A Novel Conformal Field Theory Approach to Bulk Wave Functions in the Fractional Quantum Hall Effect

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Abstract

In this thesis, a novel approach is proposed to represent bulk wave functions of fractional quantum HALL states in terms of conformal field theory correlators. It starts from the LAUGHLIN states and their generalization following JAIN's picture of composite fermions. These effective particles are naturally identified within the b/c-spin conformal field theories. The enigmatic phenomenon of fractional statistics is described by twist fields which inherently appear in the spin systems. A geometrical interpretation is obtained in which bulk wave functions are understood as holomorphic functions over a ramified covering of the complex plane. To extend JAIN's main series, the concept of composite fermions that pair to spin singlets is introduced. This is naturally adopted by the particular b/c-spin system with central charge $c = -2$ as known for the HALDANE-REZAYI state with filling fraction $\nu = 5/2$. In this way, the new conformal field theory proposal covers the set of experimentally confirmed fractional quantum HALL states in the lowest LANDAU level. Concerning their stability with respect to energy gaps of the ground states, a natural ordering is deduced where unobserved filling fractions are precisely avoided. The scheme is compatible with classifications in terms of effective CHERN-SIMONS theories. It leads to severe restrictions of the coupling K-matrices and, in addition, the b/c-spin approach can be extended to describe non-ABELIAN fractional quantum HALL states imposing physical constraints on them.

The scientific results underlying this thesis are submitted for publication to Phys. Rev. B and can be found in [72].

Zusammenfassung

In dieser Arbeit wird ein neuer Zugang zur Darstellung von Bulk-Wellenfunktionen des fraktionalen Quanten-HALL-Effekts durch Korrelatoren konformer Feldtheorien präsentiert. Beginnend mit den LAUGHLIN-Zuständen werden diese und ihre Verallgemeinerung gemäß JAINs Konzept der Komposit-Fermionen beschrieben. Diese effektiven Teilchen sind auf natürliche Weise in den konformen b/c-Spin-Feldtheorien eingebettet. Das erstaunliche Auftreten fraktionaler Statistik wird in diesem Zugang durch Twist-Felder realisiert, die inhärent in den Spin-Systemen auftreten. Auf diese Weise wird eine unmittelbare geometrische Interpretation nahegelegt, in der die Bulk-Wellenfunktionen als holomorphe Funktionen, definiert auf einer verzweigten Überlagerung der kompaktifizierten komplexen Ebene, verstanden sind. JAINs Hauptserien von Füllfaktoren werden durch die Einführung gepaarter Komposit-Fermion Spin-Singulett Zustände fortgesetzt, welche in natürlicher Weise durch das spezielle b/c-Spin-System mit zentraler Ladung $c = -2$ beschrieben werden, wie aus der Darstellung des HALDANE-REZAYI Zustandes mit Füllfaktor $\nu = 5/2$ bekannt ist. Der somit abgeleitete Zugang durch konforme Feldtheorien deckt die Menge experimentell bestätigter fraktionaler Quanten-HALL-Zustände im niedrigsten LANDAU-Niveau ab. Deren Stabilität in bezug auf Energielücken der Grundzustände wird in natürlicher Ordnung erfaßt, wohingegen unbeobachtete Füllfaktoren nicht vorhergesagt werden. Das Schema ist kompatibel im Rahmen der Klassifizierung durch effektive CHERN-SIMONS-Theorien und führt zu weitgehenden Einschränkungen der zentralen K-Matrizen, die die Kopplung zwischen verschiedenen LANDAU-Niveaus vermitteln. Ferner ist es möglich, den Zugang zum fraktionalen Quanten-HALL-Effekt durch b/c-Spin-Systeme auf die Klasse nicht-ABELscher Zustände auszuweiten und diesen gleichzeitig physikalische Zwangsbedingungen aufzuerlegen.

Contents

Introduction

The fractional quantum HALL effect is one of the most fascinating and striking phenomena in condensed matter physics [1]. Certain numbers, the filling fractions $\nu \in \mathbb{Q}$, can be observed with an extremely high precision in terms of the HALL conductivity $\sigma_{\rm H} = \nu$ in natural units. These numbers are extensively independent of many physical details, e.g., the geometry of the sample, its purity, the temperature. The enigmatic and intriguing aspect of this phenomenon is that only a certain set of these fractional numbers ν can be observed in experiments: despite ongoing attempts in varying the purity (or disorder), the external magnetic field, and various other parameters, the set of observed fractions has not changed considerably over the last few years [2, 3, 4, 5].

It was realized quite early that the fractional quantum HALL effect shows all signs of universality and large scale behavior [6, 7]. Independence of the geometrical details of the probe and of its size hint towards an effective and purely topological field theory description. Indeed, since the quantum HALL effect is essentially a (2+1)-dimensional problem, the effective theory is regarded to be dominated by the topological CHERN-SIMONS term $a \wedge \mathrm{d}a$ instead of the MAXWELL term $\mathrm{tr}\big[F^2\big]$. Suitable reviews on the theory of the fractional quantum HALL effect are [8, 9, 10, 11].

However, it is ultimately interesting to deduce a microscopic description of the fractional quantum HALL effect. The task may start from finding eigenstates of an exact microscopic HAMILTONIAN. Unfortunately, this can merely be realized for a small number of electrons. The great achievement of LAUGHLIN was to conceive how a many-particle wave function has to look like if it should respect a few reasonable symmetry constraints [12]:

$$\Psi_{\rm Laughlin}(z_1,\,\ldots,\,z_N) = \prod_{1\le i<j\le N} (z_i - z_j)^{2p+1} \exp\left(-\frac{1}{4}\sum_{1\le i\le N} |z_i|^2\right) . \tag{1}$$

It is known that LAUGHLIN's wave functions which describe fractional quantum HALL droplets with filling fractions $\nu = 1/(2p+1)$ ($p \in \mathbb{Z}_+$) are extremely good approximations to the true ground states. Furthermore, they are exact solutions for HAMILTONIANS with certain short-range electron-electron interactions. Soon after, various hierarchical schemes were developed yielding ground state wave functions for other rational filling factors [13, 14, 15, 16, 17]. It is important to note here that the ground state eigenfunctions are time-independent up to a trivial global phase. Thus, they might be regarded as solutions of a (2+0)-dimensional problem. In principle, this is the main idea behind all attempts to describe the bulk wave functions in terms of conformal field theory correlators.

The LAUGHLIN wave functions describe special incompressible quantum states of the electrons, i.e., quantum droplets. Incompressibility is connected to the existence of energy gapless excitations on the border of the quantum state [6, 7, 18, 19, 20, 21, 22, 23]. The latter can successfully be described in terms of conformal field theories with current algebras as chiral symmetries.

Furthermore, there is an exact equivalence between the (2+1)-dimensional CHERN-SIMONS theory in the bulk and the (1+1)-dimensional conformal field theory on the boundary describing the edge excitations [24]. Naturally, these edge conformal field theories have to be unitary since they describe the time evolution of spatial one-dimensional waves propagating on S^1.

However, LAUGHLIN's bulk wave functions in a static (2+0)-dimensional setting show a remarkable resemblance to correlation functions of a free EUCLIDEAN conformal field theory set on the compactified complex plane. This resemblance has motivated quite a number of works trying to find a conformal field theory description of bulk wave functions in the fractional quantum HALL effect, e.g., [25, 26, 27, 28]. Most approaches assumed from the beginning that these "bulk" theories are unitary. However, this assumption is void since the bulk wave functions to be investigated are time-independent eigenfunctions. Moreover, most approaches represented the bulk wave functions in terms of building blocks belonging to classes of conformal field theories with continuous parameters, e.g., the GAUSSIAN $c = 1$ systems. The immanent problem with these approaches is that there exists no principle selecting the wave functions for experimentally observed filling fractions. Therefore, almost all approaches so far easily accommodate arbitrary rational filling factors. On the other hand, it is not entirely surprising that the bulk wave function should have something to do with conformal field theory. As indicated above, the observable quantities of the quantum HALL system are largely independent of the precise form and size of the sample. Thus, the normalized charge distributions of the electrons should be invariant under scaling (up to an exponential factor) and area preserving changes of the shape of the sample. The first symmetry is linked to conformal invariance, the latter to the $\mathcal{W}_{1+\infty}$-algebra [29, 30]. Furthermore, in the two-dimensional case, global scaling invariance implies full conformal invariance under certain benign circumstances.

Interestingly, there exists a particularly enigmatic fractional quantum HALL state, i.e., the HALDANE-REZAYI state with $\nu = 5/2$. This is one of the very few experimentally confirmed states with an even denominator filling. Of course, attempts have been made to describe proposed bulk wave functions for this state with the help of conformal field theory correlators, e.g., [21, 25, 31, 32]. In this case, however, it turned out that this can only be achieved if the corresponding conformal field theory has central charge $c = -2$. Thus, concerning the HALDANE-REZAYI state, it is obligatory to use a non-unitary theory. This $c = -2$ theory is the *b*/*c*-spin system of two anti-commuting fields with spins one and zero, respectively. Therefore, it naturally yields the object expected to be observed in this fractional quantum HALL state, namely spin singlet states of paired electrons. In addition, the $c = -2$ conformal field theory contains a $\mathbb{Z}_2$-twisted sector which accurately describes the effect of single flux quanta piercing the quantum droplet. Thus, this theory successfully characterizes the ground state and its physically expected excitations with the correct fractional statistics without predicting arbitrary additional features.

This thesis starts from the successful bulk wave function description of the HALDANE-REZAYI fractional quantum HALL state via a non-unitary spin system conformal field theory and investigates how fractional quantum HALL state bulk wave functions can be represented in terms of conformal field theory correlators. In contrast to other approaches the assumption that these theories should be unitary will be dropped because there is no physical reason for it. By this, it is possible to concentrate on a different class of conformal field theories, namely the *b*/*c*-spin systems of two anti-commuting fields of spins j and $(1 - j)$, respectively. Locality forces

$j \in \mathbb{Z}/2$ so that the conformal field theories are confined to a discrete set. This ansatz will not only naturally explain all experimentally observed filling fractions in the range $0 \leq \nu \leq 1$, but, in addition, will not predict new unobserved series.

Besides these convenient features the approach yields a beautiful geometrical picture for the conformal field theories used to represent the bulk wave functions. Additionally, correlations of spin j (or spin $1 - j$) effective particles with flux quanta of precisely the fractional statistics which are theoretically predicted from first principles are obtained. These statistics, e.g., $1/m$, naturally manifest themselves in the presence of $\mathbb{Z}_m$-twists. These in turn have the geometrical meaning of replacing the complex plane by an m-fold ramified covering of itself. Thus, the bulk wave functions are finally recast in the language of complex analysis, i.e., j- or $(1 - j)$-differentials on $\mathbb{Z}_m$-symmetric RIEMANN surfaces.

Most of the observed filling fractions $\nu \in \mathbb{Q}$ have an odd denominator. This can be deduced from the basic fact that the elementary entities in the quantum HALL system are fermions, e.g., the composite fermions, effective particles that were conceived by JAIN. These allow to naturally link the integer and the fractional quantum HALL effect and can be successfully described by b/c systems with spin $j \in \mathbb{N}/2$. An essential part of this work is to propose a new hierarchical scheme in which filling fractions can be derived from others by forming paired singlets of composite fermions. In this way, JAIN's principal series are represented and furthermore extended to precisely cover all confirmed filling fractions. Within this approach, unobserved fractions are avoided without problem since they all lie at the far end of the hierarchical series or are characterized by series of higher order. In contrast to this feature, most other hierarchical schemes predict certain unobserved fractions, since prominent experimentally confirmed ones can only be realized at a certain order k within the hierarchy. The problem is the lack of a physical reason why the corresponding low order fractional quantum HALL state does not exist, but the higher order one derived from it. Thus, the scheme of this thesis seemingly provides a natural explanation for the completeness of the set of experimentally accessible filling fractions.

The outline of this thesis is as follows: Chapter 1 presents an introduction to the theory of the quantum HALL effect. Starting from the integer effect in the first section the basic ideas of LAUGHLIN are reviewed in the second part leading to his seminal trial wave functions. Consecutively, the appropriate generalization of them within the picture of JAIN is provided which allows to describe a wide class of fractional quantum HALL states in terms of an integer quantum HALL effect of effective particles, i.e., the composite fermions. JAIN's idea is favored since the composite fermions are naturally identified with fields of the conformal field theory approach of this work. Moreover, his picture has the advantage to realize most of the prominently observed filling fractions within the first level of its hierarchical scheme. The last part of this introductory chapter deals with more general LAUGHLIN type trial wave functions in the lowest LANDAU level representing multilayer states. These are conceptually deduced from effective CHERN-SIMONS theory which is believed to adopt many general principles of the quantum HALL effect, e.g., topological order. The central part of this formalism is a certain matrix K encoding the interactions of the layers, i.e., of different quantum fluids. Within this scheme, JAIN's main series of composite fermions are consistently reconsidered.

Assuming basic knowledge on the topic Chapter 2 introduces all relevant conformal field theory features and methods in terms of the scope of this work. These are exemplified for the b/c-spin systems. Starting from the principal structures of the fields living on the compactified complex plane in the first section their setting is naturally extended to $\mathbb{Z}_n$-symmetric RIEMANN surfaces.

The third chapter is the core of this work and develops the novel conformal field theory approach to the fractional quantum HALL effect via b/c-spin systems. Motivated by geometrical features and following the concepts and structures of the LAUGHLIN states and their natural extension, i.e., JAIN's main series, a new hierarchical scheme is deduced. It naturally links the composite fermion picture to the classification of effective CHERN-SIMONS theory. The K-matrices of the CHERN-SIMONS formalism which encode the essential information on fractional quantum HALL states, e.g., topological order and filling fraction, satisfy severe constraints demanded by evident physical properties. These constraints nicely coincide with the ones derived for the b/c-spin systems. The scheme starts from JAIN's main series which naturally generalize the LAUGHLIN states. While avoiding the principle of particle-hole duality which is not confirmed well by experiments all other observed filling fractions are consecutively obtained by pairing of composite fermions to spin singlet states. Step by step, states which include more extensive pairing structures are deduced leading to the new hierarchy. As another important consequence, the pairing scheme which is represented by tensoring the spin conformal field theories with additional b/c-spin singlet systems of central charge $\mathrm{c} = -2$ demands additional restraints concerning the CHERN-SIMONS K-matrices, i.e., essentially restricting them to block form. The approach turns out to provide all filling fractions confirmed by experimental data in one-to-one correspondence with their order of stability while ruling out controversial fractions and others which seemingly violate principles of stability. Predictions for future experiments, e.g., quasi-particle statistics of higher order states, conclude this chapter.

Chapter 4 extends the conformal field theory approach to the class of non-ABELIAN fractional quantum HALL states restricting it to discrete series. It is argued that this set reveals to possess a rather small energy gap. Therefore, proposals are made that future experimental research on non-ABELIAN statistics shall concentrate on special fractional quantum HALL states which solely exist in the non-ABELIAN form.

The fifth chapter summarizes the results and, motivated by the completeness of the approach provided in this work, tries to put them into context. Unsolved problems are stressed and some directions for possible research in the future are proposed.

CHAPTER 1

The Quantum Hall Effect

The quantum HALL effect is an incredibly intriguing as well as amazing phenomenon in the field of condensed matter physics and led to a strong interest in two-dimensional electron systems. In the last two decades, a lot of concepts have been developed in nearly all domains of modern theoretical research. The scope ranges from microscopic HAMILTONIAN theories to topological and conformal field theories. This chapter provides a short access to both the integer and the fractional quantum HALL effect. Starting from basic quantum mechanics in the first section, the integer phenomenon is discussed as a one-electron system involving disorder. The fundamental differences of the fractional effect as a strongly correlated electron system are enlightened in the second section where both phenomena are linked via JAIN's effective composite fermion model while the concluding section relates them to aspects of CHERN-SIMONS theory and introduces the concept of multi-layer states. Suitable introductions to the theory of the quantum HALL effect are provided by [8, 9, 10, 11, 35].

1.1 The Integer Quantum Hall Effect

The integer quantum HALL effect was discovered by KLAUS V. KLITZING in 1980 [36]. He studied the charge-transport behavior of high mobility two-dimensional electron gases at very low temperatures and strong magnetic fields. VON KLITZING found that — for certain values of the magnetic field B — the longitudinal resistance of the semiconductor sample becomes very small while the plot of the transverse, i.e., the HALL conductance σ_{H} over B exhibits plateaus. These plateaus turned out to be centered around integer multiples of the natural unit e^2/h. This quantization is observed with amazing precision (up to 10^{-8}). Due to experimental circumstances, e.g., macroscopic sizes and shapes of the probes, disorder, and finite temperature effects, this is even more surprising and leads to the fact that fundamental quantum physical properties are revealed. In 1985, VON KLITZING was honored with the NOBEL prize for his discovery, and the accuracy of the quantum HALL effect made it the etalon of electric resistance.

In order to understand the effect in a proper way, it is convenient to start from LANDAU's analysis of the quantum dynamics of an electron moving in a perpendicular and uniform magnetic field. The HAMILTONIAN reads

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{\mathrm{e}}{c}\vec{A}(\vec{r})\right)^2 \quad , \tag{1.1}$$

where m and e are the electron's mass and charge, c is the speed of light, and the vector potential $\vec{A}(\vec{r})$ is chosen in the LANDAU gauge

$$\vec{A}(\vec{r}) = \big(-By,\, 0,\, 0\big) \quad . \tag{1.2}$$

The electron is restricted to move in the x/y-plane and $[H, p_x] = 0$. Therefore, the natural ansatz $\Psi(\vec{r}) = \exp(\frac{\mathrm{i}}{\hbar}p_x x)\psi(y)$ inserted in the SCHRÖDINGER equation yields the differential equation of a one-dimensional harmonic oscillator

$$\psi''(y) + \frac{2m}{\hbar^2}\Big[\varepsilon - \frac{m}{2}\omega_c^2(y-y_0)^2\Big]\psi(y) = 0 \ , \tag{1.3}$$

where $\omega_c = \frac{\mathrm{e}B}{mc}$ is the cyclotron frequency and $y_0 = \frac{c\hbar k_x}{\mathrm{e}B}$ is the centre of the cyclotron orbit in classical terms. The solution is given by

$$\Psi_n(y-y_0) = \exp\Big[-\frac{m\omega_c}{2\hbar}(y-y_0)^2\Big]\mathrm{H}_n\Big(\sqrt{\tfrac{m\omega_c}{\hbar}}(y-y_0)\Big) \ , \ \varepsilon_n = \hbar\omega_c(n+\frac{1}{2}) \ . \tag{1.4}$$

Here, H_n are the HERMITE polynomials. The energy levels ε_n are called LANDAU levels and are highly degenerate due to y_0. Their degeneracy $N(n)$ is related to the total magnetic flux Φ perpendicularly piercing the electron gas and is derived to

$$N(n) = \frac{\Phi}{\Phi_0} = \frac{\Phi}{hc/e} \ . \tag{1.5}$$

Thus, the degeneracy is a constant with respect to the LANDAU levels. It depends linearly on B, and is measured in units of the magnetic flux quantum Φ_0.
If it is assumed that electron-electron interactions can be neglected, the above results can be extended to a system of n electrons. To classify this system properly, it is reasonable to define the filling fraction ν

$$\nu = \frac{\text{number of electrons}}{\text{number of LANDAU sites}} = \frac{hc}{\mathrm{e}B}n_\mathrm{e} \ , \tag{1.6}$$

where n_e is the surface density of the electrons. In order to calculate the HALL conductance σ_H, an electric field $\vec{E}(\vec{r}) = (0,\, E,\, 0)$ has to be added to (1.1). Its effect is a shift in y_0 and the energy ε_n:

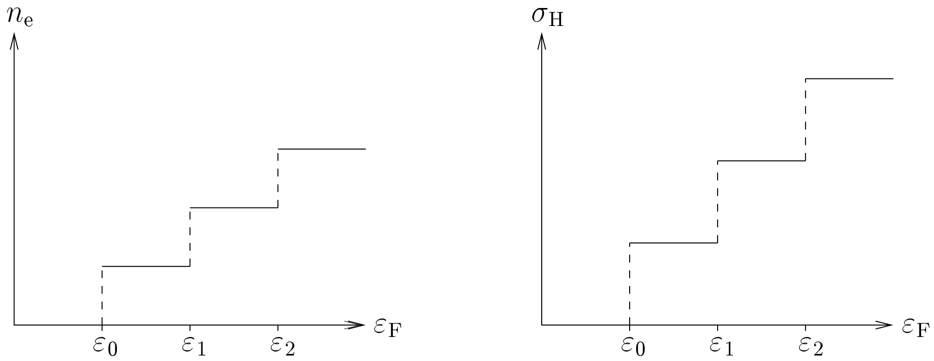
$$y_0 \longrightarrow y_0' = y_0 + \frac{\mathrm{e}E}{m\omega_c^2} \ , \quad \varepsilon_n \longrightarrow \varepsilon_n' = \varepsilon_n + \mathrm{e}Ey_0 + \frac{m}{2}\Big(\frac{cE}{B}\Big)^2 \ . \tag{1.7}$$

Deducing the expectation value $\langle v_x \rangle$ from the wave functions (1.4) by using (1.7) yields:

$$\sigma_\mathrm{H} = -\frac{n_\mathrm{e}\mathrm{e}\langle v_x\rangle}{E} = -\frac{n_\mathrm{e}c\mathrm{e}}{B} = -\nu\frac{\mathrm{e}^2}{h} \ . \tag{1.8}$$

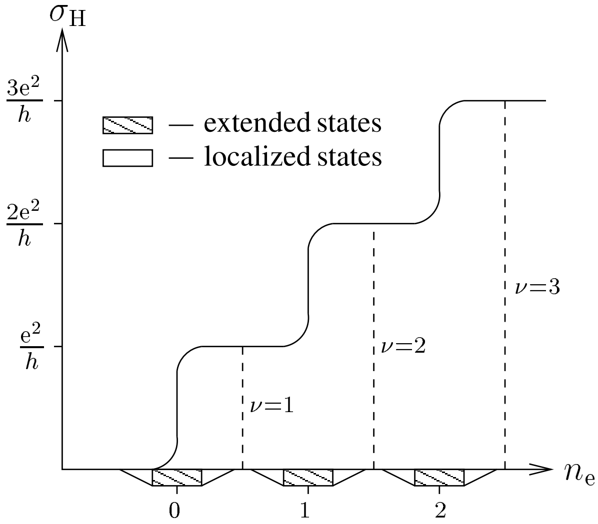
This dependence of σ_H on the electron density n_e and the filling fraction ν, respectively, has to be analyzed in detail. If the FERMI energy ε_F of the system is continuously varied, n_e remains constant until the next LANDAU level is reached. Exceeding ε_n fills the whole level. Since each electron state of the system contributes to the HALL current, σ_H shows the same behavior. The graphs are illustrated in figure 1.1. If combined, they yield the linear dependence (1.8) and no quantum HALL effect is expected. This is resolved if disorder of the system is taken into account. Impurities lift the degeneracy of the LANDAU levels which broaden into bands. These bands consist of localized states bound by defects of the probe and extended states carrying the HALL current. It is assumed and can be shown for several types of potentials

FIGURE 1.1: HALL conductance and electron density over FERMI energy



that these latter states exist in the quantum HALL regime and that they are located around the centre of the LANDAU band [18, 37]. Thus, by varying the FERMI energy, the electron density is continuously increased while in the region of localized states (separating the extended ones) σ_H remains constant. In addition, as it was first shown by PRANGE [38] for the case of impurities represented by δ-functional potentials, the total current carried by a LANDAU level is unchanged, since an extended state exactly compensates for the loss due to localization effects. As a consequence, (1.8) remains valid in the domain of a plateau. The behavior of σ_H is indicated in figure 1.2. The HALL conductance shows plateaus with centers located at integral

FIGURE 1.2: HALL conductance over electron density



filling fractions ν and is quantized in units of e^2/h. In principle, this explains the results of the experiments, but is not capable of describing the amazing accuracy of the effect. As indicated above, the quantization of σ_H remains exact. This even holds up to macroscopic length scales and more complicated types of disorder. The issue is resolved by relating the conductance to gauge covariance, first proposed by LAUGHLIN [39] and later on extended by HALPERIN [18]. It is shown that σ_{H} is a topological invariant if the FERMI level lies in a (mobility) gap, i.e., the domain of localized states. By this, it is assumed that quantum HALL states have to be incompressible. This is well supported by perturbative methods and numerical research. Therefore, the quantization of σ_{H} (1.8) is based on fundamental physical principles independent of experimental circumstances and devices.

The discussion on topological invariance of the HALL conductance exceeds the scope of this work, a detailed access is provided in the first chapter of [10] which reviews diverse approaches [40, 41] in a closed context.

There is a certainty even if a microscopic theory has not been discovered so far: the integer quantum HALL effect is to be understood as a one-electron effect involving disorder where electron-electron interactions can be neglected. The corresponding ground states have to be incompressible quantum liquids involving a non-trivial geometrical setting. This yields integer quantization if and only if the ground state is non-degenerate.

1.2 The Fractional Quantum Hall Effect

In 1982, three years before KLAUS V. KLITZING was awarded the NOBEL prize, theoretical physicists believed they understood the quantization of the HALL conductance in natural units. Therefore, it was rather surprising when TSUI, STÖRMER and GOSSARD discovered a plateau of the HALL conductance σ_{H} at $\nu = 1/3$ and indications for another one at $\nu = 2/3$ [1]. This 'anomalous' behavior of quantization was inconsistent with respect to the theory of the integer effect. It soon became obvious that fractional HALL states cannot be described by single-electron quantum mechanics. Since the FERMI energy resides within a LANDAU level, the energy gap necessary to establish a plateau is due to a strongly correlated electron movement reducing the COULOMB interaction. Therefore, the corresponding states are expected to be of completely different geometrical and topological nature.

To begin the analysis of the fractional quantum HALL effect and study its topological features it is advisable to complexify the theory introduced in the first chapter by $z = x + \mathrm{i}y$ and its complex conjugate $\bar{z}$, yielding

$$
\begin{aligned}
x &= \frac{1}{2}(z + \bar{z})\,, \qquad y = \frac{1}{2\mathrm{i}}(z - \bar{z})\ , \\
\partial_z &= \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)\,, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)\ .
\end{aligned}
\tag{1.9}
$$

Using the symmetric gauge

$$
\vec{A}(\vec{r}) = \big(-\frac{1}{2}By,\, \frac{1}{2}Bx,\, 0\big)\ , \tag{1.10}
$$

the HAMILTONIAN (1.1) becomes

$$
\mathcal{H} = -\frac{2\hbar^2}{m}\big(\partial_z - \frac{1}{4l^2}\bar{z}\big)\big(\partial_{\bar{z}} + \frac{1}{4l^2}z\big) + \frac{\hbar}{2}\omega_c\ . \tag{1.11}
$$

where $l = \sqrt{\hbar c/\mathrm{e}B}$ is the magnetic length unit.[1] Since $\hbar\omega_c/2$ is the ground state energy of (1.1), it follows from (1.11) that any wave functions $\Psi(z,\,\bar{z})$ satisfying

$$\Big(\partial_{\bar{z}} + \frac{1}{4}z\Big)\Psi(z,\,\bar{z}) = 0\ , \tag{1.12}$$

describe a lowest LANDAU level state. They are derived as

$$\Psi(z,\,\bar{z}) = \mathrm{f}(z)\exp\Big(-\frac{1}{4}\,|z|^2\Big)\ . \tag{1.13}$$

In fact, the space of lowest LANDAU level wave functions is equivalent to the space of analytic functions with inner product

$$\langle \mathrm{f}(z,\,\bar{z})|\mathrm{g}(z,\,\bar{z})\rangle = \int \mathrm{d}^2 z\,\bar{\mathrm{f}}(z,\,\bar{z})\mathrm{g}(z,\,\bar{z})\exp\Big(-\frac{1}{2}\,|z|^2\Big)\ , \tag{1.14}$$

namely, the BARGMAN space [42].

1.2.1 Laughlin States

The first big step forward in order to solve the puzzle of the geometric structure of fractional quantum HALL states was conducted by LAUGHLIN by presenting his trial wave functions [12]:

$$\Psi_{\mathrm{Laughlin}}(z_1,\ldots,z_n) = \mathcal{N}\prod_{k<l}^{n}(z_k - z_l)^{2p+1}\exp\Big(-\frac{1}{4}\sum_{i}^{n}|z_i|^2\Big)\ , \tag{1.15}$$

where $p \in \mathbb{N}$, z_i is the position of the i-th electron in unified complex coordinates (1.9), and $\mathcal{N}$ is a normalization factor. They were conceived as the variational ground state wave functions for the model HAMILTONIAN

$$\mathcal{H} = \sum_{k}^{n}\left[\frac{1}{2m}\Big(\frac{\hbar}{\mathrm{i}}\nabla_k - \frac{\mathrm{e}}{c}\vec{A}(\vec{r}_k)\Big)^2 + V_{\mathrm{bg}}(\vec{r}_k)\right] + \sum_{k<l}^{n}\frac{\mathrm{e}^2}{|\vec{r}_k - \vec{r}_l|}\ , \tag{1.16}$$

with the vector potential taken in the symmetric gauge (1.10). Here, V_{bg} is a potential of a background charge distribution that neutralizes the electrons' COULOMB repulsion. This guarantees the stability of the system. Despite their simple structure, LAUGHLIN's wave functions include amazing features. Firstly, referring to (1.13), they are an element of the BARGMAN space and thus describe a state in the lowest LANDAU level. Secondly, since $p \in \mathbb{N}$, they are completely anti-symmetric satisfying the PAULI principle, and thirdly, due to the zeroes in the polynomial factor, the electrons are widely separated from each other. This is a crucial condition for the stability of the state with respect to electron-electron interactions in strongly correlated systems. Additionally, they are exact ground states of various short ranged δ-potential HAMILTONIANS.
For further investigation it is important to realize that the modulus squared of the wave function is equivalent to the BOLTZMANN distribution of a two-dimensional one-component plasma.

$$|\Psi|^2 = \exp(-\beta\Phi)\ ,\quad \beta = \frac{1}{2p+1}\ ,$$

$$\Phi = -2(2p+1)^2\sum_{k<l}^{n}\ln|z_k - z_l| + \frac{2p+1}{2}\sum_{k}^{n}|z_k|^2\ . \tag{1.17}$$

[1] In the following, $l \equiv 1$ for reasons of simplicity.

Even if the analysis is far from being easy, the main advantage of this identification is to investigate the thermodynamic limit. It turns out that with respect to charge neutrality the electron density corresponds to filling fractions $\nu = 1/(2p+1)$. Since $p \in \mathbb{N}$, the PAULI principle is directly related to odd-denominator fractions. Furthermore, the thermodynamic behavior for small p reveals that the system is an incompressible liquid rather than a WIGNER crystal. This property yields the existence of plateaus in the HALL conductance σ_{H}. Finally, numerical calculations for systems of finite size show an excellent overlap with (1.15) of more than 99.5%.

The LAUGHLIN ground state can be extended with respect to quasi-hole excitations by introducing a simple polynomial factor

$$\Psi_{\mathrm{exc.}} = \mathcal{N}(\zeta_i) \prod_{k,l} (z_k - \zeta_l) \prod_{r<s} (z_r - z_s)^{2p+1} \exp\Big(-\frac{1}{4} \sum_i |z_i|^2 \Big) \ . \tag{1.18}$$

Here, the ζ_i denote the positions of the quasi-hole excitations. With respect to (1.17) the excited states, in contrast to the ground states, have a non-uniform charge distribution. In comparison with the two-dimensional plasma picture, a charge deficit of $\mathrm{e}/(2p+1)$ is found at the point ζ_i, which shows that the quasi-holes are fractionally charged.

In order to analyze the quasi-hole statistics more carefully, the BERRY connection has to be derived from the normalization factor. This was first stated by AROVAS et al. [43] (a detailed comment on the derivation is provided in [10], chapter 2):

$$\Psi_{\mathrm{exc}} = \mathcal{N} \prod_{k,l} (z_k - \zeta_l) \prod_{r<s} (z_r - z_s)^{2p+1} (\zeta_r - \zeta_s)^{\frac{1}{2p+1}} \exp\left(-\mathrm{F}\left(z_i,\, \zeta_i\right)\right) \ , \tag{1.19}$$

$$\mathrm{F}\left(z_i,\, \zeta_i\right) = \frac{1}{4} \sum_i \Big(|z_i|^2 + \frac{1}{2p+1} |\zeta_i|^2 \Big) \ .$$

If a quasi-particle at ζ_i encircles another one at ζ_j

$$(\zeta_i - \zeta_j) \longrightarrow (\zeta_i - \zeta_j) \exp(2\pi \mathrm{i}) \ ,$$

a phase of $2\pi/(2p+1)$ is picked up. This mapping is equivalent to exchanging them twice. Thus, they obey fractional statistics

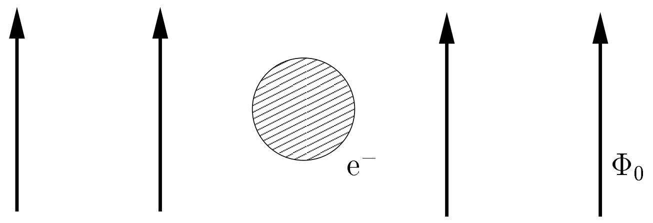
$$\theta = \frac{\pi}{2p+1} \ . \tag{1.20}$$

To stress another important feature: the non-holomorphic factors in (1.19) describing quasi-particle interactions lead to multi-valued wave functions and give rise to the complex geometry the LAUGHLIN states are built on. Despite its fundamental importance this one-to-one correspondence between statistics and analyticity is often omitted in the discussion of the fractional quantum HALL effect. However, in a suitable field theoretical description it has to be considered precisely.

1.2.2 Beyond Laughlin

A strongly correlated electron system underlies the fractional quantum HALL effect. In such systems interactions dominate the physics and long range effects take place. Well known examples are superconductivity and the HUBBARD model which can be described in terms of effective theories. A common feature of these theories is the demand for the existence of effective

FIGURE 1.3: Four flux composite fermion



particles in the system, e.g., COOPER pairs (superconductivity) or spinons and holons (HUBBARD model). Concerning the fractional quantum HALL effect one widely accepted effective theory with direct correspondence to experimental facts was developed by JAIN [16, 44, 45]. He explained the fractional effect by proposing the composite fermion model. A composite fermion consists of one electron with a number of pairs of flux quanta of the magnetic field attached to it, e.g., as in figure 1.3. JAIN showed that the fractional quantum HALL effect can be expressed in terms of an effective integer quantum HALL effect for the composite fermions.

In order to explain this in a proper way, the results of section 1.1 have to be reconsidered. Using the symmetric gauge (1.10) the single-electron ground state wave functions (1.4) expressed in unified complex coordinates z_i are classified by two suitable quantum numbers: $n, m \in \mathbb{N}_0$ labelling the LANDAU level and the angular momentum, respectively. This leads to:

$$\begin{aligned} & \Psi_{n,m}\ \mathcal{N} \exp\left(+\tfrac{1}{4}\,|z|^2\right)\partial_z^n z^m \exp\left(-\tfrac{1}{2}\,|z|^2\right) \;, \\ \text{e.g.,}\quad & \Psi_{0,m}\ \mathcal{N}_0\, z^m \exp\left(-\tfrac{1}{4}\,|z|^2\right) \;, \\ & \Psi_{1,m}\ \mathcal{N}_1\, z^{m-1}(2m-|z|^2) \exp\left(-\tfrac{1}{4}\,|z|^2\right) \;. \end{aligned} \tag{1.21}$$

The integer effect wave function Ψ_{I} (filling fraction $\nu_{\mathrm{I}} = \mathrm{I} \in \mathbb{N}$) is obtained by taking the SLATER determinant of $(\Psi_{I,0},\, \ldots,\, \Psi_{I,N-1})$ where N is the degeneracy of the LANDAU level, e.g.,

$$\begin{aligned} \Psi_1 &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \exp\Big(-\tfrac{1}{4}\sum_{i=1}^{N}|z_i|^2\Big) \\ &= \prod_{k<l}^{N}(z_k - z_l)\exp\Big(-\tfrac{1}{4}\sum_{i=1}^{N}|z_i|^2\Big) \,. \end{aligned} \tag{1.22}$$

The composite fermion trial wave functions Ψ_{CF} are obtained by multiplying the integer quantum HALL effect wave function, e.g., (1.22) with a polynomial JASTROW factor which analytically corresponds to the attachment of p pairs of flux quanta to the electron:

$$\Psi_{\mathrm{CF}} = \prod_{i<j}^{N}(z_i - z_j)^{2p}\Psi_{\mathrm{I}} \;. \tag{1.23}$$

The filling fraction of the CF state is derived as

$$\nu_{\mathrm{CF}} = \frac{\nu_{\mathrm{I}}}{2p\nu_{\mathrm{I}} + 1} \ . \tag{1.24}$$

Here, $\nu_{\mathrm{I}} = \mathrm{I}$ corresponds to the integer quantum HALL state Ψ_{I}. It can be shown that this procedure neither destroys the correlations of the system nor the incompressibility of the state. LAUGHLIN's wave functions are the simplest examples of this scheme. Starting from a $\nu = 1$ integer quantum HALL state (1.22), p pairs of flux quanta are attached. This yields:

$$\Psi_{\mathrm{Laughlin}} = \mathcal{N} \prod_{i<j}^{N} (z_i - z_j)^{2p} \underbrace{\prod_{i<j}^{N} (z_i - z_j) \exp\Big(-\tfrac{1}{4} \sum_i |z_i|^2 \Big)}_{\Psi_1}, \quad \nu = \frac{1}{2p+1} \ . \tag{1.25}$$

With respect to states beyond the main LAUGHLIN series, the crucial point in JAIN's approach is that higher LANDAU levels contribute to states with $\nu \leq 1$. This might seem confusing and has to be investigated in more detail. It is obvious from (1.21) that higher LANDAU level wave functions depend explicitly on $\bar{z}$. This makes them more complicated to deal with since a lot of numerical results and field theoretical approaches reveal to be valid solely in the lowest LANDAU level approximation. Apart from this it is naturally expected that a $\nu \leq 1$ state is dominated by its overlap with the lowest level. Following these considerations, the wave functions (1.23) have to be mapped for further analysis using the lowest LANDAU level projector:

$$\widehat{\mathrm{P}}_{\mathrm{LLL}} = \sum_{k=0}^{\infty} \frac{1}{4\pi 4^k k!} z^k \exp\big(-\tfrac{1}{4} |z|^2 \big) \int \mathrm{d}^2 z' \, (\bar{z}')^k \ldots \ , \quad \widehat{\mathrm{P}}_{\mathrm{LLL}}^2 = \widehat{\mathrm{P}}_{\mathrm{LLL}} \ . \tag{1.26}$$

Wave functions with $\mathrm{I} > 1$ consist of monomials of the form

$$\rho(z) = z^n (\bar{z})^m \exp\big(-\tfrac{1}{4} |z|^2 \big) \ . \tag{1.27}$$

These are projected to

$$\widehat{\mathrm{P}}_{\mathrm{LLL}} \big[\rho(z) \big] = 4^m \binom{n}{m} m! \, z^{n-m} \exp\big(-\tfrac{1}{4} |z|^2 \big) \ . \tag{1.28}$$

It is obvious from (1.28) that monomials with $m > n$ are identically mapped to zero.

JAIN showed that the wave functions (1.23) have a large overlap with the lowest LANDAU level for a small number of electrons, but in comparison with the LAUGHLIN series there exists no analogue that carries this argument to the thermodynamic limit. The question according to the injectivity of the projection (1.26) is even more difficult to answer. However, numerical and conceptual analyses show that purely analytic wave functions describe lowest LANDAU level states more accurately.

1.2.3 Jain's hierarchical scheme

In principle, it is possible to get any rational number $\nu \in \mathbb{Q}$ as filling fraction by applying JAIN's construction repeatedly. This forms the so-called hierarchical scheme of JAIN. Thus, instead of starting with an integer quantum HALL state, a fractional quantum HALL state obtained from JAIN's construction is taken, and new composite fermions are formed out of the old ones by attaching additional pairs of flux quanta. The new filling fractions are obtained by (1.24). Instead of Ψ_{I} the composite fermion state with ν_{CF} is taken to obtain a new filling ν'_{CF}. In this way, arbitrarily continued fractions of the form

$$\nu = [2p_1, 2p_2, \ldots, 2p_n, \nu_{\mathrm{I}}] = \cfrac{1}{2p_1 + \cfrac{1}{2p_2 + \cfrac{1}{\ddots\, 2p_n + \cfrac{1}{\nu_{\mathrm{I}}}}}} \tag{1.29}$$

can be constructed, and thus arbitrary positive rational numbers $\nu < 1$. However, this hierarchical scheme shares with all other hierarchical schemes the feature of producing way too many unobserved filling fractions. Moreover, it is necessary to invoke the principle of particle-hole duality in order to obtain some of the experimentally confirmed filling fractions within the first few levels of the hierarchy. Unfortunately, the set of all experimentally observed fractional quantum HALL states does not support particle-hole duality very well and is thus avoided in our approach.

1.3 Chern-Simons Theory and Multilayer States

The amazingly accurate quantization of the HALL conductance in the integer and, especially, the fractional quantum HALL effect is deeply linked to topological principles. The corresponding states — so-called quantum liquids — satisfy statistics which are directly related to complex geometrical structures. It is therefore crucial to characterize the fractional quantum HALL states by suitable quantum numbers in the context of an effective theory. This yields the classification in terms of universality classes and topological order. Very detailed approaches to this topic are given by [35, 46].

It is well-known that quantum electrodynamics in (2+1) dimensions consists of a MAXWELL part and a topological CHERN-SIMONS term. It is true that the latter is neglectable compared to the first one in many cases, but it was shown rigorously that the CHERN-SIMONS term dominates the fractional quantum HALL regime [7]. Therefore, this regime can be described in terms of an effective CHERN-SIMONS theory. A fractional quantum HALL system can consist of several quantum fluids which may be coupled to each other. Each fluid i in the effective field theory is described by a vector potential a_i^μ with couplings κ_i in addition to the external field A^μ. The general form of the LAGRANGIAN reads

$$\mathcal{L} = -\frac{1}{4\pi} a_{i\mu} K_{ij} \epsilon^{\mu\nu\lambda} \partial_\nu a_{j\lambda} - \frac{\mathrm{e}}{2\pi} \kappa_i A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_{i\lambda} + \ldots \ , \tag{1.30}$$

where possible other terms such as the contribution of the quasi-hole current are neglected. The complete LAGRANGIAN contains various couplings and sources which exceed the framework of this introductory section. The only important conclusion within the scope of this work is that the internal structure of a so-called m-layer fractional quantum HALL state is encoded in the invertible $m \times m$ matrix K_{ij} describing the couplings of different layers or quantum fluids with each other. This matrix contains various information of the fractional quantum HALL state, e.g., the filling fraction, the topological order, the ground state degeneracy, and the structure of the trial wave functions. This classification may seem quite ambiguous due to gauge symmetries and the variety of fields in (1.30), but it turns out that diverse approaches leading to different K-matrices can be identified with the same universality class of quantum HALL fluids. This allows a suitable ordering and classification of fractional HALL states in the unifying scheme of K-matrices based on general principles. As a result, for an electron system, K_{ij} has to satisfy the following conditions: [2]

$$K_{ij} = \begin{cases} \text{odd integer} & i = j \\ \text{integer} & i \neq j \end{cases} \quad . \tag{1.31}$$

The filling fraction is derived as

$$\nu_K = \sum_{i,j}^{m} K_{ij}^{-1} \ , \tag{1.32}$$

and the trial wave functions read:

$$\Psi_K = \prod_{i<j}^{N} \prod_{\mu}^{m} (z_i^{(\mu)} - z_j^{(\mu)})^{K_{\mu\mu}} \prod_{i,j}^{N} \prod_{\mu<\lambda}^{m} (z_i^{(\mu)} - z_j^{(\lambda)})^{K_{\mu\lambda}} \exp\Big(-\tfrac{1}{4} \sum_{i,\mu} \big| z_i^{(\mu)} \big|^2 \Big) \ . \tag{1.33}$$

These lie entirely in the lowest LANDAU level, but are not completely antisymmetrized among different layers. In the following, this is investigated in more detail.

In the K-matrix formalism integer quantum HALL effect states with filling fraction $\nu_{\mathrm{I}} = I$ are identified by an $I \times I$ identity matrix, e.g.,

$$K_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \nu = 2 \ . \tag{1.34}$$

The two layers do not interact and the corresponding wave function Ψ_2 is naively obtained as a direct product of two functions with $\nu = 1$:

$$\Psi_2 = \prod_{i<j}^{N} (z_i^{(1)} - z_j^{(1)}) \prod_{k<l}^{N} (z_k^{(2)} - z_l^{(2)}) \exp\Big(-\tfrac{1}{4} \sum_{i,\mu} \big| z_i^{(\mu)} \big|^2 \Big) \ . \tag{1.35}$$

Obviously, (1.35) is not suitable to describe the two-layer state in a meaningful way. The structure of (1.33) reveals that Ψ_K has to be understood as the lowest LANDAU level projection of the true wave function where particles of different layers, i.e., LANDAU levels, have to be distinguished. This issue is resolved by following JAIN's composite fermion picture. Starting from (1.34) the attachment of p pairs of flux quanta to all electrons is realized by adding $2p$ to each entry of (1.31):

$$K_{ij}^{\mathrm{CF}} = \begin{pmatrix} 2p+1 & 2p \\ 2p & 2p+1 \end{pmatrix} , \quad \nu = \frac{2}{4p+1} \ . \tag{1.36}$$

[2]Throughout the work, K_{ij} is represented in the symmetric electron basis of CHERN-SIMONS theory.

The corresponding wave function Ψ_{CF} reads:

$$\Psi_{\mathrm{CF}} = \prod_{i<j}^{N}\prod_{\mu=1}^{2}(z_i^{(\mu)} - z_j^{(\mu)})^{2p+1}\prod_{i,j}^{N}\prod_{\mu<\lambda}^{2}(z_i^{(\mu)} - z_j^{(\lambda)})^{2p}\exp\Big(-\tfrac{1}{4}\sum_{i,\mu}\big|z_i^{(\mu)}\big|^2\Big) \ . \tag{1.37}$$

The flux quanta introduce interactions between different layers. Hence, the layers can be interpreted as composite fermion LANDAU levels. The fractional effect for electrons is based on a non-interacting integer effect for composite fermions which is described by a diagonal K-matrix, e.g., (1.34). In order to derive a suitable lowest LANDAU level projected wave function the composite fermions of different LANDAU levels labelled by (μ) have to be distinguished between. The resulting wave function is anti-symmetric only within each LANDAU level, anti-symmetrization over different LANDAU levels is unphysical and would yield a vanishing Ψ_{K} in most cases. Furthermore, the trial wave functions (1.33) for the series (1.39) show an excellent overlap with numerical results.

The example for two layers can be directly generalized to the case of m layers. The $m \times m$ K-matrices read:

$$K_{ij} = \begin{pmatrix} 2p+1 & 2p & \cdots & \cdots & 2p \\ 2p & 2p+1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 2p+1 & 2p \\ 2p & \cdots & \cdots & 2p & 2p+1 \end{pmatrix} \ , \quad \nu_p = \frac{m}{2mp+1} \ . \tag{1.38}$$

This implies the following sequences of filling fractions, i.e., JAIN's main series:[3]

$$\begin{aligned} \nu_1 &= \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{9}{19}, \frac{10}{21}, \ldots \\ \nu_2 &= \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \frac{4}{17}, \frac{5}{21}, \frac{6}{25}, \ldots \\ \nu_3 &= \frac{1}{7}, \frac{2}{13}, \frac{3}{19}, \ldots \\ \nu_4 &= \frac{1}{9}, \frac{2}{17}, \ldots \\ &\vdots \end{aligned} \tag{1.39}$$

These are limited by the WIGNER crystal regime for $\nu \to 0$ depending on the quality of the sample. Therefore, the series for $p \geq 5$ were still not observed. On the other hand we have a cutoff if m, the number of LANDAU levels of composite fermions building the state, is increased. In terms of an effective integer quantum HALL effect this corresponds to the classical limit $B_{\mathrm{eff}} \to 0$.

[3] Solely experimentally confirmed states are indicated. Up to date experimental data is provided by [5].

CHAPTER 2

Conformal Field Theory

During the last decades two-dimensional conformal field theory has become a very powerful tool of modern theoretical physics [47]. Its origin can be traced back to string theory on the one hand and to statistical mechanics on the other. This chapter provides a short introduction to the conformal field theory of the b/c-spin systems. It is expected that the reader has basic knowledge of conformal field theory in two dimensions. Suitable introductions are found in various books, lecture notes and reviews, e.g., [33, 34, 48, 49, 50].

2.1 The b/c-Spin Systems

The chiral theories of the b/c-spin systems were analyzed in detail in 1986 by KNIZHNIK [51]. They are known to play an important role in string theory and conformal field theory due to their geometrical characteristics. Exactly these features make them the candidate to describe the bulk wave functions of the fractional quantum HALL effect in a natural way. The spin theories are described by the action

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z\, b(z,\,\bar{z})\bar{\partial}c(z,\,\bar{z}) + h.c. \tag{2.1}$$

Here, $b(z,\,\bar{z})$ and $c(z,\,\bar{z})$ are anti-commuting conformal fields of weight $j \in \mathbb{Z}/2$ and $1-j$, respectively, where $z,\,\bar{z}$ are coordinates in the complex plane.[1] Therefore, under conformal transformations $z \longrightarrow w(z)$ they behave as:

$$b(z) = b(w)\left(\frac{\mathrm{d}\,w}{\mathrm{d}z}\right)^{j} \quad , \quad c(z) = c(w)\left(\frac{\mathrm{d}\,w}{\mathrm{d}z}\right)^{1-j} \quad . \tag{2.2}$$

In mathematical terms, the fields $b(z)$ and $c(z)$ describe j- and $1-j$-differentials. Thus, they are directly related to the cohomology of the topological space they live on, i.e., the complex plane. For a general field theory, the expectation value of an arbitrary functional $F[\Phi]$ is defined by the path integral

$$\langle F[\Phi]\rangle = \int (\mathcal{D}\Phi)\exp\left(-S\left[\Phi\right]\right)F[\Phi] \quad . \tag{2.3}$$

[1] Since these theories are chiral conformal field theories, the holomorphic and the anti-holomorphic part can be treated independently. The dependence of the fields on $\bar{z}$ is suppressed in the following.

The operator valued equations of motion are obtained from the variation $\delta\langle F[\Phi]\rangle = 0$:

$$
\begin{aligned}
\bar{\partial} b(z) = \bar{\partial} c(z) &= 0 \\
(\bar{\partial} b(z)) b(z') = (\bar{\partial} c(z)) c(z') &= 0 \\
(\bar{\partial} c(z)) b(z') = (\bar{\partial} b(z)) c(z') &= 2\pi \delta^2(z - z',\ \bar{z} - \bar{z}') \quad .
\end{aligned} \tag{2.4}
$$

In classical terms the fields are expected to satisfy

$$
(\bar{\partial} c(z)) b(z') = (\bar{\partial} b(z)) c(z') = 0 \ . \tag{2.5}
$$

The normal-ordered product of the two fields is defined by the requirement (2.5). Since $\bar{\partial}(1/z) = 2\pi\delta^2(z,\ \bar{z})$, it is derived to

$$
{:}b(z)c(z'){:} = b(z)c(z') - \frac{1}{z - z'} \ . \tag{2.6}
$$

In two-dimensional conformal field theory a product of local chiral operators can be expanded in an operator valued LAURENT series with meromorphic functions as coefficients. In the evaluation of correlators these so-called operator product expansions play an important role. The operator product expansions of the two fields $b(z)$ and $c(z')$ can be read off directly from (2.6):

$$
b(z)c(z') \sim \frac{1}{z - z'} \quad , \quad c(z)b(z') \sim \frac{1}{z - z'} \ . \tag{2.7}
$$

Here, '$\sim$' denotes 'equivalent up to regular terms'. These regular terms vanish if evaluated in a correlator.

The energy-momentum tensor $T(z)$ of the theory can be derived by varying the action S with respect to the induced metric. This yields

$$
T(z) = (1 - j){:}(\partial b(z))c(z){:} - j{:}b(z)(\partial c(z)){:} \ . \tag{2.8}
$$

In principle, there are just a few facts necessary to know about a general conformal field theory: the central charge c and the set of conformal weights $\{h_i\}$ of its primary fields are two of them. They can be derived by operator product expansions involving the energy-momentum tensor using WICK's theorem:

$$
\begin{aligned}
T(z)b(w) &= {:}T(z)b(w){:} + (1 - j)c(z)b(w)\,\partial b(z) - j(\partial c(z))b(w)\,b(z) \\
&\sim \frac{1 - j}{z - w}\partial b(w) + \frac{j}{(z - w)^2}(\underbrace{b(w) + (z - w)\partial b(w)}_{b(z)}) \\
&\sim \frac{j}{(z - w)^2} b(w) + \frac{1}{z - w}\partial b(w) \ .
\end{aligned} \tag{2.9}
$$

This calculation can be done analogously for $c(w)$ and $T(w)$:

$$
T(z)c(w) \sim \frac{1 - j}{(z - w)^2} c(w) + \frac{1}{z - w}\partial c(w) \ , \tag{2.10}
$$

$$
T(z)T(w) \sim \frac{\frac{1}{2}(-12j^2 + 12j - 2)}{(z - w)^4} + \frac{2}{(z - w)^2} T(w) + \frac{1}{z - w}\partial T(w) \ . \tag{2.11}
$$

Equations (2.9) and (2.10) can be understood as the definition of a primary conformal field, the numerator of the first term of the operator product expansion yields its conformal weight h. Equation (2.11) contains a so-called anomalous term that is not proportional to the field itself or its derivatives. This term is due to the existence of a central extension of the algebra of conformal symmetries, i.e., the central charge c of the theory. In fact, in all conformal field theories the operator product expansion of $T(z)$ with itself reads

$$T(z)T(w) \sim \frac{\mathrm{c}/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \ . \tag{2.12}$$

Comparing (2.11) and (2.12), the central charge $\mathrm{c}_{b/c-\mathrm{spin}}$ can be read off directly:

$$\mathrm{c}_{b/c-\mathrm{spin}} = -2(6j^2 - 6j + 1) \ . \tag{2.13}$$

For $j \neq \frac{1}{2}$ it is negative (as $j \in \mathbb{Z}/2$). Therefore, the *b/c*-spin systems used in this work $(j \geq 1)$ are non-unitary. This may seem disturbing since the spin systems are proposed to describe the bulk regime of the fractional quantum HALL effect in a natural way. This issue is discussed in more detail in appendix A.[2]

In addition to the full set of conformal symmetries there exists another symmetry. Under the simultaneous transformation

$$b(z) \longrightarrow b(z)\exp(\mathrm{i}\alpha) \quad \text{and} \quad c(z) \longrightarrow c(z)\exp(-\mathrm{i}\alpha) \tag{2.14}$$

the action (2.1) remains unchanged. The corresponding conserved spin current $J(z)$ reads

$$J(z) = -{:}b(z)c(z){:} \tag{2.15}$$

and the conserved NOETHER charge is

$$Q_{(\mathrm{i}\alpha),J} = \frac{1}{2\pi\mathrm{i}} \oint_0 \mathrm{d}z \ (\mathrm{i}\alpha) J(z) \ . \tag{2.16}$$

Evaluating the operator product expansion of $j(w)$ with the energy-momentum tensor $T(z)$ yields

$$T(z)J(w) \sim \frac{1-2j}{(z-w)^3} + \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}\partial J(w) \ . \tag{2.17}$$

Therefore, $J(z)$ is not a primary conformal field for $j \neq \frac{1}{2}$. In fact, $j = \frac{1}{2}$ leads to the only unitary spin system $(\mathrm{c} = 1)$. It can be identified with two copies of the two-dimensional ISING model.

2.2 *b/c*-Spin Systems on Riemann Surfaces

One of the most striking results in the study of the fractional quantum HALL effect was the discovery of quasi-particles with fractional statistics $\pi/(2m+1)$ $(m \in \mathbb{N})$. A field theory describing this effect in a proper way has to be incorporated in a suitable geometrical setting.

[2] It is indicated how the space of states of the *b/c*-spin theories appropriately coincides with the HILBERT space of the (1+1)-dimensional theories describing the edge excitations.

This section briefly demonstrates the features of b/c-spin systems living on RIEMANN surfaces with global $\mathbb{Z}_n$-symmetry following the lines of KNIZHNIK [52]. $\mathbb{Z}_n$-symmetry means that every branch point of the manifold is of order n and that all monodromy matrices can be diagonalized simultaneously, e.g., a torus is a RIEMANN surface with a global $\mathbb{Z}_2$-symmetry and two branch cuts. Since this work focuses on the structure of correlators, it is sufficient to do the calculation locally for a single branch point at z_0. The results can be directly extended to m branch points. A $\mathbb{Z}_n$-symmetric RIEMANN surface M_n can be locally represented by a branched covering of the compactified complex plane $\left(\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\right)$ with the following map:

$$z:\ M_n \longrightarrow \widehat{\mathbb{C}} \quad , \quad z(y) = z_0 + y^n \ . \tag{2.18}$$

The RIEMANN surface is locally identified by n sheets of $\widehat{\mathbb{C}}$ via the inverse map of (2.18). The b/c-spin fields living on the manifold are therefore represented by an n-dimensional vector of identical copies of the b/c-fields $b^{(l)}(z)$ and $c^{(l)}(z)$ on the complex plane with boundary conditions

$$\begin{aligned} \hat{\Pi}_{z_0} b^{(l)}(z) &= b^{(l+1)}(z) \ , \quad l = 0, \ldots, n-1 \ , \quad b^{(n)}(z) = b^{(0)}(z) \ , \\ \hat{\Pi}_{z_0} c^{(l)}(z) &= c^{(l+1)}(z) \ , \quad l = 0, \ldots, n-1 \ , \quad c^{(n)}(z) = c^{(0)}(z) \ , \end{aligned} \tag{2.19}$$

where $\hat{\Pi}_{z_0}$ is the map of analytic continuation

$$\hat{\Pi}_{z_0} : (z - z_0) \longrightarrow (z - z_0) \exp(2\pi \mathrm{i}) \ . \tag{2.20}$$

For further investigation it is suitable to introduce the FOURIER basis:

$$\begin{aligned} b_k(z) &= \sum_{l=0}^{n-1} \exp\left(\frac{-2\pi \mathrm{i}(k + j(1-n))l}{n}\right) b^{(l)}(z) \ , \\ c_k(z) &= \sum_{l=0}^{n-1} \exp\left(\frac{+2\pi \mathrm{i}(k + j(1-n))l}{n}\right) c^{(l)}(z) \ , \end{aligned} \tag{2.21}$$

with $k \in \{0, \ldots, n-1\}$. This basis diagonalizes $\hat{\Pi}_{z_0}$:

$$\begin{aligned} \hat{\Pi}_{z_0} b_k(z) &= \exp\left(\frac{+2\pi \mathrm{i}(k + j(1-n))}{n}\right) b_k(z) \ , \\ \hat{\Pi}_{z_0} c_k(z) &= \exp\left(\frac{-2\pi \mathrm{i}(k + j(1-n))}{n}\right) c_k(z) \ . \end{aligned} \tag{2.22}$$

The conserved spin currents J_k become single-valued in the vicinity of the branch point. In geometrical terms this implies:

$$J_k(z) \sim \frac{\alpha_k}{z - z_0} \ , \text{ where } \quad \alpha_k = \frac{1}{2\pi \mathrm{i}} \oint_{z_0} \mathrm{d}z\ J_k(z) \ . \tag{2.23}$$

To verify (2.23), the operator product expansion of $b_k(z)$ with $c_l(w)$ has to be considered in the vicinity of the branch point located at z_0. The transformation law (2.2) and the operator product expansion of the spin fields on the complex plane (2.7) yield

$$b^{(r)}(z) c^{(s)}(w) = \frac{1}{n(z-w)} \sum_{p=0}^{n-1} \left(\frac{y^{(r)}(z)}{y^{(s)}(w)}\right)^{p + j(1-n)} \ . \tag{2.24}$$

The operator product expansion is obtained by inserting (2.24) in (2.21):

$$b_k(z)c_l(w) = \delta_{k,l}\Big(\frac{1}{z-w} + \underbrace{\frac{k+j(1-n)}{n(z-z_0)}}_{-J_k(z)}\Big) + \mathcal{O}(z-w) \ . \tag{2.25}$$

Therefore, the corresponding charge vector α_k identified with the branch point z_0 is

$$\alpha_k = -\frac{k+j(1-n)}{n} \ , \quad k \in \{0, \dots, n-1\} \ . \tag{2.26}$$

Aiming towards the analysis of the conformal structure of the geometrical features represented by the spin fields, the theory is bosonized. This means to express $b_k(z)$ and $c_k(z)$ in terms of exponentials of analytic scalar bosonic fields $\Phi_k(z)$

$$\begin{aligned} b_k(z) &= :\exp\left(+\mathrm{i}\Phi_k(z)\right): \ , \\ c_k(z) &= :\exp\left(-\mathrm{i}\Phi_k(z)\right): \ . \end{aligned} \tag{2.27}$$

The fields $\Phi_k(z)$ have conformal weights $h_k = 0$ and their operator product expansions read:

$$\Phi_k(z)\Phi_m(w) \sim -\delta_{k,m}\ln(z-w) \ . \tag{2.28}$$

The spin currents J_k are deduced from (2.27) using (2.28):

$$J_k(z) = \mathrm{i}\partial\Phi_k(z) \ . \tag{2.29}$$

Finally, the bosonized energy-momentum tensor is given by

$$\begin{aligned} T(z) &= \sum_{k=0}^{n-1} T_k(z) \ , \\ T_k(z) &= -\frac{1}{2}:\partial\Phi_k(z)\partial\Phi(z): + \mathrm{i}\beta_0\partial^2\Phi_k(z) \ , \end{aligned} \tag{2.30}$$

where β_0 is a background charge placed at infinity. This charge has to be introduced to keep the conformal structure of the spin system which by itself cannot be identified with a purely free bosonic theory. The operator product expansions involving the energy-momentum tensor $T_k(z)$ read:

$$T_k(z)T_k(w) \sim \frac{\frac{1}{2}(1-12\beta_0^2)}{(z-w)^4} + \frac{2}{(z-w)^2}T_k(w) + \frac{1}{z-w}\partial T_k(w) \ , \tag{2.31}$$

$$T_k(z)\partial\Phi_k(w) \sim \frac{2\mathrm{i}\beta_0}{(z-w)^3} + \frac{1}{(z-w)^2}\partial\Phi_k(w) + \frac{1}{z-w}\partial^2\Phi_k(w) \ . \tag{2.32}$$

Comparing (2.31) and (2.13), the background charge for a correctly bosonized *b/c*-spin system is derived to

$$\beta_0 = \frac{1}{2} - j \ . \tag{2.33}$$

It follows from (2.32) that $\partial\Phi_k(z)$ is not a primary conformal field unless the background charge vanishes. Even $\Phi_k(z)$ itself is not primary, as it is expected from (2.28) due to the logarithmic term. The remaining candidates for a primary field can thus be identified by a generalization of (2.27). They are called vertex operators and are defined by

$$V_{k,\ell}(z) = :\exp\left(\mathrm{i}\ell\Phi_k(z)\right): \ . \tag{2.34}$$

Indeed, they satisfy

$$\begin{aligned} T_k(z)V_{k,\ell}(w) &\sim \sum_{m=0}^{\infty}\frac{(\mathrm{i}\ell)^m}{m!}\left(-\frac{1}{2}:\partial\Phi_k(z)\partial\Phi_k(z): + \mathrm{i}\beta_0\partial^2\Phi_k(z)\right) :\Phi_k(w)^m: \\ &\sim -\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\mathrm{i}\ell)^m}{(m-2)!}\frac{:\Phi_k(w)^{m-2}:}{(z-w)^2} + \sum_{m=0}^{\infty}\frac{(\mathrm{i}\ell)^m}{(m-1)!}\frac{:\partial\Phi_k(w)\Phi_k(w)^{m-1}:}{(z-w)} \\ &\quad + \mathrm{i}\beta_0\sum_{m=0}^{\infty}\frac{(\mathrm{i}\ell)^m}{(m-1)!}\frac{:\Phi_k(w)^{m-1}:}{(z-w)^2} \\ &\sim \frac{\ell^2/2-\beta_0\ell}{(z-w)^2}V_{k,\ell}(w) + \frac{1}{z-w}\partial V_{k,\ell}(w) \ . \end{aligned} \tag{2.35}$$

Thus, $V_{k,\ell}(z)$ is primary with conformal weight $h_k = \ell^2/2 + (j-1/2)\ell$. In correspondence to (2.26) the branch point of the $\mathbb{Z}_n$-symmetric RIEMANN surface is represented by the vertex operator

$$V_{\vec{\alpha}}(z_0) = :\exp\left(-\mathrm{i}\sum_{k=0}^{n-1}\alpha_k\Phi_k\right): \ , \tag{2.36}$$

with weight

$$h_{\vec{\alpha}} = \sum_{k=0}^{n-1}h_{\alpha_k} = \sum_{k=0}^{n-1}\left(\frac{1}{2}\alpha_k^2 + (j-\frac{1}{2})\alpha_k\right) \ . \tag{2.37}$$

The central charge c_{RS} of the *b*/*c*-spin system living on the RIEMANN surface reads:

$$c_{\mathrm{RS}} = \sum_{k=0}^{n-1}c_{b/c-\mathrm{spin}} = -2n(6j^2-6j+1) \ . \tag{2.38}$$

From (2.26) it is apparent that the charge vector of the vertex operator is dominated by the $\mathbb{Z}_n$-symmetry of the RIEMANN surface: the spin j provides an offset which is just visible in the conformal weights of the fields since the phase is determined by $\alpha_k \bmod 1$. In addition, two different types of fields have to be distinguished. First, there exist twist fields containing the full information of the branch point. Therefore, the charge vector $\vec{\alpha}$ has to keep track of analytic continuation. For example, on a $\mathbb{Z}_3$-symmetric RIEMANN surface and for $j = 3/2$, the charge vector reads

$$\vec{\alpha}_{n=3,j=3/2} = (\,1\,,\ 2/3\,,\ 1/3\,) \ . \tag{2.39}$$

Secondly, there are projective fields. Their non-zero charge components are identical as if the branched structure was projected to an n-fold copy of $\widehat{\mathbb{C}}$. This yields charge vectors

$$\vec{\alpha}_1^{\mathrm{p}} = \ldots = \vec{\alpha}_m^{\mathrm{p}} \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots , 1\right\} \;, m \leq n \quad \text{and} \quad \vec{\alpha}_k^{\mathrm{p}} = 0 \;, k > m \;. \tag{2.40}$$

The role of $\vec{\alpha}$ is crucial. Besides local chiral fields having integer valued charge vector entries only, fractional ones (2.40) are included. The effect of the corresponding vertex operators is the action of a branch point of ramification number n. This is expected precisely from the fractional statistics of the quasi-particles. Thus, these statistics are incorporated into a geometrical setting, where the complex plane is replaced by an n-fold ramified covering of itself, created by flux quanta piercing it.

Naturally, the projective fields are assumed to describe fractional quantum HALL states in the lowest LANDAU level (LLL) projection. Since the bosons $\Phi_k(z)$ are free fields, the correlators of the corresponding vertex operators read

$$\langle\, \Omega \,| V_{\vec{\alpha}_1}(z_1) \cdot \ldots \cdot V_{\vec{\alpha}_n}(z_n) |\, 0\,\rangle = \prod_{i<j}^{n} (z_i - z_j)^{\vec{\alpha}_i \cdot \vec{\alpha}_j} \;, \tag{2.41}$$

where $\langle\, \Omega\,|$ is an out-state related to the background charge β_0 at infinity. Since charge neutrality in a correlator must be guaranteed, conformally neutral screening operators Ω_-, Ω_+ have to be introduced. These form the state $|\,\Omega\,\rangle$ by acting on the highest weight vacuum $|\,0\,\rangle$:

$$|\,\Omega\,\rangle = \Omega_+^m \Omega_-^n \;|\,0\,\rangle \;\;, m, n \in \mathbb{N} \;. \tag{2.42}$$

The set of equations (2.40) and (2.41) including their geometrical features is the main result of this chapter.

CHAPTER 3

Conformal Field Theory Approach

The fractional quantum HALL effect reveals a large variety of physical features. Strongly correlated movement of electrons, fractional charges and statistics of quasi-particles corresponding to complex geometrical aspects, and topological order are some of them. Gauge covariance arguments, hierarchical schemes, e.g., JAIN's concept of composite fermions, and effective topological theories, e.g., the CHERN-SIMONS K-matrix formalism, have been developed to adapt these features and explain them by fundamental physical principles. This chapter deals with a novel approach based on the conjecture of the correspondence between fractional quantum HALL state wave functions and correlators of conformal field theories. Motivated by geometrical aspects bulk wave functions are proposed and a new hierarchical scheme following JAIN's composite fermion model is presented. The first section starts from the LAUGHLIN states and proceeds to their natural continuation, i.e., JAIN's main series. In the second section fractional quantum HALL states built from paired composite fermions are presented which complete the scheme and order the set of experimentally confirmed filling fractions in the regime $0 \leq \nu \leq 1$ by stability in a suitable way. Up to date experimental data can be found in [2, 3, 4, 5].

3.1 Jain's Main Series

During the last two decades a multi-faceted set of lowest LANDAU level trial wave functions has been proposed which has been investigated numerically and by using methods of mean field approximation. In this connection, it often remained unconsidered to what extend the analytic structure of a certain set of wave functions fits in a unifying and natural picture. Field theoretical approaches usually follow a purely constructive principle without presenting reasonable restrictions. Reconsidering the results of section 1.3, the matrix structure (1.31) allows to construct fractional quantum HALL states with arbitrary filling fractions. Certainly, several restraints can be implemented, but often lack physical argumentation. On the other hand, even if JAIN's effective composite fermion model shows excellent correspondence with experimental data, his wave functions adopt no definite geometrical structure in their lowest LANDAU level projected form leaving the K-matrix trial wave functions (1.33) as the more reasonable candidates. The challenge is to find a scheme which naturally suits the geometrical features of the fractional quantum HALL effect and, furthermore, predicts series of states in agreement with the experimentally observed order of stability and the topological considerations from the viewpoint of effective CHERN-SIMONS theory.

It is natural to start from the LAUGHLIN states. Despite the simple appearance of their ground states (1.15), the quasi-particle excited wave functions (1.19) reveal the most important evidence for the non-trivial topological structure of fractional quantum HALL states, i.e., fractional statistics. The derivation of the BERRY connection from the complex normalization factor [43] results in a multivalued wave function. The non-holomorphic factors $(\zeta_i - \zeta_j)^{1/(2p+1)}$ lead to severe consequences for any field theoretical description. Either, they demand the loss of chirality or the fields naturally possess the topological abilities of the (quasi-)particles, i.e., the structure of a branch point with ramification number $n = 2p + 1$. This is exactly what the projective vertex operators of the b/c-spin systems presented in chapter 2 simulate. They represent particles with fractional statistics by definition. Furthermore, the b/c-spin systems are a discrete set of conformal field theories. Thus, compared with other conformal field theory approaches involving continuous parameters, e.g., GAUSSIAN $c = 1$ models, avoid arbitrariness from the beginning.

Due to the unsophisticated correspondence between statistics $\pi/(2p+1)$ and the geometrical setting of the b/c-spin fields, the approach reveals an amazing simplicity. Given a LAUGHLIN filling fraction $\nu = 1/(2p+1)$, the electron e^- and the flux quantum Φ, respectively, are identified with a $\mathbb{Z}_{2p+1}$-symmetric projective field. The charge vectors (2.40) are related to the statistics, thus $(2p+1)$-dimensional and take the form

$$\vec{\alpha}_{\mathrm{e}^-} = \Big(1, \ldots, 1 \Big) \quad , \quad \vec{\alpha}_\Phi = \Big(\frac{1}{2p+1}, \ldots, \frac{1}{2p+1} \Big) \ . \tag{3.1}$$

The correlators (2.41) yield the correct wave functions (1.15) and (1.19) up to the exponential factor:

$$\begin{aligned} \Psi_{\text{Laughlin}} &= \big\langle\, \Omega \,\big| V_{\vec{\alpha}_{\mathrm{e}^-}}(z_1) \cdot \ldots \cdot V_{\vec{\alpha}_{\mathrm{e}^-}}(z_n) \big|\, 0 \,\big\rangle = \prod_{i<l}^{n} (z_i - z_l)^{2p+1} \ , \\ \Psi_{\text{exc.}} &= \big\langle\, \Omega \,\big| V_{\vec{\alpha}_{\mathrm{e}^-}}(z_1) \cdot \ldots \cdot V_{\vec{\alpha}_{\mathrm{e}^-}}(z_n) V_{\vec{\alpha}_\Phi}(\zeta_1) \cdot \ldots \cdot V_{\vec{\alpha}_\Phi}(\zeta_k) \big|\, 0 \,\big\rangle \\ &= \prod_{r,s}^{n,k} (z_r - \zeta_s) \prod_{i<l}^{n} (z_i - z_l)^{2p+1} \prod_{p<q}^{k} (\zeta_p - \zeta_q)^{\frac{1}{2p+1}} \ . \end{aligned} \tag{3.2}$$

A comment is necessary here: in the scope of this work, the conformal field theory always lives on a ramified covering of the compactified complex plane, i.e., the RIEMANN sphere. On the other hand, the fractional quantum HALL system lives on a certain chunk of the plane, the sample. Thus, in a correct treatment, the wave functions of the fractional quantum HALL system must be elements of a suitable test space. Reconsidering the derivation yielding to (1.13), this is the BARGMANN space [42]. The elements of the BARGMANN space for N complex variables are of the form

$$\psi(\{z\}) = p(z_1, \ldots, z_N) \prod_{i=1}^{N} \exp(-c_i |z_i|^2) \ .$$

There are further restrictions on the constants c_i and on the multivariate polynomial $p(\{z\})$ whenever the function $\psi(\{z\})$ is symmetric or anti-symmetric under certain permutations of its arguments. The only effect of the exponential factor is to guarantee a sufficient fast decay of the modulus squared of the wave function if one or more of its arguments become large. It can be shown rigorously that this factor is absent if the fractional quantum HALL problem is considered in a different setting, i.e., on a sphere pierced by the field of a magnetic monopole positioned in its centre. This idea was first pointed out by HALDANE [15]. Since the sphere is a compact space, so is the support of the wave function. When computing bulk wave functions in terms of conformal field theory correlators, this is executed automatically on the latter setting, i.e., the compact sphere. Thus, it is natural to expect that the conformal field theory picture reproduces the bulk wave functions on the sphere rather than on the plane. However, for completeness it is possible to reproduce the exponential factors within the conformal field theory picture by explicitly including a homogeneous background charge distribution confining the support of the wave function as shown by MOORE and READ [25, 53].

As indicated above, the $\mathbb{Z}_n$-symmetry of the RIEMANN surface the spin fields live on has a one-to-one correspondence with the statistics and charges of the (quasi-)particles in the LAUGHLIN states. Furthermore, the scalar products of the charge vectors determine the particles' interaction, i.e., order of zeros in the polynomial terms of the wave functions. Despite the fact that the electron with elementary charge e obeys simple fermionic statistics — indicated by the integer valued components of its charge vector (3.1) — the field's nature has a geometric background in terms of the topology of the RIEMANN surface it lives on. This becomes more apparent by investigating JAIN's main series.

The complete set of states describing these main sequences of HALL fractions (1.39) is included in the b/c-spin system approach, (quasi-)particles, their charges and statistics are described in terms of $\mathbb{Z}_{2mp+1}$-symmetric projective fields. As before, p labels the number of pairs of flux quanta attached to the electron and m is the number of filled composite fermion LANDAU levels. Each layer $\mu \in \{1, \ldots, m\}$ is connected with a $(2mp+1)$-dimensional charge vector:

$$\vec{\alpha}_i^{(\mu)} = \begin{cases} 1 & 1 \leq i \leq 2p \\ 1 & i = 2mp + 2 - \mu \\ 0 & \text{otherwise} \end{cases} \quad . \tag{3.3}$$

The scalar products read:

$$\vec{\alpha}^{(\mu)} \cdot \vec{\alpha}^{(\lambda)} = 2p + \delta_{\mu,\lambda} \ . \tag{3.4}$$

Naively, a $(2p+1)m$-dimensional charge vector for an m-layer state might have been expected. However, this would demand that the flux quanta were independent for each layer. Identifying these or, equivalently, the base spaces of the m copies of the ramified complex plane immediately leads to $(2p+1)m - (m-1) = 2mp+1$ dimensions, the correct dimensionality of the charge vectors. The correlators (2.41) of the vertex operators given by (3.3) yield the trial wave functions of the K-matrices (1.38) representing JAIN's series in the CHERN-SIMONS formalism:

$$\begin{aligned} \Psi_{p,m}(z_i^{(\mu)}) &= \langle\, \Omega \,|\prod_{\mu}^{m} V_{\vec{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdot \ldots \cdot V_{\vec{\alpha}^{(\mu)}}(z_N^{(\mu)})|\, 0\,\rangle \\ &= \prod_{i<j}^{N}\prod_{\mu}^{m}(z_i^{(\mu)} - z_j^{(\mu)})^{2p+1}\prod_{i,j}^{N}\prod_{\mu<\lambda}^{m}(z_i^{(\mu)} - z_j^{(\lambda)})^{2p} \ . \end{aligned} \tag{3.5}$$

Equation (3.5) generalizes the result of (3.2). In this way, JAIN's main series (1.39) with filling fractions $\nu_p = m/(2mp+1)$ are identified.

3.2 Composite Fermion Pairing

Concerning other filling fractions all known hierarchical schemes, e.g. [13, 15, 45, 54, 55], share the problem of not being capable to predict fractional quantum HALL states by order of stability. They produce several unobserved filling fractions at low levels within hierarchy whereas some popular states are obtained at much higher order or even as descendants of unconfirmed states. To avoid this deficiency, the principle of particle-hole duality is artificially introduced, relating, for example the series

$$\nu_1 = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{9}{19}, \frac{10}{21}, \ldots \tag{3.6}$$

and

$$\nu_1^{(1)} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \frac{7}{13}, \frac{8}{15}, \frac{9}{17}, \frac{10}{19}, \ldots \ . \tag{3.7}$$

Since the complete set of experimentally confirmed states does not support this principle very well and no obvious physical motivation for its existence is provided, the scheme of this work avoids particle-hole duality.

Yet, the series (3.7) with $\nu_p^{(1)} = m/(2mp-1)$, are observed and seem to be closely related to JAIN's main series ν_p. From the topological viewpoint of CHERN-SIMONS theory, $\nu_p^{(1)}$ can be represented in terms of m-layer K-matrices

$$K_{ij} = \begin{cases} 2p-1 & i=j \\ 2p & i \neq j \end{cases} \ . \tag{3.8}$$

Reconsidering (3.4) then demands the existence of charge vectors $\vec{\alpha}$ and $\vec{\beta}$ corresponding to different layers with

$$\vec{\alpha}^2 = \vec{\beta}^2 = 2p-1 \quad \text{and} \quad \vec{\alpha}\cdot\vec{\beta} = 2p \ \lightning \ . \tag{3.9}$$

This is not possible since it contradicts SCHWARZ' inequality and indicates that these 'dual' series possess completely new physical features. The analytic structure of the wave function (1.33) for K-matrices (3.8) exhibits that composite fermions living in the same layer repulse each other with the power of $(2p-1)$ while those of different layers repulse themselves by $2p$. This suggests the existence of an effectively attractive composite fermion interaction within a LANDAU level, i.e. pairing. In a conformal field theory approach this is induced by the $c=-2$ logarithmic b/c-spin system with spin $j=1$ as shown for the HALDANE-REZAYI state with filling fraction $\nu = 5/2$ [21, 25, 31, 32].

In analogy to (2.27) the fields $b(z)$ and $c(z)$ can be bosonized on a ramified covering of the compactified complex plane locally representing the $\mathbb{Z}_n$-symmetric RIEMANN surface in terms of vertex operators:

$$\begin{aligned} b_{\vec{\gamma}}(z) &= :\exp\big(+\mathrm{i}\vec{\gamma}\vec{\Phi}(z)\big): \\ c_{\vec{\gamma}}(z) &= :\exp\big(-\mathrm{i}\vec{\gamma}\vec{\Phi}(z)\big): \end{aligned} \qquad \gamma_k \in \{0\,,1\} \ . \tag{3.10}$$

The pairing effect of the composite fermions is described by $b(z)\partial c(z')$. The operator product expansion

$$b_{\vec{\gamma}}(z)\partial c_{\vec{\gamma}}(z') \sim \frac{\vec{\gamma}^{\,2}}{(z-z')^2} \tag{3.11}$$

yields the so-called PFAFFIAN form $\mathrm{Pf}(z_i,\, z'_i)$ when the fields (3.11) are evaluated in a correlator:

$$\langle\, \Omega \,|\, \big(b_{\vec{\gamma}}(z_1)\partial_{z'_1} c_{\vec{\gamma}}(z'_1)\big)\cdot \ldots \cdot \big(b_{\vec{\gamma}}(z_N)\partial_{z'_N} c_{\vec{\gamma}}(z'_N)\big)\,|\, 0 \,\rangle = \vec{\gamma}^{\,2}\mathrm{Pf}(z_i,\, z'_i) \ ,$$

$$\mathrm{Pf}(z_i,\, z'_i) \equiv \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \frac{1}{(z_i - z'_{\sigma(i)})^2} \ . \tag{3.12}$$

In this way, the $\nu_p^{(1)}$ series can be identified by the same fields as the basic JAIN series (3.5) if additional inner-LANDAU level pairings are included. To find a physical and stable system all composite fermion LANDAU levels are expected to be paired. To obtain a proper description, each layer $\mu \in \{1, \ldots, m\}$ possesses an m-dimensional charge vector:

$$\vec{\gamma}_i^{(\mu)} = \delta_{\mu,\, i} \ \ \Rightarrow \ \ \vec{\gamma}^{(\mu)} \cdot \vec{\gamma}^{(\lambda)} = \delta_{\mu,\, \lambda} \ . \tag{3.13}$$

The composite fermions themselves correspond to the charge vectors (3.3). Thus, the wave functions read:

$$\begin{aligned}
\Psi^{(1)}_{p,m}(z_i^{(\mu)}) &= \langle\, \Omega \,|\, \prod_{\mu}^{m} V_{\vec{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdot \ldots \cdot V_{\vec{\alpha}^{(\mu)}}(z_{2N}^{(\mu)}) |\, 0 \,\rangle \\
&\quad \times \langle\, \Omega \,|\, \prod_{\mu}^{m} \Big(b_{\vec{\gamma}^{(\mu)}}(z_1^{(\mu)})\partial_{z_{N+1}} c_{\vec{\gamma}^{(\mu)}}(z_{N+1}^{(\mu)})\Big) \cdot \ldots \cdot \Big(b_{\vec{\gamma}^{(\mu)}}(z_N^{(\mu)})\partial_{z_{2N}} c_{\vec{\gamma}^{(\mu)}}(z_{2N}^{(\mu)})\Big) \,|\, 0 \,\rangle \\
&= \prod_{\mu}^{m} \mathrm{Pf}(z_i^{(\mu)},\, z_{N+i}^{(\mu)}) \underbrace{\prod_{i<j}^{2N} \prod_{\mu}^{m} (z_i^{(\mu)} - z_j^{(\mu)})^{2p+1}}_{(\star)} \prod_{i,j}^{2N} \prod_{\mu<\lambda}^{m} (z_i^{(\mu)} - z_j^{(\lambda)})^{2p} \ .
\end{aligned} \tag{3.14}$$

It is important to stress that equation (3.14) satisfies the CHERN-SIMONS approach and has to be identified with the K-matrix (3.8). Only the trial wave functions (1.33) have to be extended, since they are not capable to realize pairing effects in a proper way. However, the PFAFFIAN cancels two powers of the paired composite fermion contribution to $(\star)$. Thus, paired composite fermions repulse each other by $\big(z_i^{(\mu)} - z_j^{(\mu)}\big)^{2p-1}$ in either wave function. Additionally, both yield the same filling fractions

$$\nu^{(1)}_{p,m} = \frac{m}{2mp-1} \ . \tag{3.15}$$

The first order paired series related to JAINs main series are identified:[1]

$$
\begin{aligned}
\nu_1^{(1)} &= \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \frac{7}{13}, \frac{8}{15}, \frac{9}{17}, \frac{10}{19}, \ldots \\
\nu_2^{(1)} &= \frac{1}{3}, \frac{2}{7}, \frac{3}{11}, \frac{4}{15}, \frac{5}{19}, \frac{6}{23}, \ldots \\
\nu_3^{(1)} &= \frac{1}{5}, \frac{2}{11}, \frac{3}{17}, \ldots \\
\nu_4^{(1)} &= \frac{1}{7}, \frac{2}{15}, \ldots \\
&\vdots
\end{aligned}
\tag{3.16}
$$

This proposal can be extended in a natural way imagining that the structure of paired composite fermion singlets is not restricted to be an inner-LANDAU level effect. Two LANDAU levels of composite fermions that are completely paired among each other can form a new incompressible quantum liquid and can hence interact with other blocks or single layers of paired droplets. Therefore, two natural series of K-matrices $(K^{\mathrm{e,o}})_{ij}$ appear with an even and an odd number of layers, respectively:

$$
(K^{\mathrm{e,o}})_{ij} = \begin{cases} 2p-1 & i=j \\ 2p-2 & i \neq j, \quad 2(k-1)+1 \leq i,j \leq 2k \quad (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases} \quad . \tag{3.17}
$$

Here, b is the number of paired 2×2-blocks. The first series, given a $2b$-layer fractional quantum HALL state, read:

$$
(K^{\mathrm{e}})_{ij} = \begin{pmatrix} 2p-1 & 2p-2 & 2p & \cdots & 2p \\ 2p-2 & 2p-1 & 2p & \ddots & \vdots \\ 2p & 2p & \ddots & 2p & 2p \\ \vdots & \ddots & 2p & 2p-1 & 2p-2 \\ 2p & \cdots & 2p & 2p-2 & 2p-1 \end{pmatrix}, \quad \nu_p^{(2)\,\mathrm{e}} = \frac{2b}{4bp-3} \quad . \tag{3.18}
$$

The latter, given a $2b{+}1$-layer fractional quantum HALL state, has a remaining solely self-paired layer and corresponds to filling fractions

$$
\nu_p^{(2)\,\mathrm{o}} = \frac{2b+3}{2p(2b+3)-3} \quad . \tag{3.19}
$$

Together, they yield the second order paired series:[2]

$$
\begin{aligned}
\nu_1^{(2)\,\mathrm{e}} &= \frac{4}{5}, \left(\frac{6}{9}\right), \frac{8}{13}, \frac{10}{17}, \ldots \ , & \nu_1^{(2)\,\mathrm{o}} &= \frac{5}{7}, \frac{7}{11}, \left(\frac{9}{15}\right), \ldots \ , \\
\nu_2^{(2)\,\mathrm{e}} &= \frac{2}{5}, \frac{4}{13}, \ldots \ , & \nu_2^{(2)\,\mathrm{o}} &= \ldots \ .
\end{aligned}
\tag{3.20}
$$

[1] Solely experimentally confirmed states are indicated.

[2] Fractions in brackets are not coprime and also appear in other series. This indicates that these states can exist in different forms of quantum liquids.

The scheme can be generalized in a natural way to the case of $n \times n$ blocks of paired LANDAU levels in order to derive the n-th order series. There exist $n-1$ subseries determined by the number r of remaining solely self-paired LANDAU levels, e.g., $r = 0$ in the even case for second order and $r = 1$ in the odd case, respectively. Let b denote the number of fully paired blocks then the $m \times m$-matrices $K^{(n)}_{p,m}$, where $m = bn + r$ of the n-th order paired fractional quantum HALL states read:

$$\left(K^{(n)}_{p,m}\right)_{ij} = \begin{cases} 2p-1 & i = j \\ 2p-2 & i \neq j, \quad (k-1)n+1 \leq i,j \leq kn \quad (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases} \quad . \tag{3.21}$$

The corresponding filling fractions are derived to

$$\nu^{(n)\,r}_{p,m} = \frac{bn + r(2n-1)}{2p(bn + r(2n-1)) - (2n-1)} \quad . \tag{3.22}$$

By this, the third order states confirmed by experiment[3] are deduced (higher orders do not yield additional observed fractions):

$$\begin{aligned} &\nu^{(3)\,0}_1 = \left[\frac{6}{7}\right], \frac{9}{13}, \ldots \qquad &&\nu^{(3)\,1}_1 = \frac{8}{11}, \ldots \qquad &&\nu^{(3)\,2}_1 = \ldots \\ &\nu^{(3)\,0}_2 = \frac{3}{7}, \ldots &&\nu^{(3)\,1}_2 = \ldots &&\nu^{(3)\,2}_2 = \ldots \end{aligned} \tag{3.23}$$

Having a closer look on (3.21) the question arises to what extent the access to fractional quantum HALL pairing presented up to this point is too restrictive. More general K-matrices with band-like or even more complicated structures could be imagined yielding arbitrary ν. For example, $\nu = 4/11$, a state that was very recently confirmed by experiment [5], could be realized by

$$K_{ij} = \begin{pmatrix} 3 & 2 & 2 & 4 \\ 2 & 3 & 4 & 2 \\ 2 & 4 & 3 & 2 \\ 4 & 2 & 2 & 3 \end{pmatrix} \quad . \tag{3.24}$$

This K-matrix describes a ring of two second order blocks. Remarkably, the result of a detailed analysis of equation (3.22) shows that certain fractions do not appear, for example $7/9$, $10/13$, $5/13$, and $4/11$. In agreement, as far as experimental data is provided, there merely exist controversial data concerning the first three, indicating that if they exist they presumably have to be another kind of fractional quantum HALL fluid. The same holds for $\nu = 4/11$ that is assumed to belong to the class of non-ABELIAN states which are discussed in more detail in chapter 4. As exactly these fractions lie beyond the access of the scheme proposed in this work, the b/c-spin systems motivate a reasonable physical constraint for the CHERN-SIMONS formalism in order to classify fractional quantum HALL states. This can be directly deduced from the conformal field theory picture of the fields given by (3.10). If an off-block pairing structure was possible, there would exist a triplet of fields

$$b_{\vec{\gamma}_1}(z^{(1)}_i)\partial c_{\vec{\gamma}_1}(z^{(1)}_j) \ , \ b_{\vec{\gamma}_2}(z^{(2)}_i)\partial c_{\vec{\gamma}_2}(z^{(2)}_j) \ , \ b_{\vec{\gamma}_3}(z^{(3)}_i)\partial c_{\vec{\gamma}_3}(z^{(3)}_j) \ , \tag{3.25}$$

[3] The state $\nu = \frac{6}{7}$ has not been confirmed so far, since it falls in the domain of attraction of the $\nu = 1$ plateau. However, it is strongly expected.

with the charge vectors obeying the following set of equations:

$$\vec{\gamma_1}^{\,2} = \vec{\gamma_2}^{\,2} = \vec{\gamma_3}^{\,2} = 1 \quad , \quad \vec{\gamma_1} \cdot \vec{\gamma_2} = \vec{\gamma_1} \cdot \vec{\gamma_3} = 1 \quad \text{and} \quad \vec{\gamma_2} \cdot \vec{\gamma_3} = 0 \ . \tag{3.26}$$

Since their components are restricted to be either 0 or 1, this ends up in a contradiction:

$$\vec{\gamma_1} = \vec{\gamma_2} = \vec{\gamma_3} \quad \text{and} \quad \vec{\gamma_2} \neq \vec{\gamma_3} \quad \lightning \ . \tag{3.27}$$

As a consequence the most general K-matrix for a correct description of paired fractional quantum HALL states is restricted to have a block structure:

$$\left(K^{b,\,n_b}_{p,\,m}\right)_{ij} = \begin{cases} 2p-1 & i = j \\ 2p-2 & i \neq j, \quad 1+\sum\limits_{l=1}^{k-1} n_l \leq i,j \leq \sum\limits_{l=1}^{k} n_l \quad (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases} \ . \tag{3.28}$$

Herein, b denotes the number of blocks and n_b the corresponding sizes. Therefore, $m = \sum_{l=1}^{b} n_b$, if singly paired layers are denoted by $n_b = 1$. It is to stress that the new series of filling fractions $\nu_p^{b,\,n_b}$ obtained from (3.28) are rather unlikely to be seen in experiments as their K-matrices are by far less symmetric than the ones given in (3.21). Since it is quite difficult to derive a general formula for $\nu_p^{b,\,n_b}$, the only additional fraction that may be seen in the nearer future is provided:

$$\nu_1^{2,\,(3,2)} = \frac{19}{23} \ . \tag{3.29}$$

Therefore, the set of matrices (3.21) remains as the natural candidate to describe series of paired fractional quantum HALL states by order of stability. The corresponding bulk wave functions $\Psi^{(n)}_{p,m}$ of the n-th order paired fractional quantum HALL states can be calculated as a direct generalization of (3.14). Given the matrix $K^{(n)}_{p,m}$, an m-dimensional charge vector with respect to a paired block $B \in \{1, \ldots, b+r\}$ (either $n \times n$ or a remaining 1×1 layer) is identified with each layer μ :

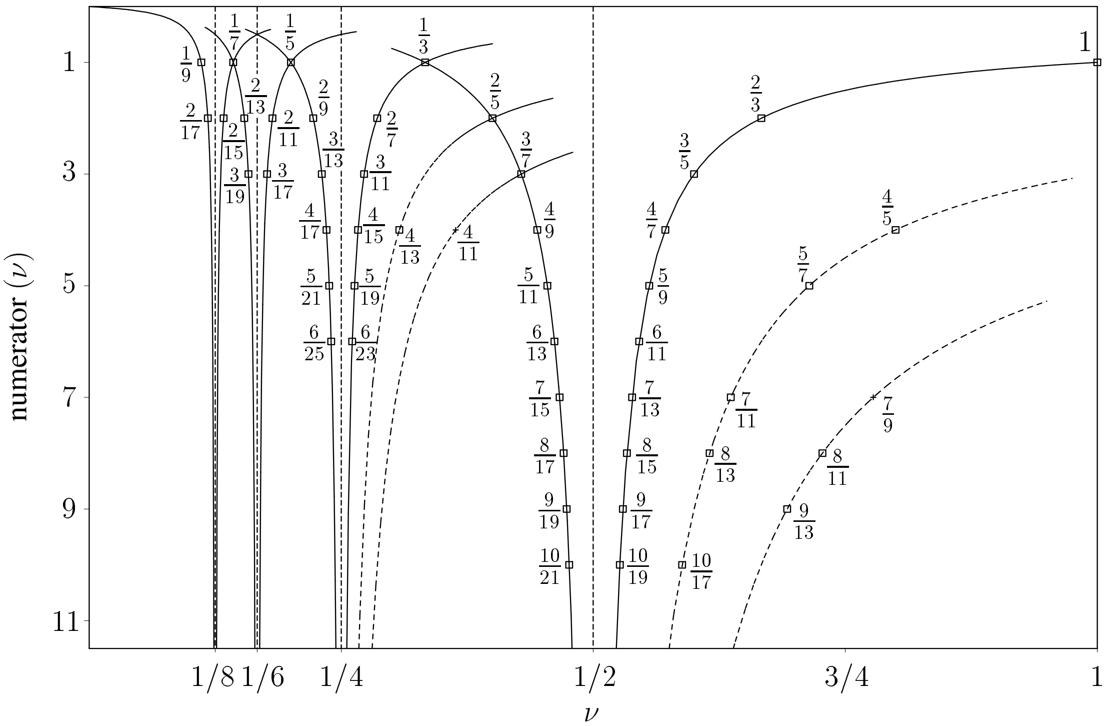
$$\vec{\gamma}^{(\mu)}_i = \delta_{B(\mu),\,i} \quad \Rightarrow \quad \vec{\gamma}^{(\mu)} \cdot \vec{\gamma}^{(\lambda)} = \delta_{B(\mu),\,B(\lambda)} \ . \tag{3.30}$$

In addition, each layer k possesses a $(2mp+1)$-dimensional charge vector for the composite fermions:

$$\vec{\alpha}^{(\mu)}_i = \begin{cases} 1 & 1 \leq i \leq 2p \\ 1 & i = 2mp+2-\mu \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad \vec{\alpha}^{(\mu)} \cdot \vec{\alpha}^{(\lambda)} = 2p + \delta_{\mu,\,\lambda} \ . \tag{3.31}$$

Let I denote the set of paired LANDAU levels, e.g., $I = \{(1,1),\,(2,2),\,(3,3),\,(1,2),\,(2,1)\}$ describes a triple-layer state with $\nu^{(2)\,1}_{p,\,3} = 5/(10p-3)$ where the first two LANDAU levels form

FIGURE 3.1: Observed HALL fractions in the interval $0 \le \nu \le 1$. Established fractions are labelled by '□'. The symbol '+' denotes cases that exceed the scheme of this work. The basic JAIN series ν_p approximate $1/2p$ from below, the corresponding first order paired series $\nu_p^{(1)}$ from above (both marked by continuous lines) as well as the higher order series $\nu_p^{(n)}$ (marked by dashed lines)



a 2×2-block while the third is solely self-paired. The wave functions read

$$
\begin{aligned}
\Psi^{(n)}_{p,m}(z_i^{(\mu)}) &= \langle\, \Omega \,|\prod_{\mu}^{m} V_{\vec{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdot \ldots \cdot V_{\vec{\alpha}^{(\mu)}}(z_{2N}^{(\mu)})|\, 0\,\rangle \\
&\quad \times \langle\, \Omega \,|\prod_{(\mu,\,\lambda)\in I} \big(b_{\vec{\gamma}^{(\mu)}}(z_1^{(\mu)})\partial_{z_{N+1}} c_{\vec{\gamma}^{(\lambda)}}(z_{N+1}^{(\lambda)})\big) \cdot \ldots \cdot \big(b_{\vec{\gamma}^{(\mu)}}(z_N^{(\mu)})\partial_{z_{2N}} c_{\vec{\gamma}^{(\lambda)}}(z_{2N}^{(\lambda)})\big)|\, 0\,\rangle \\
&= \prod_{(\mu,\,\lambda)\in I} \mathrm{Pf}(z_i^{(\mu)},\, z_{N+i}^{(\lambda)}) \underbrace{\prod_{i<j}^{2N}\prod_{\mu}^{m}(z_i^{(\mu)} - z_j^{(\mu)})^{2p+1} \prod_{i,j}^{2N}\prod_{\mu<\lambda}^{m}(z_i^{(\mu)} - z_j^{(\lambda)})^{2p}}_{\Psi_{p,\,m}(z_i^{(\mu)})}\ ,
\end{aligned}
\tag{3.32}
$$

where $\Psi_{p,m}(z_i^{(\mu)})$ is the bulk wave function of the basic JAIN series (3.5).
Combining equations (1.39), (3.16), (3.20), and (3.23), the complete set[4] of experimentally confirmed filling fractions is obtained by order of stability. A natural cutoff is found if either the number of participating composite fermion LANDAU levels m increases or if $\nu \to 0$. Series of

[4]Except for $\nu = 4/11$, which is presumably a non-ABELIAN fractional quantum HALL state falling outside the approach of this work, and controversial fractions like $\nu = 7/9$, $\nu = 10/13$, and $\nu = 5/13$.

TABLE 3.1: Expected Hall fractions

p	ν_p	$\nu_p^{(1)}$	$\nu_p^{(2)}$	$\nu_p^{(3)}$	$\nu_p^{(4)}$
1	$\frac{11}{23}$	$\frac{11}{21}$	$\frac{11}{19}, \frac{12}{21}$	$\frac{11}{17}, \frac{12}{19}, \frac{13}{21}$	$\frac{8}{9}, \frac{11}{15}, \frac{18}{29}$
2	$\frac{7}{29}$	$\frac{7}{27}$	$\frac{5}{17}, \frac{6}{21}$	$\frac{6}{19}, \frac{8}{27}$	$\frac{8}{25}$
3	$\frac{4}{25}$	$\frac{4}{23}$	$\frac{4}{21}, \frac{5}{27}$		
4	$\frac{3}{25}$	$\frac{3}{23}$	$\frac{4}{29}$		

more complicated composite fermions (larger p) are less developed, complete pairings ($r = 0$) are favored and each series precisely keeps track of the stability of the fractional quantum HALL states found in experiments whereas no unobserved fraction is predicted.

A comment has to be made on the absence of the $\nu = 7/9$ state. If the series

$$\nu = \frac{k}{2k-5} = \frac{6}{7}, \frac{7}{9}, \frac{8}{11}, \frac{9}{13}, \dots$$

was naively assumed, $\nu = 7/9$ would have to be considered to be more likely to appear than $\nu = 8/11$. Furthermore, it cannot be argued that $7/9$ is dominated by the $\nu = 1$ plateau since $\nu = 4/5$ exists. This seems rather unusual or even exceptional, but it precisely coincides with the b/c-spin system approach. Therefore, the series in figure 3.1 simply indicate where new fractions given by (3.22) will show up. Following the hierarchical scheme of this work, the subsequent filling fractions are predicted to appear if experimental circumstances are improved in the future (merely fractions with denominator $d \leq 29$ are indicated).

3.2.1 Quasi-Particle Excitations

One of the most striking results in the study of the fractional quantum HALL effect was the discovery of quasi-particles with fractional charges and statistics [12]. Experimentally it has been proven very difficult to measure them (even for the LAUGHLIN states) and a lot of effort is spent to investigate them in more detail. The two sets of wave functions (3.5) and (3.32) describe the electron ground state for a given filling fraction ν. As already shown for the LAUGHLIN series the geometric features of excitations responsible for statistics and charges are directly embedded in the b/c-spin systems and are related to the $\mathbb{Z}_n$-symmetry of the RIEMANN surface the fields live on, i.e., the dimension of the composite fermion charge vectors (3.31). However, an elementary quasi-particle excitation of an m-layer state has to be considered more carefully. First of all, a quasi-particle is expected to have trivial statistics with respect to the composite fermions:

$$\vec{\alpha}_{\Phi} \cdot \vec{\alpha}_{\mathrm{CF}}^{(\mu)} = 1 \ , \tag{3.33}$$

TABLE 3.2: Quasi-particle statistics for confirmed fractional quantum HALL states

Θ	ν	Θ	ν
$\frac{\pi}{3}$	$\frac{1}{3}$	$\frac{\pi}{15}$	$\frac{7}{15}, \frac{7}{13}, \left(\frac{9}{15}\right)$
$\frac{\pi}{5}$	$\frac{1}{5}, \frac{2}{5}, \frac{2}{3}$	$\frac{\pi}{17}$	$\frac{2}{17}, \frac{2}{15}, \frac{4}{17}, \frac{4}{15}, \frac{4}{13}, \frac{4}{9}, \frac{8}{17}, \frac{8}{15}, \frac{8}{13}$
$\frac{\pi}{7}$	$\frac{1}{7}, \frac{3}{7}, \frac{3}{5}, \frac{5}{7}$	$\frac{\pi}{19}$	$\frac{3}{19}, \frac{3}{17}, \frac{3}{13}, \frac{9}{19}, \frac{9}{17}, \frac{9}{13}$
$\frac{\pi}{9}$	$\frac{1}{9}, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{4}{9}, \frac{4}{7}, \frac{4}{5}, \frac{8}{11}$	$\frac{\pi}{21}$	$\frac{5}{21}, \frac{5}{19}, \frac{5}{11}, \frac{10}{21}, \frac{10}{19}, \frac{10}{17}$
$\frac{\pi}{11}$	$\frac{5}{11}, \frac{5}{9}, \frac{7}{11}$	$\frac{\pi}{23}$	
$\frac{\pi}{13}$	$\frac{2}{13}, \frac{2}{11}, \frac{3}{13}, \frac{3}{11}, \frac{3}{7}, \frac{6}{13}, \frac{6}{11}, \left(\frac{6}{9}\right)$	$\frac{\pi}{25}$	$\frac{6}{25}, \frac{6}{23}$

where $\vec{\alpha}_{\rm CF}^{(\mu)}$ is the charge vector of the composite fermion in the μ-th layer as given by (3.3). The naive solution

$$\vec{\alpha}_\Phi = \left(\frac{1}{2mp{+}1}, \ldots, \frac{1}{2mp{+}1} \right), \quad (\vec{\alpha}_\Phi)^2 = \frac{1}{2mp{+}1} \tag{3.34}$$

yields the value $\vec{\alpha}_\Phi \cdot \vec{\alpha}_{\rm CF}^{(\mu)} = \frac{2p+1}{2mp+1}$, which is not an integer for $m > 1$. A simple generalized solution exists, namely

$$\vec{\alpha}_{\Phi,i} = \frac{1}{2mp+1} \begin{cases} m & 1 \leq i \leq 2p \\ 1 & 2mp+2-m \leq i \leq 2mp+1 \\ 0 & \text{otherwise} \end{cases} , \tag{3.35}$$

which coincide with (3.1) for $m = 1$. By this, the desired result for all layers is obtained. Furthermore,

$$\vec{\alpha}_\Phi \cdot \vec{\alpha}_\Phi = \frac{1}{(2mp+1)^2}(2mp^2 + m) = \frac{m}{2mp+1} \ . \tag{3.36}$$

Since each of the m layers contributes $1/(2mp+1)$ this yields the correct quasi-particle statistics.

Therefore, the quasi-particle excitations of the wave functions $\Psi_{p,m}$ and $\Psi_{p,m}^{(n)}$ are predicted to carry a phase $\Theta \sim \pi/(2mp+1)$ and have the charge $q \sim \mathrm{e}/(2mp+1)$ as it is shown in Table 3.2. Since several filling fractions belong to more than one series, e.g., $2/5$, and thus exist in different forms of quantum liquids, various types of quasi-particles can be found for these states. Direct experimental observations are still difficult, and — as far as it is known so far — good indications solely exist for the LAUGHLIN series. Finally, it has to be noted that the choice (3.3) for the charge vectors of the composite fermion and (3.35) for the quasi-particles is not unique, although physically motivated, particularly simple and symmetric. The ambiguity is not disturbing since most other solutions are related by a change of basis within the tensor product of the conformal field theories. The advantage of the approach of this work is that the conformal field theories themselves are confined to a discrete series leaving not much space for arbitrariness.

CHAPTER 4

Non-Abelian Fractional Quantum Hall States

The conformal field theory approach presented in this work is capable to describe the complete set of experimentally observed filling fractions with but a few exceptions. One of these exceptions is $\nu = 4/11$, recently confirmed by PAN et al. [5]. It was named an 'odd' quantum HALL state and is assumed to belong to the class of non-ABELIAN quantum HALL states which have been analyzed in detail by [56, 57, 58]. In this chapter the scope of the b/c-spin system formalism in terms of a suitable representation of this new class of states is investigated.

4.1 Non-Abelian Spin Singlet States

The topological order of effective CHERN-SIMONS theory represented in the K-matrix formalism classifies JAIN's main series and their natural extension, the composite fermion paired fractional quantum HALL states, both obeying fractional but ABELIAN statistics. These series are described by the b/c-spin systems in a natural way putting fractional statistics in a geometrical setting and predicting the set of states by experimental order of stability. Within the last few years, a new class of fractional quantum HALL states with non-ABELIAN statistics has been discussed. These states are special insofar as their trial wave functions for quasi-hole excitations have more than one component. Therefore, braid statistics are represented by matrices acting on these wave functions if two quasi-particles are exchanged. Since matrices — in general — do not commute, the statistics were named 'non-ABELIAN'. A subset of this class is the set of non-ABELIAN spin singlet fractional quantum HALL states which are analyzed in the following.

In 1983, HALPERIN emphasized that fractional quantum HALL states do not always have to be completely spin polarized since the ZEEMAN energy is dependent on the g factor of the electrons and seems to be rather small compared to other energy scales of the system [8]. He proposed spin singlet states with the following trial wave functions:

$$\Psi_{\text{Halperin}}^{n+1,\,n+1,\,n} = \prod_{i<j}^{N} (z_i^{(\uparrow)} - z_j^{(\uparrow)})^{n+1} (z_i^{(\downarrow)} - z_j^{(\downarrow)})^{n+1} \prod_{i,\,j}^{N} (z_i^{(\uparrow)} - z_j^{(\downarrow)})^{n} \exp\big(-\tfrac{1}{4}\sum_{i,\,\mu} |z_i^{(\mu)}|^2\big)\,, \quad (4.1)$$

where $n \in \mathbb{N}$ and $z_i^{(\uparrow)}$, $z_j^{(\downarrow)}$ denote the positions of the electrons with spin up and spin down, respectively. The filling fraction is derived to $\nu = 2/(2n+1)$. For odd n a bosonic and for even n a fermionic state is obtained. The latter resembles the two-layer composite fermion state $\Psi_{p,\,2}$ of JAIN's main series (3.5) with $n = 2p$.

In 1998, READ and REZAYI studied a class of spin-polarized non-ABELIAN quantum HALL states $\Psi_{\text{NA}}^{k,\,M}$ [58] generalizing the PFAFFIAN states which successfully describe composite fermion liquids with $\nu = 1/(2p)$ where $2p$ is the number of flux quanta attached to the electrons. The non-ABELIAN wave functions were proposed to read:

$$\Psi_{\rm NA}^{k,\,M} = \Psi_{\rm para}(z_i)\prod_{i<j}(z_i - z_j)^{M+\frac{2}{k}} \ , \quad \nu = \frac{k}{Mk+2} \ . \tag{4.2}$$

Here, $\Psi_{\rm para}$ represents wave functions deduced from correlators of $\mathbb{Z}_k$-parafermionic conformal field theories.

It turned out that the features of (4.1) and (4.2) can be combined. By this, the class of non-ABELIAN spin singlet fractional quantum HALL states is conceived [56, 57]. Their wave functions are given by:

$$\begin{aligned}\Psi_{\rm NASS}^{k,\,M} = \Psi_{\rm para}^{\rm SU(3)}(z_i^{(\uparrow)},\, z_j^{(\downarrow)})\Bigg(\prod_{i<j}^{N}\prod_{\mu=\uparrow,\downarrow}(z_i^{(\mu)} - z_j^{(\mu)})^2\prod_{i,j}^{N}(z_i^{(\uparrow)} - z_j^{(\downarrow)})\Bigg)^{\frac{1}{k}} \\ \times\prod_{i<j}^{N}\prod_{\mu=\uparrow,\downarrow}(z_i^{(\mu)} - z_j^{(\mu)})^M\prod_{i,j}^{N}(z_i^{(\uparrow)} - z_j^{(\downarrow)})^M\exp\Big(-\tfrac{1}{4}\sum_{i,\mu}\big|z_i^{(\mu)}\big|^2\Big) \ ,\end{aligned} \tag{4.3}$$

where $\Psi_{\rm para}^{\rm SU(3)}$ is a wave function derived from $\mathrm{SU}(3)_k$-parafermionic conformal field theory correlators. Explicit calculations of $\Psi_{\rm para}^{\rm SU(3)}$ reveal to be rather difficult (a detailed analysis is provided by GEPNER [59]) and even though concrete K-matrices have been calculated for $\Psi_{\rm NASS}^{k,\,M}$ in [56], the one-to-one correspondence between the CHERN-SIMONS formalism and the b/c-spin systems does not hold due to the parafermionic structure. Yet, solely the non-parafermionic part of (4.3) determines the filling fraction

$$\nu_{\rm NASS}^{k,\,M} = \frac{2k}{2kM+3} \tag{4.4}$$

and the geometrical features. Therefore, it can be regarded as a two-layer state with a pseudo K-matrix

$$(K_{\rm k,\,M})_{ij} = \begin{pmatrix} M+\frac{2}{k} & M+\frac{1}{k} \\ M+\frac{1}{k} & M+\frac{2}{k}\end{pmatrix} \ . \tag{4.5}$$

It is the characteristics of (4.5) that have to be adopted by the b/c-spin systems. If this is taken for granted, the full theory is obtained by a tensor product with the $\mathrm{SU}(3)_k$-parafermionic conformal field theories.

The structure of $K_{\rm k,\,M}$ induces the one-component statistics Θ of the fractional quantum HALL state. Thus, they are expected to be proportional to $\pi/(2kM+3)$. If these statistics are to be realized by b/c-spin vertex operators, the appertaining charge vectors $\vec{\alpha}$ have to be $(2kM+3)$-dimensional. Due to the additional $\mathbb{Z}_2$-symmetry of the conformal field theory modules, i.e., the RAMOND and NEVEU-SCHWARZ sectors, the most general charge entries are restricted to

$$\vec{\alpha}_i \in \left\{0,\, \frac{1}{2(2kM+3)},\, \frac{1}{2kM+3},\, \ldots,\, 1\right\} \ . \tag{4.6}$$

This implies strong restraints on the set of states which can be represented by the spin systems since the components of (4.5) are obtained via scalar products of the charge vectors as shown in chapter 3. More precisely, this demands

$$2^2(2kM+3)^2 = nk \ , \quad k,\, n \in \mathbb{N},\ M \in \mathbb{N}_0 \ . \tag{4.7}$$

This diophantic equation is solved by prime decomposition of k. Since $2kM + 3$ is odd, the factor 2^2 on the left hand side of (4.7) has to be absorbed in k or n. From this follows

$$k = 2^p \prod_i k_i \ , \tag{4.8}$$

with $p \leq 2$ and k_i odd prime. Restricting to k_1 without loss of generality leads to

$$(2k_1M_1+3)^2 = n_1k_1 \ , \quad M_1 \in \mathbb{N}_0,\, n_1 \in \mathbb{N} \ . \tag{4.9}$$

Since k_1 is prime

$$2k_1M_1 + 3 \sim k_1 \quad \Rightarrow \quad n_2k_1 = 3 \ , \quad n_2 \in \mathbb{N} \ . \tag{4.10}$$

Therefore, the only possible prime decomposition remains to be of the form $k = 2^p \cdot 3^q$ with $p \leq 2$ and $q \in \mathbb{N}$. Inserted in (4.7) yields

$$3^2(M_2 3^{q-1}+1)^2 = n_3 3^q \ , \quad M_2 \in \mathbb{N}_0,\, n_3 \in \mathbb{N} \ . \tag{4.11}$$

This directly postulates $q \leq 2$ since $M_2 3^{q-1} + 1$ is not divisible by 3 and the prime decomposition of k reads:

$$k = 2^p \cdot 3^q \ , \quad p,\, q \leq 2 \ . \tag{4.12}$$

As a result, the non-ABELIAN fractional quantum HALL states $\Psi^{k,\,M}_{\mathrm{NASS}}$ can only be expressed in terms of b/c-spin conformal field theories if 36 is divisible by k.

Reconsidering (4.4), the filling fractions of these states coincide with the ones obtained in the pure b/c-spin approach in most cases indicating that a certain set of fractional quantum HALL states may exist in different types of quantum liquids. Yet, there are some exceptions:[1]

$$\nu = 4/11,\ 4/19,\ 8/19,\ \text{and}\ 12/25 \ , \tag{4.13}$$

which solely exist in the non-ABELIAN form of whom only the first $\nu = 4/11$ has been confirmed so far [5] giving rise to the assumption that this type of fractional quantum HALL states possesses a very small energy gap.

The requirement on k turns out to be sufficient and possible charge vectors of the corresponding Vertex operators are found to be nicely symmetric as indicated in Table 4.1.

[1] Only filling fractions with denominator $d \leq 29$ are indicated.

TABLE 4.1: Charge vectors of non-ABELIAN fractional quantum HALL states. $\vec{\alpha}^{(1)}_{\mathrm{M}=1}$ and $\vec{\alpha}^{(2)}_{\mathrm{M}=1}$ denote the charge vectors ($\mathrm{M} = 1$) of the first and the second layer of (4.5), respectively. $\vec{\alpha}^{(\mu)}_{\mathrm{M}\to\mathrm{M}+1}$ shows the components which have to be added while $\mathrm{M} \to \mathrm{M}+1$ passes through the series.

k	$\nu^{k,M}_{\mathrm{NASS}}$	$\vec{\alpha}^{(1)}_{\mathrm{M}=1,i}$ / $\vec{\alpha}^{(2)}_{\mathrm{M}=1,i}$ / $\vec{\alpha}^{(\mu)}_{\mathrm{M}\to\mathrm{M}+1,i}$
2	$\dfrac{4}{4M+3}$	$(1,\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, 0,\, 0)$ $(1,\, \frac{1}{2},\, \frac{1}{2},\, 0,\, 0,\, \frac{1}{2},\, \frac{1}{2})$ $(\ldots,\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2})$
3	$\dfrac{2}{2M+1}$	$(\frac{2}{3},\, \frac{2}{3},\, \frac{2}{3},\, \frac{1}{3},\, \frac{1}{3},\, \frac{1}{3},\, 0,\, 0,\, 0)$ $(\frac{2}{3},\, \frac{2}{3},\, \frac{2}{3},\, 0,\, 0,\, 0,\, \frac{1}{3},\, \frac{1}{3},\, \frac{1}{3})$ $(\ldots,\, 1,\, 0,\, 0,\, 0,\, 0,\, 0)$
4	$\dfrac{8}{8M+3}$	$(1,\, \frac{1}{2},\, 0,\, \frac{1}{2},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)$ $(1,\, \frac{1}{2},\, 0,\, 0,\, 0,\, 0,\, 0,\, \frac{1}{2},\, 0,\, 0,\, 0)$ $(\ldots,\, \frac{1}{2},\, \frac{1}{2},\, 0,\, 0,\, \frac{1}{2},\, \frac{1}{2},\, 0,\, 0)$
6	$\dfrac{4}{4M+1}$	$(\frac{5}{6},\, \frac{2}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{3},\, \frac{1}{3},\, \frac{1}{3},\, 0,\, 0,\, 0)$ $(\frac{5}{6},\, \frac{2}{6},\, \frac{1}{6},\, \frac{1}{3},\, \frac{1}{3},\, \frac{1}{3},\, 0,\, 0,\, 0,\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6},\, \frac{1}{6})$ $(\ldots,\, 1,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0)$
$\vdots$	$\vdots$	$\vdots$

(4.14)

As mentioned in the concluding section of chapter 3, the choice of the charge vectors is not completely fixed — not even up to a change of basis. Therefore, a more detailed analysis of non-ABELIAN statistics remains an unsolved problem and has to be investigated from an experimental as well as a theoretical point of view. Yet, the vectors in Table 4.1 are chosen in the most symmetric form in terms of geometrical aspects.

Reconsidering the results of this section, the *b*/*c*-spin approach motivates severe constraints for the set of observable non-ABELIAN spin singlet fractional quantum HALL states which can be directly applied to the class of non-ABELIAN spin polarized systems [58] as well. In this context, the recently discovered $\nu = 4/11$ state is proposed as a candidate to reveal the non-ABELIAN features in a pure form.

CHAPTER 5

Discussion

The success of the analysis of the HALDANE-REZAYI state via $c = -2$ spin systems [32, 60] stimulated the presented approach. With a few general and physically motivated assumptions restricting to a discrete set of conformal field theories a hierarchical scheme that precisely keeps track of experimental results has been constructed. After having developed these features in a natural and simple way, the complete set of filling fractions, with but a few exceptions, was consecutively derived by order of stability in the fractional quantum HALL regime of $0 \leq \nu \leq 1$ whereas no unconfirmed fractions were predicted.

More precisely, the conformal field theories used in the approach of this work provide geometrical descriptions of fractional quantum HALL states. Since odd-denominator fillings refer to fermionic statistics, the natural choices are $(j, 1-j)$ b/c-spin systems with $j \in \mathbb{N}/2$. Moreover, the statistics of the flux quanta, as suggested by JAIN's composite fermion picture, reveal to possess more general features so that RIEMANN surfaces with global $\mathbb{Z}_n$-symmetry have to be considered. Representing these surfaces as n-fold ramified covering of the complex plane, the effect of a flux quantum is geometrically the same as a branch point. Therefore, the conformal field theory correlators are sections of certain vector bundles. The bulk ground state wave function is given by a correlator of vertex operators whose twist numbers are purely fermionic resembling the quantum numbers of a composite fermion. With these ingredients the bulk wave functions for the principal main series $\nu = m/(2pm+1)$ are obtained. It turns out that the choice of conformal field theories used in this scheme not only possesses a direct geometric interpretation, but furthermore puts severe constraints on possible fractional quantum HALL states. The description of the fractional quantum HALL effect via effective CHERN-SIMONS theories leads to a classification of states in terms of so-called K-matrices. In principle, arbitrary filling fractions can be constructed this way, leaving a physically motivated classification of them as an unsolved problem. On the other hand, the access by CHERN-SIMONS theory is crucial to classify fractional quantum HALL states in terms of topological order. However, since the corresponding bulk wave functions cannot be written in factorized form in terms of conformal field theory correlators, the b/c-spin systems rule out many K-matrices and, therefore, provide a very natural restraint on them.

Besides the JAIN's main series, other filling fractions are obtained by one further principle, i.e., composite fermion pairing while the so-called particle-hole duality is explicitly avoided since it is not well confirmed by experiment. This pairing leads to a new hierarchy of states obtained from the principal series by a growing number of pairings effectively described by additional conformal field theories, namely the $c = -2$ spin singlet systems. The requirement that the bulk wave function can be written in terms of factorized conformal field theory correlators demands that only pairings leading to K-matrices in block form are possible. By this, all experimentally observed filling fractions are deduced (except for the presumably non-ABELIAN state with $\nu = 4/11$ which is discussed in appendix A, and some controversial fractions, e.g., $\nu = 7/9$, $\nu = 10/13$, and $\nu = 5/13$). However, in the scheme of this work all filling fractions which are not observed in nature are precisely avoided underlining its predictive power The ansatz yields a natural order of stability in perfect agreement with experimental data suggesting a clear picture of series which can be observed up to a given maximal numerator of ν. Thus, it is possible to denote the next members of these series, as indicated in Fig. 3.1 and Table 3.1, which might be observed under improved experimental conditions, but no other fractions.

The main advantage of this scheme is that it avoids arbitrariness. Furthermore, the concept of pairing is not exceptional as well. First of all, it precisely agrees with experimental observations for the HALDANE-REZAYI state. A nice discussion is provided by these papers [61, 62]. Moreover, pairing effects are indicated by numerical studies [63, 64], and are in analogy to similar phenomena in other fields of condensed matter physics, such as certain exactly integrable models in the context of BCS pairing [65]. Although the bulk wave functions proposed in the scope of this work which describe paired fractional quantum HALL states differ from the ones predicted by the naive K-matrix formalism, they share important asymptotic features. A check of these bulk wave functions should be done numerically, but is beyond the scope of this thesis.

The description in terms of b/c-spin systems seems to be sufficiently complete. It is even possible to incorporate fractional quantum HALL states from non-ABELIAN CHERN-SIMONS theories [56, 57] as shown in chapter 4 if the geometric principles are believed to remain unchanged. The main difference lies in the nature of the quasi-particle excitations. In the approach of this work, non-trivial statistics is a consequence of the twists introduced by the flux quanta and is – in the lowest LANDAU level – always of ABELIAN nature since all monodromies are simultaneously diagonalized. Therefore, if non-ABELIAN statistics is involved it cannot be represented within the simple conformal field theories that were used. However, it is to point out that the $c = -2$ conformal field theory which naturally describes pairing is actually a logarithmic conformal field theory and thus includes fields with non-diagonalizable monodromy action [32]. In order to understand this in more detail, it is crucial to work with the full twist fields, not only the projective ones. This immediately leads to further restrictions for the twist fields in order to be inserted in a correlator. If the twists are summed over all insertions they have to be trivial in all n copies of the b/c-spin system considered (a short discussion is provided in appendix B). However, at this stage, the full description of quasi-particle excitations remains an unsolved problem. Another one is the correct choice of the spin system, i.e., of the conformal weights $(j, 1-j)$ of the fields $b(z)$ and $c(z)$. This problem is related to the fact that the b/c-spin systems possess partition functions which are equivalent to GAUSSIAN $c = 1$ models. Unfortunately, the partition function of a $(j, 1-j)$ system is closely related to the one of any other $(j', 1-j')$ system, in particular if $j - j' \in \mathbb{Z}$. Thus, conformal field theory alone is not able to determine j. However, if the composite fermion is taken as the basic object, it might be expected that the fractional quantum HALL state involving composite fermions made out of electrons with p attached pairs of flux quanta should correspond to spin $j = \frac{1}{2}(2p+1)$ fields in the conformal field theory description. These should be elementary in the sense that the spectrum of the conformal field theory does not contain fermionic fields with smaller spin in the non-twisted sector. Moreover, the twists related to the quasi-particle excitations should have a minimal charge of $\alpha = 1/(2pm+1)$ for an m layer state, since this is the expected fractional statistics. The fractional charge is entirely determined by the geometry, i.e., the number of sheets in the covering of the complex plane. But the requirement that the composite fermions shall be the effective elementary particles fixes $j = \frac{1}{2}(2p+1)$ or $j = \frac{1}{2}(2p+3)$ due to the duality $j \leftrightarrow 1-j$. A very interesting question is, whether an effective theory of transitions between different fractional quantum HALL states could yield a mechanism for how the b/c-spin systems are mapped onto each other, e.g., along the lines of [28, 66].

Finally, it is to stress that the scheme presented in this work should be understood as a proposal. Although a stringent geometrical setting is provided which identifies the choice of conformal field theories, it is not possible to connect these conformal field theories to the full (2+1)-dimensional bulk theory via first principles. For instance, and in contrast to the (1+1)-dimensional edge theory, there is no mathematical rigorous theorem which guarantees a kind of equivalence between CHERN-SIMONS and conformal field theory. Furthermore, the expressions for the bulk wave functions in terms of conformal field theory correlators, as all existing proposals for bulk wave functions, should be understood as trial ones, since exact solutions are not known due to the fact that no microscopic HAMILTONIAN has been discovered so far. This even applies to the LAUGHLIN wave functions. Comparisons with other wave functions obtained from the numerical diagonalization of some exact HAMILTONIAN can only be drawn for a small number of electrons and not in the thermodynamic limit. On the other hand, trial wave functions such as the ones conceived by LAUGHLIN possess many special features, e.g., topological order or incompressibility, and symmetries, e.g., symmetry under area-preserving diffeomorphisms. Future research will hopefully reveal the physical nature of these properties so that the connection with conformal field theory is eventually put on firmer ground and trial wave functions are more thoroughly checked or even derived from first principles.

Appendix A

Remarks on Unitarity

It might seem disturbing that the conformal field theories proposed to describe the fractional quantum Hall bulk regime are non-unitary. It has to be stressed that these theories are not meant to yield the bulk wave functions from a dynamical principle, nor do they provide an effective Hamiltonian. Moreover, since the relevant states are stationary eigenstates of the full (2+1)-dimensional system, no time evolution is involved. In this sense, the bulk theory can be reduced to a truly Euclidean, (2+0)-dimensional one. The topological nature of the full (2+1)-dimensional system suggests the bulk theory to be at least scale invariant. Thus, the assumption that bulk wave functions should have a conformal field theory description is reasonable, but the requirement that these conformal field theories should be unitary is not obligatory and does not contain any physically relevant information. The bulk conformal field theory describes purely geometrical features, namely how the corresponding wave functions can be understood in terms of vector bundles over Riemann surfaces [67]. As it was argued in the previous chapters, the fractional statistics of the quasi-particle excitations results in a multivalued wave functions, considered on the complex plane. One of the central features of the approach proposed in this work is to replace this setting by the geometrically more natural scheme of holomorphic functions over Riemann surfaces locally represented on a ramified covering of the complex plane leading to the non-unitary $(j, 1-j)$ b/c-spin systems.

However, the question of unitarity is not irrelevant. To be consistent, it has to be demanded that the ansatz of this work fits together with the (1+1)-dimensional conformal field theories of the edge excitations. These describe waves propagating along the one-dimensional edge of the quantum droplet and hence necessarily have to be unitary.[1] Consistency requires that the space of states of either conformal field theory, the edge and the bulk one, are to be equivalent. In other terms, both should have the same partition functions. Fortunately, the b/c-spin systems have well-known partition functions which are indeed equivalent to those of certain $c = 1$ Gaussian models. These latter unitary conformal field theories are precisely the candidates for the description of the edge excitations which are most widely used:[2]

To be more explicit, a spin $(j, 1-j)$ b/c-spin system in some twisted sector with twist α is considered. The full character of this system, including the ghost number, is defined as

$$\chi^{(j,\alpha)}(q,z) \equiv \mathrm{tr}_{\mathcal{H}^{(\alpha)}}\Big[q^{L_0^{(j,\alpha)}-\frac{c_j}{24}}z^{j_0^{(\alpha)}}\Big] \quad , \tag{A.1}$$

where it is clearly indicated that the mode expansions of the Virasoro field and the ghost

[1] This also follows from the strict one-to-one correspondence of (2+1)-dimensional Chern-Simons theories on a manifold M with unitary (1+1)-dimensional conformal field theories living on the boundary ∂M [24].

[2] There are some other proposals making use of so-called minimal $\mathcal{W}_{1+\infty}$ models or $\widehat{SU}(m)$ Kac-Moody algebras for m-layer states, e.g., [7, 17, 23, 46].

current depend on the twist sector. Explicitly computed, these characters read:

$$\chi^{(j,\alpha)}(q,z) = q^{\frac{1}{2}(j+\alpha)(j+\alpha+1)+\frac{1}{12}} z^{\alpha} \prod_{n=1}^{\infty} (1 + zq^{n+(j+\alpha)-1})(1 + z^{-1}q^{n-(j+\alpha)}) \,. \tag{A.2}$$

It is evident from this formula that the characters (almost) only depend on $(j+\alpha)$. In particular, the following equivalence is obtained:

$$\chi^{(j,\alpha)}(q,z) = z^{\frac{1}{2}-j} \chi^{(\frac{1}{2},\alpha+j-\frac{1}{2})} \;\;. \tag{A.3}$$

Thus, the VIRASORO characters for $z=1$ of the *b*/*c*-spin systems are all equivalent to characters of the complex fermion with $\mathrm{c}=1$ where the twist sectors α get mapped to others with $\alpha+j-\frac{1}{2}$. Thus, all sectors which are mapped in this way keep their statistics, since $j \in \mathbb{Z}+\frac{1}{2}$ and $\alpha \equiv \alpha + j - \frac{1}{2} \bmod 1$. A more detailed analysis reveals that the partition functions are indeed equivalent. Detailed approaches are given by [68, 69, 70, 71]. This extends to the $\mathrm{c} = -2$ spin system describing pairing, which has been pointed out in [32, 60]. Therefore, the space of states of *b*/*c*-spin systems with twists $\alpha = k/m$, $k = 0, \ldots, m-1$, is equivalent to the space of states of a rational $\mathrm{c} = 1$ ($\mathbb{Z}_2$ orbifold) theory with radius of compactification $2R^2 = 1/m$. Carefully investigated, this equivalence indeed holds. Although always m copies of the *b*/*c*-spin systems are considered, the fields are represented in an ABELIAN projection where the charges (or twists) of all copies of the fields are closely related to each other. Since they are not chosen independently, solely one copy of the HILBERT space is obtained.

APPENDIX B

Twist Fields and Topology

The conformal field theory approach to lowest LANDAU level bulk wave functions of the fractional quantum HALL effect presented in this work successfully describes the complete set of experimentally observed filling fractions in the range $0 \leq \nu \leq 1$ with but a few exceptions. The spin fields $b(z)$ and $c(z)$ set on a $\mathbb{Z}_n$-symmetric RIEMANN surface naturally simulate quasi-particles with fractional charges and statistics. These features turn out to be closely related to the structure of the branch cuts of the manifold and, therefore, the n-dimensional charge vectors which define the Vertex operators in the bosonized spin conformal field theory. Reconsidering the results of chapter 3 the lowest LANDAU level bulk wave functions are deduced from the projective fields defined in (2.40). Here, the branch point which is represented by a ramified covering of the compactified complex plane $\widehat{\mathbb{C}}$ is mapped to an n-fold copy of $\widehat{\mathbb{C}}$. However, even if the analytic structure of the corresponding fractional quantum HALL states is obtained in this manner, the geometrical and conformal features are solely defined by the full theory, namely the twist fields with charge vectors

$$\vec{\alpha}^{\mathbb{Z}_n}_{j,i} = -\frac{i + j(1-n)}{n} \ , \ i \in \{0, \dots, n-1\}, j \in \mathbb{Z}/2 \ , \tag{B.1}$$

where j is the spin of the theory. This spin provides an offset in (B.1) which is irrelevant for the statistics of the quasi-particles and is only apparent in the conformal weight h of the Vertex operators (2.37). Thus, the conformal background of the spin theories is determined by the simplified charge vectors

$$\vec{\alpha}^{\mathbb{Z}_n} = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right) \ . \tag{B.2}$$

It was first shown by KNIZHNIK [51, 52] that these vectors have to satisfy severe constraints. In order to guarantee an overall fermionic system, the sum over each component $\vec{\alpha}^{\mathbb{Z}_n}_i$ of all existing quasi-particles of the fractional quantum HALL state has to be an integer. If a state with L excitations is considered, each quasi-particle is associated with a charge vector (B.2) allowing permutations of its components. This yields n diophantic equations. For example, a state with two excitations $(L=2)$ always implies the charge vectors

$$\vec{\alpha}^{\mathbb{Z}_n,(1)} = (1/n, 2/n, \dots, 0) \ , \quad \vec{\alpha}^{\mathbb{Z}_n,(2)} = ((n-1)/n, (n-2)/n, \dots, 0) \ . \tag{B.3}$$

These indeed satisfy the physical constraints since

$$\sum_{i=1}^{2} \vec{\alpha}^{\mathbb{Z}_n,(i)}_k = 1 \ , \quad \forall k \in \{0, \dots, n-1\} \tag{B.4}$$

solves the system of diophantic equations. This is the only solution up to simultaneous permutations of the vectors (B.3). Thus, there solely exists one correlation function or, more precisely, one conformal block which describes an excited bulk wave function with two quasi-particles. An interesting consequence is that there seems to be no solution for $L=1$. In the geometrical picture this would correspond to a branch cut leading to nowhere. The excitation behaves as a monodromy operation which moves the system from a sheet ℓ to $\ell+1$. The only reasonable argumentation in this case is to consider ∞ to be the second branch point. By this, the desired wave function (1.18) is obtained.

For $L>2$, there may exist more than one solution.[1] Their number N_{exc} provides deeper insight in the topological characteristics of the given fractional quantum HALL states. The intrinsic geometrical features of the *b*/*c*-spin approach allow to read off the degeneracy of the ground state for a given fractional quantum HALL state in a non-trivial setting by counting N_{exc}. On the other hand, these excitations can be interpreted to effectively generate the non-trivial topology in terms of vertex operators. In the following, this is illustrated for the degeneracy of the ground state in ($\mathbb{Z}_2$-symmetric) torus geometry which is often quoted with respect to topological order. The calculation is executed for an excited bulk wave function with $L=4$ and ends up with four independent sets of charge vectors plus the non-excited ground state, hence in total five solutions:[2]

$$(0000)\ ,\quad (\downarrow\downarrow\downarrow\downarrow)\ ,\quad (\downarrow\downarrow\uparrow\uparrow)\ ,\quad (\downarrow\uparrow\downarrow\uparrow)\quad \text{and}\quad (\downarrow\uparrow\uparrow\downarrow)\ . \tag{B.5}$$

This is the well-known degeneracy of the HALDANE-REZAYI ground state.[3]

Further calculations have been applied to global $\mathbb{Z}_3$-symmetric fractional quantum HALL systems with charge vectors

$$\vec{\alpha}^{\mathbb{Z}_3}=\left(0,\,\frac{1}{3},\,\frac{2}{3}\right) \tag{B.6}$$

or permutations of (B.6). The result is illustrated in Table B.1. The set of solutions for large L or n is far more difficult to find. The number of permutations behaves as $n!$ and there are n diophantic equations with L dependent parameters to solve. To summarize, the geometrical

TABLE B.1: Ground state degeneracy N_{GS} of $\mathbb{Z}_3$-symmetric fractional quantum HALL systems with L quasi-particle excitations.

L	2	3	4	5	6	7	8	9
N_{GS}	2	3	4	6	11	14	23	32

features of the *b*/*c*-spin conformal field theory description of the fractional quantum HALL effect

[1] The number of solutions must be always understood ‘up to simultaneous permutations of all L charge vectors’ since these global permutations do not provide additionally physical information.

[2] The symbols 0, $\downarrow$, and $\uparrow$ denote the charge vectors $(0,\,0)$, $(0,\,1/2)$, and $(1/2,\,0)$, respectively. The insertions of the excitations are at four ordered distinct points.

[3] More precisely, the full degeneracy turns out to be $N_{\mathrm{GS}}=10$ where the additional factor of two arises from a spin $1/2$ realization of an $\mathrm{SU}(2)$-symmetry via a GAUSSIAN $c=1$ conformal field theory [32, 60].

naturally encompasses quasi-particle excitations in the full, unprojected setting, leading to non-trivial constraints and furthermore revealing topological information. However, a detailed study of these excited bulk wave functions is beyond the scope of this thesis and is subject to further studies.

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