordinate transformations become

$$z \rightarrow z' = z + \varepsilon(z,\bar{z}) \ , \qquad \bar{z} \rightarrow \bar{z}' = \bar{z} + \bar{\varepsilon}(z,\bar{z}) \ . \tag{1.19}$$

The coordinates z and its complex conjugate $\bar{z}$ depend on each other, but often it is useful to regard them as independent complex coordinates. Usually this is done by extending the range of the Cartesian coordinates x^1 and x^2 to the complex plane. Then the change from $(x^1, x^2) \in \mathbb{C}^2$ to $(z,\bar{z}) \in \mathbb{C}^2$ is just a

[9] Here a short word of warning is in order: physicists often speak and write about an infinite-dimensional conformal group. This of course is (mathematically) incorrect since the conformal group is isomorphic to the group $SL(2,\mathbb{C})/\mathbb{Z}_2$. Martin Schottenloher describes this misunderstanding in Schottenloher (1997) as follows: "[...] physicists mostly think and calculate infinitesimally, while they write and talk globally. Many statements become clearer, if one replaces 'group' with 'Lie algebra' and 'transformation' by 'infinitesimal transformation' [...]".

[10] Note though that using Wick rotations is full of subtleties. One has to ensure that the considered quantities can be analytically continued to the Euclidean space.

change of variables, and thus z and $\bar{z}$ can be treated as independent variables. If needed at all a reality condition can be imposed at the end.

Changing the variables as mentioned before and using (1.19) reduces the conformal Killing equation (1.7) to

$$\partial_z \bar{\varepsilon}(z, \bar{z}) = 0 , \qquad \partial_{\bar{z}} \varepsilon(z, \bar{z}) = 0 , \tag{1.20}$$

which means that $\varepsilon = \varepsilon(z)$ does not depend on $\bar{z}$. Thus any holomorphic[11] function $\varepsilon(z)$ is an infinitesimal conformal transformation, and the same holds for the anti-holomorphic function $\bar{\varepsilon}(\bar{z})$. Since we have locally infinitely many linearly independent holomorphic functions at our disposal, this means that the dimension of the conformal algebra is infinitely large. As we pointed out in the beginning of this section this is a specialty in two dimensions, in higher dimensions the conformal algebra is finite.

More systematically we expand the infinitesimal coordinate transformations in a basis

$$z \to z' = z - a_n z^{n+1} , \qquad \bar{z} \to \bar{z}' = \bar{z} - \bar{a}_n \bar{z}^{n+1} , \quad n \in \mathbb{Z} , \tag{1.21}$$

and determine the corresponding infinitesimal generators

$$\ell_n = -z^{n+1}\partial_z , \qquad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}} , \quad n \in \mathbb{Z} . \tag{1.22}$$

These operators satisfy the *classical conformal algebra*, which is the direct sum of two isomorphic algebras, also known as *Witt algebra*[12]

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} , \qquad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m} , \tag{1.23}$$

and furthermore the holomorphic and the anti-holomorphic part decouple

$$[\ell_n, \bar{\ell}_m] = 0 . \tag{1.24}$$

We will notice the independence of these two parts in many other places as well. Nevertheless the theory is not always a simple product of the two sectors.

[11] Here again the usage of language differs between mathematics and physics. Mathematicians mean by "holomorphic" that the function is differentiable in the vicinity of some point. Most physicists use the term "holomorphic" in the sense that the function does depend on z only and not on $\bar{z}$ at all. In this sense singularities are not forbidden. Mathematicians would call such functions "meromorphic". In the scope of this thesis we use the term "holomorphic" in the physical sense.

[12] As we will see the quantized version requires adding an additional term which is also known as central extension of the Witt algebra.

In particular logarithmic conformal field theories are known for not factorizing into two sectors, cf. subsection 1.2.1.

Though the conformal algebra is infinite-dimensional not every generator is well-defined globally on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. One finds that the (global) conformal group is generated by the subalgebra of the Witt algebra which consists of $\{\ell_n, \bar{\ell}_n\}$ for $n = -1, 0, 1$. This results in global conformal transformations of the form

$$f(z) = \frac{az+b}{cz+d}\,, \qquad \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\,, \tag{1.25}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$ and the analogous conditions apply to the (independent) anti-holomorphic sector.

The transformations f, $\bar{f}$ are known as *projective conformal transformations* or *Möbius transformations*. The group of projective conformal transformations is isomorphic to $PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/\mathbb{Z}_2$ which itself is isomorphic to $SO^+(3, 1)$. The group $SO^+(3, 1)$ denotes the *proper orthochronous Lorentz group* in four dimensions.[13] As there is a holomorphic and an anti-holomorphic sector the (global) conformal group of the component containing the identity is isomorphic to $PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$.

Note though that we duplicated the number of variables in the beginning by saying that z and $\bar{z}$ are linearly independent. Removing this artifical expansion by applying the reality condition again halves the size of the conformal group, such that the (global) conformal group for the component containing the identity is $SO^+(3, 1)$.[14]

Finally we remark that the operators $\ell_{-1}, \bar{\ell}_{-1}$ are the generators of the translations, and $\ell_1, \bar{\ell}_1$ generate the special conformal transformations. It is worth keeping in mind that that $\ell_0 + \bar{\ell}_0$ generates dilatation and $K := i(\ell_0 - \bar{\ell}_0)$ generates the rotations.

The energy-momentum tensor in two dimensions

We derived a couple of properties which every energy-momentum tensor in a conformal field theory has to fulfill, cf. subsection 1.1.5. In two dimensions

[13] In notation and terms we follow Sexl and Urbantke (1992) and call $O(3, 1)$ the *Lorentz group* and $SO(3, 1)$ the *proper Lorentz group*. Finally $SO^+(3, 1)$ called the proper orthochronous Lorentz group is the component of the identity, a subgroup of $O(3, 1)$. Be careful that conventions in the literature differ widely, concerning the notation as well as the meaning of the term “Lorentz group”.

[14] In this sense the (global) conformal group for the component containing the identity is for every $d \geq 2$ isomorphic to $SO^+(p + 1, q + 1)$. Here with $p = 2$ and $q = 0$.

the tracelessness implies $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and by using (1.16) we find $\partial_z T_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}} T_{zz} = 0$. Thus the remaining two components are holomorphic and anti-holomorphic, respectively

$$T_{zz} \equiv T(z) , \qquad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) . \tag{1.26}$$

Obviously the conformal symmetry current (1.17) then becomes

$$j_z = T_{zz}\, \varepsilon(z), \qquad j_{\bar{z}} = T_{\bar{z}\bar{z}}\, \bar{\varepsilon}(\bar{z}) , \tag{1.27}$$

and we again find a separation into a holomorphic and an anti-holomorphic part.

Primary, quasi-primary and secondary fields

A field that transforms under any infinitesimal conformal transformations $z \to w(z)$ respectively $\bar{z} \to \bar{w}(\bar{z})$ as

$$\phi(z,\bar{z}) \to \phi'(w,\bar{w}) = \phi(z,\bar{z}) \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \tag{1.28}$$

is called a *primary field* of *conformal weight* (or *conformal dimension*) $(h, \bar{h})$.[15] The conformal weights h, $\bar{h}$ are real quantities, and the bar does *not* refer to complex conjugation. The sum $\Delta := h + \bar{h}$ is called *scaling dimension*, and $s := h - \bar{h}$ is often referred to as *conformal spin*. We will see later that h and $\bar{h}$ are eigenvalues of ℓ_0 and $\bar{\ell}_0$, respectively.

In the following, we will need the infinitesimal version of equation (1.28). For $w(z) = z + \varepsilon(z)$ and $\bar{w}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z})$ a primary field transforms infinitesimally as

$$\phi'(w,\bar{w}) = \phi(w,\bar{w}) - \underbrace{\left[h(\partial\varepsilon) + \varepsilon\partial + \bar{h}(\bar{\partial}\bar{\varepsilon}) + \bar{\varepsilon}\bar{\partial}\right] \phi(w,\bar{w})}_{=:\delta_{\varepsilon,\bar{\varepsilon}}\phi(w,\bar{w})} + \mathcal{O}(\varepsilon^2) , \tag{1.29}$$

where $\partial \equiv \partial_w$ and $\bar{\partial} \equiv \partial_{\bar{w}}$. We will see that these infinitesimal transformations are generated by charges that are constructed from the energy-momentum tensor.

The class of primary fields plays an outstanding role in conformal field theories. All fields that do not transform according to (1.28) are termed *secondary fields*. Sometimes, we refer to fields which transform under (global) conformal

[15] Note that the given definitions can be easily extended to arbitrary dimensions.

transformations according to (1.28) as *quasi-primary*. Thus primary fields are always quasi-primary, but secondary fields are not necessarily quasi-primary. We will learn in subsection 1.1.8 that the energy-momentum tensor is a quasi-primary field, but not a primary field. Derivatives of primary fields are typical examples for secondary fields.

1.1.7 Radial quantization

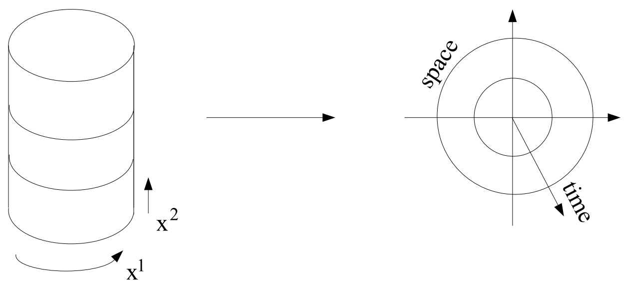
Up to now we considered a classical two-dimensional conformal field theory only. We will now quantize the system using a procedure which is called *radial quantization*. As a first step we compactify the space coordinate by the identifications

$$z \sim z + 2\pi , \qquad \bar{z} \sim \bar{z} + 2\pi , \tag{1.30}$$

which means that the worldsheet now is a cylinder. This compactification removes infrared divergences from the theory. Because of scale invariance the value of 2π is not special, but was chosen for convenience only. In order to come back to the complex plane we use the conformal map

$$w := e^{iz} = e^{x^1 + ix^2} . \tag{1.31}$$

The infinite past in the Euclidean time coordinate $x^2 = -\infty$ is mapped to $w = 0$ and the infinite future $x^2 = \infty$ is mapped to the infinite circle at $|w| = \infty$. Surfaces of equal time are mapped to circles of constant radius.



In the new coordinates the usual “time ordering” becomes a *radial ordering* which is defined[16] for two operators A, B as follows

$$R\big(A(z)B(w)\big) = \begin{cases} A(z)B(w) & , \quad \text{for } |z| > |w| \\ B(w)A(z) & , \quad \text{for } |z| < |w| . \end{cases} \tag{1.32}$$

[16]If the operators are fermionic we need to include a minus sign in the definition for $|z| < |w|$.

In (1.17) and (1.27), respectively, we identified the conserved conformal current. The corresponding conserved charge Q is calculated by integrating over the space for a fixed time, which here translates to a fixed radius and thus

$$Q_{\varepsilon,\bar{\varepsilon}} = \frac{1}{2\pi i} \oint dz\, \varepsilon(z) T(z) + \text{h.c.}\ , \tag{1.33}$$

where "h. c." denotes hermitian conjugation. The integration has to be performed counter-clockwise, and the center of the circle is the origin of the complex plane.[17] The operator $Q_{\varepsilon,\bar{\varepsilon}}$ generates infinitesimal symmetry variations of any field ϕ according to

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(w,\bar{w}) = [Q_{\varepsilon,\bar{\varepsilon}}, \phi(w,\bar{w})]\ . \tag{1.34}$$

The right hand side of the above equation can be evaluated further. Plugging in equation (1.33) and expanding the commutator results in two contour integrations where one has to be taken for $|z| > |w|$ and the other for $|z| < |w|$. Combining the integrals leads to a single contour integration around the point w, and the integration is again to be taken counter-clockwise, which leads to[18]

$$[Q_{\varepsilon,\bar{\varepsilon}}, \phi(w,\bar{w})] = \frac{1}{2\pi i} \oint_w dz\, \varepsilon(z) R\big(T(z)\phi(w,\bar{w})\big) + \text{h.c.}\ \ . \tag{1.35}$$

We assume that the radially ordered product $R(T(z)\phi(w,\bar{w}))$ is analytic in the vicinity of the point w and do a Laurent series expansion around the point w,

$$R\big(T(z)\phi(w,\bar{w})\big) = \sum_{n\in\mathbb{Z}} (z-w)^n O_n(w,\bar{w})\ , \tag{1.36}$$

where the O_n denote operators which we are going to determine in the following. The above equation is also known as *operator product expansion* (OPE).

For now we want to determine the OPE in the case of $\phi(w,\bar{w})$ being a primary field. Then we can insert (1.29) on the left hand side of equation (1.34) and compare this with the right hand side of (1.34) as given by (1.35)

$$-\left[h(\partial\varepsilon) + \varepsilon\partial\right]\phi(w,\bar{w}) = \frac{1}{2\pi i} \oint_w dz\, \varepsilon(z) \sum_n (z-w)^n O_n(w,\bar{w}) \tag{1.37}$$

where we already inserted the ansatz (1.36). The anti-holomorphic part of course leads to an analogous equation. With the help of the residue theorem

[17] Note though that that integration cannot be evaluated directly, as $Q_{\varepsilon,\bar{\varepsilon}}$ is an operator that needs to be applied to a field first.

[18] For all details in particular with respect to the ordering of the operators see Ginsparg (1988).

we can easily compute the coefficients O_n and infer that the operator product expansion is given by

$$R\big(T(z)\phi(w,\bar{w})\big) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial\phi(w,\bar{w}) + \dots\ , \tag{1.38}$$

$$R\big(\bar{T}(z)\phi(w,\bar{w})\big) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\phi(w,\bar{w}) + \dots\ . \tag{1.39}$$

From now on we will omit the symbol R for the radial ordering operator if we are dealing with operator products as is customary in the CFT literature. All operator products are understood to be radially ordered. The ellipsis in equations (1.38) and (1.39) stand for a Taylor series in $(z-w)$ which we will neglect in the future as well, as we are interested in the behavior for $z \to w$ only.

We point out that it was the transformation law (1.29) of the primary field ϕ which lead to the form of the operator product expansion. Conversely the operator product expansion, and more precisely its singular terms, include all information about the infinitesimal conformal transformation properties of a given field. Because of this we can also use (1.38) and (1.39) as alternative definition for ϕ being a primary field. In an OPE with the energy-momentum tensor the secondary fields have higher than second order singularities.

1.1.8 Central charge and Virasoro algebra

We already mentioned in subsection 1.1.6 that at the quantum level the energy-momentum tensor T does not transform like a primary field. Nevertheless we can determine the OPE of the stress-energy tensor with itself by applying two conformal transformations in succession:[19]

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{1.41}$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) \sim \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial\bar{T}(\bar{w})\ . \tag{1.42}$$

Here we use the symbol "$\sim$" which means "modulo regular terms". The constants c and $\bar{c}$ are called *central charge* or *conformal anomaly*.[20] These con-

[19] The commutator of two infinitesimal conformal transformations is given by

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\varepsilon_1 \partial \varepsilon_2 - \varepsilon_2 \partial \varepsilon_1} \tag{1.40}$$

where we considered the holomorphic part only. See also Lüst and Theisen (1989).

[20] The $(z-w)^4$ term of the conformal anomaly complies with analyticity, Bose symmetry and scale invariance.

stants cannot be fixed by the conformal properties, but are model dependent.[21] For instance the central charge is 1 for the free boson and $\frac{1}{2}$ for the free fermion. Other well-known models include the simple b-c ghost system with $c = -2$ and $c = -26$ for reparametrization ghosts. In chapter 2 we will discuss the latter two models in more detail and in particular will pay attention to logarithmic extensions.

For vanishing conformal anomaly $c = 0$ the energy-momentum tensor $T(z)$ is a primary field of weight $(2, 0)$. In presence of a conformal anomaly the field $T(z)$ is not quite primary, but quasi-primary as can be inferred from the infinitesimal conformal transformation of $T(z)$

$$\delta_{\varepsilon,\bar{\varepsilon}} T(z) = \varepsilon(z)\partial_z T(z) + 2T(z)\partial_z \varepsilon(z) + \frac{c}{12}\,\partial_z^3 \varepsilon(z)\ . \tag{1.43}$$

The above equation can be derived from the OPE of $T(z)T(w)$, cf. (1.41). Of course analogous consideration apply to the anti-holomorphic sector as well. From now on we will almost always consider the holomorphic sector only. The next step is to perform a Laurent expansion of the energy-momentum tensor:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\ , \tag{1.44}$$

where we call the modes *Virasoro generators*. These generators in turn can be written as

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)\ . \tag{1.45}$$

The hermiticity of the stress-energy tensor $T(z)$ leads to

$$L_n^\dagger = L_{-n}\ . \tag{1.46}$$

Inserting (1.45) into the commutator $[L_n, L_m]$ and performing some contour integrations leads to the famous Virasoro algebra[22]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n-1)\,n\,(n+1)\,\delta_{n,-m}\ . \tag{1.47}$$

As usual (in the absence of a boundary), the holomorphic and the anti-holomorphic sector completely decouple

$$\left[L_n, \bar{L}_m\right] = 0\ . \tag{1.48}$$

[21] We will see in subsection 1.1.9 that the constants label representations of the Virasoro algebra.

[22] Note though that the paper Virasoro (1970) does not contain the algebra, but the generators of the algebra.

Note that the Virasoro algebra (1.47) reduces to the classical conformal algebra (1.23) for $c = \bar{c} = 0$. Furthermore, note that the subalgebra generated by $\{L_{-1}, L_0, L_1\}$ remains unaffected by the central charge. In particular this means that the global conformal group $PSL(2,\mathbb{C})$ continues to be an exact symmetry group.

We need one final relation for the the following subsection. Let $\phi(z,\bar{z})$ be a primary field of conformal weight $(h,\bar{h})$. Then the equation

$$[L_n, \phi(w,\bar{w})] = h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial_w\phi(w,\bar{w}) \ , \tag{1.49}$$

can be verified by inserting (1.45) on the left hand side and then using the known OPE (1.38) to replace the appearing $T(z)\phi(w,\bar{w})$ terms. With the help of the residue theorem one immediately recovers the above result.

1.1.9 Virasoro representation theory

In this subsection our interest is to constitute representations of the Virasoro-algebra (1.47). For details about this topic we recommend Kac and Raina (1988). In the previous subsection we found that the holomorphic and anti-holomorphic parts of the Virasoro-algebra decouple, cf. (1.48). Therefore considering the holomorphic sector in the following is sufficient. We just need to keep in mind that the full representation in fact is a tensor product of two representations.

Let $|0\rangle$ be the in-vacuum of the theory. We require $T(z)|0\rangle$ to be regular at the origin ($z = 0$) and infer that

$$L_n|0\rangle = 0 \qquad \forall n \geq -1 \ . \tag{1.50}$$

Sometimes we also refer to this vacuum as the "$SL(2,\mathbb{C})$ invariant vacuum". For the out-vacuum $\langle 0|$ we find $\langle 0|L_m = 0 \ \forall m \leq 1$. Now let $\phi(z)$ be a primary field of conformal weight h. The state

$$|h\rangle := \lim_{z\to 0}\phi(z)|0\rangle \tag{1.51}$$

is called a *highest weight state*. This name is justified by the following consideration:

$$L_n\,|h\rangle = [L_n, \phi(0)]|0\rangle = 0 \qquad \forall n \geq 1 \ , \tag{1.52}$$

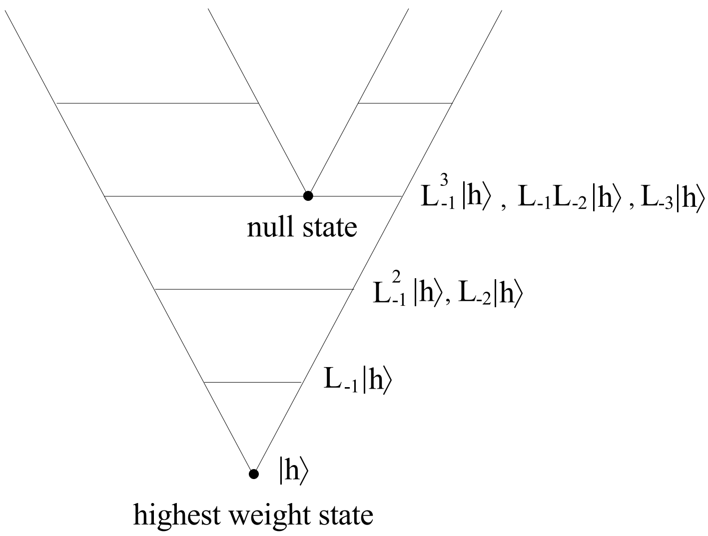
where we used (1.49) to derive the last equality. For $n = 0$ the equation (1.49) leads to $[L_0, \phi(0)] = h\phi(0)$, and therefore L_0 is diagonal with respect to the highest weight states

$$L_0 \left| h \right\rangle = h \left| h \right\rangle \ . \tag{1.53}$$

Because of the hermiticity condition (1.46) all eigenvalues h of L_0 are real. Applying the *raising operators* L_{-n} for $n > 0$ on a highest weight state can create new states.[23] The *Verma module* $\mathcal{V}_{c,h}$ of $|h\rangle$ includes all states that can be created from $|h\rangle$ by applying all combinations of the Virasoro generators[24]

$$\mathcal{V}_{c,h} = \bigoplus_{\{n_1,n_2,\ldots\}} \Big\langle L_{-n_1} \ldots L_{-n_k} \left| h \right\rangle : 1 \leq n_1 \leq \ldots \leq n_k, k \geq 0 \Big\rangle \ . \tag{1.54}$$

A state $|k\rangle \in \mathcal{V}_{c,h}$ with $|k\rangle \neq |h\rangle$ is called a *descendant state.* The primary state and its descendants form a so-called *conformal family.* By using the Virasoro algebra (1.47) one can easily show that all states of the form $L_{-n_1} \ldots L_{-n_k} |h\rangle$ are eigenstates of L_0 with eigenvalue $h + \sum_{i=1}^{k} n_i =: h + N$. The number N is called the *level of the state.* In the following picture we give a graphical representation of a Verma module $\mathcal{V}_{c,h}$:



[23]The representations of $su(2)$ have some similarities with the representations of the Virasoro algebra and are often used in the literature as an illustrative example. The $su(2)$ generators $J_\pm = J_x \pm iJ_y$ are called "raising" (J_+) and "lowering operators" (J_-). Analogously the Virasoro operators L_{-n} with $n > 0$ are referred to as *raising operators.*

[24]We do not need to include modes L_n with $n > 0$ as these can be commuted to the right such that they annihilate the highest weight state $|h\rangle$. By using an analogous argument we can also demand that the generators are ordered, i. e. $n_i \geq n_j$ if $i > j$.

Before we explain the meaning of a "null vector", we need to introduce an inner product on the space of states, which for two states

$$|\chi\rangle := L_{-n_1} \dots L_{-n_k} |h\rangle \quad \text{and} \quad |\eta\rangle := L_{-m_1} \dots L_{-m_j} |h\rangle$$

is defined by

$$\langle\chi|\eta\rangle := \langle h| L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_j} |h\rangle \ . \tag{1.55}$$

Depending on the central charge c and the conformal weight h, it is possible that states $|\chi\rangle$ in the Verma module exist, which are orthogonal to all other states within the Verma module[25]

$$\langle\chi|\psi\rangle = 0 \qquad \forall\, |\psi\rangle \in \mathcal{V}_{c,h} \ . \tag{1.56}$$

These states that completely decouple from all other states are termed *null vectors* or *singular vectors.* If the Verma module $\mathcal{V}_{c,h}$ contains null vectors, then the representation is reducible, and we speak of a reducible Verma module.[26] An irreducible representation of the Virasoro algebra can be obtained by removing all null vectors (and its descendants) from $\mathcal{V}_{c,h}$.

Null vectors play an outstanding role in conformal field theory. As we will see in subsection 1.1.10, null vectors induce differential equations for the correlation functions which may fix remaining degrees of freedom in the four-point functions. Also the study of null vectors leads to the celebrated "minimal models", which we are going to discuss succinctly in subsection 1.1.11.

The inner product induces a *semi-norm* (often just called *norm*) on the space of states by

$$||\,|\eta\rangle\,||^2 := \langle h| L_{m_j} \dots L_{m_1} L_{-m_1} \dots L_{-m_j} |h\rangle \ , \tag{1.57}$$

and it is obvious that null vectors have norm zero. Besides the appearance of zero-norm states in the Verma module it is also possible that negative-norm states occur. A representation of the Virasoro algebra containing no negative-norm states is called a *unitary representation.* By using the Virasoro algebra we can easily prove the following relation:

$$||L_{-n}|h\rangle||^2 = \langle h| L_n L_{-n} |h\rangle = \left[2nh + \frac{1}{12}cn(n^2-1)\right] \langle h|h\rangle \ . \tag{1.58}$$

[25]The structure of the Virasoro algebra (1.47) induces a grading, in the sense that the inner product of two states is zero if the two states have different levels.

[26]In the literature the Verma module $\mathcal{V}_{c,h}$ itself is often referred to as representation of the Virasoro algebra, though in fact it is the representation space.

This means that all representations with negative central charge $c < 0$ are non-unitary. Furthermore, we can infer from this equation for $n = 1$ that $h < 0$ also leads to non-unitary representations. This means that necessary conditions for a unitary representation are $c \geq 0$ and $h \geq 0$.

While unitary models and in particular minimal models are of great importance, there is also a wide range of applications for non-unitary theories. For instance the conformal b-c ghost systems which are of interest in string-theory always have a central charge $c < 0$ and thus are non-unitary. It should be mentioned, too, that all logarithmic conformal field theories are non-unitary.

1.1.10 Correlation functions

This subsection consists of three parts. In the first part we derive the so-called global conformal Ward identities (GCWIs) which put strong constraints on correlators. In the subsequent part we apply the GCWIs on 2-, 3- and 4-point correlation functions and state the results. The final part of this subsection discusses briefly how null vectors induce differential equations.

We consider the n-point correlation function G (where we as usual omit the anti-holomorphic part of the theory)

$$G(z_1, \ldots, z_n) := \langle 0| \, R\left(\phi_1(z_1) \ldots \phi_n(z_n)\right) |0\rangle \ . \tag{1.59}$$

The fields ϕ_k denote primary fields for the moment. As explained in subsection 1.1.7 the symbol R is the radial ordering operator. In the following we omit this operator and implicitly understand every sequence of fields within a correlator as radially ordered. For $i = -1, 0, 1$ we know that $\langle 0| \, L_i = 0$ and $L_i \, |0\rangle = 0$, and we can use these equations to derive the following relation:

$$\begin{aligned}
0 &= \langle 0| \, L_i \phi_1(z_1) \ldots \phi_n(z_n) \, |0\rangle \\
&= \sum_{k=1}^{n} \langle 0| \, \phi_1(z_1) \ldots \phi_{k-1}(z_{k-1}) [L_i, \phi_k(z_k)] \phi_{k+1}(z_{k+1}) \ldots \phi_n(z_n) \, |0\rangle \, + \\
&\quad \langle 0| \, \phi_1(z_1) \ldots \phi_n(z_n) L_i \, |0\rangle \\
&= \sum_{k=1}^{n} \langle 0| \, \phi_1(z_1) \ldots \phi_{k-1}(z_{k-1}) \, \delta_{\varepsilon_i} \phi_k(z_k) \, \phi_{k+1}(z_{k+1}) \ldots \phi_n(z_n) \, |0\rangle \ ,
\end{aligned} \tag{1.60}$$

where $\varepsilon_i(z) = z^{i+1}$. To show the last equality we made use of (1.34). Inserting the definition of δ_ε, cf. (1.29), for $\varepsilon(z) = 1, z, z^2$ leads to a set of three equations which are known as *(global) conformal Ward identities*,

$$L_i \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle := \sum_{k=1}^{n} z_k^i \left[z_k \partial_k + (i+1) h_k \right] \langle \ldots \rangle = 0 \, . \tag{1.61}$$

These three relations describe the constraints that global conformal invariance puts on the correlation functions. With the help of the GCWIs it is possible to determine the two- and three-point functions up to constants. We will not give a detailed derivation here and refer to Schellekens (1996) instead. As mentioned above, the fields ϕ_k are primary fields, and h_k denotes the corresponding conformal weights. The two-point function $G(z_1, z_2)$ is up to a normalization constant C given by

$$G(z_1, z_2) = C \, \frac{\delta_{h_1, h_2}}{(z_1 - z_2)^{2h_1}} \, . \tag{1.62}$$

By changing the normalization of the primary fields, the constant C can be set to 1. For the anti-holomorphic sector one gets an additional factor $(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}_1}$, where the conformal weights $\bar{h}_1$ and $\bar{h}_2$ of course have to match, too. The three-point function turns out to be[27]

$$G(z_1, z_2, z_3) = \frac{C(h_1, h_2, h_3)}{z_{12}^{h_1+h_2-h_3} \, z_{13}^{h_1+h_3-h_2} \, z_{23}^{h_2+h_3-h_1}} \, . \tag{1.63}$$

Here we introduced the abbreviation $z_{ij} := z_i - z_j$. The constants $C(h_1, h_2, h_3)$ are called *structure constants*. They cannot be determined by global conformal invariance alone. That we can determine the three-point function from global conformal transformations is clear in the following sense: these three complex transformations (translations, scaling plus rotation and special conformal transformations) can be used to fix the three variables z_1, z_2 and z_3. The same holds for the anti-holomorphic sector as well (since there exist six generators of the global conformal group L_i and $\bar{L}_i$, $i = -1, 0, 1$). This consideration also means that the global conformal Ward identities are not sufficient to fix the four-point function, but they can be used to determine the generic form:

$$G(z_1, z_2, z_3, z_4) = \prod_{i<j} z_{ij}^{\mu_{ij}} F(x) \, . \tag{1.64}$$

The exponents $\mu_{ij} = \mu_{ji}$ must satisfy the condition

$$\sum_{j \neq i} \mu_{ij} = -2h_i \tag{1.65}$$

[27] Actually exploiting the global conformal Ward identities shows that the correlator depends on z_{12} and z_{23} only, but in order to get a more symmetric form the redundant variable z_{13} was kept in the result.

and the function F depends on the *anharmonic ratio*

$$x := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \ . \tag{1.66}$$

This variable is invariant under Möbius transformations, and therefore the GCWIs cannot restrict the function F any further. On the other hand things change completely in the presence of a null vector as we briefly explain in the following.

However, we need to introduce the[28] (descendant) field that corresponds to a descendant state. The descendant field that belongs to the state $L_{-p}|h\rangle$ is given by

$$\phi^{(-p)}(z) := \frac{1}{2\pi i} \oint \frac{dw}{(w-z)^{p-1}} T(w)\phi(z) \ , \tag{1.67}$$

as one can quickly verify by evaluating $\phi^{(-p)}(0,0)|0\rangle$ with the help of equation (1.45). A slight generalization of the first part of this subsection, namely inserting $\oint \frac{dz}{2\pi i}\varepsilon(z)T(z)$ instead of L_i into the correlator leads, after some calculations, confer Schellekens (1996), to

$$\langle 0| T(z)\phi_1(z_1)\dots\phi_n(z_n)|0\rangle = \sum_{k=1}^{n}\left[\frac{h_k}{(z-z_k)^2} + \frac{1}{z-z_k}\partial_{z_k}\right]\langle\dots\rangle \ , \tag{1.68}$$

$$\langle\dots\rangle := \langle 0|\phi_1(z_1)\dots\phi_n(z_n)|0\rangle \ . \tag{1.69}$$

With the help of the two previous equations a correlator that contains a descendant can be evaluated (see the given reference for details). This results in

$$\langle 0|\phi_1(z_1)\dots\phi_{n-1}(z_{n-1})\phi_n^{(-p)}(z_n)|0\rangle = \mathcal{L}_{-p}\langle\dots\rangle \ , \tag{1.70}$$

where $\langle\dots\rangle$ is defined as in (1.69) and the newly introduced operator $\mathcal{L}_{-p}$ is given by

$$\mathcal{L}_{-p} := \sum_{k=1}^{n-1}\mathcal{L}_{-p}^{k} \ , \tag{1.71}$$

$$\mathcal{L}_{-p}^{k} := -\frac{(1-p)h_k}{(z_k-z_n)^p} + \frac{1}{(z_k-z_n)^{p-1}}\partial_{z_k} \ . \tag{1.72}$$

Equation (1.70) tells us that we can reduce any correlator containing a descendant field to an operator that acts on a correlator of primaries. This can

[28]In a well-defined conformal field theory there should be an isomorphism between the states of the Hilbert space and the fields of the theory.

also be done for correlators containing more than one descendant which then results in a more complicated formula.

The real power of equation (1.70) shows up in conjunction with null vectors. Let us assume that the Verma module $\mathcal{V}_{c,h}$ contains a level two null vector of the form $(L_{-1}^2 - tL_{-2})\,|h\rangle$, where t is some constant. Via the state field isomorphism, cf. (1.67), the descendent field corresponding to the null vector is $\phi^{(-2)} - t\phi^{(-1,-1)}$, where $\phi^{(-1,-1)}$ can be constructed from $\phi^{(-1)}$ by means of (1.67). Using this null state within a correlator results by virtue of (1.70) in a partial differential equation of the form

$$\mathcal{D}\,\langle 0|\,\phi_1(z_1)\ldots\phi_n(z_n)\,|0\rangle = 0\ , \tag{1.73}$$

with $\mathcal{D}$ being some complicated differential operator. In the case of $n = 4$ the correlator can then be replaced by the generic form of the four-point function, as given by equation (1.64). This finally translates the partial differential equation into an ordinary differential equation (ODE) of order two.[29] In some cases the ODE can explicitly be solved (a brief example is given in subsection 1.2.1) and thus the four-point function can be determined. As emphasized before this procedure requires that the Verma module contains a null state. For a more detailed description we refer the reader to di Francesco *et al.* (1997).

1.1.11 Minimal models

The models with the central charge

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{p\,q} \qquad ,\ p,q \in \mathbb{N}_*\ , \tag{1.74}$$

with p and q coprime are called *minimal models*. In Belavin *et al.* (1984) it was pointed out that these models have the very interesting feature that they only possess a finite number of primary fields and that the operator algebra closes under fusion. With the following formula it is possible to determine all conformal weights of the primary fields that can appear in the model with central charge $c_{p,q}$:

$$h_{r,s} = \frac{(rq-sp)^2 - (p-q)^2}{4pq}\ , \tag{1.75}$$

where $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. The table of weights resulting from the combinations of (r,s) is also known as *Kac table*. Without proof we note

[29]It is not just by chance that the order of the ODE and the level of the null vector match. A more general consideration shows this being always the case.

that the first null vector in the reducible Verma module with highest weight $h_{r,s}$ can be found at level $l = rs$. As the central charge as well as all conformal weights are rational numbers these models are also called *rational conformal field theories*. If one is interested in unitary minimal models only, then it is possible to determine this subset of the minimal models by setting $p = q + 1$ which leads to

$$c_q = 1 - \frac{6}{q(q+1)} \ , \qquad q = 3, 4, 5, \dots \ . \tag{1.76}$$

These models are also known as the *unitary discrete series*. The conformal weights can be calculated as in (1.75).

The minimal models play a very important role for the conformal field theory because they describe discrete statistical models at their critical points, e. g., the Ising and the Potts model. A famous example is $q = 3$ which implies $c = \frac{1}{2}$ and corresponds to the Ising model. The conformal weights in this theory are given by formula (1.75): $h_{1,1} = 0$, $h_{2,2} = 1/16$ and $h_{2,1} = 1/2$ and the corresponding fields can be interpreted as operators of the Ising model, such as the continuum Ising spin or the energy operator.

1.2 Logarithmic conformal field theory

Only three years after Belavin, Polyakov and Zamolodchikov (1984) started investigating conformal field theories in two dimensions it was noted by Knizhnik (1987) that correlation functions may also exhibit logarithmic divergences. Six years later Gurarie (1993) introduced the concept of a conformal field theory with logarithmic singularities. The basic feature of these so-called logarithmic conformal field theories (LCFTs) is that the representations of the chiral symmetry algebra may be indecomposable. It is worth pointing out that various aspects of logarithmic conformal field theories were noted in the literature before the work of Gurarie, e. g. Rozansky and Saleur (1993); Saleur (1992a,b).

In the following years almost all important structures, methods and tools, which were known from (rational) conformal field theories, were adapted and generalized to the LCFT case. This includes characters, null vectors, OPEs and correlation functions. The latter will be an important part of this thesis and is discussed in chapter 3. Nowadays the understanding of LCFT is almost at the same level as the one about (rational) conformal field theories. Furthermore, there exists a huge number of applications for LCFTs which include topics like two-dimensional conformal turbulence, fractional quantum Hall effect and

AdS/CFT correspondence. In subsection 1.2.3 we give a short overview about applications of LCFT.

Unfortunately the introductory literature into the topic is not as comprehensive as the literature for CFTs is. Probably the most extensive introductory works are Flohr (2003); Gaberdiel (2003) and Rahimi Tabar (2003). Further works that are of interest are Flohr (2002a); Kawai (2003) and Moghimi-Araghi *et al.* (2003).

In the next subsection we present one of the best studied examples of logarithmic conformal field theory and show why logarithmic operators have to be taken into account. In subsection 1.2.2 we introduce several definitions and discuss the field content of LCFTs in more detail. Finally, we give a short overview of important works and applications of LCFT.

1.2.1 A famous example

The $c = -2$ has been extensively studied, cf. Cappelli *et al.* (1999); Flohr (1997); Gurarie *et al.* (1997); Kausch (1991, 1995, 2000). For us the paper by Gurarie (1993) is of high interest, because the model is a prime example for a model with logarithmic behavior. The structure of this subsection follows Nichols (2002). By determining a correlation function in the $c_{1,2} = -2$ system we will learn that we cannot avoid logarithmic terms in the correlator. This has some important consequences for the operator product expansion and eventually forces us to include logarithmic operators in the theory.

Let us take the primary field $\phi_{1,2}$ with conformal weight $h_{1,2} = -\frac{1}{8}$ which has its first null vector on level two. While the combination $r = 1$ and $s = 2$ is not supported by (1.75) it nevertheless is part of the set of admissible irreducible highest weight representations, as shown by Wang (1998). As Knizhnik (1987) pointed out, nontrivial Riemann-surfaces, seen as a multi-sheeted covering of the complex plane, can be simulated by twist-fields inserted at the branch points. The field with conformal weight $-1/8$ simulates a $\mathbb{Z}_2$-branch point, which means that this field indeed can show up in a conformal field theory. The null vector of level two can be determined to be

$$\left(L_{-2} - 2L_{-1}^2\right)|\mu\rangle = 0\ . \tag{1.77}$$

Hereby we have adapted the notation, as it is customary in the literature, and defined $\mu \equiv \phi_{1,2}$ as well as $|\mu\rangle := \lim_{z\to 0}\mu(z)|0\rangle$. As described in subsection 1.1.10 we can determine the general form of the four-point correlation function, by making use of the global conformal invariance, which leads to

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}\,[x(1-x)]^{\frac{1}{4}}\; G(x)\ . \tag{1.78}$$

The function $G(x)$ is holomorphic and depends on the anharmonic ratio x only, cf. (1.66). The null vector of level two (1.77) induces, as explained in subsection 1.1.10, a second order differential equation in $G(x)$:

$$x(1-x)\frac{d^2G(x)}{dx^2} + (1-2x)\frac{dG(x)}{dx} - \frac{1}{4}G(x) = 0\ . \tag{1.79}$$

The differential equation has two independent solutions[30]

$$G(x) = c_1 K(x) + c_2 K(1-x)\ , \tag{1.81}$$

where c_1 and c_2 are constants and $K(x)$ is a complete elliptic integral of the first kind

$$K(x) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}\ . \tag{1.82}$$

The function $K(x)$ is regular in the vicinity of $x = 0$, but it has a logarithmic singularity at $x = 1$. To make the logarithmic singularity more obvious we rewrite the second solution $K(1-x)$ by using the following formula

$$K(1-x) = K(x)\log(x) + H(x)\ , \tag{1.83}$$

where $H(x)$ is a regular function in the vicinity of $x = 0$, for a more detailed discussion see Bhaseen (2001). This means that *all* nontrivial solutions (1.81) contain some logarithmic divergences which in turn has deep consequences for the operator product expansion as explained in the following. Let us assume the usual assumptions about OPEs are still valid, then the OPE has the form

$$B_i(z)B_j(w) = \sum_n \frac{C_n}{(z-w)^{h_i+h_j-h_n}} B_n(w)\ , \tag{1.84}$$

[30] Actually this is not the full answer since we considered for simplicity only the holomorphic sector of the theory. In order to get a single-valued correlator we must consider both sectors and then we find that the only single-valued solution for the given conformal weights is:

$$G(x,\bar{x}) = K(x)\overline{K(1-x)} + K(1-x)\overline{K(x)} \tag{1.80}$$

Also note that the logarithmic terms that show up in the following also need to be single-valued. This forces us to consider the combinations of both sectors, e. g. $\log|x|^2 = \log x + \log\bar{x}$ and this is one reason why LCFTs are often more complicated than CFTs in the sense that they do not decompose in a holomorphic and an anti-holomorphic part.

with $B_n(w)$ being a primary field or a descendent of a primary and C_n denote some coefficients. Using this OPE in the left hand side of (1.78) obviously does not produce logarithms in the correlation function. As a consequence we have to modify the ansatz of the OPE and must allow logarithmic operators in it.

A careful analysis shows that the OPE is given by

$$\mu(z,\bar{z})\mu(w,\bar{w}) \sim |z-w|^{\frac{1}{2}} \left[\tilde{\mathbb{I}}(w,\bar{w}) + \log(|z-w|^2)\,\mathbb{I}(w,\bar{w})\right] . \tag{1.85}$$

In this equation $\mathbb{I}$ is the unit operator and $\tilde{\mathbb{I}}$ is the so-called *logarithmic partner field* of the $\mathbb{I}$-field. The OPE of $T(z)$ with the two fields and in particular with the $\tilde{\mathbb{I}}$-field is highly interesting:

$$T(z)\mathbb{I}(w,\bar{w}) \sim \frac{h\,\mathbb{I}}{(z-w)^2} + \frac{\partial\mathbb{I}}{z-w} \tag{1.86}$$

$$T(z)\tilde{\mathbb{I}}(w,\bar{w}) \sim \frac{h\,\tilde{\mathbb{I}} + \mathbb{I}}{(z-w)^2} + \frac{\partial\tilde{\mathbb{I}}}{z-w} , \tag{1.87}$$

where the conformal weight h here of course is zero. This means that the $\mathbb{I}$-field is an ordinary primary field and that the $\tilde{\mathbb{I}}$-field has the basic structure of a primary, but that the OPE also gives rise to an additional $\mathbb{I}$-field. Thus we can derive that L_0 has a Jordan-block structure

$$L_0 \begin{pmatrix} \mathbb{I} \\ \tilde{\mathbb{I}} \end{pmatrix} = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix} \begin{pmatrix} \mathbb{I} \\ \tilde{\mathbb{I}} \end{pmatrix} . \tag{1.88}$$

Obviously the representation of L_0 is reducible, but not fully reducible. An indecomposable representation is a typical feature of a logarithmic conformal field theory. Note though that the LCFT we are going to present in chapter 2 has the feature that L_0 is diagonal.

1.2.2 Remarks on the field content of LCFTs

In conformal field theory we basically build everything from primary fields. Descendants of primaries were termed secondary fields and some of these secondary fields were additionally quasi-primary fields, e. g., the energy-momentum tensor. In LCFT we encounter three main classes of fields which, together with their descendants, lead to all fields of the theory.

First there are so called *proper primary fields.* These behave in all ways like primary fields of CFT: proper primary fields have an OPE with the energy-momentum tensor $T(z)$ of the form given by equation (1.38) and the OPE of two proper primary fields is a Laurent expansion of the type (1.84).

The field μ of the previous subsection belongs to the important class of *pre-logarithmic fields*. Pre-logarithmic fields behave like (Virasoro) primary fields, e. g., the twist-field $\mu \equiv \phi_{1,2}$ with conformal weight $h_{1,2} = -\frac{1}{8}$ has the following OPE with $T(z)$:

$$T(z)\mu(w) \sim \frac{-\frac{1}{8}\,\mu(w)}{(z-w)^2} + \frac{\partial\mu(w)}{z-w}\,. \qquad (1.89)$$

In difference to the proper primary fields the OPE of a pre-logarithmic field with another such field may not lead to the usual OPE. An example for this was given in equation (1.85), where we discovered that pre-logarithmic fields give rise to logarithmic fields. This behavior is typical for pre-logarithmic fields: in all known LCFTs the OPEs of certain pairs of pre-logarithmic fields generated all logarithmic fields of the theory. Furthermore, so far every known logarithmic conformal field theory also contained fields of pre-logarithmic type.

The final class consists of *logarithmic fields*. These fields have an operator product expansion with the energy-momentum tensor $T(z)$ as given by equation (1.87). Thus logarithmic fields together with proper primary fields form Jordan-cells.

Let us take an example and consider the two fields $\mathbb{I}$, $\tilde{\mathbb{I}}$ of the previous subsection. Both fields have the same conformal weight $h = 0$ and the field $\mathbb{I}$ obviously is a primary field. The OPE (1.87) shows that $\tilde{\mathbb{I}}$ is a logarithmic field and we also refer to this field as the logarithmic partner field of the $\mathbb{I}$ field. The example was simple in the sense that the fields had conformal weight zero and that the Jordan-cell was small. Now we generalize the definitions and the notation a bit as follows:

A logarithmic field with conformal weight h of Jordan-level k is denoted by $\Psi_{(h,k)}$. The *Jordan-level* of the field is defined by the position of the field in the Jordan-matrix, see (1.88). A field of Jordan-level $k = 0$ is a proper primary field $\Psi_{(h,0)}$ or sometimes simply $\Psi_{(h)}$. Pre-logarithmic fields do not possess a well-defined Jordan-level, but nevertheless there are cases, where it turns out to be sensible to assign a fractional Jordan-level to a pre-logarithmic field. However, we almost do not deal with pre-logarithmic fields in the scope of this work and thus we will neglect these fields for most of the discussions. For a more detailed description of pre-logarithmic fields the interested reader is advised to consult Flohr (2002b).

The representation space is spanned by the states $|h,k\rangle$ defined by the field-state isomorphism $|h,k\rangle := \lim_{z\to 0} \Psi_{(h,k)}(z)\,|0\rangle$. Here $|0\rangle$, as usual, denotes the

$SL(2,\mathbb{C})$ invariant vacuum. Let r denote the *Jordan-rank* of the Jordan-cell we consider. The space of states of this Jordan cell then spanned by $\{|h,r-1\rangle, \ldots, |h,1\rangle, |h,0\rangle\}$. The action of the zero mode of the Virasoro algebra L_0, is then given by

$$L_0 |h,0\rangle = h |h,0\rangle \tag{1.90}$$

$$L_0 |h,k\rangle = h |h,k\rangle + |h,k-1\rangle \qquad \text{for } k = 1, \ldots, r-1 \,. \tag{1.91}$$

Now let r be the Jordan-rank of the largest Jordan-cell. Then we can assume without loss of generality, that all Jordan-cells are of Jordan-rank r. This can be achieved by padding the smaller Jordan-cells with fields which are set to zero afterwards.

1.2.3 Tools and applications of LCFT

Since Gurarie introduced logarithmic operators in 1993 a lot of work was done to understand and to make use of logarithmic conformal field theories. Almost all important concepts that were known from (rational) conformal field theories have been adapted or generalized to logarithmic conformal field theories. This includes characters, null vectors, correlation functions, operator product expansions, partition functions, fusion rules and more. The following references deal to some extent with generalizations to LCFTs: Flohr (1996, 1997, 1998b, 2000, 2002b); Flohr and Krohn (2005a); Gaberdiel and Kausch (1996a,b, 1999); Ghezelbash and Karimipour (1997); Kausch (2000); Khorrami *et al.* (1998); Kogan and Lewis (1998); Moghimi-Araghi and Rouhani (2000); Rahimi Tabar *et al.* (1997); Rahimi Tabar and Rouhani (1998) and Rohsiepe (1996).

In the following we briefly present various topics where a connection with logarithmic conformal field theory is assumed or established and add references for further reading. This list is by no means complete but should be useful as a starting point to conduct further research.

Apart from the before mentioned $c = -2$ model which was studied in Cappelli *et al.* (1999); Flohr (1997); Gaberdiel and Kausch (1996b, 1999); Gurarie (1993); Gurarie *et al.* (1997); Kausch (1991, 1995) and Kausch (2000) there was also extensive interest in the $c_{p,q}$ models, e. g., see Flohr (1996) and Gaberdiel and Kausch (1996a), and in Wess-Zumino-Novikov-Witten (WZNW) models, cf. Bernard *et al.* (1997); Caux *et al.* (1997); Gaberdiel (2001); Kogan *et al.* (1998); Lesage *et al.* (2002); Nichols and Siwach (2001) and Nichols (2001).

The low-energy effective action of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory as found by Seiberg and Witten (1994) is related to logarithmic conformal field theory, confer Cappelli *et al.* (1998) and Flohr (1998a).

There also exist numerous applications that are related to string theory. The important AdS/CFT correspondence discovered by Maldacena (1998) describes the equivalence of a string theory on some Anti-de-Sitter space and a CFT that lives on the boundary of this space. Correspondence with LCFTs has been described by Ghezelbash *et al.* (1999); Kogan (1999); Kogan and Polyakov (2001); Moghimi-Araghi *et al.* (2001b, 2004b) and Myung and Lee (1999). Another application of LCFT in string theory is the treatment of decaying D-branes, cf. Lambert *et al.* (2003).

Stochastic Löwner evolutions (SLE) is a random growth process that was introduced by Schramm (2000) and that is believed to be linked to conformal field theory. Recent papers by Rasmussen (2004) and Moghimi-Araghi *et al.* (2004a) suggest that there exists a link between SLEs and LCFTs too.

Chapter 2

Extensions of Ghost Systems

"Ach, da kommt der Meister!
Herr, die Not ist groß!
Die ich rief, die Geister,
Werd ich nun nicht los."
— Johann Wolfgang von Goethe (1797)

2.1 Overview

Ghost fields play a very important role in modern particle physics and they show up in almost any quantum gauge theory. Let us explain this in a bit more detail in the following. One way to (covariantly) quantize a field theory is by means of path integrals. In case the considered theory has local symmetries one needs to fix them. This in general leads to the appearance of (*Faddeev-Popov*) *ghost fields* and therefore to an additional contribution to the original action which is termed *ghost field action*. The name "ghost" derives from the fact that the fields violate the spin-statistics theorem and thus do not represent any physical degrees of freedom. String theories posses reparametrization- and Weyl invariance. In order to fix these local symmetries, so called *reparametrization ghosts*, denoted by b and c in the following, are introduced.

Our own interest in ghost fields stems from the observation that the conformal b-c ghost system resembles in the case of central charge $c = -2$ very much the so called θ-$\bar{\theta}$ system, which is a logarithmic conformal field theory. In particular we investigate if and how b-c ghost systems for different central charges can be extended to LCFTs.

We conjecture that each b-c system, with central charge $c_{b,c} = 12\lambda - 12\lambda^2 - 2$, is in fact a subset of a larger logarithmic CFT with Jordan-cells of higher rank related to the spin $\lambda > 0$. We will in detail study the case $c = -26$ which is the next integer spin case, $(\lambda, 1-\lambda) = (2, -1)$, after the well-known $c = -2$ theory with spin $(\lambda, 1-\lambda) = (1, 0)$. Indeed we find a nontrivial indecomposable structure of the Virasoro modules which, however, is quite different from the Jordan-cell structure in the $c = -2$ system.

We will proceed as follows:

In the next section we describe the most important properties of the b-c ghost system. Hereby we do not restrict ourselves to a particular central charge. In section 2.3 we will briefly recall the main properties of the $c = -2$ LCFT. First we have a look at the construction via symplectic fermions. We then review an alternative approach given by Fjelstad *et al.* (2002), where the $c = -2$ LCFT is built via a deformation of the energy-momentum tensor. In the fourth section we shortly discuss the question why we need to consider LCFTs at all and present a geometrical interpretation.

Section 2.5 of this chapter deals with the generalization of the procedures to the ghost system with central charge $c = -26$. Firstly, we consider the zero mode structure of the fields by using a generalization of the symplectic fermion method. It turns out that the energy-momentum tensor cannot be constructed in a similar fashion as in the $c = -2$ case out of these fields, without running into severe difficulties. However, a generalization of the method of deformation is possible and yields a consistent representation of the Virasoro algebra. Thus, the two approaches are not equivalent in the $c = -26$ case.

Unfortunately, this Virasoro algebra does not act consistently on the Hilbert space of states of this theory. The reason for this is related to the origin of the logarithmic operators, which arise from operator product expansions of twist fields, cf. Kogan and Lewis (1998). These twist fields exist whenever the theory is put on a nontrivial Riemann surface, see Knizhnik (1987). Thus, we investigate the theory on the simplest nontrivial Riemann surfaces, the hyperelliptic ones, and find that the full theory features a consistent Virasoro algebra with the correct action on its space of states. Although this full theory turns out to be logarithmic, its structure is very different from the $c = -2$ case. For example, the zero mode of the Virasoro algebra, L_0, turns out to be diagonal, i. e., the Virasoro modules are *not* indecomposable with respect to L_0. However, other Virasoro modes definitely lead to indecomposable structures. The section concludes with building highest weight states for different conformal weights and discussing a suitable generalization of the Jordan-rank of the theory.

2.2 The *b*-*c* ghost system

In this section we give a brief overview of the anti-commuting b-c conformal ghost systems, see also Lüst and Theisen (1989) and Ketov (1995) for further details on the topic. The b-c ghost system is a quantum field theory of two Grassmann-odd fields b and c. The fields are scalars under Lorentz transformations, but they also anti-commute. This means that they clearly violate the spin-statistics theorem and hence are no physical degrees of freedom.

If the fields are not Grassmann–odd, but Grassmann-even we call this the β-γ ghost system. In this case the equations that follow below need some sign modifications as described in Ketov (1995). Bosonic β-γ ghost systems have been studied by Lesage *et al.* (2002). In this section we will deal with the anti-commuting b-c system only.

The action of the system is given by

$$S_{b,c} = \frac{1}{2\pi} \int \mathrm{d}^2 z \left[b\bar{\partial} c + \bar{b}\partial\bar{c} \right] . \tag{2.1}$$

In the following we will consider the holomorphic sector of the theory only, i.e., the ghost fields $b(z)$ and $c(z)$. These have conformal weight $(\lambda, 0)$ and $(1-\lambda, 0)$ respectively, where $\lambda \geq \frac{1}{2}$ is integer or half-integer. In this chapter, we are interested in the integer spin case only. From the action we can derive the propagator of the system, which is

$$\langle c(z)b(w)\rangle = \frac{1}{z-w} . \tag{2.2}$$

This implies that the operator product expansion is

$$c(z)b(w) \sim \frac{1}{z-w} , \tag{2.3}$$

where the symbol "$\sim$" means, as usual, "modulo regular terms". We then perform a mode expansion for the ghost fields:

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-\lambda} , \tag{2.4}$$

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-(1-\lambda)} . \tag{2.5}$$

The modes obey the hermiticity conditions $b_n^\dagger = b_{-n}$, respectively $c_n^\dagger = c_{-n}$. We denote the $SL(2,\mathbb{C})$ invariant ghost vacuum with $|0\rangle$. Regularity at $z = 0$ and $z = \infty$ requires that

$$b_n|0\rangle = 0 \qquad \forall n \geq -\lambda + 1 \ , \tag{2.6}$$

$$c_n|0\rangle = 0 \qquad \forall n \geq \lambda \ . \tag{2.7}$$

This means that the system possesses $2\lambda - 1$ *zero modes*, namely (b_i, c_i) for $i = -\lambda+1, -\lambda+2, \ldots, \lambda-1$. These modes are called zero modes for the following reason. As we can infer from the highest-weight conditions (2.6) and (2.7), the b_i modes are annihilators to the left and to the right, while the c_i modes are creators to the left and to the right. Thus, the b_i are proper zero modes, and the c_{-i} are their canonically conjugate partners.

In order to reproduce the operator product expansion (2.3), we have to fix the anti-commutation relations as follows

$$\{c_n, b_m\} = \delta_{n+m,0} \ , \tag{2.8}$$

$$\{c_n, c_m\} = \{b_n, b_m\} = 0 \ . \tag{2.9}$$

This means that the in- and out-vacuua are orthogonal to each other, since

$$\langle 0|0\rangle = \langle 0|\,\mathbb{I}\,|0\rangle = \langle 0|\,\{b_0, c_0\}\,|0\rangle = 0 \ . \tag{2.10}$$

In many cases the it is useful to introduce another out-vacuum

$$\langle\tilde{0}| := \text{const} \prod_{i=-\lambda}^{\lambda} c_i \ , \tag{2.11}$$

where the constant can be chosen such that $\langle\tilde{0}|0\rangle = 1$. Note though that this definition breaks the $SL(2,\mathbb{C})$ invariance.

The holomorphic part of the energy-momentum tensor of the b-c ghost system is

$$T_{b,c}(z) = -\lambda :b\partial c: + (1-\lambda) :(\partial b)c: \ , \tag{2.12}$$

and its mode expansion leads to

$$L_n = \sum_{l\in\mathbb{Z}} (\lambda n - l) :b_l c_{n-l}: \ . \tag{2.13}$$

It is easy to check that the Virasoro modes L_n also fulfill the hermiticity condition $L_n^\dagger = L_{-n}$, and we can also easily regain the Virasoro algebra (1.47). We then find that the central charge $c_{b,c}$ for the b-c system is given by

$$c_{b,c} = 2(-1 + 6\lambda - 6\lambda^2). \tag{2.14}$$

In the following we will study systems with central charge -2 and -26 which corresponds to $\lambda = 1$ and $\lambda = 2$, respectively.

2.3 The b-c ghost system as subset of logarithmic $c = -2$ theory

The conformal b-c system and the associated logarithmic so called θ-$\bar{\theta}$-system for central charge $c = -2$ are well-known and have been intensely studied (see e. g. Cappelli *et al.* (1999); Flohr (2000); Gurarie (1993); Gurarie *et al.* (1997); Kausch (1995); Kausch and Watts (1991)). This is the reason for us having a closer look at this system again in the hope of learning how to build such logarithmic theories in general. In the case of $c = -2$ we will briefly repeat two different ways of building a LCFT: firstly via symplectic fermions, see Gurarie *et al.* (1997); Kausch (2000), and secondly by deforming the energy-momentum tensor Fjelstad *et al.* (2002).

2.3.1 $c = -2$ LCFT via Symplectic Fermions

Following the approach described in Gurarie *et al.* (1997) the $c = -2$ theory can be represented as a pair of ghost fields, or anti-commuting fields θ, $\bar{\theta}$ of conformal weight $h = 0$, with the free action Gurarie (1993)

$$S = \int \mathrm{d}^2 z \, \partial\theta\bar{\partial}\bar{\theta} \quad . \tag{2.15}$$

(Note that θ, $\bar{\theta}$ are *not* the complex conjugate of each other, but different fields.) As described in the above mentioned reference, the vacuum $|0\rangle$ is somewhat unusual, its norm is $\langle 0|0\rangle = 0$, while the explicit insertion of the fields θ produces nonzero results, for instance $\left\langle \bar{\theta}(z)\theta(w)\right\rangle = 1$. This property of the vacuum is believed to be typical for LCFTs.

Using the results given in Gurarie *et al.* (1997) the mode expansion of the field θ (the analog holds for $\bar{\theta}$) is

$$\theta(z) = \xi + \theta_0 \log(z) + \sum_{n\neq 0} \theta_n z^{-n} \quad , \tag{2.16}$$

where ξ denotes the crucial zero modes and $n \in \mathbb{Z}$. The non-vanishing anti-commutators ($n \in \mathbb{Z}$, $n \neq 0$) are

$$\left\{\theta_n, \bar{\theta}_m\right\} = \frac{1}{n}\delta_{n,-m}\,, \quad \left\{\xi, \bar{\theta}_0\right\} = 1\,, \quad \left\{\theta_0, \bar{\xi}\right\} = -1\,, \tag{2.17}$$

and together with the highest-weight relation

$$\theta_n \left|0\right\rangle = 0 \quad \forall n \geq 0 \quad , \tag{2.18}$$

it is quite easy to see the logarithmic nature of the θ, $\bar{\theta}$ system, for instance by calculating

$$\left\langle \tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w) \right\rangle = -2\log(z-w) \tag{2.19}$$

where $\tilde{\mathbb{I}}$ is defined as $\tilde{\mathbb{I}} \equiv - :\theta\bar{\theta}:$.

The stress-energy tensor of the theory is

$$T(z) =:\partial\theta\partial\bar{\theta}: \tag{2.20}$$

and it is not hard to see that its expansion with $\tilde{\mathbb{I}}$ is indeed given by

$$T(z)\tilde{\mathbb{I}}(w) = \frac{\mathbb{I}}{(z-w)^2} + \frac{\partial\tilde{\mathbb{I}}(w)}{z-w} + \dots \tag{2.21}$$

meaning that the operator $\tilde{\mathbb{I}}$ has conformal weight 0. Also $\tilde{\mathbb{I}}$ is the logarithmic partner of $\mathbb{I}$, since $L_0\tilde{\mathbb{I}} = \mathbb{I}$. Thus, $\mathbb{I}$ and $\tilde{\mathbb{I}}$ span a Jordan-cell of rank two with respect to L_0. Indeed, the reader should convince herself that the action of L_0 cannot be diagonalized.

The most obvious differences between the b-c system and the θ-$\bar{\theta}$-system are

zero modes:	$(b_0,\, c_0)$	$(\bar{\theta}_0,\, \xi),\ (\theta_0,\, \bar{\xi})$
conformal weights:	$h(b) = 1,$	$h(\bar{\theta}) = 0,$
	$h(c) = 0$	$h(\theta) = 0$

Therefore, in order to get from the θ-$\bar{\theta}$-system to the b-c system we have to reduce the number of zero mode pairs by one and also have to increase the conformal weight of one of the fields by one. This can easily be done by defining the transformation between b, c and θ, $\bar{\theta}$ in the following way:

$$\begin{aligned} b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1} \quad &\underset{b=\partial\bar{\theta}}{\overset{\bar{\theta}=\partial^{-1}b}{\rightleftarrows}} \quad \bar{\theta}(z) = \sum_{n\neq 0} \bar{\theta}_n z^{-n} + \bar{\theta}_0 \log(z) + \bar{\xi} \\ c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n} \quad &\underset{c=\theta|_{\theta_0=0}}{\overset{\theta=c+\theta_0\log(z)}{\rightleftarrows}} \quad \theta(z) = \sum_{n\neq 0} \theta_n z^{-n} + \theta_0 \log(z) + \xi\ . \end{aligned} \tag{2.22}$$

While the derivative (respectively integration) gives the right transformation between b and $\bar{\theta}$ we artificially have to add (respectively eliminate) a zero mode, θ_0, to get the transformation between c and θ.

One might be tempted to use this method for constructing higher logarithmic CFTs, namely by putting the b, c fields on equal footing by integrating the b field $2\lambda - 1$ times where $\lambda > 0$ denotes the conformal weight of the b field. This integration leaves us with $2\lambda - 1$ new modes which then turn out to be one half of the total set of zero modes. The other half of the zero modes has to be added artificially in an analogous way, as for the c field shown above. The latter are necessary as canonically conjugate partners for the zero modes arising as integration constants. Without these conjugate partners, the action of our zero modes would be trivial.

2.3.2 $c = -2$ LCFT via logarithmic deformation

As noted in the introduction of this section there is a different way to construct logarithmic extensions of conformal field theories as described in Fjelstad *et al.* (2002). The idea of this method is to consider special deformations of the energy-momentum tensor. One defines

$$\tilde{T} := T^{\mathrm{CFT}} + T^{\mathrm{impr}} \tag{2.23}$$

where T^{impr} denotes the so called "improvement term" which extends the CFT energy-momentum tensor T^{CFT} in a way that the resulting stress tensor $\tilde{T}$ belongs to a logarithmic theory. Of course, the full stress-energy tensor must still possess the correct operator product expansion with itself.

As is well-known the CFT stress tensor is given by

$$T^{\mathrm{CFT}} = -\lambda :b(\partial c): + (1-\lambda) :(\partial b)c: = - :b\partial c: \quad , \tag{2.24}$$

for $\lambda = 1$, which yields the $c = -2$ ghost system. A careful consideration motivates the following ansatz for the improvement term:

$$T^{\mathrm{impr}} = \frac{1}{z}\theta_0 b(z) \tag{2.25}$$

with θ_0 being an additional zero mode. We have deliberately chosen to name this zero mode θ_0 to make contact to the preceding approach via symplectic fermions. Indeed, the deformed energy-momentum tensor can in this case be rewritten in a nicer form by applying a deformation to the fields as well:

$$b(z) \longrightarrow \bar{\theta}(z) = \partial^{-1} b(z) \tag{2.26}$$

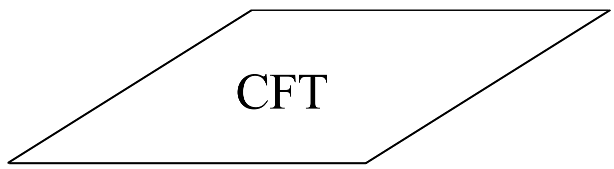
$$c(z) \longrightarrow \theta(z) = c(z) + \theta_0 \log(z) \tag{2.27}$$

which leads to the well-known result (2.20). The theory with $c = -2$ is a bit special, because (as we will see later) it is not always possible to write the energy-momentum tensor as a function of the new basic fields.

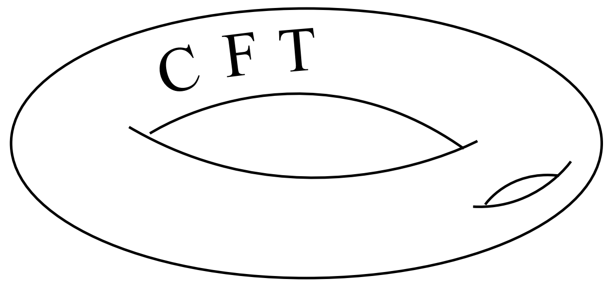
2.4 Why logarithmic conformal field theory?

We have not yet addressed the question why we need to consider logarithmic conformal field theories at all. In this section we want to give a short description, why LCFTs appear in a natural way if one considers conformal ghost systems on nontrivial Riemann surfaces. We will pick up the question again in subsection 2.5.3, where we discuss the topic in more detail.

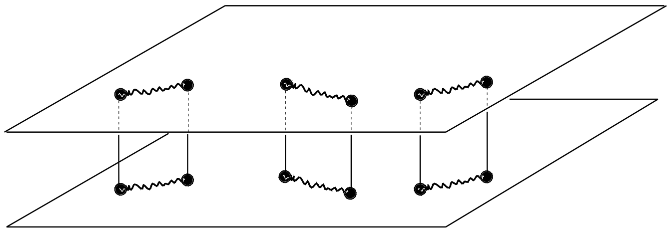
Conformal field theories can be studied on arbitrary Riemann surfaces. The most simple theory in general is a conformal field theory that "lives" on the complex plane.



While this conformal field theory has many interesting applications, there is huge interest from a string theoretical point of view to consider CFTs on nontrivial Riemann surfaces of genus g. For instance let us assume that we want to study a conformal field theory that "lives" on a Riemann surface of genus 2:

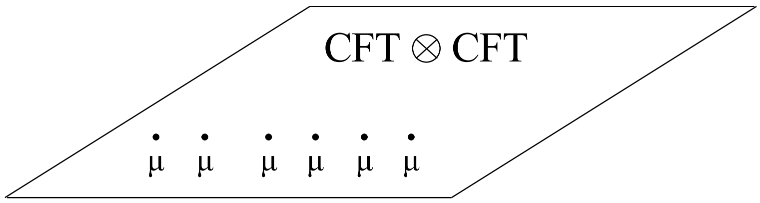


A Riemann surface can be described by a multi-sheeted ramified covering of the complex plane. The number of branch points and branch cuts is related to the genus g. In our example the Riemannian surface of genus 2 can be described by two complex planes linked by three branch cuts. The following graphical representation shows the two complex planes and the three branch cuts respectively the six branch points.



Note that the branch points have all the same ramification number and hence we also call the above a $\mathbb{Z}_2$-symmetric Riemann surface.

Knizhnik (1987) showed how we can map the problem back to the ordinary complex plane. We take a CFT for each sheet and have to add additional operators, which implement the action of the branch points on the fields. The following illustration shows the tensored CFTs that "live" on one complex plane.



The operators μ represent new vertex operators. Now it is natural to consider the operators μ not as statical operators, but as additional degrees of freedom. This means that μ now becomes $\mu(z)$, a field, which is called *twist field*.

We consider now the b-c ghost system with central charge -2, but the argument is much the same for other central charges. The operator product expansion of two twist fields is

$$\mu(z)\mu(w) \sim \tilde{\mathbb{I}}(w) - \log(z-w)\mathbb{I}(w) \ , \tag{2.28}$$

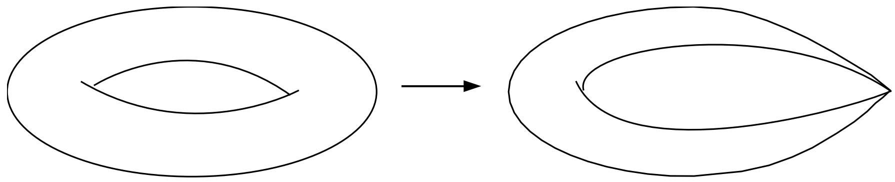
which means that the logarithmic partner of the identity operator shows up in the OPE. Now it is important to know that the vacuum representation of $\tilde{\mathbb{I}}$ is not an element of the Hilbert space of the b-c system,

$$\tilde{\mathbb{I}}|0\rangle = \xi\,\bar{\xi}\,|0\rangle \qquad \notin \mathcal{H}^{\mathrm{b,c}} \tag{2.29}$$

$$\in \mathcal{H}^{\mathrm{enlarged}} \ , \tag{2.30}$$

because there is no mode in the b-c system that corresponds to $\bar{\xi}$, as one can immediately infer from (2.22). This means that we have to extend our initial

Hilbert space in order to achieve an algebraic closure with respect to the OPE. Geometrically the OPE means that the branch points run into each other, which is illustrated for the torus below.



It should be pointed out that a $c = -2$ theory on higher genus surfaces which are hyperelliptic or $\mathbb{Z}_n$ symmetric, is somehow special, since it is possible by a base transformation to make one of the CFTs trivial. Therefore we do not have to consider tensored theories for $c = -2$ as long as we are interested in the hyperelliptic case. For $c = -26$ this is not true anymore.

2.5 A close look at $c = -26$

Motivated by the success for the $c = -2$ system we are now going to construct a logarithmic conformal field theory for $c = -26$ which basically has the same properties as the θ, $\bar{\theta}$ system in the $c = -2$ case. This construction process presumably does not only work for $c = -26$, but should work for any b-c ghost system.

One might be tempted to assume that the Jordan-cell of the LCFT for $c = -26$ has a rank greater than two since this theory possesses a larger number of zero modes, i. e., modes which annihilate the vacuum to the left as well as to the right.

As we will see we do not find higher rank Jordan-blocks: in fact, the zero mode of the Virasoro algebra L_0 turns out to be perfectly well-defined without any Jordan structure at all. The nontrivial indecomposable structure of the Virasoro modules manifests itself in the action of the Virasoro modes $L_n, n \neq 0$. Therefore, we cannot speak of a rank of a Jordan-cell anymore. We will discuss later in which way the Virasoro modules are indeed indecomposable.

Investigating such systems is interesting for string theory. The calculation of string amplitudes makes use of the computation of λ-forms on nontrivial Riemann-surfaces. In a CFT approach these are the ghost systems. As Knizhnik (1987) pointed out, nontrivial Riemann-surfaces, seen as a multi-sheeted covering of the complex plane, can be simulated by twist-fields inserted at the branch points. It has become clear by now that operator product expansions of such twist fields inevitable lead to logarithmic fields Flohr (1998a); Gaberdiel and Kausch (1996b). Therefore, computation of string amplitudes automatically involves not only the b,c system but its enlarged full LCFT. Also, there have been hints that LCFTs with higher rank Jordan-blocks play a role in the AdS/CFT correspondence Giribet (2001). Thus, it is important to learn more about LCFTs where the indecomposable structure is more involved than in the simple rank-two case. Even the simplest such higher-rank cases are very difficult to study, since the generic form of operator product expansion can only be fixed under quite restrictive assumptions Flohr (2002b).

2.5.1 Generalizing symplectic fermions

We now try to mimic what we did in the previous section, but this time for $c = -26$. Starting with the well-known b-c system for $c = -26$ and by applying the same steps as we did for $c = -2$ we get a larger system. Unfortunately building a LCFT for $c = -26$ turns out to be more complicated than for the $c = -2$ case. Basically two obstacles are in the way of constructing a LCFT for $\lambda \geq 2$:

1. The energy-momentum tensor cannot be built by combining derivatives of the generalized symplectic fermion fields.
2. LCFT is intimately linked to twist fields arising from putting the CFT on a nontrivial Riemann-surfaces. The full theory is a tensor product of the CFTs for each covering sheet. We cannot neglect this fact.

In this sense $c = -2$ is special since the above mentioned problems do not show up (as we will explain later).

The b-c system for $c = -26$ is given by the following relations if we set $\lambda = 2$:

$$b(z) = \sum_n b_n z^{-n-2}\,, \qquad c(z) = \sum_n c_n z^{-n+1}\,, \tag{2.31}$$

$$\{c_n, b_m\} = \delta_{n+m,0}\,, \qquad \{c_n, c_m\} = \{b_n, b_m\} = 0\,, \tag{2.32}$$

$$b_n\,|0\rangle = 0 \quad \forall n \geq -1\,, \qquad c_n\,|0\rangle = 0 \quad \forall n \geq 2\,, \tag{2.33}$$

$$T(z) = -2 :b\partial c: - :(\partial b)c: \;. \tag{2.34}$$

As explained in section 2.2 the b-c system for central charge $c = -26$ comes with three pairs of zero modes,[1] namely $(b_i,\, c_i)$ for $i = -1, 0, 1$. In the same way as we extended the $c = -2$ theory to a larger one by formal integration, we can try this for the $c = -26$ case by introducing the fields

$$\begin{aligned}\Lambda(z) \;&:=\; \theta_1 \log z + \theta_0(z\log z - z) + \theta_{-1}\frac{z^2}{2}\left(\log z - \frac{3}{2}\right)\\ &\quad +\xi_1 + \xi_0 z + \xi_{-1}\frac{1}{2}z^2 + \sum_{|n|>1}\theta_n\frac{z^{-n+1}}{-n+1} \qquad (2.35)\\ \bar\Lambda(z) \;&:=\; \bar\theta_1 \log z + \bar\theta_0(z\log z - z) + \bar\theta_{-1}\frac{z^2}{2}\left(\log z - \frac{3}{2}\right)\\ &\quad +\bar\xi_1 + \bar\xi_0 z + \bar\xi_{-1}\frac{1}{2}z^2 + \sum_{|n|>1}\bar\theta_n\frac{z^{-n+1}}{-n+1} \quad . \qquad (2.36)\end{aligned}$$

The field $\bar\Lambda(z) := \partial^{-3}b$ has now the same conformal weight as its partner field $\Lambda(z) := c(z) + \sum_i f_i(z)\theta_i$. We call such pairs of anti-commuting fields of identical conformal weight generalized symplectic fermions. Note that the threefold-integration adds three new modes, $\bar\xi_i$, $i = -1, 0, 1$, to the theory, which are (as we will see later) one half of the additional zero modes we have to add to the theory in order to make it logarithmic. The other half is artificially added in the Λ field. Similar to the $c = -2$ case our new fields are now on equal footing $h(\Lambda) = h(\bar\Lambda) = -1$.

Going from Λ, $\bar\Lambda$ back to b, c of course requires removing these additional modes:

$$b(z) = \partial^3\bar\Lambda(z) \qquad (2.37)$$

$$c(z) = \Lambda(z)\big|_{\theta_i=0} \quad . \qquad (2.38)$$

The relations from the b-c system can be translated to the new system and we find:

$$\left\{\theta_n, \bar\theta_m\right\} = -\frac{1}{n}\delta_{n,-m} \qquad |n|, |m| > 1 \qquad (2.39)$$

$$\left\{\xi_i, \bar\theta_{-i}\right\} = (-1)^{i+1} \qquad i = -1, 0, 1 \quad . \qquad (2.40)$$

For the new modes we require the anti-commutation relations to be

$$\left\{\bar\xi_i, \theta_{-i}\right\} := (-1)^i \qquad i = -1, 0, 1 \quad , \qquad (2.41)$$

[1] If not explicitly stated otherwise the range of i for the $c = -26$ system is -1, 0, 1

which leads to the following OPEs

$$\Lambda(z)\bar{\Lambda}(w) \sim \frac{1}{2}(z-w)^2\left[\frac{3}{4} - \log(z-w)\right] \quad (2.42)$$

$$\Lambda(z)\Lambda(w) \sim \mathcal{O}(z-w) \quad (2.43)$$

$$\bar{\Lambda}(z)\bar{\Lambda}(w) \sim \mathcal{O}(z-w) \quad . \quad (2.44)$$

The new modes indeed have the properties of zero modes, namely that all modes θ_i and $\bar{\theta}_i$ are annihilators to both sides, and the modes $\bar{\xi}_{-i}$ and ξ_{-i} are their respective conjugate modes. Therefore, the extended theory also contains twice as many zero modes compared to the original b-c ghost system.

2.5.2 Building the energy-momentum tensor

Having constructed fields which show logarithmic behavior leads us to the question how the energy-momentum tensor for $c = -26$ looks like. Therefore, we look back to the $c = -2$ case in the hope to learn from this scenario. We remember that for $\lambda = 1$ respectively $c = -2$ simply plugging in the fields (2.22) in the energy-momentum tensor

$$T = T[b,c] = -\lambda :b\partial c: + (1-\lambda) :(\partial b)c: \quad . \quad (2.45)$$

gives us the desired result (2.20). Unfortunately this does not work out in the same way for $c = -26$ and presumably neither for any other $\lambda \geq 2$.

The reason is obvious: Λ appears plainly and as first derivative in the energy-momentum tensor. Because of Λ containing $z^n \log(z)$ terms this inevitable leads to logarithmic terms in the energy-momentum tensor.

To find possible energy-momentum tensors at all we use a different approach and consider possible extensions of the stress-energy tensor on the mode level. This approach is motivated by the paper of Fjelstad *et al.* (2002), but note that our *deformation term* is slightly more general and so is our result. The deformation term in Fjelstad *et al.* (2002) is always constructed from primary fields, which we do not assume here.

$$T^{\log}(z) = T^{\mathrm{b,c}}(z) + R(z) \quad (2.46)$$

$$= \sum_n z^{-n-2}\left(L_n^{\mathrm{b,c}} + R_n\right) \quad (2.47)$$

where the modes $L_n^{\mathrm{b,c}}$ are given by

$$L_{-2}^{\mathrm{b,c}} = -\sum_{l\neq -3,\ldots,1} \frac{l(l+1)(l+4)}{l+3} :\bar{\theta}_l\theta_{-l-2}: -6 :\bar{\theta}_{-3}\xi_1: -4 :\bar{\theta}_{-2}\xi_0: -\frac{3}{2} :\bar{\theta}_{-1}\xi_{-1}: +\frac{4}{3} :\bar{\theta}_0\theta_{-2}: -\frac{5}{2} :\bar{\theta}_1\theta_{-3}: \quad (2.48)$$

$$L_{-1}^{\mathrm{b,c}} = -\sum_{l\neq -2,\ldots,1} l(l+1) :\bar{\theta}_l\theta_{-l-1}: - :\bar{\theta}_{-1}\xi_0: + :\bar{\theta}_0\xi_{-1}: -2 :\bar{\theta}_1\theta_{-2}: \quad (2.49)$$

$$L_0^{\mathrm{b,c}} = -\sum_{l\neq -1,\ldots,1} l^2 :\bar{\theta}_l\theta_{-l}: - :\bar{\theta}_1\xi_{-1}: + :\bar{\theta}_{-1}\xi_1: \quad (2.50)$$

$$L_1^{\mathrm{b,c}} = -\sum_{l\neq -1,\ldots,2} (l-2)(l+1) :\bar{\theta}_l\theta_{-l+1}: -3 :\bar{\theta}_{-1}\theta_2: -2 :\bar{\theta}_0\xi_1: -2 :\bar{\theta}_1\xi_0: \quad (2.51)$$

$$L_2^{\mathrm{b,c}} = -\sum_{l\neq -1,\ldots,3} \frac{(l-4)l(l+1)}{l-1} :\bar{\theta}_l\theta_{-l+2}: -\frac{5}{2} :\bar{\theta}_{-1}\theta_3: +4 :\bar{\theta}_0\theta_2: +6 :\bar{\theta}_1\xi_1: +12 :\bar{\theta}_2\xi_0: +6 :\bar{\theta}_3\xi_{-1}: \quad (2.52)$$

and R_n denotes the extension which may contain the new *deformation modes* θ_i, $\bar{\xi}_i$. The modes $L_n^{\log} := L_n + R_n$ of course have to obey the Virasoro Algebra, which is a strong restriction. We get two different solutions, each coming with three possible deformations of the stress tensor:

$$R_{-2} = 6A\theta_1\bar{\theta}_{-3} - 4B\theta_0\bar{\theta}_{-2} + \frac{3}{2}C\theta_{-1}\bar{\theta}_{-1} \quad (2.53)$$

$$R_{-1} = -B\theta_0\bar{\theta}_{-1} - C\theta_{-1}\bar{\theta}_0 \quad (2.54)$$

$$R_0 = -A\theta_1\bar{\theta}_{-1} + C\theta_{-1}\bar{\theta}_1 \quad (2.55)$$

$$R_1 = 2A\theta_1\bar{\theta}_0 + 2B\theta_0\bar{\theta}_1 \quad (2.56)$$

$$R_2 = -6A\theta_1\bar{\theta}_1 + 12B\theta_0\bar{\theta}_2 - 6C\theta_{-1}\bar{\theta}_3 \quad (2.57)$$

$$R_{-2} = 6A'\theta_1\bar{\theta}_{-3} - B'\theta_{-2}\bar{\xi}_0 + \frac{3}{2}C'\theta_{-1}\bar{\theta}_{-1} \quad (2.58)$$

$$R_{-1} = -C'\theta_{-1}\bar{\theta}_0 \quad (2.59)$$

$$R_0 = -A'\theta_1\bar{\theta}_{-1} + C'\theta_{-1}\bar{\theta}_1 \quad (2.60)$$

$$R_1 = 2A'\theta_1\bar{\theta}_0 \quad (2.61)$$

$$R_2 = -6A'\theta_1\bar{\theta}_1 - 3B'\theta_2\bar{\xi}_0 - 6C'\theta_{-1}\bar{\theta}_3 \quad . \quad (2.62)$$

Testing the Virasoro Algebra with the above *deformation terms* is sufficient, since all higher modes can be derived with the help of the Virasoro Algebra.

Two things are noteworthy: firstly, the second solution contains ξ modes. This is a bit unexpected since $L_n^{\log}$ should, according to what we learned from the $c = -2$ theory, only lower the zero mode content and not increase it. Secondly, both solutions look very similar. Setting $B' = 0$ in the second solution, and thus eliminating the unwanted ξ-modes, would result in a special case ($B = 0$) of the first solution. As we will see later the second solution is indeed a special case of the first one. That is why we concentrate on the first solution for now.

The extensions can be written in a nicer way, making use of the b-field:

$$T^{\log}(z) = T^{\mathrm{b,c}}(z) + A\theta_1 \frac{1}{z^0}\partial(z^0 b) + B\theta_0 \frac{1}{z^1}\partial(z^2 b) + C\theta_{-1}\frac{1}{z^2}\partial(z^4 b) \qquad (2.63)$$

which has a strikingly similarity with the energy-momentum tensor deformations described by Fjelstad, Fuchs *et al.*, but also has an important difference, namely the appearance of derivatives of the first order. The important point is that the deformations involve additional modes which are proper zero modes, i. e., annihilation operators to both sides. There are three possible "directions" to deform the energy-momentum tensor, which matches exactly the number of zero modes of our system as we might have expected. In the $c = -2$ system only one such deformation was possible. There is another difference between $c = -2$ and $c = -26$: while in the former theory it was possible to redefine the b and c fields (2.26), (2.27) in order to get an energy-momentum tensor which consists of the new fields only, this is not possible in the latter case.

Demanding that the Virasoro modes satisfy the hermiticity condition $L_n^\dagger = L_{-n}$ leads to a further restriction of the solution[2]:

$$A = C \quad . \qquad (2.64)$$

In the second solution, this requirement leads to the condition $A' = C'$.

2.5.3 Fields on nontrivial Riemann Surfaces

Up to now we have constructed fields Λ, $\bar{\Lambda}$ out of the b-c system for $c = -26$ and we have found possible deformations of the energy-momentum tensor. The Hilbert-space $\mathcal{H}^{\log}$ of the extended theory is an enlargement of the Hilbert-space of the b-c system containing the additional zero modes $\bar{\xi}_i$.

[2]Note that taking the adjoint of the modes can cause an additional constant, for instance $\xi_1^\dagger = \frac{1}{2}\xi_{-1}$, due to our normalization of the modes, which results from viewing them as integration constants.

This gives rise to another problem, namely that the constructed theory cannot be the full theory, because of $L_0^{\log}$ not being able to measure the conformal weight of all states contained in the Hilbert-space correctly. For instance $|\bar{\xi}_{-1}\rangle$ is surely an element of the Hilbert-space $\mathcal{H}^{\log}$, but $L_0^{\log}|\bar{\xi}_{-1}\rangle = 0$ gives the wrong conformal weight.

This is an extremely interesting observation. The origin of logarithmic fields is tied to the existence of so-called pre-logarithmic primary fields, whose operator product expansions contain the logarithmic fields, cf. Kogan and Lewis (1998). In fact, the first hint for the existence of the field $\tilde{\mathbb{I}}$ in the $c = -2$ theory comes from evaluating the four-point function of four $\mathbb{Z}_2$ twist fields μ of conformal weight $h = -1/8$, as has been observed in Gurarie (1993). As a result, this four-point function contains the following two conformal blocks:

$$\begin{aligned}
\langle\mu(\infty)\mu(1)\mu(x)\mu(0)\rangle &= [x(1-x)]^{\frac{1}{4}}F(x)\,,\\
F(x) &= \begin{cases} {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x)\,, & \\ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-x) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x)\log(x) & \\ \qquad + \tfrac{\partial}{\partial\epsilon}\,{}_3F_2(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon,1;1+\epsilon,1+\epsilon;x)\big|_{\epsilon=0}\,. & \end{cases}
\end{aligned} \tag{2.65}$$

In case of the ghost systems, these pre-logarithmic twist fields have a geometric meaning: these fields behave exactly as branch points of a ramified covering of the complex plane. For example, the above mentioned $\mathbb{Z}_2$ twist fields μ simulate the branch point of a hyperelliptic surface in case of the $c = -2$ theory. Whenever all branch points have the same ramification number, say n, all monodromies around these points can be diagonalized simultaneously.

As Knizhnik has shown, ghost systems on such $\mathbb{Z}_n$-symmetric Riemann surfaces can be dealt with by putting them on an n-fold sheeted covering of the complex plane where the branch points are represented by suitable constructed vertex operators. However, these vertex operators are twist fields, and thus may produce logarithmic divergences in their operator product expansions. Furthermore, to yield a local theory, we have to take the tensor product of the theories on all covering sheets.

The simplest such case is the hyperelliptic one, since then automatically all branch points are of order two. This hyperelliptic case is special since for the $c = -2$ theory, and only for this theory, one of the two copies of the conformal field theory decouples completely. This is a major difference of the $c = -2$ theory compared to other ghost systems, namely that it is possible to eliminate the theory on one of the two covering planes because after diagonalizing the monodromies the vertex operators associated to the branch cuts become trivial on one of the sheets.

Since this is a subtle point, we discuss it a bit more in detail: The twist field μ for a branch point on a hyperelliptic surface for the $c = -2$ ghost system is actually given by $\mu(z) = V_{-1/2}(z) \otimes V_0(z)$, where $V_q(z)$ denotes a vertex operator with charge q with respect to the ghost current $J = {:}bc{:}$ in a free field construction, and where we have indicated the composition of the twist field out of the two copies of the CFT. The conformal weight is, with $h(q) = \frac{1}{2}q(q+1)$, given by $h(-\frac{1}{2})+h(0) = -\frac{1}{8}$ as it should be. The background charge at infinity is for both copies $q_0 = -1/2$ such that the total sum of all charges in each copy must add up to $2q_0 = -1$. Looking at the four-point function mentioned above, we actually have to compute

$$\begin{aligned} \langle\mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle &= \langle V_{-1/2}(z_1)V_{-1/2}(z_2)V_{-1/2}(z_3)V_{-1/2}(z_4)\rangle \\ &\quad \times \langle V_0(z_1)V_0(z_2)V_0(z_3)V_0(z_4)\rangle \\ &= \langle Q_{+1}V_{-1/2}(z_1)V_{-1/2}(z_2)V_{-1/2}(z_3)V_{-1/2}(z_4)\rangle \\ &\quad \times \langle Q_{-1}V_0(z_1)V_0(z_2)V_0(z_3)V_0(z_4)\rangle\,, \end{aligned} \tag{2.66}$$

where we have indicated the necessary screening charges in the last step. Now, we can easily construct a screening current with charge $q = 1$ since $V_1(z)$ has conformal weight $h(q) = \frac{1}{2}q(q+1) = 1$ as we expect. Actually, $V_1(z)$ behaves essentially in the same way as the screening current, since $J(z)\mathrm{d}z = {:}bc{:}(z)\mathrm{d}z$ transforms exactly like a one-differential. Thus $Q_{+1} = \oint \mathrm{d}z V_1(z)$. This factor of the four-point function yields then precisely the integral representation of the hyper-geometric function appearing in (2.65). The second factor of the four-point function is more tricky, since the field V_{-1} has conformal weight $h = 0$, thus cannot serve as screening current. However, a screening current with the correct properties can be constructed in the form $Q_{-1} = \oint \mathrm{d}z \oint \mathrm{d}z' V_1(z)V_{-2}(z')$, since V_{-2} also has conformal weight $h = 1$. When inserting these two screening charges, one has to be careful with the choice of the contour for the integration. It turns out that the net result in the presence of nothing but four identity fields $V_0(z_i)$, $i = 1, \ldots, 4$, simply is the operator $:\phi V_{-1}:(0)$, where $\phi(z)$ is the free field used in the bosonization. Thus, we end up with the insertion of the *logarithmic* partner $\tilde{\mathbb{I}}(0)$ of the identity such that the second factor of (2.66) does not vanish identically, but yields simply a constant. Taken all together, we arrive at (2.65).

Repeating this computation for the $c = -26$ ghost system is a bit more involved. The twist fields for the hyperelliptic case have now the composition $\mu(z) = V_{-1/2}(z) \otimes V_{-1}(z)$, such that the second factor is not merely the identity operator. The conformal weights are now given by $h(q) = \frac{1}{2}q(q+3)$ and the background charge at infinity is now $-3/2$. The twist field has therefore conformal weight $h_\mu = h(-\frac{1}{2}) + h(-1) = -5/8 + (-1) = -13/8$. Thus, we have to satisfy

$$\begin{aligned}\langle\mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle &= \langle V_{-1/2}(z_1)V_{-1/2}(z_2)V_{-1/2}(z_3)V_{-1/2}(z_4)\rangle \\ &\quad\times\langle V_{-1}(z_1)V_{-1}(z_2)V_{-1}(z_3)V_{-1}(z_4)\rangle \\ &= \langle Q_{-1}V_{-1/2}(z_1)V_{-1/2}(z_2)V_{-1/2}(z_3)V_{-1/2}(z_4)\rangle \\ &\quad\times\langle Q_{+1}V_{-1}(z_1)V_{-1}(z_2)V_{-1}(z_3)V_{-1}(z_4)\rangle\,, \quad (2.67)\end{aligned}$$

where we have again indicated the necessary screenings. Here, the second factor is easier, since the screening charge Q_{+1} can always be taken as the contour integration of the ghost current $J_{+1}(z) \equiv J(z) = {:}bc{:}(z)$, since it transforms by construction as a one-differential. Moreover, all charges q are always defined with respect to this ghost current. This is true independent of the value of the spin λ of the ghost system considered. Thus, the screening charge Q_{+1} is always easy to construct.

For the first factor, we have to use a modified version of the screening current, since the current $\tilde{J}(z) = {:}V_1V_{-2}{:}(z)$, although it has the correct conformal weight $h = 1$ and is a local chiral field, does not yield the correct charge. It is merely an alternative representation of the screening current. Instead, we might use $J_{-1}(z) = \oint \mathrm{d}z' V_1(z)V_{-2}(z') = \oint \mathrm{d}z'(z - z')^{-2}V_{-1}(z')$. This current has the correct charge, but the wrong conformal weight $h = 0$. We arrive thus at a similar situation as with the second factor in the $c = -2$ case, namely where the effect of screening is the insertion of a nontrivial $h = 0$ field.

However, it is possible to construct a correct screening for the first factor by making use of the nontrivial $h = 5$ field of charge $q = 2$, which is part of the extended chiral symmetry algebra of the $c = -26$ ghost system. The correct screening charge reads then

$$Q_{-1} = \oint \mathrm{d}u_1 \oint \mathrm{d}u_2 \oint \mathrm{d}u_3 V_{-1}(u_1)V_{-2}(u_2)V_2(u_3)\,. \quad (2.68)$$

The integrand has total conformal weight $h = (5) + (-1) + (-1) = 3$, which after three integrations yields a conformally invariant object.

The lengthy discussion shows the following: Evaluating the four-point function of four $\mathbb{Z}_2$ twist fields in the $c = -26$ case as in (2.67) yields an expression which will exhibit logarithmic singularities just as in the $c = -2$ case. Indeed, the second factor in the $c = -26$ case is again related to an integral representation of a hyper-geometric system, ${}_2F_1(1, 0; 0; x) = {}_1F_0(1; x) = \int_{x_0}^{x} \mathrm{d}u(1 - u)^{-2}$.

The first factor, however, is much more complicated since it involves a three-fold integration

$$\oint \mathrm{d}u_1 \oint \mathrm{d}u_2 \oint \mathrm{d}u_3 \frac{(u_1 - u_2)^2}{(u_1 - u_3)^2 (u_2 - u_3)^4} \prod_{i=1}^{4} \frac{(z_i - u_1)^{1/2} (z_i - u_2)^1}{(z_i - u_3)^1} . \quad (2.69)$$

After bringing the four-point function (2.67) into standard form with $z_1 = \infty$, $z_2 = 1$, $z_3 = x$, $z_4 = 0$ with x the crossing ratio, one of the three integrations can be performed and yields a Lauricella system of D-type (see for example Exton (1976)), which is a generalized hyper-geometric system of several variables:

$$\begin{aligned} \langle V_{-1/2}(\infty) V_{-1/2}(1) V_{-1/2}(x) V_{-1/2}(0) \rangle &= \oint \mathrm{d}u_2 \oint \mathrm{d}u_3 (u_2 - u_3)^{-4} \quad (2.70) \\ \times F_D^{(3)}(\frac{3}{2}, -\frac{1}{2}, -2, 2; 3; x, u_2, u_3) & \frac{u_3(1 - u_3)(x - u_3)}{u_2(1 - u_2)(x - u_2)} . \end{aligned}$$

The system $F_D^{(3)}$ has several solutions depending on the choice of the integration contour, some of them exhibiting logarithms when expanded around $x = 0$. This is similar to the ordinary hyper-geometric case where a logarithmic solution appears whenever c in ${}_2F_1(a, b; c; x)$ is an integer. In fact, $F_D^{(3)}(\frac{3}{2}, -\frac{1}{2}, -2, 2; 3; x, u, u) = {}_2F_1(\frac{3}{2}, -\frac{1}{2}; 3; x)$, which is a hyper-geometric system with the two expansions

$$y_1 = \sum_n \frac{(\frac{3}{2})_n (-\frac{1}{2})_n}{(3)_n (1)_n} x^n , \quad (2.71)$$

$$y_2 = \log(x) \sum_n \frac{(\frac{3}{2})_n (-\frac{1}{2})_n}{(3)_n (1)_n} x^n + \sum_n \frac{\partial}{\partial \epsilon} \left(\frac{(\frac{3}{2} + \epsilon)_n (-\frac{1}{2} + \epsilon)_n}{(3 + \epsilon)_n (1 + \epsilon)_n} \right)_{\epsilon = 0} x^n \quad (2.72)$$

around $x = 0$. The full computation of this four-point functions is beyond the scope of this chapter.

We note once more that looking at n-point functions of twist fields reveals whether we should expect logarithmic operators and thus indecomposable structures in our CFT or not. The logarithmic operators get exchanged in the internal channels of the n-point functions of twist fields due to degeneracies in the moduli space of the considered Riemann surface, if branch points run into each other. The present case, $c = -26$, clearly shows all signs to be a logarithmic CFT.

This discussion motivates, however, that our logarithmic deformation of the ghost system is related to the above mentioned situation on nontrivial Riemann surfaces. For the sake of simplicity, we concentrate again on the hyperelliptic case. Doing so, we now have two sets of modes $(\theta_n^p, \bar{\theta}_n^p, \xi_i^p, \bar{\xi}_i^p, n \in \mathbb{Z}, p = 1, 2)$ and also two sets of deformation parameters: A_1, A_2, B_1, B_2.

The easiest unification of both theories is given by defining the modes of the unified theory in the following way:

$$L_n^{\text{tot}} := L_n^{\log,1} + L_n^{\log,2} \quad , \tag{2.73}$$

which means that we indeed take simply the tensor product of the two isomorphic conformal field theories. However this alone does not lead to a proper theory, since the new modes L_n^{tot} do not satisfy the Virasoro algebra. To achieve the latter we have to identify the new modes, $\bar{\xi}_i, \theta_i$, on the one plane with the ξ_i, $\bar{\theta}_i$ modes on the other covering plane, by demanding

$$\theta_i^1 \sim \bar{\theta}_i^2 \tag{2.74}$$

$$\bar{\xi}_i^1 \sim \xi_i^2 \quad . \tag{2.75}$$

and analogously for θ_i^2 and $\bar{\xi}_i^2$. Using up two more degrees of freedom by setting

$$A := A_1 = -A_2 \tag{2.76}$$

$$B := B_1 = -B_2 \tag{2.77}$$

we get

$$\left[L_n^{\log,1}, L_m^{\log,2}\right] = 0 \quad , \tag{2.78}$$

and therefore L_n^{tot} now not only fulfills the Virasoro algebra with total central charge $2 \cdot (-26) = -52$, but also acts correctly on the full space of states. It is worth mentioning that our construction automatically and naturally forces us to consider the (deformed) ghost system conformal field theory on a nontrivial Riemann surface. Moreover, we also have to slightly alter Knizhnik's prescription of constructing the full conformal field theory. A consistent Virasoro algebra with the correct action on the Hilbert space can only be obtained, if the two copies are not simply added, but only if the zero modes of the two conformal field theories are intermixed. This, in essence, encodes that the action of the monodromies cannot be fully diagonalized, leading to indecomposable structures in the conformal field theory. It is very interesting that for $c = -26$, and presumably for any other ghost system with $\lambda \neq 1$, the deformation of the Virasoro algebra inevitably leads us to consider such tensor products of these ghost systems, which do not factorize completely. As mentioned above, the $c = -2$ case appears now as particularly simple, since here the factorization of the full theory in two copies still almost holds.[3] Thus, our enlarged theory has a nice and natural geometrical interpretation.

Applying the same steps to our second solution (2.58)-(2.62) gives:

$$A' := A'_1 = -A'_2 \quad (2.79)$$
$$C' := C'_1 = -C'_2 \quad (2.80)$$
$$B' := B'_1 = B'_2 = 0 \quad . \quad (2.81)$$

This means that the second solution is already included in the first one ($B = 0$) and in particular the condition (2.78) enforces the elimination of the terms containing ξ modes. Therefore, it is sufficient to investigate the first solution though we bear in mind that $B = 0$ might be an interesting choice.

Retranslating the system to the familiar b-c system using the choice above leads to

$$R^{\text{tot}}_{-2} = -\frac{1}{2}Ab^1_{-3}b^2_1 - 2Bb^1_{-2}b^2_0 - 3Ab^1_{-1}b^2_{-1} + \frac{1}{2}Ab^2_{-3}b^1_1 + 2Bb^2_{-2}b^1_0 \quad (2.82)$$
$$R^{\text{tot}}_{-1} = -(A+B)b^1_{-1}b^2_0 - (A+B)b^1_0b^2_{-1} \quad (2.83)$$
$$R^{\text{tot}}_0 = 0 \quad (2.84)$$
$$R^{\text{tot}}_1 = (A+B)b^1_0b^2_1 + (A+B)b^1_1b^2_0 \quad (2.85)$$
$$R^{\text{tot}}_2 = 3Ab^1_1b^2_1 + 2Bb^1_2b^2_0 + \frac{1}{2}Ab^1_3b^2_{-1} + 2Bb^1_0b^2_2 + \frac{1}{2}Ab^1_{-1}b^2_3 \,. \quad (2.86)$$

Therefore, our theory is diagonal with respect to L^{tot}_0 for arbitrary A and B. Off-diagonal contributions appear in all different modes for almost all nontrivial choices of A and B. The only nontrivial exception is $A = -B$ which eliminates all off-diagonal elements for L^{tot}_{-1} and L^{tot}_1 thus leading to a theory which is as "little" as possible logarithmic, in the sense that the $\mathrm{SL}(2,\mathbb{C})$ global conformal group is not deformed at all. In particular the second solution (2.58)-(2.62) which narrowed down to the first one with $B = 0$ comes for all nontrivial choices always with a deformation term. Note that there is no physical reason forcing this choice.[4] For any nontrivial choice of A and B it is inevitable that deformations of higher modes $|n| \geq 2$ occur.

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[3] Of course, one should in principle also identify the additional zero mode for the deformation with the zero mode of the other copy of the conformal field theory for the other sheet.

[4] If we want to keep the relations we already know from other LCFTs, then $A = -B$ is mandatory. In the so far known LCFTs, the Virasoro modes can be written in the form $L_n = z^n\,(z\partial_i + (n+1)(h_i + \delta_{h_i}))$, see e.g. Flohr (2000), such that L_{-1} has no off-diagonal contribution, which one might expect for the generator of translations. This differs from the case considered here, where L_0 has no off-diagonal term. It follows then from the Virasoro algebra that L_{-1} having no logarithmic contribution implies the same for L_1 and vice versa. The choice $A = -B$ reproduces this behavior.

The next question is, what the highest weight states for our enlarged theory are, since these correspond to the primary fields. Leaving aside twisted sectors of the theory, we found the following highest weight states for $h = -2, -1, 0$ (note that there are no such states for $h = 1, 2$ and that all states for $h = -2$ are highest weight states).

$$h = -2 : \quad c^1_1 c^2_1|0\rangle \, , c^1_0 c^1_1 c^2_1|0\rangle \, , c^1_1 c^2_0 c^2_1|0\rangle \, , c^1_0 c^1_1 c^2_0 c^2_1|0\rangle$$

$$\begin{aligned} h = -1 : \quad & c^2_1|0\rangle \, , c^1_1|0\rangle \, , \\ & c^1_0 c^1_1|0\rangle \, , \big(c^1_0 c^2_1 - c^1_1 c^2_0\big)|0\rangle \, , c^2_0 c^2_1|0\rangle \, , \\ & \big(c^1_0 c^1_1 c^2_0 - 2c^1_{-1} c^1_1 c^2_1\big)|0\rangle \, , \big(c^1_0 c^2_0 c^2_1 - 2c^1_1 c^2_{-1} c^2_1\big)|0\rangle \\ & c^1_{-1} c^1_0 c^1_1 c^2_1|0\rangle \, , \big(c^1_0 c^1_1 c^2_{-1} c^2_1 - c^1_{-1} c^1_1 c^2_0 c^2_1\big)|0\rangle \, , c^1_1 c^2_{-1} c^2_0 c^2_1|0\rangle \, , \\ & -(A+B) c^1_{-1} c^1_1 c^2_1|0\rangle + c^1_{-1} c^1_0 c^1_1 c^2_0 c^2_1|0\rangle \, , \\ & (A+B) c^1_1 c^2_{-1} c^2_1|0\rangle + c^1_0 c^1_1 c^2_{-1} c^2_0 c^2_1|0\rangle \end{aligned}$$

$$\begin{aligned} h = 0 : \quad & |0\rangle \, , \\ & c^1_{-1} c^1_0 c^1_1|0\rangle \, , c^2_{-1} c^2_0 c^2_1|0\rangle \, , \\ & \big(c^1_{-1} c^1_1 c^2_0 - c^1_{-1} c^1_0 c^2_1 - c^1_0 c^1_1 c^2_{-1}\big)|0\rangle \, , \\ & \big(c^1_{-1} c^2_0 c^2_1 + c^1_1 c^2_{-1} c^2_0 - c^1_0 c^2_{-1} c^2_1\big)|0\rangle \, , \\ & \Big(\frac{A^2}{4} c^1_0 c^2_0 + \frac{1}{2} AB c^1_{-1} c^2_1 + \frac{1}{2} AB c^1_1 c^2_{-1} + B c^1_{-1} c^1_1 c^2_{-1} c^2_1 \\ & + \frac{1}{2} A c^1_{-1} c^1_0 c^2_0 c^2_1 + \frac{1}{2} A c^1_0 c^1_1 c^2_{-1} c^2_0 + c^1_{-1} c^1_0 c^1_1 c^2_{-1} c^2_0 c^2_1\Big)|0\rangle \end{aligned} \tag{2.87}$$

As we noted above $L^{\rm tot}_0$ is—in contrast to the $c = -2$ theory—diagonal. An operator for $c = -26$ which has similar properties as L_0 for $c = -2$ is $L^{\rm tot}_{-2}$. Indeed, applying this operator generates off-diagonal terms as the following example shows

$$L^{\rm tot}_{-2} \left|c^1_1 c^2_1\right\rangle = \big(c^1_1 c^2_{-1} + c^1_{-1} c^2_1\big)|0\rangle - 2\left(b^2_{-2} c^2_0 + b^1_{-2} c^1_0\right)\left|c^1_1 c^2_1\right\rangle + 3A|0\rangle \; . \tag{2.88}$$

Applying $L^{\rm tot}_{-2}$ a second time leads to further off-diagonal terms

$$\begin{aligned} \big(L^{\rm tot}_{-2}\big)^2 \left|c^1_1 c^2_1\right\rangle = & - A(12 b^1_{-2} c^1_0 + \frac{3}{2} b^1_{-3} c^1_1 + 12 b^2_{-2} c^2_0 + \frac{3}{2} b^2_{-3} c^2_1)|0\rangle \\ & + 8B(b^1_{-2} c^1_1 b^2_{-2} c^2_1)|0\rangle \; . \end{aligned} \tag{2.89}$$

While $A = -B$ in general makes the theory easier (by eliminating logarithmic contributions) this does not reduce the number of terms in this case.

If we multiply the deformation term with q ($A \to qA$, $B \to qB$) then it is interesting to note that the power in q does not go beyond a certain threshold if we consider $(L_{-2}^{\text{tot}})^m \, |\text{state}\rangle$, $m \in \mathbb{N}$. The reason for a threshold can be derived from the structure of the extension R_n^{tot} and the states: each R_n^{tot} contains at least one annihilator b_i^p ($p = 1, 2$, $i = -1, 0, 1$) while the states are words in the conjugated modes, the creators, c_i^p ($p = 1, 2, i = -1, 0, 1$) applied to the in-vacuum. By applying R_n^{tot} the number of c modes is reduced by one or two (or the term is eliminated) and most importantly there is no term in L_n^{tot} which increases the number of c modes again. Therefore, the maximum power in q which theoretically can occur is 6.

Our $c = -52$ theory comes, though logarithmic, with a non-logarithmic L_0^{tot} which is a major difference to all LCFTs we know up to now. Because of L_0^{tot} being trivial we obviously get no Jordan-cell or a Jordan-rank in the traditional sense. Nevertheless we have some properties which are the same in both types of LCFTs, the ones with and without logarithmic L_0^{LCFT}. Remember that applying L_0^{LCFT} on a highest weight state $|h, k\rangle$ leads to an extra term $|h, k-1\rangle$ for $k > 0$. Therefore, marking the logarithmic extension term with a q leads to

$$(L_0^{\log})^m \, |h, k\rangle = q^k \, |h, 0\rangle + q^{k-1}(\ldots) + \ldots + h^m \, |h, k\rangle \quad , m > k \tag{2.90}$$

where $k = 0, \ldots, \text{jrk}\,(L_0^{\log}) - 1$ and jrk denotes the rank of the Jordan-matrix. This means that the Jordan-rank can be found by applying $(L_0^{\text{LCFT}})^m$ for all $m \in \mathbb{N}$ on all $|h\rangle \in \text{HWS}$ (HWS denotes the set of all highest weight states). The highest occurring power in q plus 1 defines the rank of the Jordan-cell. This motivates the following definition (writing $L_n^{\text{LCFT}} = L_n^{\text{CFT}} + qR_n$ with R_n being deformation term):

$$\text{jrk}\,(L_n^{\log}) := \max \left\{ k = \deg_q \left((L_n^{\log})^m \, |h\rangle \right) : |h\rangle \in \text{HWS}, m \in \mathbb{N} \right\} \ , \tag{2.91}$$

where $(L_0^{\log})^m \, |h\rangle$ is to be understood as a polynomial in q after evaluation. The logarithmic behavior of this theory becomes (for $A = -B$) manifest in L_{-2}^{tot}. The Jordan-rank in the above defined sense of L_{-2}^{tot} can easily be found by examining (2.82): each term contains at least one of the modes b_i^p ($p = 1, 2, i = -1, 0, 1$). The remaining b modes are of no interest since these are creators and the zero modes are mutually distinct in each of the terms. Therefore, the only states we are interested in are words in the letters $c_{-1}^2, c_0^2, c_1^1 c_1^2, c_{-1}^1, c_0^1$. Looking at the highest weight states of conformal weight $h = 0$ in equation (2.87) shows that one highest weight state really contains a state which consists of all the above letters implying an upper bound

$$\text{jrk}\,(L_{-2}^{\text{tot}}) = 5 \tag{2.92}$$

up to accidental cancellations. Tedious and lengthy calculations reveal that the upper bound is satisfied. We note for completeness that for $B = 0, A \neq 0$ the Jordan-rank is $\text{jrk}\,(L_{-2}^{\text{tot}}) = 3$.

2.6 Summary and conclusion

The well-known b-c system with central charge $c = -26$ can be enlarged to a logarithmic CFT. In some aspects the transition is similar to the $c = -2$ case, in others it is completely different: the energy-momentum tensor cannot be built by combining derivatives of the generalized symplectic fermion fields and we also have to consider that it is not correct to neglect one half of the theory if we investigate it on hyperelliptic Riemann surfaces. On the contrary, enlarging the $c = -26$ ghost system to a logarithmic theory makes it necessary to consider this theory on nontrivial Riemann surfaces. This is natural and consistent with our understanding of the geometrical origin of logarithmic fields from operator product expansions of twist fields which simulate branch points. On the other hand, it is surprising in so far as it is possible to consider the logarithmic extension of the better known $c = -2$ theory without the need of putting it on higher genus Riemann surfaces. As we have seen, this is impossible for $c = -26$. Due to the particular structure of the vertex operators which represent the branch points, we conjecture that logarithmic extensions of other ghost systems with $\lambda \neq 1, 2$ are only possible when considered on $\mathbb{Z}_n$-symmetric Riemann surfaces.

We are confident that the presented construction scheme works not only for $c = -26$ but for all b-c ghost systems. The deformation term we used in order to obtain the new energy-momentum tensor is slightly more general than the deformation term discussed in the paper by Fjelstad *et al.* (2002), but is naturally linked to the zero mode structure of the ghost systems. Thus, we expect that the spin $(\lambda, 1-\lambda)$ ghost system has generically $2\lambda + 1$ deformation directions which presumably get restricted due to hermiticity conditions and consistency requirements for the action of the deformed Virasoro algebra of the full theory on the Hilbert space of states.

The structure of the logarithmic $c = -26$ theory is very different from what one might have expected in analogy to the $c = -2$ case: L_0^{tot} is not logarithmic at all. This is a completely new property of a LCFT. Furthermore, the special choice of the deformation parameter $A = -B$ (see equation (2.63)) leads to a theory where the whole global conformal group is non-logarithmic. This special property is not yet completely investigated. The logarithmic character of the theory becomes manifest in L_{-2}^{tot} which shows similar indecomposable properties as L_0 in a standard LCFT. A generalization of the definition of the Jordan-rank has been given which we used to find that the Jordan-rank of L_{-2}^{tot} is 5 for all nontrivial choices of A and B. This should help in identifying the proper generalization of "logarithmic partners" to primary fields.

Chapter 3

Four-Point Functions

*"A species stumped by an intractable problem
does not merely cease to compute.
It ceases to exist."*
— Seth Lloyd

3.1 Overview

Correlation functions play an important role in quantum field theories, as they are related to observables. For instance it is possible to compute the cross sections of a scattering process from the correlation functions of the theory. These cross sections are also accessible in experiments and thus are of great interest. In general one often calls a quantum field theory to be solved if all its correlation functions can be determined.

The above is the reason for us to study correlation functions in LCFT. To this aim we reconsider correlators in the framework of CFT. As we have learned in subsection 1.1.10 the global conformal Ward identities (GCWIs) are powerful tools which allow fixing the two- and three-point correlation function up to constants. Furthermore, these identities make it possible to determine the generic structure of the four-point function, see (1.64). By generic structure we mean that the result still contains an unknown function F which solely depends on the anharmonic ration x as given by (1.66). This all is a consequence of invariance under global conformal transformations.

We here again write down the GCWIs, but in difference to equation (1.61) we use the notation that was introduced in subsection 1.2.2

$$L_q \left\langle \Psi_{(h_1)}(z_1) \dots \Psi_{(h_n)}(z_n) \right\rangle := \sum_{i=1}^{n} z_i^q \left[z_i \partial_i + (q+1) h_i \right] \langle \dots \rangle = 0 \ , \tag{3.1}$$

where $\Psi_{(h)}(z)$ denotes a primary field.

In case of logarithmic conformal field theory the GCWIs need to be adapted to take into account the indecomposable representations that LCFTs typically possess. The generalized form of the GCWIs was given by Flohr (2000)

$$\sum_{i=1}^{n} z_i^q \left[z_i \partial_i + (q+1)(h_i + \delta_{h_i}) \right] \left\langle \Psi_{(h_1,k_1)}(z_1) \dots \Psi_{(h_n,k_n)}(z_n) \right\rangle = 0 \ , \tag{3.2}$$

where $\Psi_{(h_i,k_i)}(z_i)$ denotes a logarithmic field of Jordan-level k_i respectively a primary field in case $k_i = 0$. The operator δ_{h_i} acts on these logarithmic fields by reducing the Jordan-level of the field by one respectively annihilating the field in case it is a primary one: $\delta_{h_i} \Psi_{(h_i,k_i)} = \Psi_{(h_i,k_i-1)}$ for $k_i > 1$ and $\delta_{h_i} \Psi_{(h_i,k_i=0)} = 0$ otherwise (field being a primary). Note that in the above equation the additional operator δ_{h_i} vanishes for $q = -1$ meaning that the LCFT version exactly matches the CFT version for this value of q. The additional operator δ_{h_i} makes it much harder to find the generic form of the correlators, because it renders the differential equations inhomogeneous, i. e., the solution will depend on solutions of lower Jordan-level.

As before L_0 acts on the states

$$L_0 \left| h; k \right\rangle = h \left| h; k \right\rangle + \left| h; k-1 \right\rangle \quad \text{for } k > 0 \tag{3.3}$$

where we additionally define

$$\left| h; -k \right\rangle := 0 \quad \forall k > 0 \ . \tag{3.4}$$

This shows, that the fields $\Psi_{(h_i,k_i)}$ indeed correspond to Jordan cells with respect to L_0. The representation of a LCFT with the largest Jordan cell defines the rank r of the LCFT, i. e., $k_i < r$.

All states $\left| h, k \right\rangle$ are typically assumed to be quasi-primary in the sense that $L_n \left| h, k \right\rangle = 0 \ \forall n > 0$ and for all k. Thus, they almost behave as highest-weight states, up to the non-diagonal action of L_0. This is not true in general, because states to logarithmic partner fields may fail to be quasi-primary, i. e. $L_1 \left| h, k \right\rangle \neq 0$ for $k > 0$. However, under certain assumptions, this does not affect the form of correlation functions. Furthermore, from the results for 1-, 2- and 3-point functions we can expect the vacuum representation to have the maximal Jordan-rank. No counter-examples are known up to now and thus we assume that the Jordan-rank is the same for all representations without loss of generality. The latter is justified as follows: in case some smaller Jordan-rank representation does show up, we can extend this representation by adding additional fields which we set to zero. In essence, this simply means that the general results remain valid with some of the structure constants set to zero. For further details on the precise assumptions in the case of non quasi-primary fields and on the maximal rank of the vacuum representation see Flohr (2002b).

While there are generic methods to determine 2- and 3-point correlation functions Flohr (2002b); Ghezelbash and Karimipour (1997); Khorrami *et al.* (1998); Moghimi-Araghi *et al.* (2001a); Rahimi Tabar *et al.* (1997), no such method exists for 4-point correlation functions, but see Moghimi-Araghi *et al.* (2001a, 2003) for a solution in case of 4-point functions involving a level two null vector field. On the other hand all $n > 4$-point correlation functions can be reduced to 2-, 3- and 4-point correlators. Therefore one can compute all observable quantities of a CFT–at least in principle–if one knows all 2-, 3- and 4-point functions. Thus, this thesis attempts to close the remaining gap by providing the prescription to fix the generic form of 4-point correlators in the case of arbitrary rank Jordan-cells in LCFT.

While the generic form of 2- and 3-point functions is fixed up to structure constants the generic form of 4-point functions can be fixed only up to functions $F_{i_1 i_2 i_3 i_4}(x)$ of the globally conformally invariant crossing ratios x. As in the case of ordinary conformal field theory these structure functions can be computed if additional local symmetries, i. e. null vectors, exist. Indeed, such null vectors can exist in the logarithmic case Flohr (1998b), but the resulting differential equations are more difficult to solve because they are inhomogeneous in general Flohr (2000).

In this chapter we describe how the most general ansatz can be constructed and how the emerging constants can be calculated in order to find a valid ansatz for equation (3.2). Most of the constants can be fixed with the help of the global conformal Ward identities, but we will also encounter cases where some degrees of freedom are left. A necessary condition for these additional degrees of freedom is that all four fields in the four-point function are of logarithmic origin. The number of degrees of freedom very much depends on the form of the correlator. Furthermore, we find that we have to identify some of the structure functions $F_{i_1 i_2 i_3 i_4}$ that are part of the correlator.

We then will use the discussed methods to determine all correlators for

Jordan-rank $r = 2$ and $r = 3$. The results are given in a graphical representation and also we make use of permutation symmetries in order to keep the terms as short as possible. In the last section we consider the special case that only two of the four fields are logarithmic and we show how the resulting equations can be solved in this case for arbitrary Jordan-rank r.

3.2 Approaching the problem

In this section we describe how we simplify the initial problem and what algorithm we use to solve it for Jordan-rank $r = 2$ and $r = 3$. We also discuss the appearance of additional degrees of freedom that may show up if all four fields are of logarithmic type. For understanding of this section it might be helpful to have a glance at the next section which in detail discusses the most simple non-trivial case, that is Jordan-rank $r = 2$.

3.2.1 Simplification

As noted in the previous section the equation (3.2) is equal to the global conformal Ward identity in CFT for $q = -1$, meaning that the ansatz has to be translation invariant and therefore allows terms of the form $z_{ij} := z_i - z_j$ only to show up. From now on we will consider 4-point correlation functions only. After solving the first of the three equations, the ansatz has the following form

$$\left\langle \Psi_{(h_1,k_1)}(z_1) \dots \Psi_{(h_4,k_4)}(z_4) \right\rangle = \prod_{i<j} z_{ij}^{\mu_{ij}} \, f(z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}) \,. \tag{3.5}$$

Here the exponents $\mu_{ij} = \mu_{ji}$ must satisfy the conditions

$$\sum_{j \neq i} \mu_{ij} = -2h_i \,. \tag{3.6}$$

The factor $\prod_{i<j} z_{ij}^{\mu_i}$ exists to counter the h_i terms on the left hand side of equation (3.2) and therefore we can set all conformal weights to zero, $h_i = 0$ without loss of generality. Note that the full correlator of course depends on the conformal weights. The point here is that the global symmetries are not sufficient to fix the complete correlator, but they are strong enough to fix the generic form and this form has no dependence on h_i. Therefore, we can lighten

the resulting formulas by simply omitting the trivial direct dependency on the conformal weights. If we set all conformal weights to zero then (3.2) becomes

$$\sum_{i=1}^{4} z_i^q \left[z_i\partial_i + (q+1)\delta_{h_i}\right] \langle k_1k_2k_3k_4\rangle = 0 \ , \tag{3.7}$$

where we write k_i instead of the much longer form $\Psi_{(h_i,k_i)}(z_i)$. The remaining two equations for $q = 0, 1$ have a δ_{h_i} term acting on the correlator and thus lowering the sum of the Jordan-levels by one. Because of calculating the expressions recursively we can assume the predecessors $\delta_{h_i}\langle\ldots\rangle$ to be known. This leads us to the final form

$$O_0\langle\ldots\rangle := \sum_{i=1}^{4} z_i\partial_i\langle\ldots\rangle = -\sum_{i=1}^{4}\delta_{h_i}\langle\ldots\rangle \ , \tag{3.8}$$

$$O_1\langle\ldots\rangle := \sum_{i=1}^{4} z_i^2\partial_i\langle\ldots\rangle = -2\sum_{i=1}^{4} z_i\delta_{h_i}\langle\ldots\rangle \ , \tag{3.9}$$

where the correlators depend on the difference z_{ij} only. Though looking simple for given predecessors $\delta_{h_i}\langle\ldots\rangle$ at first glance, it is not easy to find an ansatz for the correlator at all. Moreover we will learn that in some cases the result is not unique. We sometimes use the sloppy term "integrating" the predecessors $\delta_{h_i}\langle\ldots\rangle$ as a shortage for finding an ansatz that fulfills the above equations.

The starting point for the recursion is given by

$$\langle k_1k_2k_3k_4\rangle = F_0(x) \quad \text{for} \quad \sum_i k_i = r-1 \quad \text{respectively} \tag{3.10}$$

$$\langle k_1k_2k_3k_4\rangle = 0 \quad \text{for} \quad \sum_i k_i < r-1 \tag{3.11}$$

where x is the anharmonic ratio, as defined in (1.66). In essence this means that a correlation function with total Jordan-level $K := \sum_i k_i = r-1$ behaves like a correlation function in ordinary conformal field theory, i. e. it depends on one function of the globally conformally invariant anharmonic ratio.

The reason for these initial conditions comes from the fact that the only non-vanishing one-point function in LCFT is the one of the highest level logarithmic partner of the identity, $\Psi_{(h=0,k=r-1)}$. Evaluating a correlation function amounts to contracting the inserted fields, in all possible ways, down to a one-point function. Therefore, it is only natural to expect that the total Jordan-level K of a non-vanishing correlator must at least be equal to $r-1$. Furthermore, since the cluster decomposition property should hold, the initial conditions must also hold for arbitrarily factorized correlators, e. g., $\langle k_1k_2k_3k_4\rangle \sim \langle k_1k_2|0\rangle\langle 0|k_3k_4\rangle$ in case that z_1, z_2 are well separated from z_3, z_4. However, some care has to be taken about the correct insertion of the "identity" channel, which formally can be thought of to be of the form $|0\rangle\langle 0| = \sum_{k=0}^{r-1} |h=0;k\rangle\langle h=0;r-1-k|$. It is easy to see that the cluster decomposition with the above identity channel implies (3.11) and that precisely one term of this identity channel survives yielding (3.10), where we made use of the results for two-point functions in Flohr (2002b).

In the beginning we mentioned that $k_i > 0$ represents a logarithmic partner field, while $k_i = 0$ is a primary field. We can subdivide the class of primary fields into two subclasses, the so called proper primary-fields and the pre-logarithmic fields. This difference between the subclasses becomes apparent if one considers the operator product expansion (OPE). In contrast to the OPE of two proper primary-fields the OPE of two pre-logarithmic shows an additional term of logarithmic behavior, cf. Kogan and Lewis (1998).

In the following we consider proper primary-fields only and use the term synonymous with primary field. Restricting to proper primary-fields is for simplicity only. It is possible to include pre-logarithmic fields into the theory, by making changes to the initial condition (3.10), (3.11). For instance in the well-known $c = -2$ example the initial-conditions for Jordan-rank $r = 2$ would be

$$\langle\phi\phi\phi\phi\rangle = 0 \tag{3.12}$$

$$\langle\mu\phi\phi\phi\rangle = 0 \tag{3.13}$$

$$\langle\mu\mu\phi\phi\rangle = F_0(x)\ , \tag{3.14}$$

where ϕ stands for a proper primary and μ denotes a twist field. Note that the same could be formally achieved by assigning rational values k_i to pre-logarithmic values, e. g., in this example assigning a value of $k_i = \frac{1}{2}$ to the twist fields and using (3.10), (3.11) would lead to the same initial conditions. A more precise analysis of this and how to assign correct values for the k_i can be found in Flohr (2002b). Apart from the initial conditions we also need slight adaption of the "connection rules" we are going to explain in subsection 3.2.4. More comments can be found in the conclusions.

3.2.2 Naming conventions

The dependence of F on the anharmonic ratio, is suppressed in the following. Further note that we do not write out the dependence on the Jordan-rank r,

e. g., $\langle 1000\rangle = F_0$ (for $r = 2$) as well as $\langle 1100\rangle = \ldots = \langle 2000\rangle = F_0$, namely for $r = 3$.

As we will see the solution for all other cases $K := \sum_i k_i > r - 1$ is always of the form

$$\langle k_1k_2k_3k_4\rangle = F_{k_1k_2k_3k_4} + (c_1 l_{12} + \ldots + c_6 l_{34})F_{k_1-1,k_2,k_3,k_4} + \ldots + \\ (\text{logarithmic degree } K - r + 1)F_0 \ , \tag{3.15}$$

where $l_{ij} := \log(z_{ij})$. The highest logarithmic powers that appear in the solution are always the factors associated with the function F_0. The degree in l_{ij} also called logarithmic degree for short, is given by

$$\deg(\{l_{ij}\}) \leq K - r + 1 =: l^{\max} \ . \tag{3.16}$$

There are cases where we will find that some of the functions $F_{j_1j_2j_3j_4}$ can be identified with each other, e. g., we will find that $F_{2100} \equiv F_{1200}$ for $r = 3$. After identification we will always use the F-term whose index represents the lowest "number". For example we write $\langle 2100\rangle = F_{1200}(x) + \ldots$ instead of using F_{2100}.

In many places we decided to use a graphical representation instead of writing long expressions of logarithms. The idea for this stems from Flohr (2003) where it was chosen in order to give a better understanding of the contractions that can appear. Reading the diagrams is straightforward, the points stand for the four vertices and each l_{ij} is represented by a line between the vertices i and j. Permutation operators P are used to further reduce the length of the expressions, for instance

$$l_{12}^2 l_{23} l_{34} - l_{23}^2 l_{12} l_{14} = (1 - P_{(13)})$$  . (3.17)

From section 3.3 on we will always use the graphical representation to present the results.

3.2.3 Properties of O_0, O_1

Both operators O_q are linear, nilpotent, act as derivatives on the function space and are invariant under any permutations $p \in S_4$.

The function space we consider is the space of polynomials in the logarithmic functions $l_{ij} := \log|z_i - z_j|$, called $\mathcal{F}_{\log} := \mathbb{C}[l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}]$.

For $q = 0, 1$ the operators O_q have a simple behavior, when acting on $\mathcal{F}_{\log}$:

$$O_0 : \begin{cases} \mathcal{F}_{\log} \to \mathcal{F}_{\log} \\ l_{i_1 j_1} \dots l_{i_n j_n} \mapsto \sum_{k=1}^{n} l_{i_1 j_1} \dots l_{i_{k-1} j_{k-1}} \, l_{i_{k+1} j_{k+1}} \dots l_{i_n j_n} \end{cases} \tag{3.18}$$

$$O_1 : \begin{cases} \mathcal{F}_{\log} \to \mathcal{F}_{\log}[\{z_{ij}\}] \\ l_{i_1 j_1} \dots l_{i_n j_n} \mapsto \sum_{k=1}^{n} l_{i_1 j_1} \dots l_{i_{k-1} j_{k-1}} (z_{i_k} + z_{j_k}) l_{i_{k+1} j_{k+1}} \dots l_{i_n j_n} \end{cases} , \tag{3.19}$$

meaning that we can replace the term by a sum, where each l_{ij} is replaced by either 1 (for $q = 0$) or by $z_i + z_j$ (for $q = 1$). Thus acting with O_q on any term obviously reduces the logarithmic degree by one and by that proves (3.16).

An obvious question is whether the map $O_q : f \to f'$ is injective: are there any non-trivial $f \in \mathcal{F}_{\log}$ with $O_0 f = 0$ and $O_1 f = 0$?

If we restrict ourselves to the function space $\mathcal{F}_{\log}$ then we find that we can exactly determine the kernel of the operator $O := (O_0, O_1)$.

As will be shown in subsection 2.6 below, the kernel is given as follows.

$$\ker_{\mathcal{F}_{\log,g}} O = \left\{ \sum_{i=0}^{g} a_i K_1^i K_2^{g-i} : a_k \in \mathbb{R} \right\} \tag{3.20}$$

$$K_1 := l_{12} + l_{34} - l_{13} - l_{24} \tag{3.21}$$

$$K_2 := l_{12} + l_{34} - l_{14} - l_{23} , \tag{3.22}$$

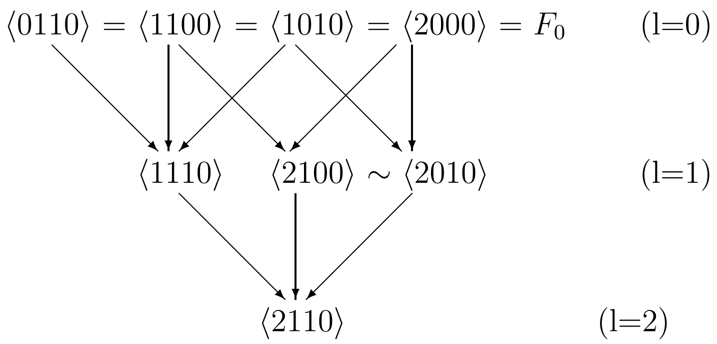
where $\mathcal{F}_{\log,g} := \{f \in \mathcal{F}_{\log} | \deg f = g\}$ denotes the space of functions with logarithmic degree g, such that $\mathcal{F}_{\log} = \bigcup_g \mathcal{F}_{\log,g}$.

3.2.4 An ansatz for the equations

As mentioned before we want to recursively solve the equations (3.8) and (3.9). Since the number of terms quickly becomes huge and calculation tedious we make use of computer algebra software for performing the calculations. In the next subsection we show that the two equations can be reduced to a set of simpler equations and in subsection 3.2.4 we present the algorithm we used for creating an ansatz.

Recursion

With recursion we mean the following: we start with the initial conditions as given in (3.10) which corresponds to logarithmic degree $l = 0$. Then we calculate all necessary correlators which contain exactly one more logarithmic field or one field whose Jordan-level is increased exactly by one. In short this means that we determine all correlators of logarithmic degree $l = 1$. The following diagram describes which correlators need to be calculated in order to determine the correlation function for $\langle 2110 \rangle$.



The effort for calculation can be reduced, since many of the correlators are related $\sim$ to others by simple permutations, e. g., $\langle 2100 \rangle = P_{23} \langle 2010 \rangle$.

Breaking down into a set of equations

The operators O_0, O_1 in equations (3.8), (3.9) are linear, they act as derivatives on the correlators $\langle \ldots \rangle$ and they are invariant under any permutation $P \in S_4$ of the indices. The ansatz as well as the term on the right hand side can, as we have seen before, be written in terms of the functions of $F_{\ldots}$, resulting in

$$\begin{aligned} O_q \big\{ F_{k_1 k_2 k_3 k_4} + (\ldots)^u F_{k_1-1,k_2,k_3,k_4} + (\ldots)^u F_{k_1,k_2-1,k_3,k_4} + \ldots \\ \ldots + (\ldots)^u F_{r-1,0,1,0} + (\ldots)^u F_{r-1,0,0,1} + (\ldots)^u F_0 \big\} \; = \\ (\ldots) F_{k_1-1,k_2,k_3,k_4} + (\ldots) F_{k_1,k_2-1,k_3,k_4} + \ldots + (\ldots) F_{r-1,0,1,0} \\ + (\ldots) F_{r-1,0,0,1} + (\ldots) F_0 \, . \end{aligned} \tag{3.23}$$

The terms $(\ldots)$ denote functions which may additionally depend on the differences $z_{12}, z_{13}, \ldots, z_{34}$ caused by the action of O_1. As usual r is the Jordan-rank of the theory. For the right hand side we can assume these terms to be known, because we will solve the equations recursively. The corresponding terms on the left hand side are unknown, they are marked with a small "u". O_q operates as a derivative and since $O_q F = 0$, we find that the problem reduces to "integrating" the following set of equations

$$O_q(\ldots)^u_{k_1,k_2,k_3,k_4} = 0 \tag{3.24}$$
$$O_q(\ldots)^u_{k_1-1,k_2,k_3,k_4} = (\ldots)_{k_1-1,k_2,k_3,k_4}$$
$$O_q(\ldots)^u_{k_1,k_2-1,k_3,k_4} = (\ldots)_{k_1,k_2-1,k_3,k_4}$$
$$\ldots$$
$$O_q(\ldots)^u_{r-1,0,0,1} = (\ldots)_{r-1,0,0,1}$$
$$O_q(\ldots)^u_0 = (\ldots)_0 \,. \tag{3.25}$$

The upper index u just reminds us that these terms are not yet known, and the lower index tells us from which part of the equation (3.23) the term stems from. Note that the first equation (3.24) and its solution is well known

$$(\ldots)^u_{k_1,k_2,k_3,k_4} = F_{k_1,k_2,k_3,k_4}(x) \,, \tag{3.26}$$

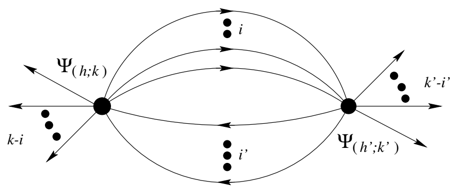
with x being the anharmonic ratio.

Description of the algorithm

Until now we did not specify what ansatz we fill in the left hand side of the equations (3.24) to (3.25). From OPE considerations, see Flohr (2002b), respectively from the structure of the operators O_0 and O_1 we expect the correlators to consist of terms of the type $l_{12}^{a_1} l_{13}^{a_2} \ldots l_{34}^{a_6}$, where each term comes with an coefficient which needs to be determined. More precisely, the generic structure of 2- and 3-point functions depends on the l_{ij} in a strictly polynomial form in such a way that the same is true for the operator product expansion. Thus, also the 4-point functions should depend only in a polynomial way on the l_{ij} since, asymptotically, a 4-point function decomposes into an operator product expansion times remaining 3-point functions, all of which are entirely polynomial in the l_{ij}. Unfortunately, the number of possible monomials in the l_{ij} grows heavily with the rank r of the LCFT, and thus the number of coefficients. Luckily we can reduce the number of possible terms that can show up in the following.

We do not have to take into account every logarithmic degree $a_1 + a_2 + \ldots + a_6$. The equations (3.18) and (3.19) tell us that the logarithmic degree is reduced by one if we apply O_0 or O_1. If we assume for a moment that in every equation of (3.24) to (3.25) the right hand side consists of terms of the same logarithmic degree l, then it is apparent that the terms on the left hand side have logarithmic degree $l + 1$. We build all correlators recursively as explained in subsection 3.2.4 and since our initial conditions only consists of one term on the right hand side, we trivially find our assumption fulfilled. Thus by induction all terms $l_{12}^{a_1} l_{13}^{a_2} \dots l_{34}^{a_6}$ in $(\dots)^u_{n_1,n_2,n_3,n_4}$ have to have the same logarithmic degree.

As described in Flohr (2003) it is helpful to use a graphical representation where each field $\Psi_{(h,k)}(z)$ in a Jordan-cell is depicted by a vertex with k outgoing lines. Contractions of logarithmic fields give rise to logarithms in the correlators, where the possible powers with which l_{ij} may occur are determined by graph combinations.



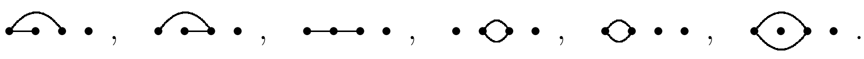
Essentially, the terms for an ansatz of logarithmic degree l are given by a sum over all admissible graphs subject to the following rules:

1. use at most k_i legs of a vertex for connections with other vertices,
2. the source i and the destination vertex j have to be different: $i \neq j$,
3. do connections with other logarithmic fields only (do not connect with primary fields) ,
4. create exactly l connections ,
5. write l_{ij} for every connection between two vertices i and j.

Let us have a look at a simple example. We consider a theory of Jordan-rank $r = 3$ and are interested in the structure of the correlator $\langle 2110 \rangle$ for the highest possible logarithmic degree, i. e. the F_0 term. The corresponding graph for $\langle 2110 \rangle$ is

 .

Altogether we have four legs to our disposal, but we also have to fix two of them leaving us with two free legs. If we want to know which terms can appear for logarithmic degree $l = 2$, then we have to create all 2-contractions according to the above rules. This results in the following six different graphs:



Note that there are only two different truly independent graphs in the sense that they are not a mere permutation of other graphs. The first three graphs and the remaining three graphs form two equivalence classes induced by permutations of S_4.

Using the algorithm results in a maximum of $\binom{l+5}{l}$ terms that can appear. Combinatorial restrictions which we will discuss in the following can reduce this number, for instance $\langle 2211 \rangle$ for $l = 3$ does not contain a l_{34}^3 term.

3.2.5 Restrictions

The analysis of the results we found shows that several restrictions reduce the number of different terms that may appear in the end result.

The first restriction naturally appears during the integration process. In some cases our method for recursively constructing “higher” correlators fails. It is not possible to repair this failure in a sensible manner by adding further terms to the ansatz, but a simple identification of different functions F immediately fixes the problem.

This behavior is a general property of the theory for $r \geq 3$, as we will see in section 3.5. For now it is sufficient to note that $F_{k_1-1,k_2,0,0} = F_{k_1,k_2-1,0,0}$, e. g., for $r = 3$ we get $F_{2100} = F_{1200}$ +5 more identifications by virtue of permutations.

The second restriction we encountered is the so called discrete symmetry of the correlators, which limits the dimension of the kernel. By discrete symmetry we mean that a correlator which contains at least two fields of the same Jordan-level should be invariant under any transposition that exchanges these fields, for instance

$$P_{(12)} \left\langle \Psi_k \Psi_k \Psi_{k_3} \Psi_{k_4} \right\rangle = \left\langle \Psi_k \Psi_k \Psi_{k_3} \Psi_{k_4} \right\rangle \ . \qquad (3.27)$$

At this point we point out again that we have set, without loss of generality, all conformal weights h_i to zero and wrote Ψ_k instead of $\Psi_{(h,k)}(z)$. In the next subsection we will discuss in more detail to what extent the above mentioned invariance limits the dimension of the kernel respectively show that in some cases no kernel term can show up at all.

The dimension of the kernel that finally shows up in the results is often smaller than the one we would expect for the given logarithmic degree and given discrete symmetry. The difference will show up especially if the logarithmic degree is close to the maximum degree $l^{\max} = K - r + 1$.

The reason for this difference is that the ansatz does not allow all terms of l_{ij} of a given degree to show up. For instance the correlator $\langle 2211 \rangle$ forbids the existence of terms of the type l_{34}^3 and by that limits the dimension of the kernel of degree 3. We also refer to this as combinatorial restriction, because the restriction depends on the form of the correlator, e. g., the term l_{34}^3 is not forbidden in $\langle 2221 \rangle$.

3.2.6 Additional constants

As we have seen in section 3.2.3 the kernel of the operator O is non-trivial. That means that the results may come with additional constants. In order to understand the meaning of these constants in the context of conformal field theory we rewrite the two basis terms K_1, K_2 which every element of the kernel consists of as follows:

$$K_1 := l_{12} + l_{34} - l_{13} - l_{24} = \log \left| \frac{z_{12}z_{34}}{z_{13}z_{24}} \right| \equiv \log |x| - \log |1 - x| \qquad (3.28)$$

$$K_2 := l_{12} + l_{34} - l_{14} - l_{23} = \log \left| \frac{z_{12}z_{34}}{z_{14}z_{23}} \right| \equiv \log |x| \qquad (3.29)$$

where $x = \frac{z_{12}z_{34}}{z_{14}z_{32}}$ is the anharmonic ratio of the four points. The anharmonic ratio x and its five possible involutions $\frac{1}{x}$, $1-x$, $1-\frac{1}{x}$, $\frac{1}{1-x}$, and $\frac{x}{x-1}$ result in four linearly independent functions. If we take the logarithm of the absolute value of these four functions, then we are left with only two independent solutions, namely $\log |x|$ and $\log |1 - x|$. The choice of the basis has no influence on the results and our choice of the basis K_1, K_2 is given as above.

We can turn around the argument and ask for all functions of the anharmonic ratio x, i. e. globally conformally invariant functions, which have the additional property to be strictly polynomial in the l_{ij}. These functions are all in the kernel of the operator O. On the other hand, there can be no other functions in the kernel if we restrict ourselves to polynomials of the l_{ij}, since every member of the kernel must be invariant under global conformal transformations and thus be a function of x. This proves the statement in section 3.2.3. However, we note that this yields only an upper bound on the size of the kernel. We will see that the size may be reduced due to further symmetries.

Equation (3.20) gives us the maximal dimension of the kernel for logarithmic degree l,

$$K^{(l)} := \left\{ \sum_{i=0}^{l} a_i K_1^i K_2^{d-i} : a_k \in \mathbb{R} \right\} \tag{3.30}$$

$$d^{\max}(l) = \; l+1 \; . \tag{3.31}$$

Up to a few exceptions we will notice that the full kernel never shows up in any equation. These restrictions on the kernel are caused by the discrete symmetry and combinatorial constraints. Examples for combinatorial constraints are shown in the next two sections.

It is worth noting that a non-trivial kernel can show up in a correlator only if there is no primary field in the correlator present. This is obvious, since both kernel elements K_1, K_2 refer to all four vertices z_1, z_2, z_3, z_4.

Discrete symmetry for invariant *F*

In this subsection we are interested in the impact on the kernel by a given symmetry. Since we consider four point correlation functions exclusively there are four interesting symmetry groups only, namely S_2, $S_2 \times S_2$, S_3 and S_4. Let us study an expression first, where the function F is invariant under any permutation, e. g., $(\ldots)F_{1111}$.

We start with the smallest symmetry group $S_2 = \{1, P_{(12)}\}$, P being, as usual, a permutation of the indices. One immediately remarks that $P_{(12)}K_1 = K_2$, $P_{(12)}K_2 = K_1$ and thus a S_2 invariant kernel of logarithmic degree l has the form

$$K_{S_2}^{(l)} := \left\{ \sum_{i=0}^{l} a_i K_1^i K_2^{d-i} : a_k \in \mathbb{R}, a_k = a_{l-k} \right\} \; . \tag{3.32}$$

Therefore the maximum number of constants $d_{S_2}^{\max}$ that could appear for logarithmic degree l is

$$d_{S_2}^{\max}(l) = \left\lfloor \tfrac{l}{2} \right\rfloor + 1 \; , \tag{3.33}$$

where $\lfloor . \rfloor$ denotes the floor function.

If we replace the transposition $P_{(12)}$ by $P_{(34)}$ all statements stay true. Thus when restricting to kernel space $K := \bigcup_l K^{(l)}$ we have $P_{(12)} \equiv P_{(34)}$. This in turn means that the kernel is not only S_2 invariant, but automatically has full $S_2 \times S_2 = \{1, P_{(12)}, P_{(34)}, P_{(12)(34)}\}$ invariance:

$$K_{S_2\times S_2}^{(l)} \equiv K_{S_2}^{(l)} \tag{3.34}$$

$$d_{S_2\times S_2}^{\max}(l) = \left\lfloor \tfrac{l}{2} \right\rfloor + 1 \; . \tag{3.35}$$

For the S_3 symmetry we note, that a S_3 invariance extends to S_4 invariance. This is because S_3 invariance in particular means $P_{(12)}$ invariance which, as explained above, also means $P_{(34)}$ invariance. By this we immediately obtain full S_4 invariance:

$$K_{S_3}^{(l)} \equiv K_{S_4}^{(l)} \,. \tag{3.36}$$

No linear combination of K_1, K_2 is S_4 invariant, but higher terms in K_1, K_2 have this property. The first two dimensional kernel $d = 2$ can be found for logarithmic degree $l = 6$:

$$\begin{aligned}
K_{S_4}^{(2)} &:= K_1^2 - K_1K_2 + K_2^2 \\
K_{S_4}^{(3)} &:= (2K_1 - K_2)(2K_2 - K_1)(K_1 + K_2) \\
K_{S_4}^{(4)} &:= (K_{S_4}^{(2)})^2 \\
K_{S_4}^{(5)} &:= K_{S_4}^{(3)}K_{S_4}^{(2)} \\
K_{S_4}^{(6,(2,2,2))} &:= (K_{S_4}^{(2)})^3 \\
K_{S_4}^{(6,(3,3))} &:= (K_{S_4}^{(3)})^2 \\
K_{S_4}^{(7)} &:= (K_{S_4}^{(2)})^2K_{S_4}^{(3)} \,.
\end{aligned} \tag{3.37}$$

These results are unique, up to constants and linear combinations. Of course any combination of the form $(K_{S_4}^{(2)})^i(K_{S_4}^{(3)})^j$ leads to a kernel of logarithmic degree $2i + 3j$ and we believe that the kernel space is not larger than this, though it is not important since we consider kernels up to logarithmic degree $l = 6$ only in the further course of this chapter.

We expect the dimension of the S_4 invariant kernel to be the number of possible partitions of the of the degree in the numbers 2 and 3, e. g., $6 = 2+2+2$ as well $6 = 3 + 3$. This in turn means that every integer 6 can be represented in two different ways, leading to the following number of degrees of freedom that could appear at most for logarithmic degree l:

$$d_{S_4}^{\max}(l) = \begin{cases} \left\lfloor \frac{l}{6} \right\rfloor & ; l = 6k + 1,\ k \in \mathbb{N}_0 \\ \left\lfloor \frac{l}{6} \right\rfloor + 1 & ; \text{else} \,. \end{cases} \tag{3.38}$$

Discrete symmetry and non-invariant $\boldsymbol{F}$

In the previous subsection we analyzed the structure of the kernel under symmetry groups and found that we have to consider S_2 and S_4 symmetry groups only. This holds if the function $F_{\dots}$ itself is invariant under permutations.

Things get more complicated if F_a is mapped to F_b by the permutation. For example we know that the S_2 invariant kernel for one F (F_1 for short) is one-dimensional, namely $K^{(1)}_{S_2} = \{a(K_1 + K_2) : a \in \mathbb{R}\}$. But if we have two F, which are related by the permutation, e. g., F_{1022} and F_{0122}, then the dimension of the kernel for F_{1022} becomes larger. The kernel would be $(cK_1 + c'K_2)F_{1022} + (c'K_1 + cK_2)F_{0122}$, or $\mathcal{P}_{S_2}(K^{(2)}F_{1022})$, $\mathcal{P}_{S_2} = 1 + P_{(12)}$ for short.

The kernel dimension therefore not only depends on the symmetry group and the logarithmic degree, but also on the size of the equivalence class of functions F which are involved. The size of the equivalence class is noted by F_n and the results for the logarithmic degrees $l = 1, 2, \ldots 5$ are listed in appendix A.

The simple rule that S_2 corresponds to $S_2 \times S_2$ respectively that S_3 corresponds to S_4 does not hold for $n > 1$, therefore we have to discuss all four symmetry groups in the appendix. The dimension of the kernel decreases with increasing size of the symmetry group and increases with increasing size of the equivalence class. It is interesting though not surprising, that the full kernel $K^{(l)}$ is recovered, if the size of the equivalence class $|F|$ equals the size of the symmetry group $|S|$.

3.3 Results for Jordan-rank $r = 2$

In this section we present and discuss the results for a logarithmic conformal field theory with Jordan-level $r = 2$. We have used the algorithm described in subsection 3.2.4 to obtain these results and though known, e. g., Flohr (2000), we can write them in a more appealing form. Also we will discuss the appearance of an additional degree of freedom, which shows up for $\langle 1111 \rangle$.

We start with simply writing down the first three expressions that our algorithm provides:

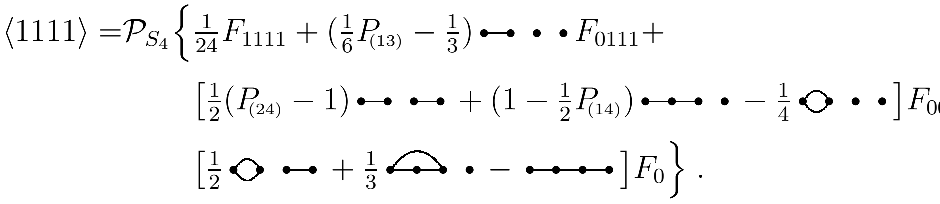
$$\langle 1000 \rangle = F_0 \tag{3.39}$$

$\langle 1100 \rangle = \mathcal{P}_{S_2}\Big\{ \frac{1}{2} F_{1100} -$ $F_0 \Big\}$ (3.40)

$\langle 1110 \rangle = \mathcal{P}_{S_3}\Big\{ \frac{1}{6} F_{1110} + (\frac{1}{2} P_{(13)} - 1)$  $F_{0110} + \big[$ $- \frac{1}{2}$  $\big] F_0 \Big\}$ (3.41)

with $\mathcal{P}_X = \sum_X P_{(x)}$. Writing the results this way makes the discrete symmetry manifest, that is S_2 for $\langle 1100 \rangle$ and S_3 invariance for $\langle 1110 \rangle$.

For the correlator $\langle 1111 \rangle$ we have a logarithmic partner field at every vertex which means that we can expect getting a non-trivial kernel for the first time. The result without kernel is given by



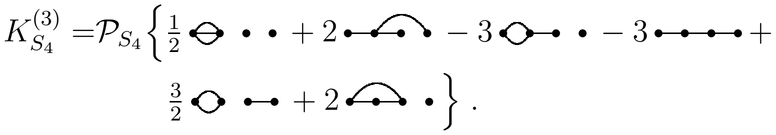
(3.42)

the contribution to the kernel is

$$\mathrm{Ker}_{\langle 1111 \rangle} = \mathcal{P}_{S_4}\left\{ K^{(2)}_{S_2} F_{0011} \right\} . \tag{3.43}$$

That we get a two-dimensional kernel for F_{0011} is not surprising, since there are 6 functions F belonging to the equivalence class of F_{0011} and the resulting $K^{(2)}_{S_2}$ can be read of the table for logarithmic degree 2 from the appendix.

The inverse question is more interesting, namely we are interested in understanding, why no other kernel term shows up at all. For logarithmic degree $l = 1$ the equivalence class of F_{0111} is four and thus there is no kernel term showing up. According to (3.38) the S_4 invariant kernel of logarithmic degree $l = 3$ should be one-dimensional. We can immediately understand why this kernel term does not show up, by looking at the graphical representation:



(3.44)

This shows us that terms of the form l_{12}^3 appear, which is impossible for a Jordan-rank $r = 2$ theory. Though three free legs are available the threefold connection between vertices i and j is forbidden for $r = 2$. Of course higher Jordan-rank LCFT $r > 2$ are allowed to include such terms, but similar combinatorial restrictions will show up for $r = 3$ as well.

3.4 Results for Jordan-rank $r = 3$

While the general structure of the correlators for Jordan-rank $r = 2$ has been known before, nobody so far has studied the form of correlators for LCFTs beyond the case of $r = 2$. With what we have learned we can apply our methods to the case $r = 3$ in order to determine the form of all correlators for a theory of Jordan-rank $r = 3$.

Analogously to $r = 2$ the starting point for the recursion is given by

$$\langle 1100\rangle = F_0 \quad , \quad \langle 2000\rangle = F_0 \ . \tag{3.45}$$

As before the missing correlators result from applying a permutation to the correlators.

$$\langle 0012\rangle = \mathcal{P}_{S_2}\Big\{\tfrac{1}{2}F_{0012} - \bullet \;\bullet\; \bullet\!\!-\!\!\bullet\, F_0\Big\} \tag{3.46}$$

$$\langle 1110\rangle = \mathcal{P}_{S_3}\Big\{\tfrac{1}{6}F_{1110} - \tfrac{1}{2}\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_0\Big\} \tag{3.47}$$

$$\langle 2200\rangle = \mathcal{P}_{S_2\times S_2}\Big\{\tfrac{1}{4}F_{2200} - \tfrac{1}{2}\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{1200} + \tfrac{1}{2}\,\bullet\!\!\bigcirc\!\!\bullet\;\bullet\;\bullet\, F_0\Big\} \tag{3.48}$$

$$\begin{aligned}\langle 1120\rangle = \mathcal{P}_{S_2}\Big\{&\tfrac{1}{2}F_{1120} - (1+P_{(23)}+P_{(13)})\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{0120} + (\tfrac{1}{2}-P_{(23)})\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{1110}+\\ &\big[(1+\tfrac{3}{2}P_{(23)})\,\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\;\bullet - (\tfrac{1}{4}+\tfrac{1}{2}P_{(23)})\,\bullet\!\!\bigcirc\!\!\bullet\;\bullet\;\bullet\big]F_0\Big\}\end{aligned} \tag{3.49}$$

where as before $\mathcal{P}_X = \sum_{x\in X} P_x$.

The above correlators do not have an additional degree of freedom because they contain at least one primary field. The simplest correlator with no primaries is

$$\langle 1111\rangle = \mathcal{P}_{S_4}\Big\{\tfrac{1}{24}F_{1111} + (\tfrac{1}{6}P_{(13)} - \tfrac{1}{3})\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{0111} + (\tfrac{1}{4}\,\bullet\!\!-\!\!\bullet\;\;\bullet\!\!-\!\!\bullet)F_0\Big\} \ . \tag{3.50}$$

This is the first correlator for $r = 3$ which has a non-trivial kernel, namely

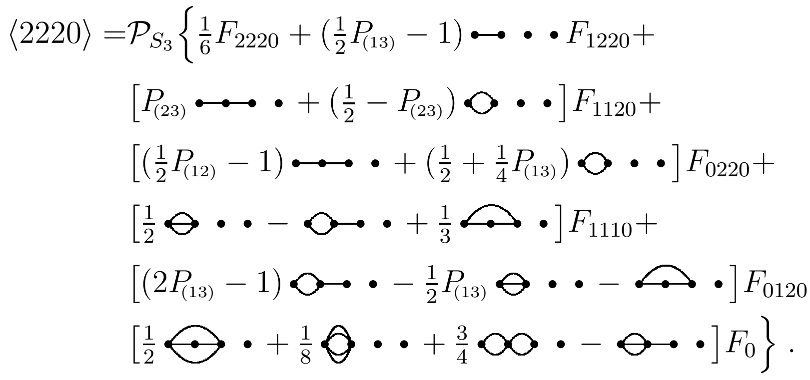
$$\mathrm{Ker}_{\langle 1111\rangle} = c_1 K^{(2)}_{S_4} F_0 \ . \tag{3.51}$$

The restriction that the expression needs to be invariant under S_4 permutations is very strong and forbids any kernel terms of degree one to show up.

The remaining correlators containing at least a primary field are

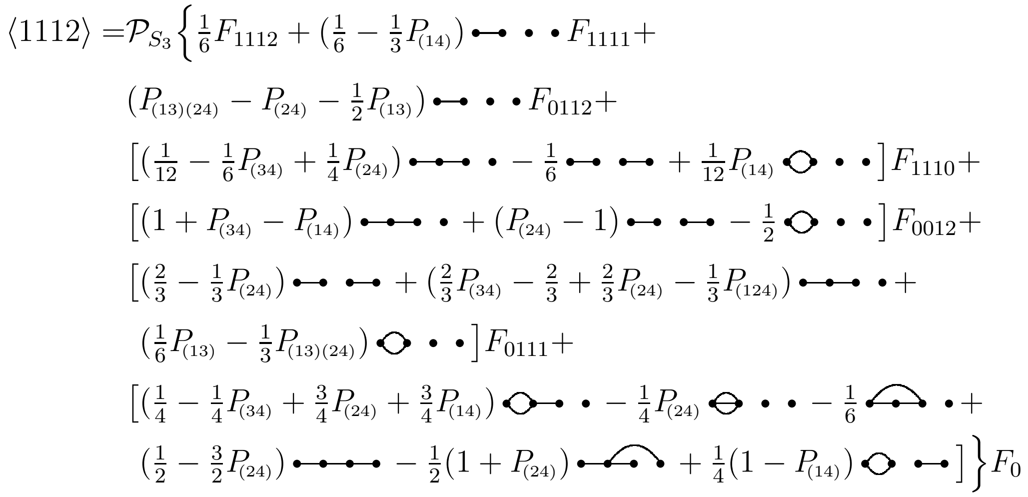
$$\begin{aligned}\langle 2210\rangle = \mathcal{P}_{S_2}\Big\{&\tfrac{1}{2}F_{2210} + (\tfrac{1}{2}-P_{(13)})\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{2200} - (1+P_{(23)}-P_{(13)})\,\bullet\!\!-\!\!\bullet\;\bullet\;\bullet\, F_{1210}+\\ &\big[2\,\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\;\bullet - \bullet\!\!\bigcirc\!\!\bullet\;\bullet\;\bullet\big]F_{1200} + \big[P_{(23)}\,\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\;\bullet + (\tfrac{1}{2}-P_{(13)})\,\bullet\!\!\bigcirc\!\!\bullet\;\bullet\;\bullet\big]F_{1110}+\\ &\big[(-1-P_{(23)}+P_{(12)})\,\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\;\bullet + \tfrac{1}{2}(1+P_{(23)}+P_{(13)})\,\bullet\!\!\bigcirc\!\!\bullet\;\bullet\;\bullet\big]F_{0120}+\\ &\big[(P_{(13)}-2)\,\bullet\!\!\bigcirc\!\!\bullet\!\!-\!\!\bullet\;\bullet + \tfrac{1}{2}\,\bullet\!\!\ominus\!\!\bullet\;\bullet\;\bullet - \overset{\frown}{\bullet\;\bullet\;\bullet}\;\bullet\big]F_0\Big\}\end{aligned} \tag{3.52}$$

and



(3.53)

Finally there are, up to permutations, four correlators without primary field and at least one field being of Jordan-level 2.



(3.54)

This correlator has 6 additional degrees of freedom:

$$\mathrm{Ker}_{\langle 1112\rangle} = \mathcal{P}_{S_3}\Big\{c_1(2K_2 - K_1)F_{0112} + K^{(2)}_{S_4}F_{1110} + \big[c_3K_1^2 + c_4(K_2^2 - K_1K_2)\big]F_{0111} + \big[c_5K_1^2 + c_6(K_2^2 - K_1K_2)\big]F_{1002}\Big\}. \quad (3.55)$$

The set of $\{F_{0112}, F_{1012}, F_{1102}\}$ allows a one dimensional kernel, namely $2K_2 - K_1$. Note that this kernel is not S_3 invariant.

For the self-invariant terms F_{1110} and F_0 we remarked in subsection 3.2.6 that S_3 invariance implies S_4 variance. The dimension of the S_4-invariant kernel is $n^{S_4}_{\max} = 1$ for $d = 2, 3$. The corresponding kernel term for F_{1110} shows up, combinatorial restrictions forbid the same for F_0. The only kernel of degree

3 that would have been possible is $K_{S_4}^{(3)}$, but this one includes l_{ij}^3 terms for all $1 \leq i < j \leq 4$, which is not compatible with the contraction rules as described in subsection 3.2.4–$\langle 2111 \rangle$ cannot contain l_{34}^3 terms, cf. 3.44.

For the 3-element sets $\{F_{0111}, F_{1011}, F_{1101}\}$ respectively $\{F_{0012}, F_{0102}, F_{1002}\}$ we know from the kernel analysis in appendix A that there is a two-dimensional kernel.

It should be noted that $\langle 1211 \rangle$ is generated by applying $P_{(12)}$ to $\langle 2111 \rangle$. The same holds for the additional terms of the kernel. This means that the degrees of freedom we have for $\langle 2111 \rangle$ are not available for the permutations of this correlator, e. g., $\langle 1211 \rangle$.

The correlator $\langle 2211 \rangle$ comes with a high number of additional degrees of freedom, some of these are restricted by combinatorial constraints. The correlator without kernel terms has the form

$$
\begin{aligned}
\langle 2211 \rangle =& \mathcal{P}_{S_2 \times S_2} \Big\{ \tfrac{1}{4} F_{2211} + (\tfrac{1}{2} - P_{(13)}) \bullet\!-\!\bullet\;\bullet\;\bullet\, F_{2201} + \\
& (\tfrac{1}{2} P_{(13)(24)} - P_{(23)}) \bullet\!-\!\bullet\;\bullet\;\bullet\, F_{1211} + \\
& \big[\tfrac{1}{2} P_{(23)} \bullet\!-\!\bullet\;\bullet\!-\!\bullet + (\tfrac{1}{2} P_{(14)} - 1) \bullet\!-\!\bullet\!-\!\bullet\;\bullet + \tfrac{1}{4} \bullet\!\bigcirc\!\bullet\;\bullet\;\bullet \big] F_{2200} + \\
& \big[(\tfrac{1}{2} P_{(23)} - \tfrac{1}{6}) \bullet\!-\!\bullet\;\bullet\!-\!\bullet + (\tfrac{1}{3} - \tfrac{1}{2} P_{(14)}) \bullet\!-\!\bullet\!-\!\bullet\;\bullet + \tfrac{1}{12} \bullet\!\bigcirc\!\bullet\;\bullet\;\bullet \big] F_{1111} + \\
& \big[(P_{(243)} + P_{(24)} + P_{(12)} - P_{(12)(34)} - P_{(124)} + P_{(142)}) \bullet\!-\!\bullet\!-\!\bullet\;\bullet + \\
& (1 - P_{(23)}) \bullet\!-\!\bullet\;\bullet\!-\!\bullet - P_{(24)} \bullet\!\bigcirc\!\bullet\;\bullet\;\bullet \big] F_{1201} + \\
& \big[(\tfrac{1}{2} + \tfrac{1}{2} P_{(23)} - \tfrac{1}{2} P_{(24)} + \tfrac{1}{2} P_{(12)} + \tfrac{1}{2} P_{(124)} + \tfrac{3}{4} P_{(142)} + \tfrac{1}{4} P_{(14)}) \bullet\!-\!\bullet\!-\!\bullet\;\bullet + \\
& (-\tfrac{1}{2} - P_{(24)}) \bullet\!-\!\bullet\;\bullet\!-\!\bullet - (\tfrac{1}{4} + \tfrac{1}{2} P_{(14)}) \bullet\!\bigcirc\!\bullet\;\bullet\;\bullet \big] F_{0211} + \\
& \big[2P_{(12)} \bullet\!\bigcirc\!\bullet\!-\!\bullet\;\bullet - P_{(12)} \bullet\!-\!\bullet\frown\bullet\!-\!\bullet - P_{(13)} \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet - \tfrac{1}{2} \bullet\!\ominus\!\bullet\;\bullet\;\bullet \big] F_{1200} + \\
& \big[\tfrac{1}{2}(P_{(34)} - 1 + P_{(243)} - P_{(24)} - 2P_{(1243)} - P_{(124)} - 2P_{(143)} - P_{(14)}) \bullet\!\bigcirc\!\bullet\!-\!\bullet\;\bullet + \\
& (1 + P_{(24)}) \bullet\!-\!\bullet\frown\bullet\!-\!\bullet - (\tfrac{1}{2} + \tfrac{1}{2} P_{(24)}) \bullet\!\bigcirc\!\bullet\;\bullet\!-\!\bullet + P_{(34)} \overset{\frown}{\bullet\!-\!\bullet\!-\!\bullet}\;\bullet + \\
& (P_{(34)} + P_{(243)} - P_{(12)(34)} - P_{(13)}) \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet + \tfrac{1}{2} P_{(14)} \bullet\!\ominus\!\bullet\;\bullet\;\bullet \big] F_{0102} + \\
& \big[\tfrac{1}{3}(P_{(23)} - 2 + P_{(34)} + 2P_{(234)} - P_{(134)} + 4P_{(13)} + 3P_{(14)} - P_{(143)}) \bullet\!\bigcirc\!\bullet\!-\!\bullet\;\bullet + \\
& (\tfrac{1}{6} - \tfrac{1}{3} P_{(13)}) \bullet\!\ominus\!\bullet\;\bullet\;\bullet - (\tfrac{4}{3} + \tfrac{2}{3} P_{(23)}) \bullet\!-\!\bullet\frown\bullet\!-\!\bullet + (\tfrac{1}{3} P_{(34)} - \tfrac{4}{3}) \overset{\frown}{\bullet\!-\!\bullet\!-\!\bullet}\;\bullet + \\
& (\tfrac{2}{3} P_{(23)} - 2P_{(243)} + \tfrac{4}{3} P_{(13)}) \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet - (\tfrac{1}{3} + \tfrac{1}{3} P_{(23)} + P_{(24)}) \bullet\!\bigcirc\!\bullet\;\bullet\!-\!\bullet \big] F_{1101} +
\end{aligned}
$$

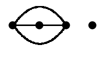
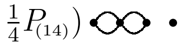
$\Big[\big(\frac{2}{3}P_{(23)} - \frac{5}{4} + \frac{7}{12}P_{(24)} - \frac{7}{12}P_{(12)} + \frac{3}{4}P_{(124)} + \frac{7}{12}P_{(132)} + \frac{2}{3}P_{(142)} + \frac{5}{4}P_{(13)} +$

$\frac{1}{12}P_{(1423)} - \frac{1}{6}P_{(14)} - \frac{1}{12}P_{(14)(23)}\big)$  $- \big(\frac{4}{3}P_{(24)} + \frac{2}{3}P_{(12)}\big)$  $+$

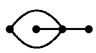
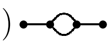
$\big(\frac{5}{12} - \frac{5}{12}P_{(24)} - \frac{1}{4}P_{(14)}\big)$  $- \big(\frac{4}{3} + \frac{2}{3}P_{(24)} + \frac{1}{2}P_{(14)}\big)$  $+$

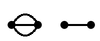
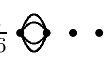
$\big(\frac{4}{3}P_{(23)} - \frac{7}{6} + \frac{7}{6}P_{(24)} + \frac{1}{6}P_{(12)} + \frac{7}{6}P_{(132)}\big)$ $+$

$\big(\frac{11}{12} - \frac{5}{4}P_{(24)} - \frac{7}{12}P_{(13)} + \frac{1}{4}P_{(13)(24)}\big)$  $\Big]F_{0111} +$

$\Big[\big(P_{(24)} - \frac{1}{2}P_{(23)} - \frac{3}{4}P_{(14)}\big)$  $+ \frac{1}{2}\big(\frac{1}{2} - P_{(23)} + P_{(24)} - \frac{1}{4}P_{(14)}\big)$  $+$

$\frac{1}{2}\big(1 + P_{(23)} - 4P_{(24)} - P_{(14)} - 4P_{(13)}\big)$  $- \big(\frac{1}{8} - \frac{5}{8}P_{(24)}\big)$  $+$

$\frac{1}{2}\big(1 + P_{(23)} - P_{(14)}\big)$  $+ \big(\frac{3}{4}P_{(24)} - P_{(23)} + \frac{5}{4}P_{(13)}\big)$  $+$

$\big(\frac{1}{2}P_{(23)} - \frac{1}{2}P_{(24)} - 1\big)$  $+ \frac{1}{2}P_{(23)}$  $+ \frac{3}{16}$  $+$

$\big(2P_{(13)} + P_{(14)} - P_{(13)(24)}\big)$  $+$  $\Big]F_0\Big\}$, (3.56)

where $\mathcal{P}_{S_2\times S_2} = 1 + P_{(12)} + P_{(34)} + P_{(12)(34)}$. The kernel of $\langle 2211\rangle$ has a dimension of 18:

$$\begin{aligned}\mathrm{Ker}_{\langle 2211\rangle} =& \mathcal{P}_{S_2\times S_2}\Big\{K^{(1,d=1)}_{S_2}F_{2201} + K^{(1,d=1)}_{S_2}F_{1211} + \\ & K^{(2,d=3)} + K^{(2,d=2)}_{S_2}(F_{2200} + F_{0211} + F_{1111}) + \\ & (K_1 - K_2)^2(K_1 + K_2)(F_{1101} + F_{0111} + F_{1200}) + \\ & \big[c(K_1 - K_2)^2K_1 + c'(K_1 - K_2)^2K_2 + c''(K_1 - K_2)K_1K_2\big]F_{0102} + \\ & K^{(4,d=1)}_{S_4}F_0\Big\}\ ,\end{aligned} \tag{3.57}$$

where we used a somewhat condensed notation and left out almost all constants. If multiple F in brackets show up you should add the necessary constants in your mind, for instance, $K^{(2,d=2)}_{S_2}(F_{2200} + F_{0211} + F_{1111})$ stands for $2\cdot 3 = 6$ degrees of freedom. For better orientation we added a small "$d = \ldots$" index to the common kernel terms which notes their dimension.

For the sets $\{F_{2201}, F_{2210}\}$ and $\{F_{1211}, F_{2111}\}$ we get the expected one-dimensional kernel $K^{(1)}_{S_2}$. Also the results for logarithmic degree 2 are not surprising.

Things are more complicated for higher logarithmic degrees. For $d = 3$ we would have been expecting a two-dimensional kernel $K^{(3)}_{S_2}$ for F_{1101}, F_{1200}, F_{0111} and a full 4-dimensional kernel $K^{(3)}$ for F_{0102}. But here we have to take into account the combinatorial restrictions again. The basis elements of $K^{(3)}_{S_2}$, $K^{(3,a)}_{S_2} \sim K_1^3 + K_2^3$ contains $l_{12}^3, l_{13}^3, \ldots, l_{34}^3$ contributions and $K^{(3,b)}_{S_2} \sim K_1^2K_2 + K_1K_2^2$ comes with l_{12}^3, l_{34}^3 terms. Thus both basis elements are not allowed due to the l_{34}^3 term, but a linear combination is, namely $(K_1 - K_2)^2(K_1 + K_2)$, which contains no l_{34}^2 term. For the set $\{F_{0102} \equiv F_{0201}, F_{1002} \equiv F_{2001}, F_{0120} \equiv F_{0210}, F_{1020} \equiv F_{2010}$ the four dimensional kernel reduces by the combinatorial constraint to a three dimensional one.

The reasoning for $d = 4$ goes along the same line. We expect from (3.35) a three dimensional kernel space, but also we have two restrictions. No l_{ij}^4 term may show up, not even l_{12}^4, because of the $S_2 \times S_2$ invariance. And no l_{34}^3 term is allowed. These two restrictions limit the kernel to $K = K_1K_2(K_1 - K_2)^2$, leaving us with a one-dimensional kernel.

$$\begin{aligned}
\langle 2221\rangle = \mathcal{P}_{S_3}\Big\{ & \tfrac{1}{6}F_{2221} + (\tfrac{1}{6} - \tfrac{1}{3}P_{(14)}) \; F_{2220} + (\tfrac{1}{2}P_{(13)} - 1) \; F_{1221} + \\
& \big[(2 - P_{(34)} - \tfrac{1}{2}P_{(12)} + 2P_{(12)(34)} + P_{(14)}) \; - (1 + P_{(24)}) \; + \\
& (-\tfrac{1}{2} - \tfrac{3}{4}P_{(13)}) \; \big]F_{0221} + \\
& \big[(2 - P_{(24)}) \; + (2 - 2P_{(34)} - 2P_{(12)} + 2P_{(12)(34)} - P_{(124)}) \; + \\
& (P_{(13)(24)} - \tfrac{1}{2}P_{(13)}) \; \big]F_{1220} + \\
& \big[(1 - \tfrac{1}{2}P_{(23)} + \tfrac{1}{2}P_{(243)} + P_{(34)} + P_{(234)} + P_{(24)} - P_{(14)}) \; - \; + \\
& (-P_{(24)} - \tfrac{1}{2}P_{(13)(24)}) \; \big]F_{1121} + \\
& \big[\tfrac{2}{3} \; - \tfrac{2}{3} \; - \tfrac{2}{3} \; - \tfrac{1}{3} \; - \tfrac{1}{6} \; + \\
& (\tfrac{1}{3} + \tfrac{2}{3}P_{(34)}) \; \big]F_{1111} + \\
& \big[\tfrac{1}{2} \; - 2P_{(23)} \; + P_{(23)} \; + \; + \\
& (P_{(134)} - P_{(34)} - P_{(13)}) \; \big]F_{1120} + \\
& \big[\tfrac{1}{2}(1 - P_{(34)} - 2P_{(13)} - P_{(134)}) \; + \; + (1 - P_{(12)}) \; + \\
& \tfrac{1}{4}P_{(13)} \; - \tfrac{1}{2} \; + \tfrac{1}{2} \; \big]F_{0220} + \\
& \big[\tfrac{1}{2}(P_{(23)} - 2P_{(34)} + 2P_{(234)} - P_{(243)} + P_{(13)(24)} + P_{(12)(34)} + 3P_{(1243)} + P_{(14)(23)} + \\
& 3P_{(124)} - P_{(12)} + 2P_{(132)} + P_{(1342)} + P_{(24)} - P_{(1324)} + 4P_{(143)} - P_{(1423)}) \; + \\
& (1 - 2P_{(34)} + P_{(23)} + 2P_{(234)} + P_{(243)} - P_{(12)} + 2P_{(12)(34)} + P_{(1243)} - P_{(13)} + \\
& 2P_{(132)}) \; - (1 + 2P_{(34)} + P_{(24)} + P_{(14)}) \; + \\
& (\tfrac{1}{2} - \tfrac{3}{2}P_{(23)} - \tfrac{1}{2}P_{(24)} - \tfrac{3}{2}P_{(14)}) \; - (\tfrac{1}{2}P_{(23)} + P_{(14)}) \; + \\
& (-P_{(23)} - 2P_{(24)} - P_{(12)}) \; \big]F_{0121} + \\
& \big[\tfrac{2}{3} \; - \tfrac{2}{3}P_{(14)} \; - (\tfrac{1}{3} + P_{(24)}) \; + (\tfrac{2}{3}P_{(34)} - \tfrac{1}{3}) \; + \\
& (\tfrac{4}{3} + \tfrac{2}{3}P_{(34)}) \; - \tfrac{1}{6} \; + \tfrac{2}{3} \; - \; + \\
& (\tfrac{4}{3}P_{(14)} - \tfrac{2}{3}) \; - \tfrac{1}{3}P_{(24)} \; - \tfrac{2}{3} \; \big]F_{1110} +
\end{aligned}$$

$$
\begin{aligned}
&\big[\big(\tfrac{1}{4}P_{(34)} + \tfrac{1}{2}P_{(23)} - \tfrac{1}{2}P_{(234)} - \tfrac{1}{4} + \tfrac{1}{4}P_{(14)}\big)\; + \big(P_{(23)} - P_{(34)}\big)\; + \\
&\big(P_{(23)} - \tfrac{1}{2} + \tfrac{1}{2}P_{(243)} + \tfrac{1}{2}P_{(24)}\big)\; + \big(P_{(23)} - 1\big)\; + \; + \\
&\tfrac{1}{2}\big(P_{(234)} - 3P_{(34)} - 3 - P_{(23)} + 2P_{(13)} - 2P_{(134)} + P_{(143)} - P_{(14)}\big)\; + \\
&\big(\tfrac{1}{2}P_{(234)} - 1 - P_{(34)} - \tfrac{1}{2}P_{(23)} + \tfrac{1}{2}P_{(13)} - \tfrac{1}{2}P_{(134)}\big)\; + \tfrac{3}{8}\; + \\
&\big(\tfrac{3}{2}P_{(23)} - \tfrac{1}{2} - \tfrac{1}{2}P_{(143)} + \tfrac{1}{2}P_{(14)}\big)\; + \big(\tfrac{3}{4} + \tfrac{1}{4}P_{(24)}\big)\; + \\
&\big(1 - P_{(13)} - P_{(14)} + P_{(1324)}\big)\;\big]F_{0012} + \\
&\big[\tfrac{1}{2}\big(P_{(34)} + P_{(23)} - 3P_{(243)} - P_{(12)(34)} - 6P_{(123)} - 6P_{(1234)} + P_{(132)} - 2P_{(13)(24)} + \\
&2P_{(1324)} - 3P_{(1432)} - 2P_{(14)} + 2P_{(1423)}\big)\; + \big(\tfrac{1}{2}P_{(24)} + \tfrac{3}{2}P_{(13)}\big)\; + \\
&\tfrac{1}{2}\big(3P_{(24)} - P_{(23)} - 3P_{(234)} - P_{(12)} + P_{(12)(34)} + P_{(123)}\big)\; + \tfrac{3}{2}P_{(24)}\; + \\
&\tfrac{1}{2}\big(2P_{(23)} - P_{(34)} + P_{(12)(34)} - P_{(132)} - P_{(14)}\big)\; + \tfrac{1}{2}\big(P_{(243)} - P_{(23)} - P_{(234)} + \\
&P_{(124)}\big)\; + \big(2P_{(1234)} - P_{(23)} - P_{(234)} + P_{(243)} - P_{(124)}\big)\; + \\
&\tfrac{1}{2}\big(P_{(124)} + P_{(13)} - P_{(134)} - P_{(13)(24)} - P_{(14)} + P_{(1423)}\big)\; + \\
&\big(1 + P_{(34)} + P_{(24)} + P_{(13)}\big)\; + \big(2 - P_{(34)}\big)\;\big]F_{0120} + \\
&\big[\tfrac{1}{3}\big(\tfrac{17}{2}P_{(34)} - 1 + P_{(12)} + \tfrac{11}{2}P_{(124)} + \tfrac{5}{2}P_{(1342)} + 2P_{(13)} - P_{(13)(24)} + \tfrac{3}{2}P_{(14)}\big)\; + \\
&\big(\tfrac{1}{2} - \tfrac{5}{3}P_{(34)} + \tfrac{2}{3}P_{(24)} + \tfrac{1}{2}P_{(12)(34)} + \tfrac{1}{3}P_{(13)} - \tfrac{1}{3}P_{(12)}\big)\; + \tfrac{1}{3}\; + \\
&\big(\tfrac{5}{6} - \tfrac{1}{2}P_{(34)} - \tfrac{5}{6}P_{(24)} - \tfrac{2}{3}P_{(12)} + \tfrac{2}{3}P_{(12)(34)} - \tfrac{1}{6}P_{(124)} + \tfrac{1}{3}P_{(14)}\big)\; + \\
&\big(\tfrac{5}{6}P_{(34)} - \tfrac{5}{6} + \tfrac{1}{3}P_{(134)} + \tfrac{1}{2}P_{(124)} - \tfrac{1}{2}P_{(13)(24)} + \tfrac{1}{3}P_{(143)} - \tfrac{1}{3}P_{(14)}\big)\; + \\
&\big(\tfrac{1}{2}P_{(12)(34)} - \tfrac{1}{2}P_{(34)} - \tfrac{7}{6}P_{(12)} + \tfrac{5}{6}P_{(142)}\big)\; - \tfrac{1}{12}P_{(13)}\; + \\
&\tfrac{1}{3}\big(1 - P_{(34)} - P_{(24)} - P_{(13)} - 2P_{(134)} - 4P_{(14)}\big)\; - \tfrac{11}{6}\; + \\
&\big(1 - P_{(34)} + \tfrac{1}{2}P_{(24)} + \tfrac{1}{6}P_{(134)} - \tfrac{1}{2}P_{(124)}\big)\; + \\
&\big(-\tfrac{5}{6} - \tfrac{1}{3}P_{(13)} - \tfrac{1}{2}P_{(13)(24)}\big)\;\big]F_{0111} + \\
&\big[\big(\tfrac{1}{4} - P_{(24)} + \tfrac{3}{4}P_{(34)}\big)\; + \big(1 - P_{(34)}\big)\; + \big(P_{(24)} - P_{(34)}\big)\; + \\
&\big(\tfrac{3}{4}P_{(24)} + \tfrac{5}{4}\big)\; - \big(\tfrac{3}{4} + \tfrac{1}{4}P_{(24)}\big)\; + \big(1 - P_{(34)} - 2P_{(14)}\big)\; + \\
&\big(P_{(24)} - \tfrac{7}{4} + \tfrac{3}{4}P_{(34)} - P_{(14)}\big)\; + \big(P_{(34)} - 1 + \tfrac{5}{2}P_{(24)} - \tfrac{3}{2}P_{(14)}\big)\; + \\
&\tfrac{1}{4}\big(1 + P_{(34)} - 2P_{(14)}\big)\; + \tfrac{1}{4}\big(P_{(34)} - 1\big)\; + \big(P_{(34)} - 1\big)\; + \\
&\tfrac{1}{4}\big(P_{(24)} - 1\big)\; + \tfrac{3}{2}\big(1 + P_{(34)}\big)\; - \tfrac{1}{2}\big(P_{(34)} + P_{(14)}\big)\; + \\
&\tfrac{1}{2}\big(P_{(14)} - 1\big)\; + \big(\tfrac{1}{2}P_{(34)} - \tfrac{3}{2}P_{(14)}\big)\; - \tfrac{1}{2}\;\big]F_0\Big\} \qquad (3.58)
\end{aligned}
$$

Interestingly the dimension of the kernel for $\langle 2221 \rangle$ is also 18 and by that not larger than the kernel for $\langle 2211\rangle$. Naively one would expect that the kernel dimension increases with growing Jordan-level $K := \sum_i k_i$. On the other hand the larger symmetry group (S_3 instead of $S_2 \times S_2$) reduces the kernel size which can even lead to a smaller kernel, as we will see in the case of $\langle 2222\rangle$.

$$\begin{aligned}\mathrm{Ker}_{\langle 2221\rangle} = &\mathcal{P}_{S_3}\Big\{(2K_2 - K_1)F_{1221}|^{d=1} + \\ &\big[cK_1^2 + c'(K_2^2 - K_1K_2)\big](F_{1220}|^{d=2} + F_{2111}|^{d=2} + F_{0221}|^{d=2}) + \\ &K^{(3)}F_{0121}|^{d=4} + K_{S_4}^{(3)}F_{1111}|^{d=1} + \\ &\big[cK_2(K_1 - K_2)(K_1 - 2K_2) + c'K_1^2(K_1 - 2K_2)\big]F_{0220}|^{d=2} + \\ &\big[cK_2(K_1 - K_2)(K_1 - 2K_2) + c'K_1^2(K_1 - 2K_2)\big]F_{2110}|^{d=2} + \\ &K_1^2K_2(K_1 - K_2)F_{0111}|^{d=1} + K_1^2K_2(K_1 - K_2)F_{0120}|^{d=1} + \\ &K_1^2K_2(K_1 - K_2)F_{1002}|^{d=1}\Big\} \end{aligned} \tag{3.59}$$

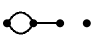
There is not much surprise for most results. For logarithmic degree 1 and 2 we get for the sets containing three F a $d = 1$ respectively a $d = 2$ kernel. For degree 3 we have the self-invariant term F_{1111} with a S_4 symmetry and two $d = 2$ kernels for $\{F_{0220}, F_{2020}, F_{2200}\}$ and $\{F_{2110}, F_{1210}, F_{1120}\}$. There is also a set containing six F, which results in a full four dimensional $K^{(3)}$ kernel.

As expected combinatorial constraints show up the first time for degree four, because of the last vertex having one leg only and thus disallowing any l_{i4}^4 $(i = 1, 2, 3)$ term. For degree actually a $d = 3$ kernel would have been possible, but eliminating all l_{i4}^4 terms means that the kernel has to be reduced to a one-dimensional kernel each. Also note that the only possible kernel term of degree 5 would have been $K_{S_4}^{(5)}$ which does not show up, because of the same combinatorial restriction.

$\langle 2222\rangle = \mathcal{P}_{S_4}\Big\{\frac{1}{24}F_{2222} + (\frac{1}{6}P_{(13)} - \frac{1}{3})$ $F_{1222} +$

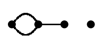
$\big[(\frac{1}{3}P_{(12)} + \frac{1}{6}P_{(14)})$ $- \frac{1}{3}$ $- \frac{1}{12}P_{(13)}$  $\big]F_{0222} +$

$\big[\frac{1}{2}P_{(24)}$ $+ (\frac{1}{2}P_{(23)} - P_{(24)})$ $+ \frac{1}{4}P_{(13)(24)}$  $\big]F_{1122} +$

$\big[(3P_{(34)} + 3 + P_{(14)})$  $- 5$  $- (\frac{13}{6} + \frac{5}{2}P_{(34)})$  $+$

$(5 + 3P_{(24)})$ $-$  $- (3 + \frac{3}{2}P_{(14)})$  $)F_{1112} +$

$\big[\frac{1}{2}(P_{(124)} - 11 - 9P_{(12)} - 7P_{(123)} - P_{(132)} - 3P_{(142)} - 7P_{(14)} - 7P_{(13)(24)} +$

$P_{(13)} - 8P_{(14)(23)})$  $+ (5 + \frac{3}{2}P_{(24)} + 3P_{(14)} + \frac{9}{2}P_{(13)(24)})$  $+$

$(6 + 7P_{(23)} + 5P_{(12)})$  $+ (\frac{3}{2} + P_{(13)} + \frac{5}{4}P_{(13)(24)})$  $+$

$$(-10 - 2P_{(23)} - P_{(24)} - 8P_{(12)} - 5P_{(123)} - 2P_{(132)}) +$$

$$(9 + 4P_{(24)} + \tfrac{9}{2}P_{(14)}) \Big] F_{0122} +$$

$$\Big[- \tfrac{1}{3} - \tfrac{1}{2} - - \tfrac{1}{24} +$$

$$\tfrac{5}{12} + \tfrac{1}{6} + \tfrac{1}{3} \Big] F_{1111} +$$

$$\Big[\tfrac{1}{2}(P_{(13)} - P_{(23)}) - \tfrac{1}{4}(1 + P_{(13)(24)}) + (\tfrac{1}{8}P_{(23)} - \tfrac{3}{8}) +$$

$$\tfrac{1}{2}(1 - P_{(23)} + P_{(24)} - \tfrac{1}{2}P_{(14)}) - (\tfrac{1}{4} + \tfrac{1}{8}P_{(14)}) - \tfrac{1}{2} +$$

$$\tfrac{1}{2}(3 + P_{(23)} - P_{(24)} + 3P_{(13)(24)}) + (P_{(23)} - 1 - P_{(1324)}) +$$

$$\tfrac{1}{2}(\tfrac{1}{2}P_{(24)} - 1 - P_{(23)} - \tfrac{1}{2}P_{(13)}) - \tfrac{1}{16} + \tfrac{1}{2} \Big] F_{0022} +$$

$$\Big[\tfrac{1}{4}P_{(24)} + \tfrac{1}{2}(1 - P_{(34)} + 2P_{(24)} + \tfrac{1}{2}P_{(12)} - \tfrac{1}{2}P_{(124)} + P_{(142)}) +$$

$$\tfrac{1}{2}(P_{(134)} - 2P_{(24)} - P_{(13)} - P_{(1243)} + P_{(124)} - 2P_{(142)} + P_{(143)} - P_{(14)}) +$$

$$\tfrac{1}{2}(3P_{(34)} - 1 - 3P_{(24)} - P_{(12)(34)} + 2P_{(124)} + P_{(12)} - 2P_{(13)(24)} - 3P_{(142)}) +$$

$$(P_{(24)} - P_{(34)} - 1 + P_{(13)} - 2P_{(14)} - 2P_{(13)(24)}) + (P_{(34)} + 2) +$$

$$\tfrac{1}{2}(3P_{(34)} - 1 - P_{(24)} + P_{(12)} - 2P_{(142)}) + \tfrac{1}{2}(P_{(24)} - P_{(13)}) +$$

$$\tfrac{1}{2}(P_{(34)} - P_{(12)} - P_{(134)} - 2P_{(13)} + P_{(14)} + 2P_{(143)}) - \tfrac{1}{2} +$$

$$\tfrac{1}{2}(1 - P_{(34)} + 3P_{(24)} + P_{(13)} + 3P_{(1243)} + P_{(12)}) \Big] F_{0112} +$$

$$\Big[\tfrac{1}{12}P_{(13)} - \tfrac{4}{3} - \tfrac{1}{12}P_{(14)}) + (\tfrac{5}{12} + \tfrac{11}{12}P_{(13)}) + \tfrac{4}{3}P_{(14)} +$$

$$(\tfrac{13}{12} - \tfrac{1}{4}P_{(14)}) + \tfrac{1}{12}(P_{(13)} - P_{(14)}) + \tfrac{3}{4}(P_{(12)} - 1) +$$

$$(\tfrac{1}{6}P_{(14)} - \tfrac{5}{6}P_{(13)}) + (\tfrac{1}{6}P_{(13)} - \tfrac{7}{6}P_{(14)}) + \tfrac{1}{4}P_{(14)} +$$

$$(\tfrac{1}{2} - \tfrac{1}{6}P_{(13)}) + \tfrac{5}{6}(1 - P_{(14)}) - (\tfrac{5}{4} + \tfrac{7}{12}P_{(12)}) +$$

$$(\tfrac{3}{2}P_{(12)} - \tfrac{7}{6} - \tfrac{2}{3}P_{(14)}) + \tfrac{2}{3} + (\tfrac{1}{3} - \tfrac{2}{3}P_{(12)}) +$$

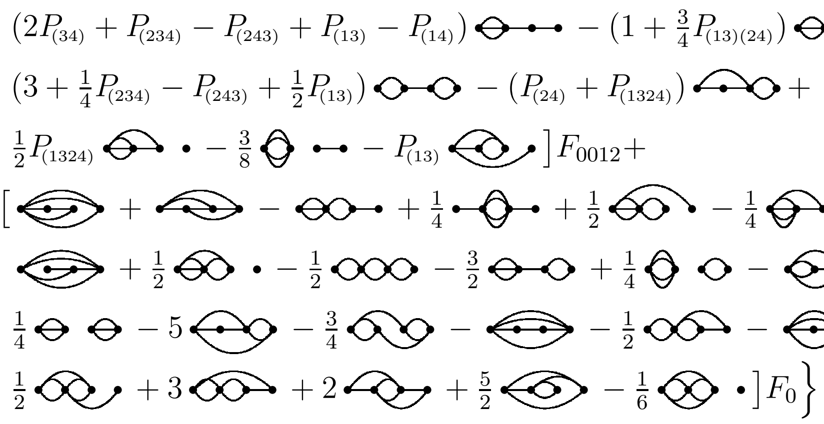
$$(\tfrac{4}{3}P_{(14)} - \tfrac{1}{6} - \tfrac{1}{2}P_{(12)}) + (\tfrac{1}{12}P_{(14)} - \tfrac{5}{6} + \tfrac{3}{4}P_{(13)}) \Big] F_{0111} +$$

$$\Big[\tfrac{1}{2}(1 - P_{(243)} - P_{(24)} - \tfrac{1}{2}P_{(14)}) + \tfrac{1}{2}(1 - P_{(24)} + P_{(23)} - P_{(13)}) +$$

$$\tfrac{1}{2}(P_{(243)} - 1 + P_{(234)} + P_{(134)} - P_{(143)}) + (P_{(243)} - \tfrac{1}{2} - P_{(13)}) +$$

$$\tfrac{1}{2}(P_{(234)} - P_{(243)} + P_{(13)} - P_{(14)}) + (P_{(23)} - P_{(34)} - \tfrac{1}{2}P_{(234)}) +$$

$$\tfrac{1}{2}(P_{(243)} - P_{(24)} - P_{(13)} + P_{(14)}) + (P_{(34)} - P_{(13)} + P_{(13)(24)}) +$$



(3.60)

We saw that the transition from $\langle 2211\rangle$ to $\langle 2221\rangle$ did not increase the dimension of the kernel mainly because of the increase of the discrete symmetry group from $S_2 \times S_2$ to S_3. This transition to $\langle 2222\rangle$ enlarges the symmetry group from S_3 to S_4 and by that even reduces the dimension of the kernel to 13.

$$
\begin{aligned}
\mathrm{Ker}_{\langle 2222\rangle} =&\mathcal{P}_{S_4}\Big\{K_{S_2}^{(2)} F_{1122}|^{d=2} + \\
&\big[cK_2(K_1-K_2)(K_1-2K_2) + c'K_1^2(K1-2K_2)\big]F_{0122}|^{d=2} + \\
&K_{S_4}^{(3)} F_{1112}|^{d=1} + K_{S_2}^{(4)} F_{0022}|^{d=3} + K_{S_4}^{(4)} F_{1111}|^{d=1} + \\
&\big[cK_1^4 + c'K_2^2(K_1-K_2)^2 + c''(K_1^3K_2 - 2K_1K_2^3 + K_2^4)\big]F_{0112}|^{d=3} + \\
&K_1K_2(K_1-K_2)^2(K_1+K_2)F_{0012}|^{d=1}\Big\} \qquad (3.61)
\end{aligned}
$$

There is no kernel of logarithmic degree one because at most we have four F in a set and the S_4 symmetry then is forbidden according to appendix A. Additional combinatorial constraints start with logarithmic degree 5: no l_{ij}^5 for $1 \leq i < j \leq 4$ is allowed to show up.

For F_{0012} plus the five permutations there would have been a $d = 3$ kernel, but the given linear combination is the only one which eliminates all l_{ij}^5 terms. For $\{F_{0111}, F_{1011}, F_{1101}, F_{1110}\}$ only $K_{S_4}^{(5,d=1)}$ would be possible, but is ruled by the combinatorial restriction.

That there is a kernel for $d = 6$ is a bit unexpected. On the one hand we have (3.37) a two dimensional kernel, but on the other there are two constraints which need to be satisfied, namely all l_{ij} to the power of 5 and to the power of 6 have to be eliminated. The given combination fulfills both restrictions and thus gives us an additional degree of freedom for F_0.

3.5 Exact results for two logarithmic fields

The most easiest non-trivial case is the one, where we have two logarithmic fields and two primaries. For this case the correlator $\langle k_1 k_2 00\rangle$ for $k_1, k_2 > 0$ can be solved exactly for arbitrary Jordan-rank r.

The correlator for Jordan-rank r has the following form

$$\langle k_1 k_2 00\rangle = F_{k_1,k_2,0,0} + c_1 l_{12} F_{k_1-1,k_2,0,0} + c_2 l_{12} F_{k_1,k_2-1,0,0} + \dots \ . \tag{3.62}$$

As described in subsection 3.2.5 it is possible to identify some of the appearing F-terms with each other. In this case it turns out that it is easy to find the identifications that stems from the integration process by inserting the above ansatz in equation (3.9). This leads to

$$O_1 \langle k_1 k_2 00\rangle = -2z_1 \langle k_1 - 1, k_2, 0, 0\rangle - 2z_2 \langle k_1, k_2 - 1, 0, 0\rangle \ , \tag{3.63}$$

and considering the terms of the lowest order in $\{l_{ij}\}$ only we get

$$\begin{aligned}(z_1 + z_2)(c_1 F_{k_1-1,k_2,0,0} + c_2 F_{k_1,k_2-1,0,0}) + \mathcal{O}(l_{12}) \\ = -2z_1 F_{k_1-1,k_2,0,0} - 2z_2 F_{k_1,k_2-1,0,0} + \mathcal{O}(l_{12}) \ .\end{aligned} \tag{3.64}$$

We immediately see that these equations do not have a solution. As before we can circumvent the problem by reducing the complexity of the equations, which can be accomplished by identification of some of the functions F. Here we can solve equation (3.64) by using the following identifications

$$F_{k_1-1,k_2,0,0} \equiv F_{k_1,k_2-1,0,0} \ . \tag{3.65}$$

This in perfect agreement with the results presented so far for $r = 3$. Because of having the the above identifications we are left with only one function F for each logarithmic degree of $\langle k_1 k_2 00\rangle$. Using (3.8) yields after a short calculation the full result for a correlator of Jordan-rank r with two primary fields:

$$\langle k_1 k_2 00\rangle = \sum_{n=0}^{k_1+k_2-(r-1)} \frac{(-2)^n}{n!} l_{12}^n F_{k_1+k_2-(r-1)-n,r-1,0,0} \ . \tag{3.66}$$

As a consistency check we can compare the above result with the one presented in Flohr (2002b), respectively Rahimi Tabar *et al.* (1997). For the two-point correlation function the first paper gives the following result

$$\langle \Psi_{k_1}(z_1)\Psi_{k_2}(z_2)\rangle = \sum_{\ell=0}^{k_1+k_2} \frac{(-2)^\ell}{\ell!} l_{12}^\ell D_{(h_1=0,h_2=0,k_1+k_2-\ell)} \ . \tag{3.67}$$

where we slightly adapted the notation and have set the conformal weights h_1, h_2 to zero. The $D_{(\dots)}$ are called "structure constants" and have the property that $D_{(h,h;k)} = 0$ for $k < r - 1$. In other words the index ℓ in (3.67) effectively runs from 0 to $k_1 + k_2 - (r - 1)$ and thus (3.66) and (3.67) are of identical structure. This means, that the polynomial dependence on the logarithms l_{ij} is exactly the same and that precisely the same number of free structure constants $D_{(\dots)}$ or structure functions $F_{\dots}(x)$ are needed.

3.6 Summary and conclusion

In the scope of this thesis we analyzed the influence of the global conformal symmetries in form of the global conformal Ward identities on 4-point correlation functions in arbitrary logarithmic conformal field theory. While it is not possible to completely determine the correlators, this does not even work in the CFT case, it is possible to fix the generic structure of the correlators.

The presented algorithm can be used to calculate the generic structure of 4-point correlators. Within this thesis we restricted ourselves to combinations of proper primary and logarithmic fields, but did mention how to adjust the algorithm in order to extend the algorithm to pre-logarithmic fields.

We explicitly gave the results for, up to permutations, all correlators of Jordan-rank $r = 2, 3$. In some of the results we found additional constants which were identified as elements of the kernel O. Furthermore, we discussed various restrictions which limit the number of terms that can appear in an ansatz or which lead to lesser degrees of freedom in the kernel. Also we found that integration sometimes requires that some functions F need to be identified with each other.

Finally we gave explicit results for the case of exactly two logarithmic fields for arbitrary Jordan-rank r. Studying this very simple case showed us why we need to identify some of the functions F with each other. Also we did a consistency check of the result and showed that equation (3.66) is equivalent to the one presented in Flohr (2002b).

The comparison can be extended to three-point correlators. For instance we can consider the terms of logarithmic degree $l = 2$ of the correlator $\langle 2110 \rangle$ in a Jordan-rank $r = 3$ theory, cf. equation (3.49):

$$\langle 2110 \rangle |_{l=2} = \big[-\tfrac{1}{2}(l_{12}^2 + l_{13}^2 + l_{23}^2) + 3l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23} \big] F_0 \ . \qquad (3.68)$$

As a comparison we evaluate formula (3.11) in Flohr (2002b) and get for $l = 2$ the same result, except that F_0 has to be replaced by the structure constant $C_{(h_1,h_2,h_3;k=0)}$. We once more note that we suppressed any direct but trivial dependence on the conformal weights, so actually, we should compare with $C_{(0,0,0;k=0)}$. However, our results are, up to the omitted pre-factor $\prod_{i<j} z_{ij}^{\mu_{ij}}$, valid and independent of the values of the conformal weights h_i. For the other correlators like $\langle 2210 \rangle$ et cetera we also confirmed that the results match if we restrict us to the highest logarithmic degree, which corresponds to $l^{\max} = k_1 + k_2 + k_3 - r + 1$. As we will see in the following it is interesting to study the case where $l < l^{\max}$. We use $\langle 2110 \rangle$ as an example again, but this time we consider the term of $l = 1$ only:

$$\begin{aligned}\langle 2110 \rangle \,|_{l=1} =& F_{1020}(l_{13} - l_{12} - l_{23}) + F_{1110}(l_{23} - l_{12} - l_{13}) + \\ & F_{1200}(l_{12} - l_{13} - l_{23}) \ . \end{aligned} \tag{3.69}$$

We remind the reader that the above result includes the usual identifications such as $F_{2100} \equiv F_{1200}$. The structure of the formula in Flohr (2002b) makes it obvious that for $l = 1$ only one structure constant shows up and thus the corresponding term is

$$\langle 211 \rangle \,|_{l=1} = -(l_{12} + l_{13} + l_{23})\, C_{(h_1=0,h_2=0,h_3=0;k=1)} \ , \tag{3.70}$$

where we again set the conformal weights to zero and slightly adjusted the notation. Though looking differently at first glance we can achieve the same form of the result if we demand that the following extended identifications hold too, namely

$$F_{1200} \equiv F_{1020} \equiv F_{1110} \ . \tag{3.71}$$

This means that we do not only regain the F_0 terms, but that we can reclaim all information, provided that we do all necessary identifications. With "necessary" we mean that we have to identify all $F_{\dots}$ terms of the same logarithmic degree.

We already encountered one situation where we had to identify several functions F with each other: the initial conditions (3.10) where we identified $F_0 \equiv F_0' \equiv \dots$ by virtue of the cluster decomposition argument.

This evokes the question whether this form of massive identifications of functions F is necessary or useful in the context of some physical theory respectively what conditions could force us to massively reduce the number of functions F. It is clear that the special case where all conformal weights h_i

are equal to each other has an additional symmetry, since we can freely exchange the fields. In this case, we definitely expect that a large number of such identifications should take place.

Furthermore, one can quickly check that the given solutions remain valid after identifying remaining free structure functions because any remaining such function can be arbitrarily chosen as long as no further constraints such as local conformal symmetry are invoked. Due to the recursive dependence of the solutions for total Jordan-level K on the ones for level $K' < K$, identifications are consistent only if restricted to functions $F_{k_1k_2k_3k_4}$, $F_{k_1'k_2'k_3'k_4'}$ with $k_1+k_2+k_3+k_4 = k_1'+k_2'+k_3'+k_4'$. However, a more detailed analysis which identifications should be present in the general case, i. e. for arbitrary values of the conformal weights h_i, will be left to future work.

Of course, when all four fields in the 4-point function are logarithmic, we cannot expect that the resulting polynomials in the l_{ij} can be matched with the ones of 2- and 3-point functions. But one might attempt to make the following comparison.

The four-point functions $\langle k_1k_2k_3k_4\rangle$ are ultimately composed out of (a suitable generalization of) conformal blocks which depend on the internal propagator in the 4-point function. Crossing symmetry of the 4-point function imply that the 4-point function $\left\langle\Psi_{(h_1,k_1)}(\infty)\Psi_{(h_2,k_2)}(1)\Psi_{(h_3,k_3)}(x)\Psi_{(h_4,k_4)}(0)\right\rangle$, viewed as an analytical function of the anharmonic ratio x, possesses for each asymptotic region $|x| < 1$, $|1-x| < 1$, or $1/|x| < 1$ expansions of the schematic form

$$\langle(h_1,k_1)(h_2,k_2)(h_3,k_3)(h_4,k_4)\rangle \sim \sum_{(h,k)} C^{(h,k)}_{(h_i,k_i)(h_j,k_j)} C_{(h,k)(h_l,k_l)(h_m,k_m)} + \dots \tag{3.72}$$

for all permutations $\{i,j,l,m\}$ of $\{1,2,3,4\}$, which must all be expansions of the same analytical function. These expansions involve the 3-point structure constants as well as the OPE structure constants. In the logarithmic case, these structure "constants" are matrix valued with coefficients in $\mathbb{C}[\{l_{ij}\}]$. In the notation used in this paper, $C_{(h_1,k_1)(h_2,k_2)(h_3,k_3)} = \langle k_1k_2k_3\rangle$ where on both sides all terms of the form $z_{ij}^{\mu_{ij}}$ depending in the canonical way on the conformal weights are omitted. In the r-dimensional Jordan-cell space, this defines matrices $(C_{k_1})_{k_2k_3}$ labeled by the first Jordan-level and with indices given by the second and third Jordan-level. In the same way, the propagator defines a matrix $(D)_{k_1k_2} = \langle k_1k_2\rangle$. The OPE structure "constants" are then given by the matrix product

$$(C_{k_1})_{k_2}^{k_3} = (C_{k_1})_{k_2 k}(D^{-1})^{kk_3} \tag{3.73}$$

involving the inverse propagator. Now, one can compute the leading orders of the different expansions of the 4-point structure functions which will yield different polynomials in the l_{ij} with coefficients given by rational functions of the 2- and 3-point structure constants $D_{(h,h;p)}$ and $C_{(h_i h_j,h;q)}$. Two observations can now be made:

Firstly, the three expansions for the s-, t- and u-channel, i. e. for $|x| < 1$, $|1 - x| < 1$ and $1/|x| < 1$ all differ. They lead to different polynomials. It is easy to check in simple examples that certain monomials in the l_{ij} may appear only in one of the expansions. This always happens for 4-point functions of the form $\langle k_1 k_2 k_3 k_4 \rangle$ with all $k_i > 0$ but not all k_i equal.

Secondly, the polynomials in l_{ij} with coefficients given by the structure functions $F_{k_1 k_2 k_3 k_4}(x)$ cannot be matched to any of the three expansions. On the contrary, the 4-point functions will involve all the different monomials in the l_{ij} and in particular all the ones which do not appear in all the expansions, but in only one of them. It is therefore much more difficult to match the 4-point structure functions to expressions in the 3- and 2-point structure constants or to suggest further identifications as they can easily be read off in the case of 4-point functions of type $\langle k_1 k_2 00 \rangle$ or $\langle k_1 k_2 k_3 0 \rangle$. In fact, it is not straightforward how the three different expansions should be combined for a comparison of coefficients in case all four fields are logarithmic. A further complication is given by the freedom to change the polynomials in the l_{ij} by elements in the kernel of the operator O or, equivalently, by a redefinition of the structure function coefficients. But we believe that it would be very interesting to investigate the consequences of crossing symmetry for the structure functions of LCFT 4-point functions, because this might yield severe restrictions on the number of functions which have to be determined by other means, for example local conformal invariance. This is an important task for future work in order to greatly ease the full computation of 4-point correlation functions in LCFT of rank $r > 2$.

Appendix A

Overview of the kernel terms

The following tables contain the kernel terms that can show up for a logarithmic degree from one to five. In addition to the logarithmic degree the kernel depends on the number of functions F that are involved and also on the discrete symmetry. As we are considering four point functions only we are left with four different symmetry groups.

The format of the entries is the same as in subsection 3.2.6 with the small addition of the dimension d of the kernel. It is interesting how similar the entries for the different logarithmic degrees are, the only exception being the entry for the pair (F_3, S_3) respectively (F_{12}, S_4). Also note that each column contains the full kernel, namely if and only if $|F| = |S|$, where S denotes the symmetry group and $|S|$ its cardinality .

"$\leftrightarrow$" means that these entries have to be identical as shown in (3.34), (3.36).

Log.deg 1	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(1),d=1}$	$\leftrightarrow$	$K_{S_2}^{(1),d=1}$	0	$\leftrightarrow$	0
F_2	$K^{(1),d=2}$		$K_{S_2}^{(1),d=1}$	—		—
F_3	—		—	$(*)\|^{d=1}$		—
F_4	—		$K^{(1),d=2}$	—		0
F_6	—		—	$K^{(1),d=2}$		$K_{S_2}^{(1),d=1}$
F_{12}	—		—	—		$(*)\|^{d=1}$
F_{24}	—		—	—		$K^{(1),d=2}$

$(*) = 2K_2 - K_1|^{d=1}$

Log.deg 2	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(2),d=2}$	$\leftrightarrow$	$K_{S_2}^{(2),d=2}$	$K_{S_4}^{(2),d=1}$	$\leftrightarrow$	$K_{S_4}^{(2),d=1}$
F_2	$K^{(2),d=3}$		$K_{S_2}^{(2),d=2}$	—		—
F_3	—		—	$(*)\vert^{d=2}$		—
F_4	—		$K^{(2),d=3}$	—		$K_{S_4}^{(2),d=1}$
F_6	—		—	$K^{(2),d=3}$		$K_{S_2}^{(2),d=2}$
F_{12}	—		—	—		$(*)\vert^{d=2}$
F_{24}	—		—	—		$K^{(2),d=3}$

$(*) = cK_1^2 + c'(K_2^2 - K_1K_2)|^{d=2}$

Log.deg 3	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(3),d=2}$	$\leftrightarrow$	$K_{S_2}^{(3),d=2}$	$K_{S_4}^{(3),d=1}$	$\leftrightarrow$	$K_{S_4}^{(3),d=1}$
F_2	$K^{(3),d=4}$		$K_{S_2}^{(3),d=2}$	—		—
F_3	—		—	$(*)\vert^{d=2}$		—
F_4	—		$K^{(3),d=4}$	—		$K_{S_4}^{(3),d=1}$
F_6	—		—	$K^{(3),d=4}$		$K_{S_2}^{(3),d=2}$
F_{12}	—		—	—		$(*)\vert^{d=2}$
F_{24}	—		—	—		$K^{(3),d=4}$

$(*) = cK_1^2(K_1 - 2K_2) + c'K_2(K_1 - K_2)(K_1 - 2K_2)|^{d=2}$

Log.deg 4	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(4),d=3}$	$\leftrightarrow$	$K_{S_2}^{(4),d=3}$	$K_{S_4}^{(4),d=1}$	$\leftrightarrow$	$K_{S_4}^{(4),d=1}$
F_2	$K^{(4),d=5}$		$K_{S_2}^{(4),d=3}$	—		—
F_3	—		—	$(*)\vert^{d=3}$		—
F_4	—		$K^{(4),d=5}$	—		$K_{S_4}^{(4),d=1}$
F_6	—		—	$K^{(4),d=5}$		$K_{S_2}^{(4),d=5}$
F_{12}	—		—	—		$(*)\vert^{d=3}$
F_{24}	—		—	—		$K^{(4),d=5}$

$(*) = cK_1^4 + c'K_2^2(K_1 - K_2)^2 + c''(K_1^3K_2 - 2K_1K_2^3 + K_2^4)$

Log.deg 5	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(5),d=3}$	$\leftrightarrow$	$K_{S_2}^{(5),d=3}$	$K_{S_4}^{(5),d=1}$	$\leftrightarrow$	$K_{S_4}^{(5),d=1}$
F_2	$K^{(5),d=6}$		$K_{S_2}^{(5),d=3}$	—		—
F_3	—		—	$(*)\vert^{d=3}$		—
F_4	—		$K^{(5),d=6}$	—		$K_{S_4}^{(5),d=1}$
F_6	—		—	$K^{(5),d=6}$		$K_{S_2}^{(5),d=3}$
F_{12}	—		—	—		$(*)\vert^{d=3}$
F_{24}	—		—	—		$K^{(5),d=6}$

$$\begin{aligned}(*) = {} & c(-2K_1^3K_2^2 + 8K_1^2K_2^3 - 11K_1K_2^4 + 5K_2^5) + \\ & c'(-K_1^4K_2 + 4K_1^3K_2^2 - 6K_1^2K_2^3 + 4K_1K_2^4) + \\ & c''(8K_1^5 + 1K_1^4K_2 - 10K_1^2K_2^3 + 20K_1K_2^4 - 20K_2^5)|^{d=3}\end{aligned}$$

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List of Publications

This work is based on the following publications: Krohn and Flohr (2003), Flohr and Krohn (2004, 2005a,b).

M. Krohn and M. Flohr
Ghost Systems Revisited: Modified Virasoro Generators and Logarithmic Conformal Field Theories
J. High Energy Phys. JHEP01(2003)020, [hep-th/0212016]

We study the possibility of extending ghost systems with higher spin to a logarithmic conformal field theory. In particular we are interested in $c = -26$ which turns out to behave very differently to the already known $c = -2$ case. The energy momentum tensor cannot be built anymore by a combination of derivatives of generalized symplectic fermion fields. Moreover, the logarithmically extended theory is only consistent when considered on nontrivial Riemann surfaces. This results in a LCFT with some unexpected properties. For instance the Virasoro mode L_0 is diagonal and for certain values of the deformation parameters even the whole global conformal group is non-logarithmic.

M. Flohr and M. Krohn
Operator Product Expansion and Zero Mode Structure in Logarithmic CFT
Fortschr. Phys. 52 (2004) 503-508, [hep-th/0312185]

The generic structure of 1-, 2- and 3-point functions of fields residing in indecomposable representations of arbitrary rank are given. These in turn determine the structure of the operator product expansion in logarithmic conformal field theory. The crucial role of zero modes is discussed in some detail.

M. Flohr and M. Krohn
A Note on Four-Point Functions in Logarithmic Conformal Field Theory
to appear in Fortschr. Phys., [hep-th/0501144]

The generic structure of 4-point functions of fields residing in indecomposable representations of arbitrary rank is given. The presented algorithm is illustrated with some nontrivial examples and permutation symmetries are exploited to reduce the number of free structure-functions, which cannot be fixed by global conformal invariance alone.

M. Flohr and M. Krohn
Four-Point Functions in Logarithmic Conformal Field Theories
submitted to Nucl. Phys. B, [hep-th/0504211]

The generic structure of 4-point functions of fields residing in indecomposable representations of arbitrary rank is given. The used algorithm is described and we present all results for Jordan-rank $r = 2$ and $r = 3$ where we make use of permutation symmetry and use a graphical representation for the results. A number of remaining degrees of freedom which can show up in the correlator are discussed in detail. Finally we present the results for two-logarithmic fields for arbitrary Jordan-rank.

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als die, Dank zu sagen."
— Marcus Tullius Cicero

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Lebenslauf

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