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Supersymmetric Generalizations of Matrix Models

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Supersymmetrische Verallgemeinerungen von Matrix Modellen

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der supersymmetrischen Verallgemeinerung von Matrix– und Eigenwertmodellen. Nach einer kurzen Einführung in das Hermitesche Ein–Matrix–Modell wenden wir uns dem $c = -2$ Matrix–Modell zu. In seiner Formulierung durch ein matrixwertiges Superfeld ist dieses Modell invariant unter Supersymmetrietransformationen auf Matrixebene. Wir zeigen die Existenz einer Nicolai–Abbildung dieses Modells auf ein freies Hermitesches Matrix–Modell und diskutieren seine diagrammatische Entwicklung. Korrelationsfunktionen für quartische Potentiale und beliebigen Genus werden berechnet, welche die Stringsuszeptibilität von $c = -2$ Liouville–Theorie im Skalenlimes aufweisen. Wir zeigen auf, wie sich diese Ergebnisse zum Zählen supersymmetrischer Graphen verwenden lassen.
Daraufhin studieren wir das Supereigenwertmodell, der bis heute einzige erfolgreiche diskrete Zugang zur Quantisierung von 2d Supergravitation. Das Modell wird in einer superkonformen Feldtheorie Formulierung durch Forderung von Super–Virasoro Bedingungen konstruiert. Die vollständige Lösung wird mit Hilfe der Momentenmethode hergeleitet. Diese ermöglicht die Berechnung der freien Energie und aller Multi–Loop–Korrelatoren auf beliebigem Genus und für allgemeine Potentiale. Die Lösung wird im diskreten Fall und im Doppelskalenlimes präsentiert. Explizite Resultate bis Genus zwei werden angegeben.
Es folgt eine Diskussion der supersymmetrischen Verallgemeinerung des externen Feld Problems. Die diskreten Super–Miwa–Transformationen des Supereigenwertmodells können im Eigenwert– und im Matrixfall angegeben werden. Eigenschaften von externen Supereigenwertmodellen werden diskutiert, die genaue Form des zum gewöhnlichen Modell korrespondierenden externen Supereigenwertmodells konnte jedoch noch nicht hergeleitet werden.

Referent: Prof. Dr. O. Lechtenfeld
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Supersymmetric Generalizations of Matrix Models

Abstract

In this thesis generalizations of matrix and eigenvalue models involving supersymmetry are discussed. Following a brief review of the Hermitian one matrix model, the $c = -2$ matrix model is considered. Built from a matrix valued superfield this model displays supersymmetry on the matrix level. We stress the emergence of a Nicolai–map of this model to a free Hermitian matrix model and study its diagrammatic expansion in detail. Correlation functions for quartic potentials on arbitrary genus are computed, reproducing the string susceptibility of $c = -2$ Liouville theory in the scaling limit. The results may be used to perform a counting of supersymmetric graphs.
We then turn to the supereigenvalue model, today's only successful discrete approach to 2d quantum supergravity. The model is constructed in a superconformal field theory formulation by imposing the super–Virasoro constraints. The complete solution of the model is given in the moment description, allowing the calculation of the free energy and the multi–loop correlators on arbitrary genus and for general potentials. The solution is presented in the discrete case and in the double scaling limit. Explicit results up to genus two are stated.
Finally the supersymmetric generalization of the external field problem is addressed. We state the discrete super–Miwa transformations of the supereigenvalue model on the eigenvalue and matrix level. Properties of external supereigenvalue models are discussed, although the model corresponding to the ordinary supereigenvalue model could not be identified so far.

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Contents

Introduction

> Nature is simple if you look at it in the right way. For example I believe that God created two dimensions. One time and one space dimension. Nothing is simpler.
> [1]

Random matrices have been studied in physics since the work of E. Wigner in the 1950's. Initially proposed as an effective model for higher excitations in nuclei, they have found numerous applications in various fields throughout the years. In high–energy and mathematical physics matrix models experienced a great renaissance following the discovery of their relevance for the quantization of two dimensional gravity and bosonic string theory in 1985. Only recently the supersymmetric generalization of matrix models has been addressed. This thesis reports on the construction, solution and interpretation of such generalized models, focusing on the work of the author.

Matrix models have the appealing property that they are exactly solvable in the limit of infinite matrix size N. Subleading corrections to arbitrary order in $1/N$ may be computed by iterative means, while results for finite N are few. The analysis of the diagrammatic expansion of the Hermitian matrix model reveals that solving the model is equivalent to performing the sum over random equilateral triangulations of two–dimensional surfaces of fixed genus.

String theory on the other hand is today's most popular candidate for a unification of quantum field theory and gravity. It is built on the attractive idea of replacing particles by string–like one–dimensional objects, which sweep out a two–dimensional "world–sheet" as they evolve in time. According to Polyakov this theory is to be interpreted as two–dimensional quantum gravity, in which the world–sheet replaces two–dimensional space–time, and where the coordinates of the string form matter fields coupled to the two–dimensional gravity theory. Interactions correspond to topology changes and are encoded in the genus of the world–sheet, which may be viewed as an "inflated" Feynman graph. Thus the task in a path–integral quantization is to perform an integral over the two dimensional geometries of the world–sheet and sum over their genera.

Precisely this integration may be performed with the help of the matrix model by taking a "continuum limit" of the above–mentioned exact solution, i.e. when the triangles of the discretized surfaces become dense. However, this rather unconventional method of performing a path integral only works for toy models of bosonic strings with world–sheets of Euclidean signature living in $D = c \leq 1$ dimensions. Extending these techniques to dimensions greater than one is a hard problem still unsolved. Due to the exact solvability of matrix models new non–perturbative techniques were introduced to these low–dimensional string theories. We will review the Hermitian matrix model and its correspondence to random surfaces in chapter I.

Of potential phenomenological relevance, however, are superstring theories, which in contrast to the bosonic string have bosons and fermions in their spectrum. The Neveu–Schwarz–Ramond superstring is to be interpreted as two dimensional supergravity coupled to a matter action of superfields. It is thus very desirable to find a supersymmetric generalization of the matrix model technique in order to perform the corresponding path integral over super–geometries. There are (at least) three different ways to proceed.

In a geometrical approach one puts superstrings on a random lattice and then tries to establish a map to an adequate supersymmetric matrix model. But due to the known problems with supersymmetry and fermions in lattice field theory this seems to be rather hopeless. However, the alternative superstring theory of Green and Schwarz does not suffer from these problems, as here the fields on the world–sheet remain bosonic and supersymmetry is only present on the external space–time level. A lattice version of this superstring theory has been formulated and studied. Still, the relation to a solvable supersymmetric matrix model is an open problem (for more details see refs. [2]). Even classically the Green–Schwarz string is supersymmetric only in $D \geq 4$ dimensions, a domain that seems to be hardly reachable for solvable matrix models.

Alternatively one can consider supersymmetric generalizations of matrix models and hope that their critical behaviour reveals properties of two–dimensional supergravity. However, no example is known in the literature where this is the case. Nevertheless, the analysis of these models is interesting in its own right from the viewpoint of random matrices. We shall therefore study the $c = -2$ matrix model in chapter II, a model displaying supersymmetry on the level of matrices. In its "continuum limit" this model describes the propagation of bosonic strings in minus two–dimensional space–time.

The final and successful approach seeks for a generalization of the integrable structure of the Hermitian matrix model. This structure at the heart of the Hermitian matrix model resides in a set of Virasoro constraints. It turns out that, by imposing a generalization in the form of super–Virasoro constraints, one obtains the supereigenvalue model of Alvarez–Gaumé, Itoyama, Mañes and Zadra, which generalizes the eigenvalue formulation of the Hermitian matrix model. Due to the eigenvalue (rather than matrix) character of this model there is no geometrical interpretation in terms of triangulated super–Riemann surfaces at hand. The supereigenvalue model is exactly solvable for general polynomial potentials in the limit of an infinite number N of eigenvalues. Moreover, all subleading corrections in $1/N$ are determined through an iterative process. We will describe this solution in chapter III in the "discrete" and "continuum" cases. The continuum results reveal that the supereigenvalue model indeed describes the coupling of

Introduction

minimal superconformal theories to two dimensional quantum supergravity.

As we shall see, the supereigenvalue model has a large number of similarities to the Hermitian matrix model. This motivated the author to study the external field problem in this context as well. In chapter IV we review the external Hermitian matrix model and present some preliminary results of a generalization to the supersymmetric case.

The presentation of this thesis is intended to be self contained, only basic knowledge of conformal field theory is assumed. For a review on this topic see e.g. ref. [3]. There is a large number of reviews on matrix models and two–dimensional quantum gravity, e.g. refs. [4], but only the recent ref. [5] contains the supereigenvalue model.

The parts of this thesis containing the solution of the supereigenvalue model have been published priorly in refs. [6, 7].

Chapter I: The Hermitian Matrix Model

Generally speaking matrix models are quantum field theories where the field is a $N \times N$ real or complex matrix $\boldsymbol{M}(x)$. We shall consider the simple case of the Hermitian one matrix model in $D = 0$ dimensions, which due to its simplicity in exactly solvable in the limit of infinite matrix size N. This model has been intensively studied in the literature. In the following we give a basic introduction to the model, as well as a review on a collection of more detailed aspects which will be relevant for the generalization to the supersymmetric case.

1. The Model

The Hermitian one matrix model is defined by the partition function

$$\mathcal{Z}_N[g_k] = e^{N^2 F[g_k]} = \int \mathcal{D}\boldsymbol{M} \, \exp\Big[-N \,\mathrm{Tr}\, V(\boldsymbol{M})\Big], \tag{1.1.1}$$

where $\boldsymbol{M}_{ij}$ is a $N \times N$ Hermitian matrix. $F[g_k]$ is the free energy. The measure for the "path integral" of this zero dimensional theory is given by

$$\mathcal{D}\boldsymbol{M} = \prod_{i<j} d\mathfrak{Re}(\boldsymbol{M}_{ij})\, d\mathfrak{Im}(\boldsymbol{M}_{ij}) \prod_i d\boldsymbol{M}_{ii}, \tag{1.1.2}$$

and we consider the most general polynomial matrix potential with coupling constants g_k

$$V(\boldsymbol{M}) = \sum_{k=0}^{\infty} g_k \, \boldsymbol{M}^k . \tag{1.1.3}$$

Note the $U(N)$ invariance $\boldsymbol{M} \mapsto U^\dagger \boldsymbol{M}\, U$ of the "action" $\mathrm{Tr}\, V(\boldsymbol{M})$. This may be used to diagonalize the Hermitian matrix $\boldsymbol{M} = U^\dagger \boldsymbol{D}\, U$, where $\boldsymbol{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ is the diagonal matrix. By performing the change of variables from $\boldsymbol{M}$ to λ_i and U, the integral over the unitary group factors out and we are left with the integration over the eigenvalues λ_i. The Jacobian of this transformation is $\prod_{i<j}(\lambda_i - \lambda_i)^2$.

This may be seen by considering the norm of the infinitesimal variation of $\boldsymbol{M} = U^\dagger \boldsymbol{D}\, U$

$$\begin{aligned}
|\,\delta\boldsymbol{M}\,|^2 &= \sum_{i,j} \delta\boldsymbol{M}_{ij}\, \delta\boldsymbol{M}_{ji} \;=\; \mathrm{Tr}\,(\delta\boldsymbol{M})^2 \\
&= \mathrm{Tr}\,\Big(-U^\dagger\, \delta U\, U^\dagger\, \boldsymbol{D}\, U + U^\dagger\, \delta\boldsymbol{D}\, U + U^\dagger\, \boldsymbol{D}\, \delta U\Big)^2 \\
&= \mathrm{Tr}\,(\delta\boldsymbol{D})^2 - 2i\, \mathrm{Tr}\,[\delta\boldsymbol{D}, \boldsymbol{D}]\, \delta u + 2\, \mathrm{Tr}\,(-\delta u\, \boldsymbol{D}\, \delta u\, \boldsymbol{D} + (\delta u)^2\, \boldsymbol{D}^2\,),
\end{aligned}$$

where we have introduced $\delta u = i\, \delta U\, U^\dagger = \delta u^\dagger$. The second term in the last expression vanishes as $\delta\boldsymbol{D}$ and $\boldsymbol{D}$ are diagonal. We then find

$$|\,\delta\boldsymbol{M}\,|^2 = \sum_i (\delta\lambda_i)^2 + \sum_{i,j} (\lambda_i - \lambda_j)^2\, |\delta u_{ij}|^2 .$$

Note that the independent variables are the variation of the eigenvalues $\delta\lambda_i$ and $\mathfrak{Re}\delta u_{ij}$, $\mathfrak{Im}\delta u_{ij}$ for $i < j$. From the last equation we obtain the Jacobian of $\boldsymbol{M} \mapsto \lambda_i, U$ which is $\sqrt{G}$ where G is the metric tensor, explicitly

$$G = \prod_{i \neq j} (\lambda_i - \lambda_j)^2 \quad \Rightarrow \quad \sqrt{G} = \prod_{i<j} (\lambda_i - \lambda_j)^2 \equiv \Delta^2(\lambda).$$

The Hermitian one matrix model (1.1.1) may then be written in the eigenvalue representation

$$\mathcal{Z}_N[g_k] = c_N \int_{-\infty}^{\infty} (\prod_{i=1}^{N} d\lambda_i) \prod_{i<j} (\lambda_i - \lambda_j)^2 \, \exp\Big[-N \sum_{i=1}^{N} V(\lambda_i)\Big], \tag{1.1.4}$$

where c_N is the volume of the $U(N)$ group. Note the relation of the Jacobian $\Delta^2(\lambda)$ to the van der Monde determinant $\Delta(\lambda) = (-)^{N(N-1)/2}\, \mathrm{Det}\,(\lambda_i^{j-1})$.

Loop Insertion Operator

Expectation values in the Hermitian one matrix model are defined in the usual way as

$$\langle\, \mathcal{O}(\boldsymbol{M})\, \rangle = \frac{1}{\mathcal{Z}_N} \int \mathcal{D}\boldsymbol{M}\, \mathcal{O}(\boldsymbol{M}) \, \exp\Big[-N \,\mathrm{Tr}\, V(\boldsymbol{M})\Big]. \tag{1.1.5}$$

A similar expression holds in the eigenvalue picture. It is very convenient to work with the one–loop correlator

$$W(p) = \frac{1}{N} \sum_{k=0}^{\infty} \frac{\langle \operatorname{Tr} \boldsymbol{M}^k \rangle}{p^{k+1}}, \tag{1.1.6}$$

which acts as a generating functional for the amplitudes $\langle \operatorname{Tr} \boldsymbol{M}^k \rangle$. Similarly we may define the generating functional for higher point amplitudes, the n–loop correlators

$$W(p_1, \ldots, p_n) = N^{n-2} \sum_{k_1, \ldots, k_n = 1}^{\infty} \frac{\langle \operatorname{Tr} \boldsymbol{M}^{k_1} \ldots \operatorname{Tr} \boldsymbol{M}^{k_n} \rangle_{\text{conn.}}}{{p_1}^{k_1+1} \ldots {p_n}^{k_n+1}}, \tag{1.1.7}$$

where “conn.” refers to the connected part. The last two equations may be rewritten as

$$W(p_1, \ldots, p_n) = N^{n-2} \left\langle \operatorname{Tr} \frac{1}{p_1 - \boldsymbol{M}} \ldots \operatorname{Tr} \frac{1}{p_n - \boldsymbol{M}} \right\rangle_{\text{conn.}}. \tag{1.1.8}$$

A useful object to define is the loop insertion operator [9]

$$\frac{\delta}{\delta V(p)} \equiv - \sum_{j=1}^{\infty} \frac{1}{p^{j+1}} \frac{\partial}{\partial g_k}, \tag{1.1.9}$$

because we may now obtain the n–loop correlators from the free energy F by applying the loop insertion operators:

$$W(p_1, \ldots, p_n) = \frac{\delta}{\delta V(p_1)} \frac{\delta}{\delta V(p_2)} \cdots \frac{\delta}{\delta V(p_n)} F. \tag{1.1.10}$$

Hence once the free energy for a general potential is known, all observables may be calculated. The same holds true for the one–loop correlator $W(p_1)$, as with

$$W(p_1, \ldots, p_n) = \frac{\delta}{\delta V(p_2)} \cdots \frac{\delta}{\delta V(p_n)} W(p_1) \tag{1.1.11}$$

all multi–loop correlators follow from it. We thus see that solving for $W(p)$ with a general potential really means completely solving the Hermitian one matrix model. We shall see in section 4 that this is most efficiently done by considering the loop equations of the model.

It is known since the work of t'Hooft [10] that the free energy F and all correlators admit an expansion in $1/N^2$, which may be seen by looking at the perturbative evaluation of the matrix model (1.1.1).

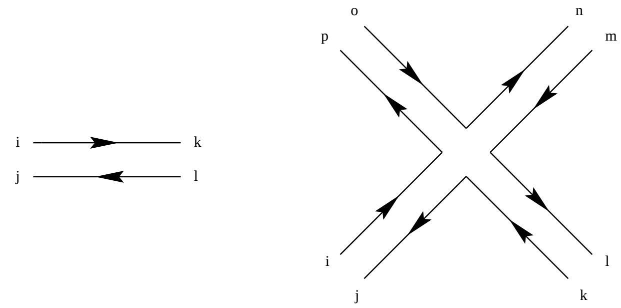


Figure 1: The propagator and a four point vertex of the Hermitian matrix model

Feynman Diagrams

The propagators of the $\boldsymbol{M}_{ij}$ fields of eq. (1.1.1) may be represented diagrammatically by double lines each one corresponding to the separate propagation of the matrix indices. For Hermitian matrices the lines should be oriented in opposite directions (cf. Figure 1). The propagator then simply is

$$\langle \boldsymbol{M}_{ij} \boldsymbol{M}_{kl} \rangle_0 = \frac{1}{N} \delta_{ik} \delta_{jl}, \tag{1.1.12}$$

where the subscript 0 denotes the average taken in the free theory, i.e. $g_2 = 1/2$ and $g_k = 0$ for $k \neq 2$. With the general interactions of eq. (1.1.3) there will be three–point vertices, four–point vertices, etc..., each n–point vertex contributing a factor of $(-g_n N)$. Moreover each loop of internal index will yield the factor $N = \delta_{ii}$.

Consider for instance a connected vacuum diagram built out of P propagators, L closed loops of internal index and V_3 three–point vertices, V_4 four–point vertices, etc..., and let

$$V = V_3 + V_4 + V_5 + \ldots. \tag{1.1.13}$$

Each loop of internal index may be considered as a face of a polyhedron, and the Euler relation then gives

$$V - P + L = 2 - 2g, \tag{1.1.14}$$

in which g is the genus of the surfaces on which the polyhedron (or Feynman diagram) is drawn (0 for a sphere, 1 for a torus, etc...). The contribution of this diagram is

$$N^{V-P+L} (-g_3)^{V_3} (-g_4)^{V_4} \ldots = N^{2-2g} \prod_k (-g_k)^{V_k}. \tag{1.1.15}$$

As the free energy F is the sum of all connected diagrams we find the genus expansion

$$N^2\,F[g_k] = \sum_{g=0}^{\infty} N^{2-2g}\,F_g[g_k], \tag{1.1.16}$$

where F_g is the sum of all diagrams that can be drawn on a surface of genus g. The Hermitian one–matrix model can be solved rather easily in the "planar" limit of infinite matrix size $N\to\infty$ as we shall see in section 4. Here only the planar diagrams of F_0 survive, the Hermitian matrix model thus represents a powerful tool for planar graph counting [11].

2. Discretized Surfaces and 2d Quantum Gravity

We learned in the previous section that the closed Feynman diagrams of the Hermitian matrix model may be viewed as polyhedrons. Moreover the matrix size N appeared as the parameter controlling the genus of such a polyhedron. The nontrivial combinatorial problem of how many inequivalent polyhedrons of genus g with V_3 3–point vertices, V_4 4–point vertices, etc. ...exist, may be answered by expanding $F_g[g_k]$ of eq. (1.1.16) in the g_k's. But as a polyhedron may be interpreted as a discrete approximation to a smooth two dimensional surface, this result may be exploited to study an entirely different problem: The quantization of two dimensional Euclidean gravity!

In two dimensions the Einstein–Hilbert action forms a topological invariant, the Euler characteristic of the underlying manifold. So for fixed topologies only the cosmological term will be dynamic. If we consider a specific manifold the action of 2d Euclidean gravity is

$$S = \mu \int dx^2\,\sqrt{g} - \frac{1}{4\pi G}\int dx^2\,\sqrt{g}\,R = \mu\,A - \frac{2-2g}{G} \tag{1.2.1}$$

where G denotes the gravitational constant, μ the cosmological constant and A the area of the surface. In a path integral quantization of this theory the integration over the metric may be split up into separate integrals over topologies and areas

$$\mathcal{Z}_{\rm QG} = \int \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol}(\mathit{Diff})}\,e^{-S} = \sum_{g=0}^{\infty}\int_0^{\infty} dA\; e^{(2-2g)/G\,-\,\mu\,A}\int_{\Sigma_{g,A}}\frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol}(\mathit{Diff})}, \tag{1.2.2}$$

where we have formally divided out the volume of the group of diffeomorphisms. In general the volume of the moduli space $\mathrm{Vol}(\Sigma_{g,A}) = \int_{\Sigma_{g,A}}\mathcal{D}g_{\mu\nu}/\mathrm{Vol}(\mathit{Diff})$ is difficult to calculate. The most progress in quantizing 2d gravity in the continuum has been made via the Liouville approach [12]. If we discretize the surface, on the other hand, it turns out that (1.2.2) is much easier to calculate, even before removing the finite cutoff. We consider in particular a "random triangulation", in which the surface is constructed from equilateral triangles. [1] Assign the unit area ϵ to each triangle, so that the total area of this simplicial manifold built out of F triangles is given by $\epsilon\,F$. The path integral over the metric $\int\mathcal{D}g_{\mu\nu}$ is now replaced by a sum over triangulations, so that the discretized partition function $\mathcal{Z}_{\rm QG}$ of eq. (1.2.2) reads

$$\mathcal{Z}_{\rm QG}^{\rm discr.} = \sum_{g=0}^{\infty}\sum_{F=1}^{\infty} e^{(2-2g)/G}\,e^{-\mu\,\epsilon F}\;\mathcal{N}_{g,F} \tag{1.2.3}$$

where $\mathcal{N}_{g,F}$ denotes the total number of inequivalent triangulations of genus g built out of F triangles.

In the above, triangles do not play an essential role and may be replaced by any set of polygons leading to a similar expression as in eq. (1.2.3).

As we saw in the previous section $\mathcal{N}_{g,F}$ is calculated by the free energy of the Hermitian matrix model. In particular to establish contact to random triangulations consider the matrix model of eq. (1.1.1) with a cubic interaction, i.e. $g_2 = 1/2$, $g_3 = \lambda$ and $g_k = 0$ for $k > 3$. The dual to a Feynman diagram of this model (in which each face, edge and vertex is associated respectively to a dual vertex, edge and face) is identical to a random triangulation of some orientable Riemann surface (cf. Figure 2). Hence there is a one–to–one correspondence between Feynman diagram vertices and the faces of the random triangulation. By looking at the free energy of this cubic matrix model

$$N^2\,F[\lambda] = \sum_{g=0}^{\infty}\sum_{V=0}^{\infty} N^{2-2g}\,(-\lambda)^V\,F_{g,V}\,, \tag{1.2.4}$$

we see that by formally identifying

$$e^{1/G} = N \qquad \text{and} \qquad e^{-\mu\,\epsilon} = -\lambda \tag{1.2.5}$$

in eqs. (1.2.3) and (1.2.4) the free energy of the matrix model is actually the partition function of discrete 2d gravity

$$N^2\,F[g_k] = \mathcal{Z}_{\rm QG}^{\rm discr.}\,. \tag{1.2.6}$$

Needless to say that this argument goes through for general matrix potentials of eq. (1.1.3) as well, then describing "random polygonulations" of surfaces. In the continuum limit the result should not depend of the particular discretization used. However the different possibilities for generating vertices constitute additional degrees of freedom that can be realized as the coupling of 2d gravity to different varieties of matter. We shall see how this works in chapter III in the context of the m'th multicritical points [13].

[1] This constitutes the basic difference to Regge calculus, where the link lengths are the geometrical degrees of freedom.

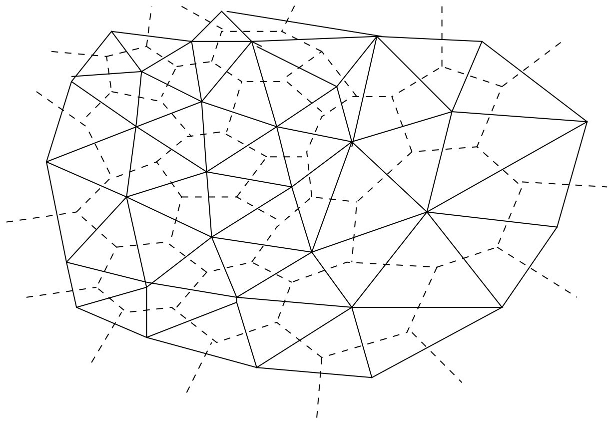


Figure 2: A piece of a random triangulation of a surface. Each of the triangular faces is dual to a three point vertex of a Hermitian matrix model.

Continuum Limit

In order to discuss the continuum limit we have to anticipate a result which will be derived in later chapters. The genus g contributions to the free energy of the matrix model are analytic functions of the couplings g_k around the origin. Singular behaviour arises at a critical surface in the space of couplings. Specializing again to the cubic model of eq. (1.2.4) this means that by moving the constant λ to its critical value λ_c the free energy becomes singular. In the vicinity of the critical point one finds the scaling behaviour

$$F_g[\lambda] \sim (\lambda - \lambda_c)^{(2-\gamma_0)\,(1-g)}, \tag{1.2.7}$$

in the case of the cubic potential the critical exponent γ_0 takes the value $\gamma_0 = -1/2$. We can extract the continuum limit of the random triangulation by tuning $\lambda \to \lambda_c$. This is because the average value of the area of the surface is given by

$$\langle A \rangle = \epsilon \, \langle V \rangle = \epsilon \, \lambda \, \frac{\partial}{\partial \lambda} \ln F_g[\lambda] \; \sim \; \frac{\epsilon}{\lambda - \lambda_c} \tag{1.2.8}$$

(recall that the area is proportional to the number of vertices V, appearing as the power of λ associated to each graph which has to be averaged with respect to $F_g[\lambda]$, as the free energy of the matrix models represents the partition function of 2d gravity, cf. eq. (1.2.6)). As $\lambda \to \lambda_c$, we see that $A \to \infty$ so that we may rescale the area of an individual triangle ϵ to zero, thus giving a continuum surface with finite area. Intuitively this is clear, by tuning the coupling constant to the point where the perturbation series diverges the sum becomes dominated by diagrams with infinite number of vertices, which is precisely what we need for the continuum limit.

The above reasoning constitutes by no way a proof that the matrix model in the continuum limit coincides with continuum 2d–gravity. However, one is able to compare properties of the partition function and correlation functions calculated by matrix models with results directly computable in the continuum formulation of Liouville theory. This gives implicit confirmation that the matrix model approach is sensible.

One such quantity computable in Liouville theory is the scaling behaviour of the Volume of $\Sigma_{g,A}$ of eq. (1.2.2)

$$\mathrm{Vol}(\Sigma_{g,A}) = e^{\mu_c \, A} \, A^{(1-g)\,(\gamma_{\mathrm{str.}}-2)-1} \, W_g, \tag{1.2.9}$$

where μ_c and W_h are undetermined constants. Here the critical exponent $\gamma_{\mathrm{str.}}$ is called the string susceptibility. The above formula even holds true if one couples conformal field theories with central charge c to Euclidean gravity. The Liouville theory prediction for the string susceptibility then is [12]

$$\gamma_{\mathrm{str.}} = \frac{1}{12}\,(c - 1 - \sqrt{(c-1)\,(c-25)}\,). \tag{1.2.10}$$

By plugging eq. (1.2.9) into eq. (1.2.2) and integrating over the area A we find [2].

$$\mathcal{Z}_{\mathrm{QG}} = \sum_{g=0}^{\infty} e^{(2-2g)/G} \, (\mu - \mu_c)^{(2-\gamma_{\mathrm{str.}})\,(1-g)} \; \Gamma[\,(g-1)\,(2-\gamma_{\mathrm{str.}})\,]\, W_g. \tag{1.2.11}$$

Thus we see that the scaling behaviours of the matrix model in eq. (1.2.7) and that of quantum gravity in eq. (1.2.11) are identical with $\gamma_o = \gamma_{\mathrm{str.}} = -1/2$. Comparing this with eq. (1.2.10), shows that we in fact are describing the $c = 0$ theory, i.e. “pure” 2d-Euclidean gravity. This result may be further confirmed by comparing planar correlation functions.

Double Scaling Limit

There is yet another limit we have to discuss. As a matter of fact the matrix model can only be solved in the $N \to \infty$ limit, as we shall see in the next chapters. But then due to eq. (1.1.16) only the planar contribution to the free energy survives. However, the successive contributions F_g all diverge at the critical value for the couplings $\lambda = \lambda_c$. This suggests that if we take the limits $N \to \infty$ and $\lambda \to \lambda_c$ not independently, but together in a correlated manner, we may compensate the large N high genus suppression with a $\lambda \to \lambda_c$ enhancement. This results in a coherent contribution from all genus surfaces.

[2] Simply use the definition of the gamma function $\Gamma(x) = \int_0^\infty dt\, e^{-t}\, t^{x-1}$.

We already saw the leading singular behaviour of F_g in eq. (1.2.7), i.e.

$$F_g[\lambda] = f_g\,(\lambda-\lambda_c)^{(2-\gamma_0)(1-g)}. \tag{1.2.12}$$

Then in terms of

$$\alpha = N^{-2}\,(\lambda-\lambda_c)^{\gamma_0-2} \tag{1.2.13}$$

the genus expansion (1.1.16) may be rewritten as

$$N^2\,F[\lambda] = \sum_{g=0}^{\infty} \alpha^{g-1}\,f_g. \tag{1.2.14}$$

The double scaling [14] limit is thus obtained by taking the limits $N\to\infty$, $\lambda\to\lambda_c$ while keeping fixed the "renormalized" string coupling constant α of eq. (1.2.13).

The above limit may be performed in the case of general potentials as well. In this case the additional degrees of freedom can be used to fine tune the couplings in such a manner as to adjust an alternative string susceptibility, i.e. at the m'th "multicritical" point we find $\gamma_0 = -1/m$. It may be shown that these theories then correspond to the $(2, 2m-1)$ non–unitary, minimal conformal field theories [3] coupled to quantum gravity and having the central charge $c = 1-3\,(3-2m)^2/(2m-1)$.

3. Virasoro Constraints and the Loop Equation

We now derive a set of constraint equations for the Hermitian matrix model. The generators of these constraints obey the Virasoro algebra, alluding at the integrability of the model as well as a conformal field theory formulation of it. This structure sets the basis for a supersymmetric generalization of the Hermitian one matrix model to the supereigenvalue model.

Consider the eigenvalue model (1.1.4) under the shift of integration variables $\lambda_i \mapsto \lambda_i + \epsilon\,{\lambda_i}^{n+1}$, with ϵ infinitesimal and $n\geq -1$ free. The resulting expression proportional to ϵ must equal zero. One finds

$$\epsilon\,\Big\langle\, N\sum_{k\geq 0} k\,g_k\,\sum_i {\lambda_i}^{k+n} + \sum_{k=0}^{n}\big(\sum_i {\lambda_i}^{n-k}\big)\big(\sum_j {\lambda_j}^{k}\big)\Big\rangle = 0 \qquad n\geq -1. \tag{1.3.1}$$

Of course this analysis may also be performed in the matrix formulation of eq. (1.1.1).

To see this look at the following variations

$$\delta\big(\prod_i d\lambda_i\big) = \sum_j (n+1)\,{\lambda_j}^n\,\prod_i d\lambda_i,$$

and

$$\delta\,V(\lambda_i) = \sum_{k\geq 0} k\,g_k\,{\lambda_i}^{k+n}.$$

For the van der Monde determinant we have $\delta\,\Delta(\lambda) = \Delta(\lambda)\,\sum_{i<j}\frac{\lambda_i^{n+1}-\lambda_j^{n+1}}{\lambda_i-\lambda_j}$, and thus

$$\delta\,\Delta^2(\lambda) = \Delta^2(\lambda)\sum_{i\neq j}\frac{\lambda_i^{n+1}-\lambda_j^{n+1}}{\lambda_i-\lambda_j} = \Delta^2(\lambda)\sum_{k=0}^{n}\big(\sum_i \lambda_i^{n-k}\big)\big(\sum_j \lambda_j^k\big) - \Delta^2(\lambda)\,(n+1)\sum_i \lambda_i^n.$$

Putting these three equations together yields eq. (1.3.1).

The Schwinger–Dyson equation (1.3.1) may be recast in the form [15]

$$\mathcal{L}_n\,\mathcal{Z}_N = 0 \qquad n\geq -1, \tag{1.3.2}$$

where $\mathcal{L}_n$ is a differential operator in the coupling constants

$$\mathcal{L}_n = \sum_{k\geq 0} k\,g_k\,\partial_{g_{k+n}} + \frac{1}{N^2}\sum_{k=0}^{n}\partial_{g_k}\,\partial_{g_{n-k}}. \tag{1.3.3}$$

In the case $n=-1$ the second sum simply vanishes. The operators $\mathcal{L}_n$ are generators of a closed subset of the Virasoro algebra

$$[\,\mathcal{L}_m\,,\,\mathcal{L}_n\,] = (m-n)\,\mathcal{L}_{m+n} \tag{1.3.4}$$

without central extension (remember that $n, m\geq -1$). This is the reason why (1.3.2) are called the Virasoro constraints, in bosonic string theory physical states obey an analogous set of constraints.

Conformal Field Theory Formulation

Given a complete set of constraints on a partition function which form a closed algebra, one might now ask the inverse question: What is the integral representation of the partition function obeying these constraints? In the case of the Virasoro algebra it is natural to look for an answer to this question in form of a correlation function in a conformal field theory [16]. The methods introduced in this subchapter can be easily generalized to the case of the super–Virasoro algebra considered in chapter III and lead to the construction of the supereigenvalue model.

Just as the generators $\mathcal{L}_n$ of eq. (1.3.2) the modes T_n of the energy–momentum tensor of a free scalar field obey the Virasoro algebra. Let us thus consider the simplest possible conformal field theory, a holomorphic scalar field

$$\phi(z) = \widehat{q} + \widehat{p}\,\ln z + \sum_{k\neq 0}\frac{\widehat{J}_{-k}}{k}\,z^k, \tag{1.3.5}$$

[3] These are classified by the pair (p,q) with $c = 1-6(p-q)^2/pq$.

with the commutation relations

$$[\widehat{J}_n, \widehat{J}_m] = n\,\delta_{n+m,0}, \qquad [\widehat{q}, \widehat{p}] = 1. \tag{1.3.6}$$

Define the vacuum states

$$\begin{aligned} \widehat{J}_k\,|0\rangle = 0 \qquad \langle N|\,\widehat{J}_{-k} = 0 \qquad k > 0 \\ \widehat{p}\,|0\rangle = 0, \end{aligned} \tag{1.3.7}$$

where $|N\rangle \equiv \exp[\sqrt{2}\,N\,\widehat{q}\,]\,|0\rangle$ and $\widehat{q}$ is anti–Hermitian. The energy–momentum tensor is given by

$$T(z) = \frac{1}{2}\,:[\partial\phi(z)]^2 : = \sum_{n\in\mathbb{Z}} T_n\,z^{-n-2},$$

$$T_n = \sum_{k>0} \widehat{J}_{-k}\,\widehat{J}_{k+n} + \frac{1}{2}\sum_{a+b=n} \widehat{J}_a\,\widehat{J}_b \qquad n \geq 0, \tag{1.3.8}$$

and we define a Hamiltonian by

$$H(g_k) = \frac{1}{\sqrt{2}}\sum_{k>0} g_k\,\widehat{J}_k = \frac{1}{\sqrt{2}}\oint_{C_0}\frac{dz}{2\pi i}\,V(z)\,\partial\phi(z), \tag{1.3.9}$$

where $V(z) = \sum_{k>0} g_k\,z^k$. With these definitions one shows that

$$\mathcal{L}_n\,\langle N|\,\exp[\,H(g_k)\,]\ldots = \langle N|\,\exp[\,H(g_k)\,]\,T_n\;\ldots. \tag{1.3.10}$$

So any operator $\mathcal{G}$ satisfying

$$[\,T_n, \mathcal{G}\,] = 0, \qquad n \geq -1, \tag{1.3.11}$$

will lead to [4]

$$\mathcal{L}_n\,\langle N|\,\exp[\,H(g_k)\,]\,\mathcal{G}\,|0\rangle = 0, \tag{1.3.12}$$

as $T_k\,|0\rangle = 0$. Therefore any nonvanishing correlator of this form is a candidate for the partition function

$$\mathcal{Z}_N = \langle N|\,\exp[\,H(g_k)\,]\,\mathcal{G}\,|0\rangle, \tag{1.3.13}$$

obeying the Virasoro constraints. Finding the operators $\mathcal{G}$ satisfying eq. (1.3.11) is an internal problem of conformal field theory. The solution is given by an arbitrary function of the screening charges $\mathcal{Q}_\pm$ which are of conformal dimension zero

$$\mathcal{Q}_\pm = \oint_C d\omega\;:\exp[\,\pm\sqrt{2}\,\phi(\omega)\,]:. \tag{1.3.14}$$

[4] Note that we have extended the definition of T_n to $n=-1$ by dropping the second term in eq. (1.3.8).

Choose $\mathcal{G} = \mathcal{Q}_+^N$ in order to get a nonvanishing correlator in eq. (1.3.13) [5]. We then have

$$\mathcal{Z}_N = \langle N|\;:\exp[\,\frac{1}{\sqrt{2}}\oint_{C_0} V(z)\,\partial\phi(z)\,]:\;\prod_{i=1}^N \oint_{C_i} dz_i\;:\exp[\,\sqrt{2}\,\phi(z_i)\,]:\;|0\rangle. \tag{1.3.15}$$

Using the operator product expansion $\phi(z)\,\phi(\tilde{z}) \sim \ln(z-\tilde{z})$ this is evaluated to

$$\mathcal{Z}_N = \prod_{i=1}^N \oint_{C_i} dz_i\;\exp[\sum_i V(z_i)\,]\,\prod_{i<j}(z_i - z_j)^2. \tag{1.3.16}$$

After deforming the contour integrals to integrals on the real line we thus recover the Hermitian matrix model in the eigenvalue representation of eq. (1.1.4).

To obtain eq. (1.3.16) from eq. (1.3.15) use $:e^A::e^B: = e^{\overbrace{AB}}\;:e^{A+B}:$, therefore

$$\begin{aligned}
\mathcal{Z}_N &= \Big(\prod_{i=1}^N \oint_{C_i} dz_i\Big)\,\langle N|\;:\exp[\,\frac{1}{\sqrt{2}}\oint_{C_0} V(z)\,\partial\phi(z) + \sqrt{2}\,\phi(z_1)\,]:\;\prod_{i=2}^N :\exp[\,\sqrt{2}\,\phi(z_i)\,]:\;|0\rangle\,e^{V(z_1)} \\
&= \prod_{i=1}^N \oint_{C_i} dz_i\,\langle N|\;:\exp[\,\frac{1}{\sqrt{2}}\oint_{C_0} V(z)\,\partial\phi(z) + \sqrt{2}\,\{\phi(z_1)+\phi(z_2)\}\,]:\;\prod_{i=3}^N :\exp[\,\sqrt{2}\,\phi(z_i)\,]:\;|0\rangle \\
&\quad\cdot e^{V(z_1)+V(z_2)}\,(z_1 - z_2)^2 \\
&\;\;\vdots \\
&= \prod_{i=1}^N \oint_{C_i} dz_i\,\langle N|\;:\exp[\,\frac{1}{\sqrt{2}}\oint_{C_0} V(z)\,\partial\phi(z) + \sqrt{2}\sum_i \phi(z_i)\,]:\;|0\rangle\,\exp[\sum_i V(z_i)\,]\,\prod_{i<j}(z_i - z_j)^2 \\
&= \prod_{i=1}^N \oint_{C_i} dz_i\,\langle N|\;:\exp[\,\sqrt{2}\,N\,\widehat{q}\,]:\;|0\rangle\,\exp[\sum_i V(z_i)\,]\,\prod_{i<j}(z_i - z_j)^2,
\end{aligned}$$

and as $\langle N|\;:\exp[\,\sqrt{2}\,N\,\widehat{q}\,]:\;|0\rangle = 1$ we have proven eq. (1.3.16).

Loop Equations

An immediate consequence of the Virasoro constraints is the loop equation, an integral equation for the one–loop correlator $W(p)$ of eq. (1.1.6). Multiply eq. (1.3.1) for n with

[5] More generally one could take $\mathcal{Q}_+^{N+M}\,\mathcal{Q}_-^M$, for more details see [16].

$1/p^{n+2}$ and sum up the obtained equations. The resulting geometric series [6] may be performed to give

$$\epsilon \Big\langle N \sum_i \frac{V'(\lambda_i)}{p-\lambda_i} - \Big[\sum_i \frac{1}{p-\lambda_i} \Big]^2 \Big\rangle = 0. \tag{1.3.17}$$

Note that generally $\langle \sum_i \mathcal{O}(\lambda_i)^2 \rangle_{\text{conn.}} = \langle \sum_i \mathcal{O}(\lambda_i)^2 \rangle - \langle \sum_i \mathcal{O}(\lambda_i) \rangle^2$ and hence

$$\Big\langle \Big[\sum_i \frac{1}{p-\lambda_i} \Big]^2 \Big\rangle = W(p,p) + N^2 W(p)^2, \tag{1.3.18}$$

with the definition of the loop–correlators of eq. (1.1.8). In order to rewrite the first term of eq. (1.3.17) in an integral form introduce the eigenvalue density

$$\rho(\lambda) = \frac{1}{N} \sum_i \langle \delta(\lambda - \lambda_i) \rangle. \tag{1.3.19}$$

The eigenvalue density has the remarkable property that it vanishes outside a support of $[y,x]$ on the real line in the limit of infinite matrix size, $N \to \infty$. This can be seen by studying the saddlepoint evaluation of the eigenvalue model (1.1.4) in this limit [11]. It is precisely this limit which we consider in the following, keeping all subleading terms in $1/N^2$ of the genus expansion. In this sense we will be able to solve the matrix model perturbatively by expanding around the point $N = \infty$. Each term of this expansion will be some function of the couplings g_k, which can be determined exactly. So the Hermitian matrix model will turn out to be solvable nonperturbatively in couplings but perturbatively in $1/N^2$.

By using the eigenvalue density of eq. (1.3.19) the first term of eq. (1.3.17) becomes

$$\begin{aligned} \langle N \sum_i \frac{V'(\lambda_i)}{p-\lambda_i} \rangle &= N^2 \int d\lambda\, \rho(\lambda)\, \frac{V'(\lambda)}{p-\lambda} \\ &= N^2 \int d\lambda\, \rho(\lambda) \oint_C \frac{d\omega}{2\pi i} \frac{1}{\omega-\lambda} \frac{V'(\omega)}{p-\omega}. \end{aligned} \tag{1.3.20}$$

Here the curve C of the contour integral runs around the cut $[x,y]$. Moreover p lies outside the curve. The integral over λ may now be performed in eq. (1.3.20) yielding the one–loop correlator $W(\omega)$. Combining this with the result of eq. (1.3.18) finally leads us to the loop equation of the Hermitian one–matrix model [17, 18, 19]

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} W(\omega) - W(p)^2 = \frac{1}{N^2} W(p,p). \tag{1.3.21}$$

[6] The subtlety of p larger or smaller than λ_i may be removed by directly considering a shift $\lambda_i \mapsto \lambda_i + \frac{\epsilon}{p-\lambda_i}$.

This integral equation basically captures the whole Hermitian matrix model. Due to eq. (1.1.11) we have $W(p,p) = \delta/\delta V(p)\, W(p)$. So the loop equation (1.3.21) is a closed equation for the one–loop correlator $W(p)$, determining this quantity. As explained in section 1 knowing $W(p)$ then means knowing all correlators of the model.

4. The Solution

There are various methods to solve the Hermitian one matrix model. Two of the most prominent ones are the saddlepoint evaluation [11] and the method of orthogonal polynomials [20]. The most effective solution, however, is based on an iterative procedure to solve the loop equation (1.3.21) genus by genus [9]. This method is the one which has found a generalization in the supereigenvalue model, so let us briefly review its basic concepts.

The loop equation (1.3.21) may be solved by making use of the genus expansion of $W(p)$

$$W(p) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(p). \tag{1.4.1}$$

Plugging this expansion into the loop equation (1.3.21) and comparing terms of common order in $1/N^2$ leads to a coupled hierarchy of equations for the $W_g(p)$. To leading order in $1/N^2$ one simply has

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} W_0(\omega) = W_0(p)^2. \tag{1.4.2}$$

From the definition of $W(p)$ in eq. (1.1.6) we know that asymptotically $W_0(p) = 1/p + \mathcal{O}(p^{-2})$ for $p \to \infty$. If one additionally assumes that the singularities of $W_0(p)$ consist of only one cut on the real axis [7] one finds [17]

$$W_0(p) = \frac{1}{2} \oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \Big[\frac{(p-x)(p-y)}{(\omega-x)(\omega-y)} \Big]^{1/2}, \tag{1.4.3}$$

where the endpoints of the eigenvalue distribution x and y are determined through the matrix potential in the following way:

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}} = 0, \tag{1.4.4}$$

$$\oint_C \frac{d\omega}{2\pi} \frac{\omega\, V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}} = 2, \tag{1.4.5}$$

[7] This follows from the relation of $W_0(p)$ to the eigenvalue density of eq. (1.3.19): $\rho(z) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} [W_0(z-i\epsilon) - W_0(z+i\epsilon)]$.

which are a direct consequence of $W(p) = 1/p + \mathcal{O}(p^{-2})$.

Let us verify the planar solution (1.4.3):

$$W_0(p)^2 = \frac{1}{4}\oint_{C_1}\frac{d\omega}{2\pi}\oint_{C_2}\frac{dz}{2\pi}\frac{V'(\omega)\,V'(z)}{(p-\omega)\,(p-z)}\,\frac{(p-x)\,(p-y)}{[\,(\omega-x)\,(\omega-y)\,(z-x)\,(z-y)\,]^{1/2}}.$$

Now rewrite $(p-x) = (p-\omega) + (\omega - x)$ in the numerator. By eq. (1.4.4) only the $(\omega - x)$ terms survives. Doing the same for $(p-y)$ gives

$$W_0(p)^2 = \frac{1}{4}\oint_{C_1}\frac{d\omega}{2\pi}\oint_{C_2}\frac{dz}{2\pi}\frac{V'(\omega)\,V'(z)}{(p-\omega)\,(p-z)}\,\frac{(\omega-x)\,(z-y)}{[\,(\omega-x)\,(\omega-y)\,(z-x)\,(z-y)\,]^{1/2}}. \tag{1.4.6}$$

Using

$$\frac{1}{p-\omega}\,\frac{1}{p-z} = \frac{1}{\omega-z}\,(\frac{1}{p-\omega} - \frac{1}{p-z})$$

in eq. (1.4.6) and renaming $\omega \leftrightarrow z$ in the resulting second term one finds

$$W_0(p)^2 = \frac{1}{4}\oint_{C_1}\frac{d\omega}{2\pi}\oint_{C_2}\frac{dz}{2\pi}\frac{V'(\omega)\,V'(z)}{(p-\omega)\,(\omega-z)}\,\frac{(\omega-x)\,(z-y)+(\omega-y)\,(z-x)}{[\,(\omega-x)\,(\omega-y)\,(z-x)\,(z-y)\,]^{1/2}},$$

(The process of pulling the curve C_2 over C_1 gives no contribution). Now we do the same trick again and write $(z-y) = (z-\omega) + (\omega - y)$ as well as $(z-x) = (z-\omega) + (\omega - x)$ to get

$$W_0(p)^2 = \frac{1}{2}\oint_{C_1}\frac{d\omega}{2\pi}\oint_{C_2}\frac{dz}{2\pi}\frac{V'(\omega)\,V'(z)}{(p-\omega)\,(\omega-z)}\left[\frac{(\omega-x)\,(\omega-y)}{(z-x)\,(z-y)}\right]^{1/2} = \oint_{C_1}\frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}\,W(\omega),$$

which proves eq. (1.4.2).

An interesting consequence of the planar solution (1.4.3) is the rather universal form of the planar two–loop correlator $W_0(p,p)$

$$W_0(p,p) = \frac{(x-y)^2}{16\,(p-x)^2\,(p-y)^2}, \tag{1.4.7}$$

depending on the matrix potential only through the endpoints of the eigenvalue distribution.

Let us now turn to the higher genus contributions of $W(p)$. By plugging the genus expansion (1.4.1) into the loop equation (1.3.21) it appears that the $W_g(p)$ for $g \geq 1$ obey the equation [9]

$$\widehat{\mathcal{V}'} \circ W_g(p) = \sum_{g'=1}^{g-1} W_{g'}(p)\,W_{g-g'}(p) + \frac{\delta}{\delta V(p)}\,W_{g-1}(p), \tag{1.4.8}$$

where we have introduced the linear operator $\widehat{\mathcal{V}'}$ by

$$\widehat{\mathcal{V}'} \circ f(p) = \oint_C \frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}\,f(\omega) - 2\,W_0(p)\,f(p). \tag{1.4.9}$$

In eq. (1.4.8) $W_g(p)$ is expressed entirely in terms of the $W_{g'}$, $g' < g$. This makes it possible to develop an iterative procedure to solve for $W_g(p)$ as done by Ambjørn, Chekhov, Kristjansen and Makeenko [9]. One basically has to find a way to invert the operator $\widehat{\mathcal{V}'}$ of the left hand side of eq. (1.4.8). It turns out that this is effectively done by expanding the $W_g(p)$ in a set of basis functions of $\widehat{\mathcal{V}'}$ augmented by a change of variables from coupling constants to moments of the potential. As we shall encounter precisely the same problem in our discussion of the iterative solution of the superloop equations for the supereigenvalue model in chapter III, let us postpone the analysis of this point.

Chapter II:

The $c = -2$ Matrix Model

To construct supersymmetric versions of matrix models it is necessary to augment the one matrix model of chapter I by fermionic degrees of freedom. There are various ways to do this. On the level of matrices one could think of replacing the Hermitian matrices $\boldsymbol{M}$ by supermatrices $\mathcal{M}$ of type $(N|M)$

$$\mathcal{M}_{AB} = \begin{pmatrix} X_{ij} & \Psi_{i\alpha} \\ \bar{\Psi}_{\alpha i} & Y_{\alpha\beta} \end{pmatrix} \qquad \begin{matrix} i,j = 1,\dots,N \\ \alpha,\beta = N+1,\dots,N+M, \end{matrix} \tag{2.1}$$

where the X_{ij} and $Y_{\alpha\beta}$ are Grassmann even and $\Psi_{i\alpha}$ and $\bar{\Psi}_{\alpha i}$ are Grassmann odd quantities. A supermatrix model is then naturally defined by the partition function of eq. (1.1.1) where one replaces the trace by a supertrace. This model has been studied in refs. [21], but the outcome is rather disappointing: The partition function $\mathcal{Z}_S(N|M)$ of such a supermatrix model of type $(N|M)$ turns out to be proportional to the partition function $\mathcal{Z}_S(N-M|0)$, which is nothing but an ordinary matrix model built out of $(N-M)\times(N-M)$ matrices [1].

In this chapter we shall address an alternative approach to this problem. Guided by the superspace formulation of supersymmetric field theories, one augments the zero dimensional space of the Hermitian matrix model by two anticommuting coordinates. The fields living in this space will be matrix valued superfields, whose component expansion consists of two bosonic and two fermionic matrices. Thinking of the interpretation of anticommuting coordinates as negative dimensions, this model should be considered as a $d=-2$ matrix model. The planar triangulations of closed surfaces in $d=-2$ dimensions was first studied on combinatorial grounds by Kazakov, Kostov and Mehta [22]. The matrix model approach to this problem was introduced by David [23] and further studied in refs. [24, 25]. And in fact in its scaling limit the matrix model does describe the coupling of $c=-2$ matter to 2d gravity.

[1] A negative $(N-M)$ corresponds to a change of sign in the exponent of eq. (1.1.1).

In our analysis of the model we shall stress its supersymmetric structure and discuss the emergence of a Nicolai–map to a Gaussian $d=0$ Hermitian matrix model. The diagrammatic interpretation of the model is investigated, leading us to its equivalence to random surfaces decorated by fermion loops. The one–, two– and three–point functions are computed in the case of a quartic potential, which has not appeared in literature so far. In the critical regime of this solution we recover the scaling exponents of $c=-2$ Liouville theory. Finally as a nice application of the solution we present the counting of the first few supersymmetric graphs, similar in fashion to the counting of bosonic planar diagrams in ref. [11].

1. The Model

In chapter I we studied Hermitian matrices living in zero dimensional space. As a method of supersymmetrization of matrix models, let us consider matrices living in superspace. In order to maintain exact solvability we introduce two fermionic coordinates $\bar{\theta}$ and θ, but keep the zero dimensional bosonic space. A matrix valued superfield Φ will then have the following component expansion

$$\Phi = \boldsymbol{\Phi} + \bar{\boldsymbol{\Psi}}\,\theta + \bar{\theta}\,\boldsymbol{\Psi} + \theta\bar{\theta}\,\boldsymbol{F}, \tag{2.1.1}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{F}$ are Hermitian $N\times N$ matrices and the $\boldsymbol{\Psi}$ and $\bar{\boldsymbol{\Psi}}$ are $N\times N$ matrices with complex Grassmann odd entries. We have $\boldsymbol{\Psi}^\dagger = \bar{\boldsymbol{\Psi}}$ and $\theta^* = \bar{\theta}$. Note that Φ is Hermitian.

The matrix model built out of Φ is given by the partition function

$$\mathcal{Z}_\Phi[\,g_k\,] = \int_{N\times N} \mathcal{D}\Phi\,\exp\Big[-N\,\mathrm{Tr}\,S[\,\Phi\,]\Big]. \tag{2.1.2}$$

As measure we take $\mathcal{D}\Phi = \mathcal{D}\boldsymbol{\Phi}\,\mathcal{D}\boldsymbol{F}\,\mathcal{D}\bar{\boldsymbol{\Psi}}\,\mathcal{D}\boldsymbol{\Psi}$, where the measure of the Hermitian matrices $\boldsymbol{\Phi}$ and $\boldsymbol{F}$ are as in eq. (1.1.2) and for the fermionic matrices we take

$$\mathcal{D}\bar{\boldsymbol{\Psi}}\,\mathcal{D}\boldsymbol{\Psi} = \prod_{\alpha,\beta=1}^{N} d\bar{\boldsymbol{\Psi}}_{\alpha\beta}\,d\boldsymbol{\Psi}_{\beta\alpha}. \tag{2.1.3}$$

The action of the Φ–matrix model of eq. (2.1.2) reads

$$S(\Phi) = \int d\theta d\bar\theta \Big\{ -\mathcal{D}\Phi\, \bar{\mathcal{D}}\Phi \,+\, \sum_{k=0}^{\infty} g_k\, \Phi^k \Big\}, \tag{2.1.4}$$

with the "superspace" derivatives $\mathcal{D} = \partial/\partial\theta$ and $\bar{\mathcal{D}} = \partial/\partial\bar\theta$. Bearing in mind the interpretation of anticommuting coordinates as negative dimensions it should be clear why eq. (2.1.2) is considered as a $d = -2$ matrix model.

The obvious extension of this model to one bosonic dimension, i.e. considering component fields $\boldsymbol{\Phi}(x)$, $\bar{\boldsymbol{\Psi}}(x)$, $\boldsymbol{\Psi}(x)$ and $\boldsymbol{F}(x)$ depending on one variable x, is known as the Marinari–Parisi superstring [26] [2] .

After performing the integrals over θ and $\bar\theta$ in the action (2.1.4) and using cyclicity under the trace one finds

$$\mathrm{Tr}\, S[\,\Phi\,] = -\mathrm{Tr}\, \boldsymbol{F}^2 + \mathrm{Tr}\, V'(\boldsymbol{\Phi})\, \boldsymbol{F} + \sum_{k=0}^{\infty} k\, g_k \sum_{a+b=k-2} \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^b\, \boldsymbol{\Psi}, \tag{2.1.5}$$

using the general matrix potential

$$V(\boldsymbol{\Phi}) = \sum_{g=0}^{\infty} g_k\, \boldsymbol{\Phi}^k. \tag{2.1.6}$$

As the auxiliary matrix $\boldsymbol{F}$ enters in the action $\mathrm{Tr}\, S[\,\Phi\,]$ of eq. (2.1.5) only quadratically, a shift in integration variables $\boldsymbol{F}_{\alpha\beta} \mapsto \boldsymbol{F}_{\alpha\beta} + 1/2\, V'(\boldsymbol{\Phi})_{\alpha\beta}$ lets the integral over $\boldsymbol{F}$ decouple. The Jacobian associated with this shift is unity, hence the integral over the auxiliary matrix $\boldsymbol{F}$ can be performed yielding an N dependent constant c_N. We are thus led to the effective action

$$\mathrm{Tr}\, S_{\text{eff}}[\,\Phi\,] = \frac{1}{4}\, \mathrm{Tr}\, V'(\boldsymbol{\Phi})^2 \,+\, \sum_{k=0}^{\infty} k\, g_k \sum_{a+b=n-2} \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^b\, \boldsymbol{\Psi}, \tag{2.1.7}$$

and the partition function now has the form

$$\mathcal{Z}_\Phi[\,g_k\,] = c_N \int_{N\times N} \mathcal{D}\boldsymbol{\Phi}\, \mathcal{D}\bar{\boldsymbol{\Psi}}\, \mathcal{D}\boldsymbol{\Psi}\; \exp\Big[-N\, \mathrm{Tr}\, S_{\text{eff}}[\,\Phi\,] \Big]. \tag{2.1.8}$$

The next obvious thing to do is to integrate out the fermionic matrices. Let us, however, first investigate the symmetries of this effective action.

[2] The question whether the continuum limit of this model describes a theory with target space supersymmetry is still unsettled [27].

Supersymmetry Transformations

After integrating out the auxiliary matrix $\boldsymbol{F}$ we find that the effective action of eq. (2.1.7) is invariant under the following set of (global) supersymmetry transformations

$$\begin{aligned} \delta\boldsymbol{\Phi}_{\alpha\beta} &= \bar\epsilon\, \boldsymbol{\Psi}_{\alpha\beta} + \bar{\boldsymbol{\Psi}}_{\alpha\beta}\, \epsilon \\ \delta\boldsymbol{\Psi}_{\alpha\beta} &= -\frac{1}{2}\, \epsilon\, V'(\boldsymbol{\Phi})_{\alpha\beta} \\ \delta\bar{\boldsymbol{\Psi}}_{\alpha\beta} &= -\frac{1}{2}\, \bar\epsilon\, V'(\boldsymbol{\Phi})_{\alpha\beta}, \end{aligned} \tag{2.1.9}$$

where ϵ and $\bar\epsilon$ are anti–commuting parameters.

This is easily verified. Consider the variation of eq. (2.1.7) under the transformations (2.1.9), then

$$\delta\Big[\, \frac{1}{4}\, \mathrm{Tr}\, [\, V'(\boldsymbol{\Phi})\,]^2 \,\Big] = \frac{1}{2}\, \mathrm{Tr}\, V'(\boldsymbol{\Phi})\, V''(\boldsymbol{\Phi})\, \Big(\, \bar\epsilon\boldsymbol{\Psi} + \bar{\boldsymbol{\Psi}}\, \epsilon \,\Big)$$

cancels with

$$\delta\Big[\sum_k g_k\, k \sum_{a+b=n-2} \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^b\, \boldsymbol{\Psi} \,\Big] = -\frac{1}{2}\, \mathrm{Tr}\, \bar\epsilon\, \boldsymbol{\Psi}\, V'(\boldsymbol{\Phi})\, V''(\boldsymbol{\Phi}) - \frac{1}{2}\, \mathrm{Tr}\, V'(\boldsymbol{\Phi})\, V''(\boldsymbol{\Phi})\, \bar{\boldsymbol{\Psi}}\, \epsilon + \{\, \delta\boldsymbol{\Phi}\text{'s}\,\}.$$

except for the variations $\{\,\delta\boldsymbol{\Phi}\text{'s}\,\}$, which cancel by themselves. See this by looking at the part of the transformations (2.1.9) proportional to $\bar\epsilon$

$$\begin{aligned} \{\,\delta\boldsymbol{\Phi}\text{'s}\,\}_{\bar\epsilon} &= \sum_{a+b=n-2} \mathrm{Tr}\, \delta_{\bar\epsilon}(\boldsymbol{\Phi}^a)\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^b\, \boldsymbol{\Psi} + \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar{\boldsymbol{\Psi}}\, \delta_{\bar\epsilon}(\boldsymbol{\Phi}^b)\, \boldsymbol{\Psi} \\ &= \sum_{a+b+c=n-3} \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar\epsilon\boldsymbol{\Psi}\, \boldsymbol{\Phi}^b\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^c\, \boldsymbol{\Psi} + \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \bar{\boldsymbol{\Psi}}\, \boldsymbol{\Phi}^b\, \bar\epsilon\boldsymbol{\Psi}\, \boldsymbol{\Phi}^c\, \boldsymbol{\Psi} = 0, \end{aligned}$$

the calculation for the part proportional to ϵ works analogously.

If one goes back to the form of the action including the auxiliary matrix $\boldsymbol{F}$ of eq. (2.1.5), the supersymmetry transformations become linear in the fields:

$$\begin{aligned} \delta\boldsymbol{\Phi}_{\alpha\beta} &= \bar\epsilon\, \boldsymbol{\Psi}_{\alpha\beta} + \bar{\boldsymbol{\Psi}}_{\alpha\beta}\, \epsilon \\ \delta\boldsymbol{\Psi}_{\alpha\beta} &= -\epsilon\, \boldsymbol{F}_{\alpha\beta} \\ \delta\bar{\boldsymbol{\Psi}}_{\alpha\beta} &= -\bar\epsilon\, \boldsymbol{F}_{\alpha\beta} \\ \delta\boldsymbol{F}_{\alpha\beta} &= 0. \end{aligned} \tag{2.1.10}$$

Note that the measure $\mathcal{D}\boldsymbol{\Phi}\, \mathcal{D}\bar{\boldsymbol{\Psi}}\, \mathcal{D}\boldsymbol{\Psi}$ is invariant under the supersymmetry transformation (2.1.9). This immediately follows from the form of the Jacobian of the transformations (2.1.9) which is zero in linear order of ϵ and $\bar\epsilon$.

Integrating out the Fermions

Just as in the Hermitian matrix model the action $\mathrm{Tr}\, S_{\mathrm{eff}}$ of eq. (2.1.7) is invariant under the simultaneous $U(N)$ transformations $\boldsymbol{\Phi} \mapsto U^\dagger \boldsymbol{\Phi} U$, $\boldsymbol{\Psi} \mapsto U^\dagger \boldsymbol{\Psi} U$ and $\bar{\boldsymbol{\Psi}} \mapsto U^\dagger \bar{\boldsymbol{\Psi}} U$. This may be employed to diagonalize the Hermitian matrix $\boldsymbol{\Phi} = U^\dagger \boldsymbol{\Phi}_{\boldsymbol{D}} U$, with $\boldsymbol{\Phi}_{\boldsymbol{D}} = \mathrm{diag}(\varphi_1, \ldots, \varphi_N)$. We again pick up the van der Monde determinant $\prod_{\alpha<\beta}(\varphi_\alpha - \varphi_\beta)^2$ as the Jacobian of this transformation. As the transformation matrix U does not depend on the fermionic matrices $\boldsymbol{\Psi}$ and $\bar{\boldsymbol{\Psi}}$, the Jacobian of their transformation is unity. We thus obtain the semi–eigenvalue picture of the Φ–model

$$\mathcal{Z}_\Phi[g_k] = \tilde{c}_N \int \prod_{\alpha=1}^{\infty} d\varphi_\alpha \prod_{\alpha,\beta=1}^{N} d\bar{\boldsymbol{\Psi}}_{\alpha\beta}\, d\boldsymbol{\Psi}_{\beta\alpha} \prod_{\alpha<\beta} (\varphi_\alpha - \varphi_\beta)^2 \exp\Big[-N\, S_{\mathrm{eff}}[g_k]\Big], \quad (2.1.11)$$

where the constant $\tilde{c}_N$ now contains the integral over $U(N)$ and the constant c_N of eq. (2.1.8). The partially diagonalized effective action reads

$$S_{\mathrm{eff}}[g_k] = \frac{1}{4} \sum_\alpha V'(\varphi_\alpha)^2 + \sum_k k\, g_k \sum_{a+b=k-2} \sum_{\alpha\beta} \varphi_\alpha^a\, \bar{\boldsymbol{\Psi}}_{\alpha\beta}\, \varphi_\beta^b\, \boldsymbol{\Psi}_{\beta\alpha}. \quad (2.1.12)$$

In this form the integration over the fermions may be performed rather easily. To do this introduce the abbreviation

$$C_{\alpha\beta} = N \sum_k k\, g_k \sum_{a+b=k-2} \varphi_\alpha^a\, \varphi_\beta^b = \begin{cases} N\, \frac{V'(\varphi_\alpha) - V'(\varphi_\beta)}{\varphi_\alpha - \varphi_\beta} & \text{if } \alpha \neq \beta \\ N\, V''(\varphi_\alpha) & \text{if } \alpha = \beta \end{cases}. \quad (2.1.13)$$

We then have

$$\begin{aligned} &\int \prod_{\alpha,\beta=1}^{N} d\bar{\boldsymbol{\Psi}}_{\alpha\beta}\, d\boldsymbol{\Psi}_{\alpha\beta} \exp\Big[-\sum_{\alpha,\beta} C_{\alpha\beta}\, \bar{\boldsymbol{\Psi}}_{\alpha\beta}\, \boldsymbol{\Psi}_{\beta\alpha}\Big] \\ &\qquad = \int \prod_{\alpha,\beta=1}^{N} d\bar{\boldsymbol{\Psi}}_{\alpha\beta}\, d\boldsymbol{\Psi}_{\alpha\beta} \prod_{\alpha,\beta}\Big[1 + C_{\alpha\beta}\, \boldsymbol{\Psi}_{\alpha\beta}\, \bar{\boldsymbol{\Psi}}_{\beta\alpha}\Big] \\ &\qquad = \prod_{\alpha,\beta}\Big[C_{\alpha\beta}\Big] \\ &\qquad = N^{N^2} \prod_\alpha [V''(\varphi_\alpha)] \prod_{\alpha<\beta} \Big[\frac{V'(\varphi_\alpha) - V'(\varphi_b)}{\varphi_\alpha - \varphi_\beta}\Big]^2. \end{aligned} \quad (2.1.14)$$

This is a rather astonishing result, as the van der Monde determinant will now drop out of the partition function. Plugging eq. (2.1.14) into eq. (2.1.11) yields the purely bosonic partition function

$$\mathcal{Z}_\Phi[g_k] = \hat{c}_N \prod_{\alpha=1}^{N} \Big(\int_{-\infty}^{\infty} d\varphi_\alpha\, V''(\varphi_\alpha)\Big) \prod_{\alpha<\beta} \Big[V'(\varphi_\alpha) - V'(\varphi_\beta)\Big]^2 \exp\Big[-\frac{N}{4} \sum_\alpha V'(\varphi_a)^2\Big]. \quad (2.1.15)$$

So by making the N independent substitutions

$$\lambda_\alpha = V'(\varphi_\alpha) \quad \Rightarrow \quad d\lambda_\alpha = V''(\varphi_\alpha)\, d\varphi_\alpha \qquad \alpha = 1, \ldots, N, \quad (2.1.16)$$

the partition function $\mathcal{Z}_\Phi[g_k]$ takes the Gaussian form

$$\mathcal{Z}_\Phi[g_k] = \hat{c}_N \prod_{\alpha=1}^{N} \Big(\int_{V'(-\infty)}^{V'(\infty)} d\lambda_\alpha\Big) \prod_{\alpha<\beta} (\lambda_\alpha - \lambda_\beta)^2\, e^{-\frac{N}{4}\sum_{\alpha=1}^{N} \lambda_a^2}, \quad (2.1.17)$$

note that this is true for any continuous function $V'(\varphi_\alpha)$.

We conclude that in the case of unbounded potentials $V'(-\infty) = V'(\infty)$ the partition function $\mathcal{Z}_\Phi$ vanishes. On the other hand for potentials with a lower bound, i.e. $V'(-\infty) = -\infty$ and $V'(\infty) = \infty$, the Φ–matrix model is proportional to a pure Gaussian Hermitian matrix model

$$\mathcal{Z}_\Phi[g_k] = \check{c}_N \int \mathcal{D}\boldsymbol{\Lambda} \exp\Big[-\frac{N}{4}\, \mathrm{Tr}\, \boldsymbol{\Lambda}^2\Big] = 2^N\, \pi^{N^2}, \quad (2.1.18)$$

and is completely independent of the coupling constants g_k! The result $2^N\, \pi^{N^2}$ is obtained by collecting all N dependent constants and making use of the result for the Gaussian partition function [20].

But this means that the partition function can obviously not have any critical behaviour. Moreover any integrated supersymmetric expectation value vanishes

$$\Big\langle \int d\theta\, d\bar{\theta}\, \mathrm{Tr}\, \Phi^n \Big\rangle = 0, \quad (2.1.19)$$

which is a direct consequence of differentiating $\mathcal{Z}_\Phi$ of eq. (2.1.2) with respect to g_n. On the level of the effective action (2.1.7) this Ward identity reads

$$\frac{1}{2} \Big\langle \mathrm{Tr}\, V'(\boldsymbol{\Phi})\, \boldsymbol{\Phi}^{n-1} \Big\rangle = \sum_{a+b=n-2} \Big\langle \mathrm{Tr}\, \boldsymbol{\Phi}^a\, \boldsymbol{\Psi}\, \boldsymbol{\Phi}^b\, \bar{\boldsymbol{\Psi}} \Big\rangle. \quad (2.1.20)$$

Nevertheless non–supersymmetric correlation functions like $\langle \mathrm{Tr}\, \phi^k \rangle$ are non–trivial and display critical behaviour as we shall see in the subsequent sections.

Nicolai–Map

The observed situation is very reminiscent to the Nicolai–map [28] of globally supersymmetric field theories: Integrating out the fermions in a supersymmetric field theory yields a determinant and an effective bosonic action. The vanishing of the vacuum energy of these theories alludes at their equivalence to a free theory. As a matter of fact one can show that there always exists a map from a free, Gaussian action to the effective bosonic action of the supersymmetric theory, whose Jacobian is identical to the determinant of the fermionic integration. On the level of correlation functions this implies that an operator expectation value of the supersymmetric theory may be calculated as a generically rather complicated operator expectation value in a simple free theory.

This scenario directly translates to the Φ–matrix model. The correlator $\langle \operatorname{Tr} \boldsymbol{\Phi}^k \rangle$ may be evaluated in the Hermitian matrix model with Gaussian measure $S_0 = -N/4 \operatorname{Tr} \boldsymbol{\Lambda}^2$ by using the inverse map of eq. (2.1.16)

$$\left\langle \operatorname{Tr} \boldsymbol{\Phi}^k \right\rangle = \left\langle\!\!\left\langle \operatorname{Tr} [\, V'^{\,-1}(\boldsymbol{\Lambda})\,]^k \right\rangle\!\!\right\rangle, \tag{2.1.21}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ denotes the expectation value in the free theory

$$\left\langle\!\!\left\langle\, \mathcal{O}(\boldsymbol{\Lambda}) \right\rangle\!\!\right\rangle = \frac{1}{\mathcal{Z}_0} \int_{N\times N} \mathcal{D}\boldsymbol{\Lambda}\, \mathcal{O}(\boldsymbol{\Lambda})\, \exp[-\frac{N}{4} \operatorname{Tr} \boldsymbol{\Lambda}^2\,]. \tag{2.1.22}$$

We shall exploit this relation in sections 3 and 4 for the calculation of bosonic one–, two– and three–point correlators.

2. Feynman Diagrams

Let us derive the Feynman rules for the Φ-matrix model. This will lead to a geometrical interpretation in terms of "fat" graphs, as we saw in the Hermitian matrix model in chapter I.

To be specific we consider a quartic potential of the form

$$V(\Phi) = \frac{1}{2}\,\Phi^2 + \frac{1}{4}\,g\,\Phi^4. \tag{2.2.1}$$

Note that the quartic potential is the simplest non–trivial bounded potential to consider, i.e. satisfying $V'(-\infty) = -\infty$ and $V'(\infty) = \infty$ for g positive. To our mind this point has not found adequate attention in the literature [23, 24, 25] where a cubic potential is assumed and the boundaries of integration in eq. (2.1.17) are set by hand to $-\infty$ and ∞.

With this quartic potential we are led to the effective action for the matrices $\boldsymbol{\Phi}$, $\bar{\boldsymbol{\Psi}}$ and $\boldsymbol{\Psi}$ via eq. (2.1.7)

$$\begin{aligned} \operatorname{Tr} S_{\text{eff.}} = \frac{1}{4} \operatorname{Tr} \boldsymbol{\Phi}^2 + \frac{1}{2}\, g \operatorname{Tr} \boldsymbol{\Phi}^4 + \frac{1}{4}\, g^2 \operatorname{Tr} \boldsymbol{\Phi}^6 \\ + \operatorname{Tr} \bar{\boldsymbol{\Psi}}\boldsymbol{\Psi} + g \left\{ \operatorname{Tr} \boldsymbol{\Phi}^2 \bar{\boldsymbol{\Psi}}\boldsymbol{\Psi} + \operatorname{Tr} \boldsymbol{\Phi}\bar{\boldsymbol{\Psi}}\boldsymbol{\Phi}\boldsymbol{\Psi} + \operatorname{Tr} \bar{\boldsymbol{\Psi}}\boldsymbol{\Phi}^2\boldsymbol{\Psi} \right\}. \end{aligned} \tag{2.2.2}$$

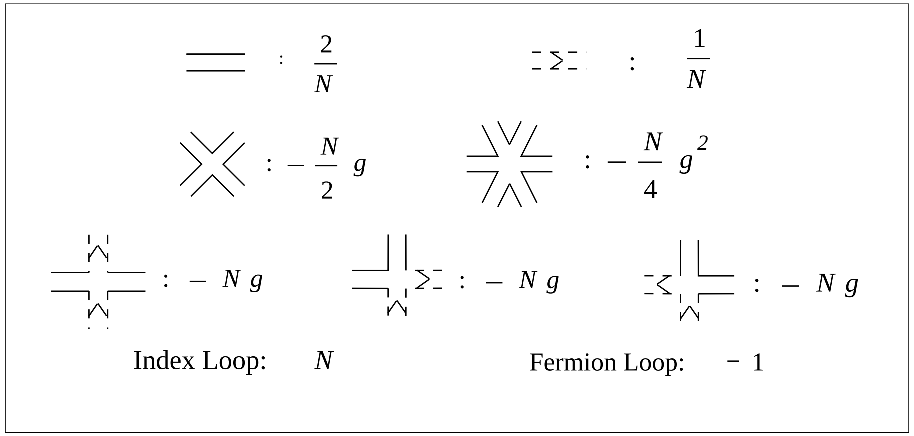


Figure 3: The Feynman rules for the quartic Φ–matrix model.

The bosonic and fermionic propagators read

$$\langle\, \boldsymbol{\Phi}_{\alpha\beta}\, \boldsymbol{\Phi}_{\gamma\delta}\, \rangle_{g=0} = \frac{2}{N}\, \delta_{\alpha\delta}\, \delta_{\beta\gamma}\,, \qquad \langle\, \boldsymbol{\Psi}_{\alpha\beta}\, \bar{\boldsymbol{\Psi}}_{\gamma\delta}\, \rangle_{g=0} = \frac{1}{N}\, \delta_{\alpha\delta}\, \delta_{\beta\gamma}\,, \tag{2.2.3}$$

and again they are represented by double lines each one corresponding to the separate propagation of matrix indices. Moreover one has to assign a direction of propagation to the fermions due to a necessary canonical ordering of the matrices $\boldsymbol{\Psi}$ and $\bar{\boldsymbol{\Psi}}$.

The form of the propagators in eq. (2.2.3) is directly derived by

$$\langle \boldsymbol{\Phi}_{\alpha\beta}\, \boldsymbol{\Phi}_{\gamma\delta}\, \rangle_{g=0} = \mathcal{Z}_\Phi^{-1} \int \mathcal{D}\boldsymbol{\Phi}\, \mathcal{D}\bar{\boldsymbol{\Psi}}\, \mathcal{D}\boldsymbol{\Psi}\, \left(-\frac{2}{N}\right) \boldsymbol{\Phi}_{\alpha\beta}\, \frac{\delta}{\delta \boldsymbol{\Phi}_{\delta\gamma}}\, \exp\!\left[-\frac{N}{4}\, \boldsymbol{\Phi}_{ij}\, \boldsymbol{\Phi}_{fi} - N\, \bar{\boldsymbol{\Psi}}_{ij}\, \boldsymbol{\Psi}_{ji}\right] = \frac{2}{N}\, \delta_{\alpha\delta}\, \delta_{\beta\gamma}$$

and

$$\langle\, \boldsymbol{\Psi}_{\alpha\beta}\, \bar{\boldsymbol{\Psi}}_{\gamma\delta}\, \rangle_{g=0} = \mathcal{Z}_\Phi^{-1} \int \mathcal{D}\boldsymbol{\Phi}\, \mathcal{D}\bar{\boldsymbol{\Psi}}\, \mathcal{D}\boldsymbol{\Psi}\, \left(\frac{1}{N}\right) \boldsymbol{\Psi}_{\alpha\beta}\, \frac{\delta}{\delta \boldsymbol{\Psi}_{\delta\gamma}}\, \exp\!\left[-\frac{N}{4}\, \boldsymbol{\Phi}_{ij}\, \boldsymbol{\Phi}_{fi} - N\, \bar{\boldsymbol{\Psi}}_{ij}\, \boldsymbol{\Psi}_{ji}\right] = \frac{1}{N}\, \delta_{\alpha\delta}\, \delta_{\beta\gamma}$$

via partial integration.

With the interactions of eq. (2.2.2) there will be a bosonic four–point vertex contributing $(-N/2\, g)$, a bosonic six–point vertex contributing $(-N/4\, g^2)$ and four–point Yukawa vertices contributing $(-N\, g)$. Again each loop of internal index will yield a factor of $N = \delta_{\alpha\alpha}$. Moreover there is a factor of (-1) for every closed fermionic propagator loop. To see this consider a fermionic loop in an arbitrary diagram built out of n Yukawa

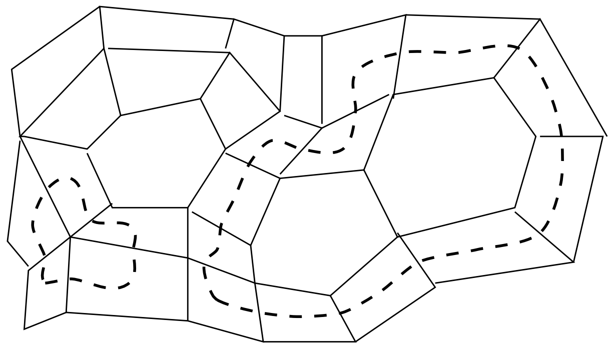
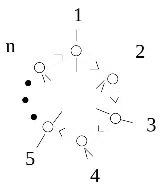


Figure 4: A dual graph of the Φ–matrix model.

vertices. We then have to evaluate the following expectation value, where the dots stand for bosonic propagators running out of the Yukawa vertices and all matrix indices are suppressed



$$\{\ldots\bar{\Psi}_1\ldots\Psi_1\ldots\bar{\Psi}_2\ldots\Psi_2\ldots\bar{\Psi}_3\ldots\Psi_3\ldots\bar{\Psi}_4\ldots\Psi_4\ldots\ldots\ldots\bar{\Psi}_n\ldots\Psi_n\ldots\}$$
$$= (-)\{\ldots\Psi_n\ldots\bar{\Psi}_1\ldots\Psi_1\ldots\bar{\Psi}_2\ldots\Psi_2\ldots\bar{\Psi}_3\ldots\Psi_3\ldots\ldots\ldots\bar{\Psi}_n\ldots\}.$$

One thus gets a factor of (-1) for every closed fermionic propagator loop.

Hence a diagram consisting of n_4 4–point and n_6 6–point bosonic vertices, n_F Yukawa–vertices, E_B boson propagators, E_F fermion propagators, L index loops and l fermion propagator loops contributes the factor

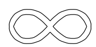
$$\mathrm{Diag} = (-)^l \left(-\frac{g}{2}\right)^{n_4} \left(-\frac{g^2}{4}\right)^{n_6} (-g)^{n_F}\, N^{n_4+n_6+n_F} \left(\frac{1}{N}\right)^{E_B+E_F} 2^{E_B}\, N^L \tag{2.2.4}$$

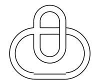
to the partition function. Using the Euler relation one again has the overall genus dependent N^{2-2g} weight for a graph of genus g. One may envisage the partition function $\mathcal{Z}_\Phi$ as coming from a Hermitian matrix model with a $(4,6)$ vertex interaction, where one draws l loops running over n_F 4–point vertices on the diagrams and weights each resulting diagram by a factor of $(-1)^l$. (This is true as n_F boson propagators are turned into fermion propagators yielding a factor of 2^{n_F}, which is exactly the factor needed to convert n_F bosonic 4–point vertices into Yukawa–vertices). Note that the loops drawn on the graph are not allowed to intersect or touch. Hence

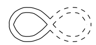
$$\mathcal{Z}_\Phi = \sum_{l=0}^{\infty} (-)^l\, \mathcal{Z}^{(l)}_{\mathrm{herm}}(g_2 = 1/4, g_4 = g/2, g_6 = g^2/4) \tag{2.2.5}$$

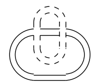
in the notation of eq. (1.1.3). Interestingly enough the loop drawing lets the sum in eq. (2.2.5) trivialize. On the level of the dual graphs we see that each diagram of $\mathcal{Z}_\Phi$ represents a discretized surface built from squares and hexagons with loops painted across squares.

As a nice check of the triviality of the partition function $\mathcal{Z}_\Phi$ of eq. (2.1.18) let us evaluate the $\mathcal{O}(g)$ vacuum graphs:

 : $[2]\left(-\frac{N}{2}g\right)\left(\frac{2}{N}\right)^2 N^3 = -4\,g\,N^2$

 : $[1]\left(-\frac{N}{2}g\right)\left(\frac{2}{N}\right)^2 N = -2\,g$

 : $-[2]\left(-N\,g\right)\left(\frac{2}{N}\right)\left(\frac{1}{N}\right)N^3 = 4\,g\,N^2$

 : $-[1]\left(-N\,g\right)\left(\frac{2}{N}\right)\left(\frac{1}{N}\right)N = 2\,g$

Keep in mind that the six–point vertex is of order g^2. The numbers in $[\ldots]$ denote the combinatorial factors. We see that the sum of all vacuum diagrams to first order g vanishes, as they should. Note the genus controlling factor of N^2 in the above graphs.

3. Bosonic One–Point Functions

As mentioned in section 1 bosonic one–point functions of the form $\langle\, \mathrm{Tr}\,\boldsymbol{\Phi}^k\,\rangle$ might display critical behaviour despite of the triviality of the partition function $\mathcal{Z}_\Phi$. To calculate these quantities one makes use of the Nicolai–map of eq. (2.1.21) to map these correlators onto correlators of a Hermitian matrix model with Gaussian potential.

Let us again assume the quartic matrix potential of eqs. (2.2.1) and (2.2.2). As it is symmetric all odd bosonic one–point correlators $\langle\, \mathrm{Tr}\,\boldsymbol{\Phi}^{2n+1}\,\rangle$ vanish. For the even correlators one has to solve the equation

$$V'(\varphi) = \varphi + g\,\varphi^3 = \lambda \tag{2.3.1}$$

for $\varphi(\lambda) = V'^{-1}(\lambda)$. For $g > 0$ there is only one real solution to eq. (2.3.1), whereas for $g < 0$ we have three. Picking the branch selected by smooth continuation of $g \to -g$ we find

$$V'^{-1}(\lambda) = \frac{1}{\sqrt{|g|}} \Big[\sqrt[3]{\frac{s\sqrt{|g|}}{2}\lambda + \sqrt{\frac{|g|}{4}\lambda^2 + \frac{s}{27}}} + \sqrt[3]{\frac{s\sqrt{|g|}}{2}\lambda - \sqrt{\frac{|g|}{4}\lambda^2 + \frac{s}{27}}} \Big], \quad (2.3.2)$$

where we have introduced $s = \mathrm{sign}(g)$.

As all the correlators $\langle\!\langle\, \mathrm{Tr}\, \boldsymbol{\Lambda}^{2k} \,\rangle\!\rangle$ in the free theory are computable, all one has to do in order to calculate the expectation values $\langle\, \mathrm{Tr}\, \boldsymbol{\Phi}^{2n} \,\rangle$ is to take the $2n$'th power of the result (2.3.2) and expand the resulting expression in λ. The outcome of this straightforward calculation is

$$\langle\!\langle\, \mathrm{Tr}\, V'^{-1}(\boldsymbol{\Lambda})^{2n} \,\rangle\!\rangle = \sum_{k=1}^{n} (-)^k \binom{2n}{n-k} \sum_{i=n-1}^{\infty} \left\{ \prod_{l=0}^{i} [\,(3l)^2 - k^2\,] \right\} \cdot 2\, \frac{(-3g)^{i+1-n}}{(2i+2)!} \langle\!\langle\, \mathrm{Tr}\, \boldsymbol{\Lambda}^{2i+2} \,\rangle\!\rangle, \quad (2.3.3)$$

which is true for positive and negative g.

In order to obtain this result, introduce the abbreviations

$$(+) \equiv \sqrt[3]{\frac{s\sqrt{|g|}}{2}\lambda + \sqrt{\frac{|g|}{4}\lambda^2 + \frac{s}{27}}}$$

$$(-) \equiv \sqrt[3]{\frac{s\sqrt{|g|}}{2}\lambda - \sqrt{\frac{|g|}{4}\lambda^2 + \frac{s}{27}}}.$$

Then $V'^{-1}(\lambda)^{2n} = [\,(+) + (-)\,]^{2n}/|g|^n$. Using $(+) \cdot (-) = -s/3$ we have

$$\Big[(+) + (-)\Big]^{2n} = \sum_{k=1}^{n} \binom{2n}{n+k} \left(-\frac{s}{3}\right)^{n-k} \Big[(+)^{2k} + (-)^{2k}\Big] + \binom{2n}{n} \left(-\frac{s}{3}\right)^{n}. \quad (2.3.4)$$

Consider the case $s = 1$. With the expansion [29]

$$(x + \sqrt{1+x^2})^q + (x - \sqrt{1+x^2})^q = 2(1 + \sum_{i=0}^{\infty} \frac{q^2\,[\,q^2 - 2^2\,] \ldots [\,q^2 - (2i)^2\,]}{(2i+2)!} x^{2i+2}),$$

and eq. (2.3.4) one arrives at

$$V'^{-1}(\lambda)^{2n} = \sum_{k=1}^{n} 2\,(-)^{n-k} \binom{2n}{n+k} \Big[(3g)^{-n} + \sum_{i=0}^{\infty} \left\{ \prod_{l=0}^{i} [\,k^2 - (3l)^2\,] \right\} \frac{(3g)^{i+1-n}}{(2i+2)!} \lambda^{2i+2} \Big] + \binom{2n}{n} (-)^n\, (3g)^{-n}. \quad (2.3.5)$$

With the help of the identities

$$\sum_{k=1}^{n} \binom{2n}{n+k} (-)^k = -\frac{1}{2} \binom{2n}{n}$$

and

$$\sum_{k=1}^{n} \binom{2n}{n+k} (-)^k\, k^{2q} = 0 \qquad \text{for} \qquad n > q > 0,$$

which one may prove, one recovers the result of eq. (2.3.3) from eq. (2.3.5). A similar analysis is possible for $s = -1$ yielding the same result as eq. (2.3.3).

Gaussian One–Point Functions

In order to calculate the one–point functions of the Gaussian matrix model $\langle\!\langle\, \mathrm{Tr}\, \boldsymbol{\Lambda}^{2k} \,\rangle\!\rangle$ one can use the iterative solution of the loop equations of ref. [9] sketched in chapter I. However, due to the simplicity of the Gaussian model it is more convenient to employ the method of orthogonal polynomials [20]. Here the evaluation of $\langle\!\langle\, \mathrm{Tr}\, \boldsymbol{\Lambda}^{2k} \,\rangle\!\rangle$ for all genera reduces to an integral involving Hermite polynomials. This problem was solved by Kostov and Mehta [24] and we simply cite their result.

For the matrix potential $1/4\, \boldsymbol{\Lambda}^2$ of eq. (2.1.22) the result reads

$$\frac{1}{N} \left\langle\!\!\left\langle\, \mathrm{Tr}\, \boldsymbol{\Lambda}^{2i+2} \,\right\rangle\!\!\right\rangle = 2^{i+1} \frac{(2i+2)!}{(i+1)!\,(i+2)!} \mathcal{P}_{i+1}(2N), \quad (2.3.6)$$

where $\mathcal{P}_n(2N)$ is a polynomial in $(2N)^{-2}$

$$\mathcal{P}_n(2N) = \sum_{j=0}^{[n/2]} a_{nj} \frac{1}{(2N)^{2j}}, \quad (2.3.7)$$

whose coefficients a_{nj} are defined by the recursion relation

$$a_{n+1,\,j} = \sum_{k=2j-1}^{n} k\,(k+1)\, a_{k-1,\,j-1}, \quad (2.3.8)$$

with $a_{n0} = 1$. Note the genus expansion recovered in eq. (2.3.7). The fact that one has a recursive solution for higher genera contributions is not very astounding if one recalls the general iterative solution sketched in chapter I.

The first few a_{nj} read

$$a_{n0} = 1$$

$$
\begin{aligned}
a_{n1} &= \frac{1}{3}\,(n+1)\,n\,(n-1) \\
a_{n2} &= \frac{1}{90}\,(n+1)\,n\,(n-1)\,(n-2)\,(n-3)\,[5n-2] \\
a_{n3} &= \frac{1}{5670}\,(n+1)\,n\,\ldots\,(n-5)\,[35n^2-77n+12] \\
a_{n4} &= \frac{1}{340200}\,(n+1)\,n\,\ldots\,(n-7)\,[175n^3-945n^2+1094n-72] \\
&\ \vdots \\
a_{nj} &= (n+1)\,n\,\ldots\,(n-2j+1)\,[\text{Polynomial of degree}\, n^{j-1}\,].
\end{aligned}
\tag{2.3.9}
$$

Results for all Genera

Putting the results of the previous subsections together, i.e. plugging eq. (2.3.6) into eq. (2.3.3), yields the bosonic one–point correlators of the Φ–matrix model with quartic potential for all genera

$$
G_n = \frac{1}{N}\left\langle \operatorname{Tr}\boldsymbol{\Phi}^{2n}\right\rangle = \sum_{h=0}^{\infty} G_n^h\,\frac{1}{N^{2h}}.
\tag{2.3.10}
$$

Let us define

$$
A_j^n = \sum_{k=1}^{n}(-)^k\binom{2n}{n-k}\prod_{l=0}^{j+n-1}[\,(3l)^2-k^2\,].
\tag{2.3.11}
$$

Note that A_j^1 simplifies to $A_j^1 = 3^{-j}\,(3j+1)!/(j!)$. Then we have the genus h correlators G_n^h(g):

$$
\begin{aligned}
G_n^0 &= \sum_{j=0}^{\infty} A_j^n\,\frac{2^{\,j+n+1}}{(j+n)!\,(j+n+1)!}\,(-3g)^j \\
G_n^1 &= \sum_{j=\kappa(n,h)}^{\infty} A_j^n\,\frac{2^{\,j+n-1}}{(j+n)!\,(j+n-2)!}\,\frac{1}{3}\,(-3g)^j \\
G_n^2 &= \sum_{j=\kappa(n,h)}^{\infty} A_j^n\,\frac{2^{\,j+n-3}}{(j+n)!\,(j+n-4)!}\,\frac{[\,5(j+n)-2\,]}{90}\,(-3g)^j \\
G_n^3 &= \sum_{j=\kappa(n,h)}^{\infty} A_j^n\,\frac{2^{\,j+n-5}}{(j+n)!\,(j+n-6)!}\,\frac{[\,35(j+n)^2-77(j+n)+12\,]}{5670}\,(-3g)^j \\
G_n^4 &= \sum_{j=\kappa(n,h)}^{\infty} A_j^n\,\frac{2^{\,j+n-7}}{(j+n)!\,(j+n-8)!} \\
&\quad\cdot\frac{[\,175(j+n)^3-945(j+n)^2+1094(j+n)-72\,]}{340200}\,(-3g)^j \\
&\ \vdots \\
G_n^h &= \sum_{j=\kappa(n,h)}^{\infty} A_j^n\,\frac{2^{\,j+n-(2h-1)}}{(j+n)!\,(j+n+1)!}\,a_{j+n,h}\,(-3g)^j\,,
\end{aligned}
\tag{2.3.12}
$$

where we have introduced $\kappa=\max(0,2h-n)$ as the lower bound of summations. The correlators G_n^h all have the same radius of convergence of $|g_c|=1/54$. Moreover the planar $G_n^0(g=g_c)$ are seen to converge at the critical point, whereas the higher genus correlators $G_n^{h>0}(g=g_c)$ diverge at the critical value of the coupling constant.

Let us now compute the asymptotic form of the one–point correlators $G_n^h(g)$. The factors A_j^n of eq. (2.3.11) behave as

$$
A_j^n \sim 9^{\,j+n-1}\Big[\,(j+n-1)!\,\Big]^2,
\tag{2.3.13}
$$

for $j\gg 1$. Using this we compute the scaling behaviour of the one–point functions

$$
\begin{aligned}
G_n^h &\simeq \sum_{j\gg 1} A_j^n\,\frac{2^{j+n-2h+1}}{(j+n)!\,(j+n-2h)!}\,j^{h-1}\,(-3g)^j \\
&\simeq 2^{n-2h+1}\,9^{n-1}\sum_{j\gg 1}\frac{(j+n-1)\,(j+n-2)\,\ldots\,(j+n-2h+1)}{(j+n)}\,j^{h-1}\,(-54g\,)^j \\
&\sim \sum_{j\gg 1} j^{3h-3}\left(-\frac{g}{g_c}\right)^j,
\end{aligned}
\tag{2.3.14}
$$

with $g_c=1/54$. Hence the correlators scale like

$$
G_n^h(g)\sim (\,g_c-g\,)^{2-3h},
\tag{2.3.15}
$$

and are independent of n. The bosonic one–point correlators G_n^h are to be interpreted as sum over random surfaces of genus h with an n–gon hole as a boundary. In fact the form of the boundary becomes irrelevant at the critical point of eq. (2.3.15). One now argues that the sum over surfaces without boundaries is obtained by integrating eq. (2.3.15) over g. The resulting scaling behaviour

$$
\int dg\,G_n^h(g)\sim(\,g_c-g\,)^{[\,2-(-1)\,]\,(1-h)}
\tag{2.3.16}
$$

may then with the help of eq. (1.2.7) be associated with a string susceptibility of $\gamma_{\text{str.}}=-1$, which via eq. (1.2.10) corresponds to a central charge of $c=-2$. In this sense we have verified the claim that the Φ–matrix model of eq. (2.1.2) describes random surfaces embedded in -2 dimensions. The double scaling limit of this theory and the connection to the Liouville approach of this problem is studied in refs. [25]. Let us remark that despite the supersymmetric structure of this model it does not show any correspondence to two dimensional supergravity.

The results for G_n^h encode as well all the fermionic one–point correlators $\sum_{a+b=n}\operatorname{Tr}\boldsymbol{\Phi}^a\boldsymbol{\Psi}\boldsymbol{\Phi}^b\bar{\boldsymbol{\Psi}}$ if one makes use of the Ward identities presented in eq. (2.1.20).

4. Planar Two– and Three–Point Functions

The calculation of higher point functions goes along the same lines. However, in this section we shall be less ambitious and study only planar contributions. We keep the quartic potential of eq. (2.2.1).

For the two–point functions there are now two nonvanishing types of correlators possible: The doubly even powers in $\langle\operatorname{Tr}\boldsymbol{\Phi}^{2k}\,\boldsymbol{\Phi}^{2l}\rangle$ and the doubly odd ones in $\langle\operatorname{Tr}\boldsymbol{\Phi}^{2k-1}\,\boldsymbol{\Phi}^{2l-1}\rangle$. For the computation of the last type we will have to know the power expansion of $V'^{-1}(\lambda)^{2n+1}$ in order to apply the Nicolai–map. This is just another exercise in elementary algebra, take the $(2n+1)$'st power of eq. (2.3.2) and expand the result in λ. The outcome of this calculation is

$$\operatorname{Tr}V'^{-1}(\boldsymbol{\Lambda})^{2n+1}=\sum_{k=0}^{n}(-)^k\,(2k+1)\,\binom{2n+1}{n-k}\sum_{l=n}^{\infty}\left\{\prod_{i=1}^{l}[\,3i-k-2\,]\,[\,3i+k-1\,]\right\}$$
$$\cdot\,2\,\frac{(-3g)^{l-n}}{(2l+1)!}\operatorname{Tr}\boldsymbol{\Lambda}^{2l+1}. \tag{2.4.1}$$

The next ingredient needed are the two point functions of the Gaussian matrix model. Here there is no nice closed form known in the literature. The planar result, however, may be obtained rather easily from the planar one–loop correlator of eq. (1.4.3) through application of the loop insertion operator as discussed in eq. (1.1.11) of chapter I. Specializing to the potential $1/4\operatorname{Tr}\boldsymbol{\Lambda}^2$ one finds the connected, planar two point functions

$$\left\langle\!\!\left\langle\operatorname{Tr}\boldsymbol{\Lambda}^{2i}\operatorname{Tr}\boldsymbol{\Lambda}^{2l}\right\rangle\!\!\right\rangle_{\text{conn.}}=2^{\,i+l}\,\frac{i\cdot l}{i+l}\,\binom{2i}{i}\,\binom{2l}{l}$$
$$\left\langle\!\!\left\langle\operatorname{Tr}\boldsymbol{\Lambda}^{2i-1}\operatorname{Tr}\boldsymbol{\Lambda}^{2l-1}\right\rangle\!\!\right\rangle_{\text{conn.}}=2^{\,i+l-3}\,\frac{i\cdot l}{i+l-1}\,\binom{2i}{i}\,\binom{2l}{l}, \tag{2.4.2}$$

through an expansion of $W_0(p,q)$ in the loop parameters p and q. From this one may now compose the planar bosonic–two point functions

$$\left\langle\operatorname{Tr}\boldsymbol{\Phi}^{2m}\operatorname{Tr}\boldsymbol{\Phi}^{2n}\right\rangle_{\text{conn.}}=\left\langle\!\!\left\langle\operatorname{Tr}V'^{-1}(\boldsymbol{\Lambda})^{2m}\operatorname{Tr}V'^{-1}(\boldsymbol{\Lambda})^{2n}\right\rangle\!\!\right\rangle_{\text{conn.}}$$
$$\left\langle\operatorname{Tr}\boldsymbol{\Phi}^{2m-1}\operatorname{Tr}\boldsymbol{\Phi}^{2n-1}\right\rangle_{\text{conn.}}=\left\langle\!\!\left\langle\operatorname{Tr}V'^{-1}(\boldsymbol{\Lambda})^{2m-1}\operatorname{Tr}V'^{-1}(\boldsymbol{\Lambda})^{2n-1}\right\rangle\!\!\right\rangle_{\text{conn.}}, \tag{2.4.3}$$

by plugging eqs. (2.3.3), (2.4.1) and (2.4.2) into the right hand sides of eq. (2.4.3). We do not write out the results explicitly, as they are lengthy and not too instructive.

For the evaluation of the planar three–point functions take $W_0(p,q,r)$ and expand in p, q and r to read off the planar, connected three–point functions of the Gaussian model [30]

$$\left\langle\!\!\left\langle\operatorname{Tr}\boldsymbol{\Lambda}^{2i}\operatorname{Tr}\boldsymbol{\Lambda}^{2j-1}\operatorname{Tr}\boldsymbol{\Lambda}^{2k-1}\right\rangle\!\!\right\rangle_{\text{conn.}}=2^{\,i+j+k-3}\,i\cdot j\cdot k\,\binom{2i}{i}\,\binom{2j}{j}\,\binom{2k}{k}$$
$$\left\langle\!\!\left\langle\operatorname{Tr}\boldsymbol{\Lambda}^{2i}\operatorname{Tr}\boldsymbol{\Lambda}^{2j}\operatorname{Tr}\boldsymbol{\Lambda}^{2k}\right\rangle\!\!\right\rangle_{\text{conn.}}=2^{\,i+j+k}\,i\cdot j\cdot k\,\binom{2i}{i}\,\binom{2j}{j}\,\binom{2k}{k}. \tag{2.4.4}$$

And we may again compose the three–point functions of the Φ–matrix model by plugging these results into the appropriate combinations of eqs. (2.3.3) and (2.4.1).

Needless to mention that these results may be employed for the calculation of higher–point amplitudes involving fermions by making use of higher–point Ward identities. These are obtained from the one–point Ward identity of eq. (2.1.20) by appropriate differentiation with respect to g_m, e.g.

$$\frac{1}{2}\left\langle\operatorname{Tr}\boldsymbol{\Phi}^{n+m-2}\right\rangle+\frac{1}{4}\left\langle\operatorname{Tr}V'(\boldsymbol{\Phi})\,\boldsymbol{\Phi}^{m-1}\operatorname{Tr}V'(\boldsymbol{\Phi})\,\boldsymbol{\Phi}^{n-1}\right\rangle=$$
$$\frac{1}{2}\sum_{a+b=n-2}\left\langle\operatorname{Tr}V'(\boldsymbol{\Phi})\,\boldsymbol{\Phi}^{m-1}\operatorname{Tr}\boldsymbol{\Phi}^a\,\boldsymbol{\Psi}\,\boldsymbol{\Phi}^b\,\bar{\boldsymbol{\Psi}}\right\rangle+\frac{1}{2}\sum_{a+b=m-2}\left\langle\operatorname{Tr}V'(\boldsymbol{\Phi})\,\boldsymbol{\Phi}^{n-1}\operatorname{Tr}\boldsymbol{\Phi}^a\,\boldsymbol{\Psi}\,\boldsymbol{\Phi}^b\,\bar{\boldsymbol{\Psi}}\right\rangle$$
$$-\sum_{\substack{a+b=m-2\\c+d=n-2}}\left\langle\operatorname{Tr}\boldsymbol{\Phi}^a\,\boldsymbol{\Psi}\,\boldsymbol{\Phi}^b\,\bar{\boldsymbol{\Psi}}\operatorname{Tr}\boldsymbol{\Phi}^c\,\boldsymbol{\Psi}\,\boldsymbol{\Phi}^d\,\bar{\boldsymbol{\Psi}}\right\rangle. \tag{2.4.5}$$

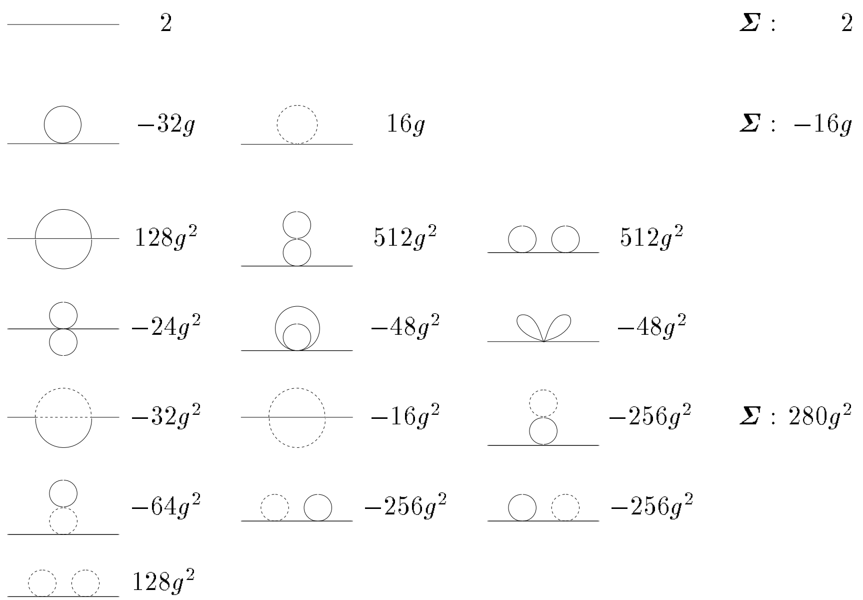
5. Supersymmetric Graph Counting

Quite similar to the classical paper on planar diagram counting of Brézin, Itzykson, Parisi and Zuber [11] the results of eq. (2.3.12) may be viewed as a solution to the combinatorial problem of supersymmetric graph counting with the Feynman rules derived in section 2.

Let us first consider the planar diagrams of this theory. For the counting of all diagrams with 2–bosonic legs we have to write down $G_1^0(g)$ of eq. (2.3.12) in the first few orders of g

$$G_1^0(g) = \lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} \boldsymbol{\Phi}^2 \rangle = 2 - 16\,g + 280\,g^2 + \ldots . \tag{2.5.1}$$

These terms correspond to the planar diagrams



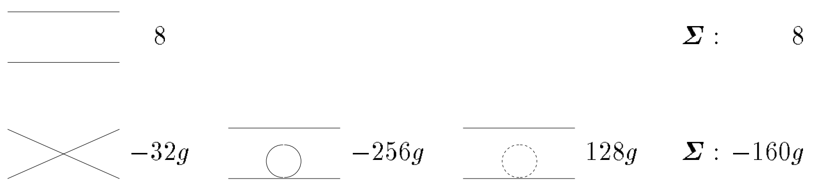
(2.5.2)

This constitutes an independent combinatorial check of our results.

Similarly one can look at the amplitudes with 4–bosonic legs of planar topology by expanding G_2^0(g)

$$G_2^0(g) = \lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} \boldsymbol{\Phi}^4 \rangle = 8 - 160\,g + 4032\,g^2 + \ldots , \tag{2.5.3}$$

whose first two terms stem from the planar graphs



(2.5.4)

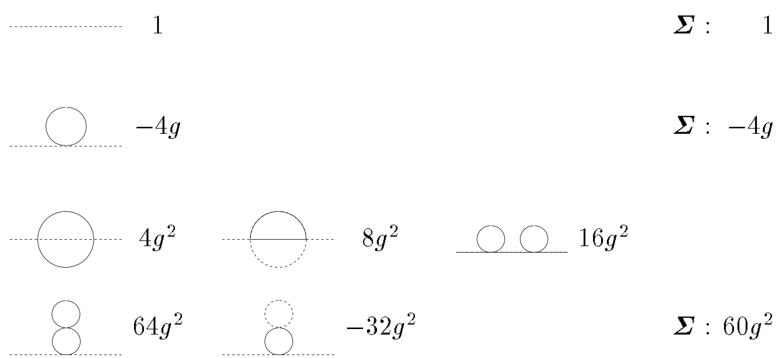
Planar diagrams with 2–fermionic legs are counted with the help of the observed Ward identities of eq. (2.1.20). For our quartic potential we have

$$\langle \operatorname{Tr} \boldsymbol{\Psi} \bar{\boldsymbol{\Psi}} \rangle = \frac{1}{2} \langle \operatorname{Tr} \boldsymbol{\Phi}^2 \rangle + \frac{g}{2} \langle \operatorname{Tr} \boldsymbol{\Phi}^4 \rangle . \tag{2.5.5}$$

Hence with the results of eqs. (2.5.1) and (2.5.3) the sum of the planar amplitudes containing 2–fermionic legs reads

$$\lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} \boldsymbol{\Psi} \bar{\boldsymbol{\Psi}} \rangle = 1 - 4\,g + 60\,g^2 + \ldots , \tag{2.5.6}$$

and may be seen to originate from the diagrams



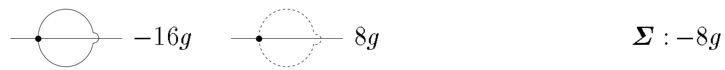
(2.5.7)

confirming our calculations.

Of course our results allow the counting of non–planar diagrams as well. For example the toroidal diagrams with 2–bosonic legs are contained in $G_1^1(g)$, i.e.

$$G_1^1(g) = \lim_{N\to\infty} N \langle \operatorname{Tr} \boldsymbol{\Phi}^2 \rangle = -8\,g + 560\,g^2 + \ldots , \tag{2.5.8}$$

whose first term stems from the toroidal diagrams



(2.5.9)

This concludes our analysis of the Φ–matrix model.

6. Conclusions

In this chapter we have studied the Φ–matrix model, defined by a matrix valued superfield in (0,2)-dimensional superspace. After integrating out the auxiliary and fermionic degrees of freedom its partition function was shown to be identical to the partition function of a Hermitian matrix model with pure Gaussian potential. These two models are related to each other through a Nicolai–map. The analysis of the Feynman rules of the Φ–matrix model revealed that its dual graphs may be interpreted as discretized random surfaces decorated with sign producing loops. Subsequently the one point–functions of the Φ–matrix model with quartic potential were calculated for all genera, as well as the planar two– and three–point functions. Despite of the triviality of the partition function, these correlators are non–trivial and display critical behaviour. The scaling limit of these correlators reproduced the string susceptibility exponent of $c=-2$ Liouville theory. Finally we showed how these results represent the solution to the combinatorial problem of supersymmetric graph counting.

Chapter III:
The Supereigenvalue Model

Following the successful application of Hermitian matrix models to 2d gravity and lower dimensional bosonic strings, it was natural to ask how these methods could be generalized to the supersymmetric case. Unfortunately the description of discretized 2d random surfaces by the Hermitian matrix model to date has no analogue in terms of some supersymmetric matrix model describing a discretization of super–Riemann surfaces. In a sense that might not be too astounding as there exists no geometrical picture of superspace.

Nevertheless the supereigenvalue model proposed by Alvarez–Gaumé, Itoyama, Mañes and Zadra [31] appears to have all the virtues of a discrete approach to 2d quantum supergravity. Guided by the prominent role the Virasoro constraints (cf. eq. (1.3.2)) played for the Hermitian matrix model, the authors constructed a partition function built out of N Grassmann even and odd variables (the "supereigenvalues") obeying a set of super–Virasoro constraints. In order to formulate these constraints it is necessary to augment the bosonic coupling constants g_k by a set of anticommuting couplings usually denoted by $\xi_{k+1/2}$. As it is unknown whether there exists a matrix–based formulation of this model, we do not have a geometric interpretation of it at hand. Nevertheless many of the well known features of the Hermitian matrix model, such as the genus expansion, the loop equations, the double scaling limit, the moment description and the loop insertion operators, find their supersymmetric counterparts in the supereigenvalue model. From this point of view the supereigenvalue model appears as the natural supersymmetric generalization of the Hermitian one matrix model.

The supereigenvalue model is solvable nonperturbatively in coupling constants but perturbatively in its genus expansion. Again the problem of solving the model may be reformulated in a set of superloop equations obeyed by the superloop correlators. Away from the double scaling limit these equations were first solved for general potentials in the planar limit in ref. [32]. We were able to develop a complete iterative solution of these equations for all genera and general potentials in ref. [6]. The key point in this scheme is the change of variables from coupling constants to the so–called moments of the bosonic and fermionic potentials, thus generalizing the approach of ref. [9] for the Hermitian matrix model. An alternative approach was pursued by the authors of refs. [33, 34] who managed to directly integrate out the Grassmann odd variables on the level of the partition function. No generalization of a solution based on orthogonal polynomials is known. The supereigenvalue model also displays a connection to supersymmetric integrable hierarchies [35].

In order to make contact to continuum physics the supereigenvalue model has to be studied in its double scaling limit. This was done in refs. [31, 36, 32] for planar (and partially for higher genus) topologies. These studies revealed that in its continuum limit the supereigenvalue model describes the coupling of minimal superconformal field theories (of type $(2, 4m)$) to 2d supergravity. Moreover Abdalla and Zadra [37] were able to develop a dictionary between $N = 1$ super–Liouville amplitudes and supereigenvalue correlators. It turns out that the moment description is very useful for the identification of the critical points in the space of coupling constants. In ref. [7] we applied this method to the scaling limit, and presented an improved iterative scheme to directly compute only the higher genus terms relevant in the double scaling limit.

In the following we derive the supereigenvalue model directly from the super–Virasoro constraints in a superconformal field theory formulation of the problem. After some short remarks on the general structure of the free energy, we introduce superloop correlators and the superloop insertion operators. The superloop equations are derived and solved through the iterative procedure discussed above. We then turn to the double scaling limit, identify the scaling of moments and basis functions and present our improved iterative scheme. Finally some remarks on the identification of the double scaled supereigenvalue model with $N = 1$ super–Liouville theory are made.

1. Super–Virasoro Generators

As we have discussed, we wish to regard the super–Virasoro constraints as the fundamental property of a supersymmetric generalization of the Hermitian one matrix model. The Virasoro generators of the Hermitian matrix model in eq. (1.3.3) are differential operators in the coupling constants g_k. In order to write down an analogous set of super–Virasoro generators it will be necessary to introduce anticommuting coupling constants $\xi_{k+1/2}$ as the "superpartners" of the bosonic couplings g_k. The super–Virasoro generators are then given by [31]:

$$\begin{aligned}
\mathcal{G}_{n+1/2} &= \sum_{k\geq 0} k\, g_k\, \partial_{\xi_{k+n+1/2}} + \sum_{k\geq 0} \xi_{k+1/2}\, \partial_{g_{k+n+1}} + \frac{1}{N^2}\sum_{k=0}^{n} \partial_{\xi_{k+1/2}}\, \partial_{g_{n-k}}, \qquad n\geq -1,\\
\mathcal{L}_n &= \sum_{k\geq 0} k\, g_k\, \partial_{g_{k+n}} + \frac{1}{2N^2}\sum_{k=0}^{n} \partial_{g_k}\, \partial_{g_{n-k}} + \sum_{k\geq 0}\Big(k+\frac{n+1}{2}\Big)\, \xi_{k+1/2}\, \partial_{\xi_{k+n+1/2}}\\
&\quad + \frac{1}{2N^2}\sum_{k=0}^{n-1} k\, \partial_{\xi_{n-k-1/2}}\, \partial_{\xi_{k+1/2}}, \qquad n\geq -1. \qquad (3.1.1)
\end{aligned}$$

Note that in the above we define $\sum_{k=0}^{-1}\ldots\equiv 0$. The super–Virasoro generators satisfy

$$\begin{aligned}
[\,\mathcal{L}_n\,,\,\mathcal{L}_m\,] &= (n-m)\,\mathcal{L}_{n+m}\\
[\,\mathcal{L}_n\,,\,\mathcal{G}_{m+1/2}\,] &= (n/2-1/2-m)\,\mathcal{G}_{n+m+1/2}\\
\{\mathcal{G}_{n+1/2}\,,\,\mathcal{G}_{m+1/2}\} &= 2\,\mathcal{L}_{n+m+1}, \qquad (3.1.2)
\end{aligned}$$

which constitutes a closed subalgebra of the $N=1$ superconformal algebra in the Neveu–Schwarz sector (remember that $n,m\geq -1$).

2. Construction of the Model

The supereigenvalue model will now be constructed from the requirement that its partition function $\mathcal{Z}_S[\,g_k,\xi_{k+1/2}]$ is annihilated by the $\{\mathcal{G}_{n+1/2},\mathcal{L}_n;\ n\geq -1\}$. However, equation (3.1.2) shows that it is sufficient to impose

$$\mathcal{G}_{n+1/2}\,\mathcal{Z}_S = 0, \qquad n\geq -1, \qquad (3.2.1)$$

since the constraints $\mathcal{L}_n\,\mathcal{Z}_S=0$ for $n\geq -1$ come out of the consistency conditions of eq. (3.2.1) using the algebra (3.1.2).

The method to construct $\mathcal{Z}_S[\,g_k,\xi_{k+1/2}]$ is based on a comment made in ref. [38] to generalize the conformal field theory formulation of matrix models discussed in chapter I. We seek for a representation of $\mathcal{Z}_S[\,g_k,\xi_{k+1/2}]$ in the form of a correlator in a superconformal field theory. Let us consider a free, holomorphic superfield $\mathcal{X}(z,\theta)=\phi(z)+\theta\,\psi(z)$ with the mode expansions

$$\begin{aligned}
\phi(z) &= \widehat{q}+\widehat{p}\,\ln z - \sum_{k\neq 0}\frac{\widehat{J}_k}{k}\,z^{-k},\\
\psi(z) &= \sum_{k\in\mathbb{Z}} \widehat{\psi}_{k+1/2}\,z^{-k-1}, \qquad (3.2.2)
\end{aligned}$$

whose modes obey the (anti)–commutation relations

$$\begin{aligned}
&[\,\widehat{J}_n\,,\,\widehat{J}_m\,] = n\,\delta_{n+m,0}, && [\,\widehat{q}\,,\,\widehat{p}\,]=1,\\
&\{\,\widehat{\psi}_{n+1/2}\,,\,\widehat{\psi}_{m-1/2}\,\} = \delta_{n+m,0}, && [\,\widehat{J}_n\,,\,\widehat{\psi}_m\,]=[\,\widehat{q}\,,\,\widehat{\psi}_m\,]=[\,\widehat{p}\,,\,\widehat{\psi}_m\,]=0.
\end{aligned} \qquad (3.2.3)$$

Define the vacuum states

$$\begin{aligned}
&\widehat{J}_k\,|0\rangle = 0 && \langle 2N|\,\widehat{J}_{-k}=0 && k>0\\
&\widehat{p}\,|0\rangle = 0 && &&\\
&\widehat{\psi}_{n+1/2}\,|0\rangle = 0 && \langle 2N|\,\widehat{\psi}_{-n-1/2}=0 && n\geq 0, \qquad (3.2.4)
\end{aligned}$$

where $|2N\rangle\equiv\exp[\,2N\,\widehat{q}\,]\,|0\rangle$ and $\widehat{q}$ is anti–Hermitian. The super–energy–momentum tensor of this theory is given by

$$\mathcal{T}(z,\theta) = T^{\mathcal{F}}(z) + 2\,\theta\,T^{\mathcal{B}}(z) = \psi\,\partial\phi + \theta : (\partial\phi\,\partial\phi + \psi\,\partial\psi):, \qquad (3.2.5)$$

whose fermionic part has the mode expansion

$$\begin{aligned}
T^{\mathcal{F}}(z) &= \sum_{n\in\mathbb{Z}} T^{\mathcal{F}}_{n+1/2}\,z^{-n-2}\\
T^{\mathcal{F}}_{n+1/2} &= \sum_{k>0}\Big(\widehat{J}_{-k}\,\widehat{\psi}_{k+n+1/2}+\widehat{\psi}_{-k+1/2}\,\widehat{J}_{k+n}\Big) + \sum_{a+b=n}\widehat{\psi}_{a+1/2}\,\widehat{J}_b, \qquad n\geq 0. \qquad (3.2.6)
\end{aligned}$$

Similar to the bosonic case we introduce a Hamiltonian built from Grassmann even and odd coupling constants

$$H(g_k,\xi_{k+1/2}) = \sum_{k\geq 0}\Big(g_k\,\widehat{J}_k + \xi_{k+1/2}\,\widehat{\psi}_{k+1/2}\Big) = \oint_{C_0}\frac{dz}{2\pi i}\Big(V(z)\,\partial\phi(z)+\Psi(z)\,\psi(z)\Big), \qquad (3.2.7)$$

where $V(z)=\sum_{k\geq 0}g_k\,z^k$ and $\Psi(z)=\sum_{k\geq 0}\xi_{k+1/2}\,z^k$. Using these definitions one shows that

$$\mathcal{G}_{n+1/2}\,\langle 2N|\,\exp[\,H(g_k,\xi_{k+1/2})\,]\,\ldots = \langle 2N|\,\exp[\,H(g_k,\xi_{k+1/2})\,]\,T^{\mathcal{F}}_{n+1/2}\,\ldots . \qquad (3.2.8)$$

Hence any operator $\mathcal{O}$ satisfying [1]

$$[\,T^{\mathcal{F}}_{n+1/2}\,,\,\mathcal{O}\,]=0, \qquad n\geq -1, \qquad (3.2.9)$$

[1] We again extend the definition of $T^{\mathcal{F}}_{n+1/2}$ to $n=-1$ by dropping the second term in the second eq. of (3.2.6)

will give us a correlator obeying the super–Virasoro constraints

$$\mathcal{G}_{n+1/2}\,\langle 2N|\,\exp[\,H(g_k,\xi_{k+1/2})\,]\,\mathcal{O}\,|0\rangle = 0, \tag{3.2.10}$$

as $T^{\mathcal{F}}_{n+1/2}\,|0\rangle = 0$. Thus every nonvanishing correlator of this form may be used for the definition of the partition function

$$\mathcal{Z}_S = \langle 2N|\,\exp[\,H(g_k,\xi_{k+1/2})\,]\,\mathcal{O}\,|0\rangle. \tag{3.2.11}$$

The problem of finding the operators $\mathcal{O}$ obeying eq. (3.2.9) is solved in superconformal field theory. It turns out that any function of the super screening charges $\mathcal{Q}_\pm$ will do

$$\mathcal{Q}_\pm = \oint_C dz \int d\theta\ :\exp[\,\pm\mathcal{X}(z,\theta)\,]:, \tag{3.2.12}$$

where $\mathcal{X}(z,\theta)$ again denotes the holomorphic superfield. We choose $\mathcal{Q}_+^{2N}$ to get a nonvanishing correlator in eq. (3.2.11) [2]. Note that we take an even $2N$ in order to have a Grassmann even partition function $\mathcal{Z}_S$. The model is then defined by

$$\mathcal{Z}_S = \langle 2N|\ :\exp[\,H(g_k,\xi_{k+1/2})\,]\prod_{i=1}^{2N}\oint_{C_i} dz_i \int d\theta_i\ :\exp[\,\phi(z_i)+\theta_i\,\psi(z_i)\,]:\ |0\rangle. \tag{3.2.13}$$

This correlator may be evaluated by using the operator product expansions $\phi(z)\,\phi(\omega)\sim\ln(z-\omega)$ and $\psi(z)\,\psi(\omega)\sim(z-\omega)^{-1}$. The result is

$$\mathcal{Z}_S = \prod_{i=1}^{2N}\oint_{C_i} dz_i \int d\theta_i\ \exp\Big[\sum_{i=1}^{2N}\Big(V(z_i)+\theta_i\,\Psi(z_i)\Big)\Big]\prod_{i<j}(z_i-z_j-\theta_i\theta_j). \tag{3.2.14}$$

Note the resemblance of this model to the eigenvalue description of the Hermitian matrix model of eqs. (1.1.4) and (1.3.16). This is the reason for calling $\mathcal{Z}_S$ the "supereigenvalue model" despite of the fact that the z_i and θ_i are no eigenvalues of any matrix[3]. It was first constructed in ref. [31] using different methods.

The derivation of eq. (3.2.14) goes along the same lines as the bosonic case of eq. (1.3.16). Using $:e^A::e^B:=e^{\underbrace{A\,B}}\ :e^{A+B}:$ we have

$$\begin{aligned}
\mathcal{Z}_S &= \Big(\prod_{i=1}^{2N}\oint_{C_i} dz_i \int d\theta_i\Big)\,\langle 2N|\ :\exp[\oint_{C_0}\{\,V(z)\,\partial\phi(z)+\Psi(z)\,\psi(z)\,\}+\phi(z_1)+\theta_1\,\psi(z_1)\,]:\\
&\quad\cdot\prod_{i=2}^{2N}:\exp[\,\phi(z_i)+\theta_i\,\psi(z_i)\,]:\ |0\rangle\ \cdot\ e^{V(z_1)+\theta_1\,\Psi(z_1)}\\
&= \Big(\prod_{i=1}^{2N}\oint_{C_i} dz_i \int d\theta_i\Big)\,\langle 2N|\ :\exp[\oint_{C_0}[\,V(z)\,\partial\phi(z)+\Psi(z)\,\psi(z)\,]+\sum_{j=1}^{2}\phi(z_j)+\theta_j\,\psi(z_j)\,]:\\
&\quad\cdot\prod_{i=3}^{2N}:\exp[\,\phi(z_i)+\theta_i\,\psi(z_i)\,]:\ |0\rangle\, e^{\ln(z_1-z_2)-\frac{\theta_1\theta_2}{z_1-z_2}}\ \exp[\sum_{j=1}^{2}V(z_j)+\theta_j\,\Psi(z_j)]\\
&\ \ \vdots\\
&= \Big(\prod_{i=1}^{2N}\oint_{C_i} dz_i \int d\theta_i\Big)\,\langle 2N|\ :\exp[\oint_{C_0}[\,V(z)\,\partial\phi(z)+\Psi(z)\,\psi(z)\,]+\sum_{j=1}^{2N}\phi(z_j)+\theta_j\,\psi(z_j)\,]:\ |0\rangle\\
&\quad\cdot\prod_{i<j}(z_i-z_j-\theta_i\theta_j)\ \exp[\sum_{j=1}^{2N}V(z_j)+\theta_j\,\Psi(z_j)]\\
&= \Big(\prod_{i=1}^{2N}\oint_{C_i}\int d\theta_i\Big)\prod_{i<j}(z_i-z_j-\theta_i\theta_j)\ \exp[\sum_{j=1}^{2N}V(z_j)+\theta_j\,\Psi(z_j)]\ \underbrace{\langle 2N|\ :\exp[\,2N\hat{q}\,]:\ |0\rangle}_{=1}
\end{aligned}$$

and we recover eq. (3.2.14).

3. Structure of the Free Energy

After deforming the contours of integration in eq. (3.2.14) to the real line we recover the supereigenvalue model proposed by Alvarez–Gaumé, Itoyama, Mañes and Zadra [31]. After a slight change of notation the partition function is given by

$$\mathcal{Z}_S = \prod_{i=1}^{2N}\Big(\int_{-\infty}^{\infty} d\lambda_i \int d\theta_i\Big)\prod_{i<j}(\,\lambda_i-\lambda_j-\theta_i\theta_j\,)\ \exp\Big[-2N\sum_{i=1}^{2N}\Big(\,V(\lambda_i)-\theta_i\,\Psi(\lambda_i)\,\Big)\Big], \tag{3.3.1}$$

built from a set of $2N$ bosonic and fermionic variables λ_i and θ_i respectively. The bosonic and fermionic potentials read

$$V(\lambda_i) = \sum_{k=0}^{\infty} g_k\,{\lambda_i}^k, \qquad\text{and}\qquad \Psi(\lambda_i) = \sum_{k=0}^{\infty}\xi_{k+1/2}\,{\lambda_i}^k, \tag{3.3.2}$$

with the Grassmann even and odd coupling constants g_k and $\xi_{k+1/2}$ respectively. The free energy of the supereigenvalue model is naturally defined by

$$\mathcal{Z}_S(g_k,\xi_{k+1/2};\,2N) = e^{\,(2N)^2\,F(g_k,\xi_{k+1/2};\,2N)}. \tag{3.3.3}$$

[2] More generally one could take $\mathcal{Q}_+^{2N+M}\,\mathcal{Q}_-^{M}$, supereigenvalue models constructed in this fashion have not been analyzed so far.

[3] The only attempt in literature to formulate eq. (3.3.1) in terms of matrices may be found in ref. [39], where an explicit solution to this problem was found only for low N.

Before developing an iterative procedure to solve the supereigenvalue model, let us discuss the fermionic structure of the free energy. Consider the supereigenvalue model of eq. (3.3.1) in the absence of fermionic couplings, i.e. $\xi_{k+1/2} = 0$. The integration over the fermionic variables θ_i may then be performed [4] giving rise to the effective bosonic eigenvalue model

$$\mathcal{Z}_S(g_k, 0; 2N) = \prod_{i=1}^{2N}\Big(\int_{-\infty}^{\infty} d\lambda_i\Big) \prod_{i<j}(\lambda_i - \lambda_j)\,\mathrm{Pfaff}\,(\lambda_{ij}^{-1})\,\exp\Big[-2N\sum_{i=1}^{2N} V(\lambda_i)\Big], \tag{3.3.4}$$

where $\mathrm{Pfaff}\,(\lambda_{ij}^{-1})$ is the Pfaffian of the antisymmetric matrix

$$\lambda_{ij}^{-1} = \frac{1}{\lambda_i - \lambda_j}, \qquad i \neq j$$

$$\mathrm{Pfaff}\,(A) = \sqrt{\det A} = \frac{1}{2^N\, N!}\,\epsilon^{i_1 i_2 \ldots i_{2N}}\, A_{i_1 i_2} A_{i_3 i_4} \ldots A_{i_{2N-1} i_{2N}}. \tag{3.3.5}$$

There exists an interesting identity between the effective bosonic model of eq. (3.3.4) and the Hermitian matrix model

$$\mathcal{Z}_S(g_k, 0; 2N) = c(N)\,\Big[\,\mathcal{Z}_B(2g_k; N)\Big]^2, \tag{3.3.6}$$

where $\mathcal{Z}_B(2g_k; N)$ is the partition function of the purely bosonic $N \times N$ Hermitian matrix model of eq. (1.1.1). We prove this identity in the following.

The statement we wish to prove is

$$\int\Big(\prod_{i=1}^{2N} d\mu(\lambda_i)\Big)\,\mathrm{Pfaff}\,(\lambda_{ij}^{-1})\,\Delta(\lambda_1,\ldots,\lambda_{2N}) =$$

$$c(N)\int\Big(\prod_{i=1}^{2N} d\mu(\lambda_i)\Big)\,\Delta^2(\lambda_1,\ldots,\lambda_N)\,\Delta^2(\lambda_{N+1},\ldots,\lambda_{2N}), \tag{3.3.7}$$

where $\mu(\lambda_i)$ is a measure factor depending only on the eigenvalue λ_i, $\Delta(\lambda_1,\ldots,\lambda_M)$ denotes the van der Monde determinant and $c(N)$ some irrelevant N dependent factor.

We prove eq. (3.3.7) by induction in N. First note that due to the antisymmetry of $\Delta(\lambda_1,\ldots,\lambda_{2N})$ under the exchange of two eigenvalues we have

$$\int\Big(\prod_{i=1}^{2N} d\mu(\lambda_i)\Big)\,\mathrm{Pfaff}\,(\lambda_{ij}^{-1})\,\Delta(\lambda_1,\ldots,\lambda_{2N}) =$$

$$\int\Big(\prod_{i=1}^{2N} d\mu(\lambda_i)\Big)\,\Delta(\lambda_1,\ldots,\lambda_{2N})\,\frac{1}{\lambda_1-\lambda_2}\,\frac{1}{\lambda_3-\lambda_4}\cdots\frac{1}{\lambda_{2N-1}-\lambda_{2N}}, \tag{3.3.8}$$

up to irrelevant factors depending on N. Moreover from $\Delta(\lambda) = (-)^{N(N-1)/2}\det(\lambda_i{}^{j-1})$ we have the identity

$$\int\Big(\prod_{i=1}^{N} d\mu(\lambda_i)\Big)\,\Delta^2(\lambda_1,\ldots,\lambda_N) = \int\Big(\prod_{i=1}^{N} d\mu(\lambda_i)\Big)\,\Delta(\lambda_1,\ldots,\lambda_N)\prod_{i=1}^{N}\lambda_i{}^{i-1}. \tag{3.3.9}$$

Let us now show that eq. (3.3.7) is true for $N = 2$

$$\begin{aligned} Z_4 &= \int\Big(\prod_{i=1}^{4} d\mu(\lambda_i)\Big)\,(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4) \\ &= \int\Big(\prod_{i=1}^{4} d\mu(\lambda_i)\Big)\,\det\begin{pmatrix}1 & \lambda_1\\ 1 & \lambda_3\end{pmatrix}\det\begin{pmatrix}1 & \lambda_2\\ 1 & \lambda_4\end{pmatrix}(\lambda_2-\lambda_3)(\lambda_1-\lambda_4) \\ &= 2\int\Big(\prod_{i=1}^{4} d\mu(\lambda_i)\Big)\Big[\det\begin{pmatrix}1 & \lambda_1\\ \lambda_3 & \lambda_3{}^2\end{pmatrix}\det\begin{pmatrix}1 & \lambda_2\\ \lambda_4 & \lambda_4{}^2\end{pmatrix} - \det\begin{pmatrix}\lambda_1 & \lambda_1{}^2\\ \lambda_3 & \lambda_3{}^2\end{pmatrix}\det\begin{pmatrix}1 & \lambda_2\\ 1 & \lambda_4\end{pmatrix}\Big], \end{aligned}$$

the second term in the last expression vanishes. Applying eq. (3.3.9) to the first term then proves the assumption (3.3.7) for $N = 2$.

Now by using the induction hypothesis the left hand side of eq. (3.3.7) for $(N+1)$ becomes

$$\begin{aligned} Z_{2N+2} &= \int\Big(\prod_{i=1}^{2N+2} d\mu(\lambda_i)\Big)\,\Delta^2(\lambda_1,\ldots,\lambda_N)\,\Delta^2(\lambda_{N+1},\ldots,\lambda_{2N})\prod_{i=1}^{2N}(\lambda_i-\lambda_{2N+1})(\lambda_i-\lambda_{2N+2}) \\ &= \int\Big(\prod_{i=1}^{2N+2} d\mu(\lambda_i)\Big)\,\Delta(\lambda_1,\ldots,\lambda_N)\prod_{i=1}^{N}\lambda_i{}^{i-1}\;\Delta(\lambda_{N+1},\ldots,\lambda_{2N})\prod_{i=N+1}^{2N}\lambda_i{}^{i-N-1} \\ &\quad\cdot\prod_{i=1}^{2N}(\lambda_i-\lambda_{2N+1})(\lambda_i-\lambda_{2N+2}) \\ &= \int\Big(\prod_{i=1}^{2N+2} d\mu(\lambda_i)\Big)\,\Delta(\lambda_1,\ldots,\lambda_N,\lambda_{2N+1})\prod_{i=1}^{N}\lambda_i{}^{i-1}\;\Delta(\lambda_{N+1},\ldots,\lambda_{2N},\lambda_{2N+2}) \\ &\quad\cdot\prod_{i=N+1}^{2N}\lambda_i{}^{i-N-1}\prod_{i=1}^{N}(\lambda_{i+N}-\lambda_{2N+1})(\lambda_i-\lambda_{2N+2}). \end{aligned} \tag{3.3.10}$$

Note that

$$\Delta(\lambda_1,\ldots,\lambda_N,\lambda_{2N+1})\prod_{i=1}^{N}\lambda_i{}^{i-1} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1{}^2 & \ldots & \lambda_1{}^{N-1} \\ \lambda_2 & \lambda_2{}^2 & \lambda_2{}^3 & \ldots & \lambda_2{}^N \\ \vdots & & & & \\ \lambda_N{}^{N-1} & \lambda_N{}^N & \lambda_N{}^{N+1} & \ldots & \lambda_N{}^{2N-2} \\ 1 & \lambda_{2N+1} & \lambda_{2N+1}{}^2 & \ldots & \lambda_{2N+1}{}^{N-1} \end{pmatrix}$$

as well as an analogue expression for the second van der Monde determinant in eq. (3.3.10). One may then convince oneself that in the product

[4] Simply write $\prod_{i<j}(\lambda_i - \lambda_j - \theta_i\theta_j) = \prod_{i<j}(\lambda_i - \lambda_j)\exp[\sum_{i<j}\frac{\theta_i\theta_j}{\lambda_i-\lambda_j}]$.

$$\prod_{i=1}^{N}(\lambda_{i+N}-\lambda_{2N+1})(\lambda_i-\lambda_{2N+2})=$$

$$\sum_{l,k=0}^{N}\sum_{\substack{i_1\neq\ldots\neq i_l\\ \in(1,\ldots,N)}}\sum_{\substack{j_1\neq\ldots\neq j_k\\ \in(N+1,\ldots,2N)}}(-)^{k+l}\,\lambda_{2N+1}{}^{N-k}\,\lambda_{i_1}\ldots\lambda_{i_l}\,\lambda_{2N+2}{}^{N-l}\,\lambda_{j_1}\ldots\lambda_{j_k},$$

only the following terms contribute to Z_{2N+2} of eq. (3.3.10)

$$\lambda_{2N+1}{}^N\,\lambda_{2N+2}{}^N+\sum_{r=1}^{N}\lambda_{2N+1}{}^{r-1}\,\lambda_{2N+2}{}^{r-1}\,\lambda_r\,\lambda_{r+1}\ldots\lambda_N\,\lambda_{N+r}\,\lambda_{N+r+1}\ldots\lambda_{2N}$$

Hence after some relabeling of indices and via eq. (3.3.9) we arrive at

$$\begin{aligned}Z_{2N+2}=&\int\Big(\prod_{i=1}^{2N+2}d\mu(\lambda_i)\Big)\,\Delta(\lambda_1,\ldots,\lambda_N,\lambda_{2N+1})\prod_{i=1}^{N}\lambda_i{}^{i-1}\,\lambda_{2N+1}{}^N\\ &\Delta(\lambda_{N+1},\ldots,\lambda_{2N},\lambda_{2N+2})\prod_{i=N+1}^{2N}\lambda_i{}^{i-N-1}\,\lambda_{2N+2}{}^N\\ =&\int\Big(\prod_{i=1}^{2N+2}d\mu(\lambda_i)\Big)\,\Delta^2(\lambda_1,\ldots,\lambda_N,\lambda_{2N+1})\,\Delta^2(\lambda_{N+1},\ldots,\lambda_{2N},\lambda_{2N+2}),\end{aligned}$$

and we have proven eq. (3.3.7).

Eq. (3.3.6) was first proposed in ref. [33]. It implies that up to irrelevant additive constants the part of the free energy independent of the fermionic couplings obeys

$$F_S^{(0)}[g_k,\xi_{k+1/2}=0;2N]=2\,F_B[g_k;N],\tag{3.3.11}$$

a relation which we will recover in the explicit solution of the supereigenvalue model.

Including the fermionic couplings $\xi_{k+1/2}$ in eq. (3.3.1) and performing the fermionic integrations to obtain the complete effective bosonic eigenvalue model is a more complicated task. Nevertheless McArthur [34] succeeded in doing so. The outcome important to us is the remarkable result that the free energy contains contributions only up to second order in the fermionic couplings, i.e.

$$F_S[g_k,\xi_{k+1/2}]=F_S^{(0)}[g_k]+\sum_{k,l}\xi_{k+1/2}\,\xi_{l+1/2}\,F_{k,l}^{(2)}[g_k],\tag{3.3.12}$$

confirming a conjecture made in ref. [33]. This observation will be the key to the solution of the superloop equation which we will describe in the following section.

4. Superloop Equations

Our solution of the supereigenvalue model is based on an integral form of the superloop equations, which are the analogue of the loop equations of the Hermitian matrix model discussed in chapter I. Generalizing the approach of Ambjørn, Chekhov, Kristjansen and Makeenko [9] for the Hermitian one matrix model we develop an iterative procedure which allows us to calculate the genus g contribution to the $(n|m)$-superloop correlators for (in principle) any g and any $(n|m)$ and (in practice) for any potential. The possibility of going to arbitrarily high genus is provided by the superloop equations, whereas the possibility of obtaining arbitrary $(n|m)$-superloop results is due to the superloop insertion operators introduced below. A change of variables from the coupling constants to moments allows us to explicitly present results for arbitrary potentials. The remainder of this chapter is essentially an enlarged version of the authors papers [6] on the general and [7] on the double scaled solution of the supereigenvalue model.

Superloop Insertion Operators

For notational simplicity let us from now on write the supereigenvalue model with N taken to be even as

$$\mathcal{Z}_S=e^{N^2F}=\int\Big(\prod_{i=1}^{N}d\lambda_i\,d\theta_i\Big)\prod_{i<j}(\lambda_i-\lambda_j-\theta_i\theta_j)\,\exp\Big(-N\sum_{i=1}^{N}[V(\lambda_i)-\theta_i\Psi(\lambda_i)]\Big),\tag{3.4.1}$$

with the potentials $V(\lambda_i)=\sum_{k=0}^{\infty}g_k\lambda_i{}^k$ and $\Psi(\lambda_i)=\sum_{k=0}^{\infty}\xi_{k+1/2}\lambda_i{}^k$ of eq. (3.3.2). Expectation values are defined in the usual way by

$$\Big\langle\mathcal{O}(\lambda_j,\theta_j)\Big\rangle=\frac{1}{\mathcal{Z}_S}\int\Big(\prod_{i=1}^{N}d\lambda_i\,d\theta_i\Big)\,\Delta(\lambda_i,\theta_i)\,\mathcal{O}(\lambda_j,\theta_j)\,\exp\Big(-N\sum_{i=1}^{N}[V(\lambda_i)-\theta_i\Psi(\lambda_i)]\Big),\tag{3.4.2}$$

where we write $\Delta(\lambda_i,\theta_i)=\prod_{i<j}(\lambda_i-\lambda_j-\theta_i\theta_j)$ for the measure. We introduce the one–superloop correlators

$$\widehat{W}(p\,|\,)=N\Big\langle\sum_i\frac{\theta_i}{p-\lambda_i}\Big\rangle\qquad\text{and}\qquad\widehat{W}(\,|\,p)=N\Big\langle\sum_i\frac{1}{p-\lambda_i}\Big\rangle,\tag{3.4.3}$$

which act as generating functionals for the one–point correlators $\langle\sum_i\lambda_i{}^k\rangle$ and $\langle\sum_i\theta_i\lambda_i{}^k\rangle$ upon expansion in p. This easily generalizes to higher–point correlators with the $(n|m)$–superloop correlator

$$\begin{aligned}&\widehat{W}(p_1,\ldots,p_n\,|\,q_1,\ldots,q_m)=\\ &\quad N^{n+m}\Big\langle\sum_{i_1}\frac{\theta_{i_1}}{p_1-\lambda_{i_1}}\cdots\sum_{i_n}\frac{\theta_{i_n}}{p_n-\lambda_{i_n}}\sum_{j_1}\frac{1}{q_1-\lambda_{j_1}}\cdots\sum_{j_m}\frac{1}{q_m-\lambda_{j_m}}\Big\rangle.\end{aligned}\tag{3.4.4}$$

Quite analogously to the bosonic case these correlators may be obtained from the partition function $\mathcal{Z}_S$ by application of the superloop insertion operators $\delta/\delta V(p)$ and $\delta/\delta\Psi(p)$:

$$\widehat{W}(p_1,\ldots,p_n\mid q_1,\ldots,q_m)=\frac{1}{\mathcal{Z}_S}\frac{\delta}{\delta\Psi(p_1)}\cdots\frac{\delta}{\delta\Psi(p_n)}\frac{\delta}{\delta V(q_1)}\cdots\frac{\delta}{\delta V(q_m)}\,\mathcal{Z}_S, \tag{3.4.5}$$

where

$$\frac{\delta}{\delta V(p)}=-\sum_{k=0}^{\infty}\frac{1}{p^{k+1}}\frac{\partial}{\partial g_k}\quad\text{and}\quad\frac{\delta}{\delta\Psi(p)}=-\sum_{k=0}^{\infty}\frac{1}{p^{k+1}}\frac{\partial}{\partial\xi_{k+1/2}}. \tag{3.4.6}$$

In particular equation (3.4.3) can now be written as $\widehat{W}(p\mid)=\delta\ln\mathcal{Z}_S/\delta\Psi(p)$ and $\widehat{W}(\mid p)=\delta\ln\mathcal{Z}_S/\delta V(p)$.

However, it is convenient to work with the connected part of the $(n|m)$–superloop correlators, denoted by W. They may be obtained from the free energy $F=N^{-2}\ln\mathcal{Z}_S$ through

$$W(p_1,\ldots,p_n\mid q_1,\ldots,q_m)=\frac{\delta}{\delta\Psi(p_1)}\cdots\frac{\delta}{\delta\Psi(p_n)}\frac{\delta}{\delta V(q_1)}\cdots\frac{\delta}{\delta V(q_m)}\,F=$$

$$N^{n+m-2}\Big\langle\sum_{i_1}\frac{\theta_{i_1}}{p_1-\lambda_{i_1}}\cdots\sum_{i_n}\frac{\theta_{i_n}}{p_n-\lambda_{i_n}}\sum_{j_1}\frac{1}{q_1-\lambda_{j_1}}\cdots\sum_{j_m}\frac{1}{q_m-\lambda_{j_m}}\Big\rangle_{\mathrm{conn}}, \tag{3.4.7}$$

in complete analogy to the bosonic case of eq. (1.1.10). Note that $n\le 2$ due to the maximally quadratic dependence of F on the $\xi_{k+1/2}$ mentioned in section 3.

With the normalizations chosen above, one assumes that these correlators enjoy the genus expansion

$$W(p_1,\ldots,p_n\mid q_1,\ldots,q_m)=\sum_{g=0}^{\infty}\frac{1}{N^{2g}}W_g(p_1,\ldots,p_n\mid q_1,\ldots,q_m). \tag{3.4.8}$$

Similarly one has the genus expansion

$$F=\sum_{g=0}^{\infty}\frac{1}{N^{2g}}F_g \tag{3.4.9}$$

for the free energy. In contrast to the Hermitian matrix model there is no geometrical argument based on Feynman diagrams and the Euler relation available to justify the genus expansion. This is of course due to a missing formulation of the supereigenvalue model in terms of some generalized matrix model. However, the genus expansion is motivated from the structure of the superloop equations, which we derive in the following subsection.

Superloop Equations

The superloop equations of our model are two Schwinger–Dyson equations, which are derived through a shift in integration variables λ_i and θ_i. They were first stated in [31, 32], and we present them in an integral form for the loop correlators $W(p\mid)$ and $W(\mid p)$.

The Grassmann–odd superloop equation reads

$$\oint_C\frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}W(\omega\mid)+\oint_C\frac{d\omega}{2\pi}\frac{\Psi(\omega)}{p-\omega}W(\mid\omega)=W(p\mid)\,W(\mid p)+\frac{1}{N^2}W(p\mid p) \tag{3.4.10}$$

and its counterpart, the Grassmann–even superloop equation, takes the form

$$\oint_C\frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}W(\mid\omega)+\oint_C\frac{d\omega}{2\pi}\frac{\Psi'(\omega)}{p-\omega}W(\omega\mid)-\frac{1}{2}\frac{d}{dp}\oint_C\frac{d\omega}{2\pi}\frac{\Psi(\omega)\,W(\omega\mid)}{p-\omega}=$$

$$\frac{1}{2}\Big[W(\mid p)^2-W(p\mid)\,W'(p\mid)+\frac{1}{N^2}\Big(W(\mid p,p)-\frac{d}{dq}W(p,q\mid)\Big|_{p=q}\Big)\Big]. \tag{3.4.11}$$

In the derivation we have assumed that the loop correlators have one–cut structure, i.e. in the limit $N\to\infty$ we assume that the eigenvalues λ_i are contained in a finite interval $[x,y]$. Moreover C is a curve around the cut.

In order to derive the first of the two superloop equations for the model of eq. (3.4.1) consider the shift in integration variables

$$\lambda_i\to\lambda_i+\theta_i\frac{\epsilon}{p-\lambda_i}\quad\text{and}\quad\theta_i\to\theta_i+\frac{\epsilon}{p-\lambda_i}$$

where ϵ is an odd constant. Under these we find that

$$\prod_i d\lambda_i\,d\theta_i\to(1-\epsilon\sum_i\frac{\theta_i}{(p-\lambda_i)^2})\prod_i d\lambda_i\,\theta_i$$

and the measure transforms as

$$\Delta(\lambda_i,\theta_i)\to(1-\epsilon\sum_{i\neq j}\frac{\theta_i}{(p-\lambda_i)(p-\lambda_j)})\,\Delta(\lambda_i,\theta_i).$$

The vanishing of the terms proportional to ϵ then gives us the Schwinger–Dyson equation

$$\Big\langle N\Big\{\sum_i\frac{1}{p-\lambda_i}\Big(V'(\lambda_i)\theta_i+\Psi(\lambda_i)\Big)\Big\}-\sum_i\frac{\theta_i}{p-\lambda_i}\sum_j\frac{1}{p-\lambda_j}\Big\rangle=0. \tag{3.4.12}$$

Note that $\langle\sum_i\theta_i\,(p-\lambda_i)^{-1}\sum_j(p-\lambda_j)^{-1}\rangle=N^{-2}\,\widehat{W}(p\mid p)$ with the definitions of the previous subsection. In order to transform eq. (3.4.12) into an integral equation we define the bosonic and fermionic density operators

$$\rho(\lambda)=\frac{1}{N}\sum_i\langle\,\delta(\lambda-\lambda_i)\,\rangle\quad\text{and}\quad r(\lambda)=\frac{1}{N}\sum_i\langle\,\theta_i\,\delta(\lambda-\lambda_i)\,\rangle.$$

With these the first sum in eq. (3.4.12) may be written as

$$N^2 \int d\lambda \left(r(\lambda) \frac{V'(\lambda)}{p-\lambda} + \rho(\lambda) \frac{\Psi(\lambda)}{p-\lambda} \right) =$$

$$N^2 \int d\lambda\, r(\lambda) \left[\oint_C \frac{d\omega}{2\pi} \frac{1}{\omega - \lambda} \frac{V'(\omega)}{p-\omega} \right] + N^2 \int d\lambda\, \rho(\lambda) \left[\oint_C \frac{d\omega}{2\pi} \frac{1}{\omega-\lambda} \frac{\Psi(\omega)}{p-\omega} \right] \tag{3.4.13}$$

where we assume that the real eigenvalues λ_i are contained within a finite interval $x < \lambda_i < y$, $\forall i$ in the limit $N \to \infty$. Moreover C is a curve around the cut $[x, y]$ and p lies outside this curve. Performing the λ integrals in eq. (3.4.13) gives us the one–superloop correlators. The full Schwinger–Dyson equation (3.4.12) may then be expressed in the integral form

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \widehat{W}(\omega\,|\,) + \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega} \widehat{W}(\,|\,\omega) = N^{-2}\, \widehat{W}(p\,|\,p).$$

Rewriting this in terms of the connected superloop correlators W of eq. (3.4.7) yields eq. (3.4.10).

The derivation of the second superloop equation goes along the same lines by performing the shift

$$\lambda_i \to \lambda_i + \frac{\epsilon}{p-\lambda_i} \quad \text{and} \quad \theta_i \to \theta_i + \frac{1}{2} \frac{\epsilon\, \theta_i}{(p-\lambda_i)^2}$$

with ϵ even and infinitesimal. Similar steps as the ones discussed above then lead us to the second superloop equation

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)\, \widehat{W}(\,|\,\omega) + \Psi'(\omega)\, \widehat{W}(\omega\,|\,)}{p-\omega} - \frac{1}{2} \frac{d}{dp} \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)\, \widehat{W}(\omega\,|\,)}{p-\omega} =$$

$$\frac{1}{2} \frac{1}{N^2} \widehat{W}(\,|\,p,p) - \frac{1}{2} \frac{1}{N^2} \frac{d}{dq} \widehat{W}(p,q\,|\,)\Big|_{p=q}$$

which, after rephrasing in connected quantities, gives eq. (3.4.11).

Note the similarity to the loop equation for the hermitian matrix model (1.3.21). The eqs. (3.4.10) and (3.4.11) are equivalent to the super–Virasoro constraints. The constraints follow from the superloop equations by expanding them in p and evaluating the resulting contour integrals. The resulting equations at each power of p then correspond to $\mathcal{G}_{n+1/2}\, \mathcal{Z}_S = 0$ and $\mathcal{L}_n\, \mathcal{Z}_S = 0$ for $n \geq -1$.

The key to the solution of these complicated equations order by order in N^{-2} is the observation stated in eq. (3.3.12) that the free energy F depends at most quadratically on fermionic coupling constants [33, 34]. Via eq. (3.4.7) this directly translates to the one-loop correlators, which we from now on write as

$$\begin{aligned} W(p\,|\,) &= v(p) \\ W(\,|\,p) &= u(p) + \widehat{u}(p). \end{aligned} \tag{3.4.14}$$

Here $v(p)$ is of order one in fermionic couplings, whereas $u(p)$ is taken to be of order zero and $\widehat{u}(p)$ of order two in the fermionic coupling constants $\xi_{k+1/2}$. This observation allows us to split up the two superloop equations (3.4.10) and (3.4.11) into a set of four equations, sorted by their order in the $\xi_{k+1/2}$'s. Doing this we obtain

Order 0:

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} u(\omega) = \frac{1}{2} u(p)^2 + \frac{1}{2} \frac{1}{N^2} \frac{\delta}{\delta V(p)} u(p) - \frac{1}{2} \frac{1}{N^2} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v(q)\Big|_{p=q} \tag{3.4.15}$$

Order 1:

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} v(\omega) + \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega} u(\omega) = v(p)\, u(p) + \frac{1}{N^2} \frac{\delta}{\delta V(p)} v(p) \tag{3.4.16}$$

Order 2:

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \widehat{u}(\omega) + \oint_C \frac{d\omega}{2\pi} \frac{\Psi'(\omega)}{p-\omega} v(\omega) - \frac{1}{2} \frac{d}{dp} \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega} v(\omega) =$$

$$u(p)\, \widehat{u}(p) - \frac{1}{2} v(p) \frac{d}{dp} v(p) + \frac{1}{2} \frac{1}{N^2} \frac{\delta}{\delta V(p)} \widehat{u}(p) \tag{3.4.17}$$

Order 3:

$$\oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega} \widehat{u}(\omega) = v(p)\, \widehat{u}(p). \tag{3.4.18}$$

It is the remarkable form of these four equations which allows us to develop an iterative procedure to determine $u_g(p), v_g(p), \widehat{u}_g(p)$ and F_g genus by genus. Plugging the genus expansions into these equations lets them decouple partially, in the sense that the equation of order 0 at genus g only involves u_g and lower genera contributions. The order 1 equation then only contains v_g, u_g and lower genera results and so on. The first thing to do, however, is to find the solution for $g = 0$.

5. The Planar Solution

In the following the planar solution for the superloop correlators is given for a general potential. It was first obtained in [32]. We present it in a very compact integral form augmented by the use of new variables to characterize the potentials, the moments.

Solution for $u_0(p)$ and $v_0(p)$

In the limit $N \to \infty$ the order 0 equation (3.4.15) becomes

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} u_0(\omega) = \frac{1}{2} u_0(p)^2. \tag{3.5.1}$$

This equation is well known, as up to a factor of $1/2$ it is nothing but the planar loop equation of the hermitian matrix model (1.4.2). With the above assumptions on the one–cut structure and by demanding that $u(p)$ behaves as $1/p$ for $p \to \infty$ one finds [17]

$$u_0(p) \;=\; \oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \left[\frac{(p-x)(p-y)}{(\omega-x)(\omega-y)} \right]^{1/2}, \tag{3.5.2}$$

as we verified in chapter I section 4. The endpoints x and y of the cut on the real axis are determined by the following requirements:

$$0 = \oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}}, \qquad 1 = \oint_C \frac{d\omega}{2\pi} \frac{\omega\, V'(\omega)}{\sqrt{(\omega-x)(\omega-y)}}, \tag{3.5.3}$$

deduced from our knowledge that $W(\,|\,p) = 1/p + \mathcal{O}(p^{-2})$.

The order 1 equation (3.4.16) in the $N \to \infty$ limit determining the odd loop correlator $v_0(p)$ reads

$$\oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega}\, v_0(\omega) \;+\; \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega}\, u_0(\omega) \;=\; v_0(p)\, u_0(p). \tag{3.5.4}$$

It is solved by

$$v_0(p) \;=\; \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{p-\omega} \left[\frac{(\omega-x)(\omega-y)}{(p-x)(p-y)} \right]^{1/2} \;+\; \frac{\chi}{\sqrt{(p-x)(p-y)}}. \tag{3.5.5}$$

Here χ is a Grassmann odd constant not determined by eq. (3.5.4), in fact $\chi = N^{-1} \langle \sum_i \theta_i \rangle$ in the planar limit. It will be determined in the analysis of the two remaining equations (3.4.17) and (3.4.18).

One verifies the above solution by direct computation. Using eqs. (3.5.2) and (3.5.5) the right hand side of eq. (3.5.4) reads

$$\begin{aligned} u_0(p)\, v_0(p) = &\oint_{C_1} \frac{d\omega}{2\pi} \oint_{C_2} \frac{dz}{2\pi} \frac{V'(\omega)\, \Psi(z)}{(p-\omega)(p-z)} \left[\frac{(z-x)(z-y)}{(\omega-x)(\omega-y)} \right]^{1/2} \\ &+ \oint_{C_1} \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \frac{\chi}{[(\omega-x)(\omega-y)]^{1/2}}, \end{aligned} \tag{3.5.6}$$

and the left hand side becomes

$$\begin{aligned} &\oint_{C_1} \frac{d\omega}{2\pi} \oint_{C_2} \frac{dz}{2\pi} \frac{V'(\omega)\, \Psi(z)}{(p-\omega)(\omega-z)} \left[\frac{(z-x)(z-y)}{(\omega-x)(\omega-y)} \right]^{1/2} + \oint_{C_1} \frac{d\omega}{2\pi} \frac{V'(\omega)}{p-\omega} \frac{\chi}{[(\omega-x)(\omega-y)]^{1/2}} \\ &+ \oint_{C_1} \frac{d\omega}{2\pi} \oint_{C_2} \frac{dz}{2\pi} \frac{\Psi(\omega)\, V'(z)}{(p-\omega)(\omega-z)} \left[\frac{(\omega-x)(\omega-y)}{(z-x)(z-y)} \right]^{1/2}. \end{aligned}$$

Now in the last term pull the contour integral C_2 over the curve C_1. One can show that the contribution from the extra pole vanishes. After renaming $\omega \leftrightarrow z$ and combining the first and third terms one gets eq. (3.5.6). Thus eq. (3.5.4) is verified.

Moments and Basis Functions

Let us now define new variables characterizing the potentials $V(p)$ and $\Psi(p)$. Instead of the couplings g_k we introduce the bosonic moments M_k and J_k defined by [9]

$$\begin{aligned} M_k &= \oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{(\omega-x)^k} \frac{1}{[\,(\omega-x)\,(\omega-y)\,]^{1/2}}, \quad k \geq 1 \\ J_k &= \oint_C \frac{d\omega}{2\pi} \frac{V'(\omega)}{(\omega-y)^k} \frac{1}{[\,(\omega-x)\,(\omega-y)\,]^{1/2}}, \quad k \geq 1, \end{aligned} \tag{3.5.7}$$

and the couplings $\xi_{k+1/2}$ are replaced by the fermionic moments

$$\begin{aligned} \Xi_k &= \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{(\omega-x)^k} [\,(\omega-x)(\omega-y)\,]^{1/2}, \quad k \geq 1 \\ \Lambda_k &= \oint_C \frac{d\omega}{2\pi} \frac{\Psi(\omega)}{(\omega-y)^k} [\,(\omega-x)(\omega-y)\,]^{1/2}, \quad k \geq 1. \end{aligned} \tag{3.5.8}$$

These moments depend on the coupling constants, both explicitly and through x and y:

$$\begin{aligned} M_k &= (k+1)\, g_{k+1} + \Big[\, (k+\frac{1}{2})x + \frac{1}{2}y \,\Big]\, (k+2)\, g_{k+2} + \ldots \\ J_k &= (k+1)\, g_k + \Big[\, \frac{1}{2}x + (k+\frac{1}{2})y \,\Big]\, (k+2)\, g_{k+2} + \ldots \\ \Xi_k &= (\,1-\delta_{1,k}\,)\, \xi_{k-3/2} + \Big[\, (k-\frac{1}{2})x - \frac{1}{2}y \,\Big]\, \xi_{k-1/2} + \ldots \\ \Lambda_k &= (\,1-\delta_{1,k}\,)\, \xi_{k-3/2} + \Big[\, (k-\frac{1}{2})y - \frac{1}{2}x \,\Big]\, \xi_{k-1/2} + \ldots \end{aligned} \tag{3.5.9}$$

The main motivation for introducing these new variables is that, for each term in the genus expansion of the free energy and the correlators, the dependence on an infinite number of coupling constants arranges itself nicely into a function of a *finite* number of moments.

We further introduce the basis functions $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$ recursively

$$\chi^{(n)}(p) = \frac{1}{M_1}\left(\phi_x^{(n)}(p) - \sum_{k=1}^{n-1}\chi^{(k)}(p)\, M_{n-k+1}\right),$$
$$\Psi^{(n)}(p) = \frac{1}{J_1}\left(\phi_y^{(n)}(p) - \sum_{k=1}^{n-1}\Psi^{(k)}(p)\, J_{n-k+1}\right), \tag{3.5.10}$$

where

$$\phi_x^{(n)}(p) = (p-x)^{-n}\,[\,(p-x)(p-y)\,]^{-1/2},$$
$$\phi_y^{(n)}(p) = (p-y)^{-n}\,[\,(p-x)(p-y)\,]^{-1/2}, \tag{3.5.11}$$

following ref. [9].

It is easy to show that for the linear operator $\widehat{\mathcal{V}}'$ defined by

$$\widehat{\mathcal{V}}' \circ f(p) = \oint_C \frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}\, f(\omega) - u_0(p)\, f(p) \tag{3.5.12}$$

and appearing in the superloop equations we have

$$\widehat{\mathcal{V}}' \circ \chi^{(n)}(p) = \frac{1}{(p-x)^n}, \quad n \geq 1,$$
$$\widehat{\mathcal{V}}' \circ \Psi^{(n)}(p) = \frac{1}{(p-y)^n}, \quad n \geq 1. \tag{3.5.13}$$

Moreover, $\phi_x^{(0)}(p) = \phi_y^{(0)}(p) \equiv \phi^{(0)}(p)$ lies in the kernel of $\widehat{\mathcal{V}}'$.

Solution for $\widehat{u}_0$ and χ

Next consider the order 2 equation (3.4.17) at genus 0

$$\widehat{\mathcal{V}}' \circ \widehat{u}_0 = \frac{1}{2}\frac{d}{dp}\oint_C \frac{d\omega}{2\pi}\frac{\Psi(\omega)}{p-\omega}\, v_0(\omega) - \oint_C \frac{d\omega}{2\pi}\frac{\Psi'(\omega)}{p-\omega}\, v_0(\omega) - \frac{1}{2}\, v_0(p)\,\frac{d}{dp}\, v_0(p). \tag{3.5.14}$$

Plugging eq. (3.5.5) into the right hand side of this equation yields after a somewhat lengthy calculation

$$\widehat{\mathcal{V}}' \circ \widehat{u}_0 = \frac{1}{2}\frac{\Xi_2\,(\Xi_1-\chi)}{(x-y)}\frac{1}{p-x} - \frac{1}{2}\frac{\Lambda_2\,(\Lambda_1-\chi)}{(x-y)}\frac{1}{p-y}. \tag{3.5.15}$$

With eq. (3.5.13) this immediately tells us that

$$\widehat{u}_0(p) = \frac{1}{2}\frac{\Xi_2\,(\Xi_1-\chi)}{(x-y)}\chi^{(1)}(p) - \frac{1}{2}\frac{\Lambda_2\,(\Lambda_1-\chi)}{(x-y)}\Psi^{(1)}(p). \tag{3.5.16}$$

There can be no contributions proportional to the zero mode $\phi^{(0)}(p)$, as we know that $\widehat{u}(p)$ behaves as $\mathcal{O}(p^{-2})$ for $p \to \infty$.

Finally we determine the odd constant χ. This is done by employing the order 3 equation (3.4.18) for $g = 0$, i.e.

$$\oint_C \frac{d\omega}{2\pi}\frac{\Psi(\omega)}{p-\omega}\,\widehat{u}_0(\omega) - v_0(p)\,\widehat{u}_0(p) = 0. \tag{3.5.17}$$

After insertion of eqs. (3.5.5) and (3.5.16) one can show that

$$0 = \widehat{\Psi} \circ \widehat{u}_0(p) = \frac{1}{2}\frac{\Xi_2\,(\Xi_1-\chi)\,(\Lambda_1-\chi)}{M_1\,(x-y)^3\,(p-y)} - \frac{1}{2}\frac{\Lambda_2\,(\Lambda_1-\chi)\,(\Xi_1-\chi)}{J_1\,(x-y)^3\,(p-x)}, \tag{3.5.18}$$

where we have defined the linear operator $\widehat{\Psi}$ by

$$\widehat{\Psi} \circ f(p) = \oint_C \frac{d\omega}{2\pi}\frac{\Psi(\omega)}{p-\omega}\, f(\omega) - v_0(p)\, f(p), \tag{3.5.19}$$

in accordance to $\widehat{\mathcal{V}}'$. The result (3.5.18) lets us finally read off the coefficient χ as

$$\chi = \frac{1}{2}(\,\Xi_1 + \Lambda_1\,). \tag{3.5.20}$$

Putting it all together, we may now write down the complete genus 0 solution for the one–superloop correlators $W(\,|\,p)$ and $W(p\,|\,)$:

$$\begin{aligned} W_0(\,|\,p) &= \oint_C \frac{d\omega}{2\pi}\frac{V'(\omega)}{p-\omega}\left[\frac{(p-x)(p-y)}{(\omega-x)(\omega-y)}\right]^{1/2} \\ &\quad + \frac{1}{4}\frac{\Xi_2\,(\Xi_1-\Lambda_1)}{M_1\,(x-y)}\phi_x^{(1)}(p) + \frac{1}{4}\frac{\Lambda_2\,(\Xi_1-\Lambda_1)}{J_1\,(x-y)}\phi_y^{(1)}(p) \\ W_0(p\,|\,) &= \oint_C \frac{d\omega}{2\pi}\frac{\Psi(\omega)}{p-\omega}\left[\frac{(\omega-x)(\omega-y)}{(p-x)(p-y)}\right]^{1/2} + \frac{1}{2}\frac{\Xi_1+\Lambda_1}{[\,(p-x)(p-y)\,]^{1/2}}. \end{aligned} \tag{3.5.21}$$

One can show that this solution is equivalent to the less compact one obtained in ref. [32].

We shall make use of the following rewriting of the purely bosonic part of $W_0(\,|\,p)$

$$u_0(p) = V'(p) - \frac{1}{2}[(p-x)(p-y)]^{1/2}\sum_{q=1}^{\infty}\Big\{(p-x)^{q-1}\, M_q + (p-y)^{q-1}\, J_q\Big\}, \tag{3.5.22}$$

derived by deforming the contour integral in eq. (3.5.21) into one surrounding the point p and the other encircling infinity. To take the residue at infinity one rewrites $(p-\omega)^{-1}$ as

$$\frac{1}{p-\omega}=\frac{1}{2}\frac{1}{(p-x)-(\omega-x)}+\frac{1}{2}\frac{1}{(p-y)-(\omega-y)}, \tag{3.5.23}$$

and expands in $\left(\frac{p-x}{\omega-x}\right)$ and $\left(\frac{p-y}{\omega-y}\right)$ respectively. Doing this for the fermionic $W_0(p\,|\,)$ yields

$$v_0(p)=\Psi(p)-\frac{1}{2}\left[(p-x)(p-y)\right]^{-1/2}\sum_{q=2}^{\infty}\Big\{(p-x)^{q-1}\,\Xi_q\,+\,(p-y)^{q-1}\,\Lambda_q\Big\}. \tag{3.5.24}$$

It is important to realize that the bracketed terms in eq. (3.5.22) as well as in eq. (3.5.24) are actually identical. Here we see that the planar solution is special in the sense that it depends on the full set of moments. Interestingly enough this is not the case for higher genera.

6. The Iterative Procedure

Our iterative solution of the superloop equations results in a certain representation of the free energy and the loop correlators in terms of the moments and basis functions defined in section 5. We will show that it suffices to know $u_g(p)$ and $v_g(p)$ only up to a zero mode in order to calculate F_g. We give explicit results for genus one.

The Iteration for u_g and v_g

The correlators $u_g(p)$ and $v_g(p)$ are determined by the order 0 and order 1 equations (3.4.15) and (3.4.16) after insertion of the genus expansions (3.4.8) of these operators. We find

$$\widehat{\mathcal{V}}'\circ u_g(p)=\frac{1}{2}\sum_{g'=1}^{g-1}u_{g'}(p)\,u_{g-g'}(p)\,+\,\frac{1}{2}\frac{\delta}{\delta V(p)}\,u_{g-1}(p)\,-\,\frac{1}{2}\frac{d}{dq}\frac{\delta}{\delta\Psi(p)}\,v_{g-1}(q)\Big|_{p=q} \tag{3.6.1}$$

and

$$\widehat{\mathcal{V}}'\circ v_g(p)=-\widehat{\Psi}\circ u_g(p)\,+\,\sum_{g'=1}^{g-1}v_{g'}(p)\,u_{g-g'}(p)\,+\,\frac{\delta}{\delta V(p)}\,v_{g-1}(p) \tag{3.6.2}$$

at genus $g\geq 1$. From the structure of these equations we directly deduce that $u_g(p)$ and $v_g(p)$ will be linear combinations of the basis functions $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$. By eq. (3.5.13) the coefficients of this linear combination may be read off the poles $(p-x)^{-k}$ and $(p-y)^{-k}$ of the right hand sides of eqs. (3.6.1) and (3.6.2) after a partial fraction decomposition.

Let us demonstrate how this works for $g=1$. According to eq. (3.6.1) for $u_1(p)$ we first calculate $\delta\,u_0/\delta V(p)$. We then need to know the derivatives $\delta\,x/\delta V(p)$ and $\delta\,y/\delta V(p)$. They can be obtained from eq. (3.5.3) and read

$$\frac{\delta\,x}{\delta V(p)}=\frac{1}{M_1}\,\phi_x^{(1)}(p),\qquad \frac{\delta\,y}{\delta V(p)}=\frac{1}{J_1}\,\phi_y^{(1)}(p). \tag{3.6.3}$$

Using the relation

$$\frac{\delta}{\delta V(p)}\,V'(\omega)=\frac{d}{dp}\frac{1}{p-\omega} \tag{3.6.4}$$

one finds [5]

$$\frac{\delta}{\delta V(p)}\,u_0(p)=\frac{1}{8}\frac{1}{(p-x)^2}+\frac{1}{8}\frac{1}{(p-y)^2}-\frac{1}{4\,d}\frac{1}{(p-x)}+\frac{1}{4\,d}\frac{1}{(p-y)}, \tag{3.6.5}$$

where $d=x-y$.

Next we determine $\delta\,v_0(q)/\delta\Psi(p)$. Using the relation

$$\frac{\delta}{\delta\Psi(p)}\,\Psi(q)=-\frac{1}{p-q} \tag{3.6.6}$$

and the result

$$\begin{aligned}\frac{\delta\,\Xi_k}{\delta\Psi(p)}&=\delta_{k1}\;-\;\frac{[\,(p-x)(p-y)\,]^{1/2}}{(p-x)^k}\\ \frac{\delta\,\Lambda_k}{\delta\Psi(p)}&=\delta_{k1}\;-\;\frac{[\,(p-x)(p-y)\,]^{1/2}}{(p-y)^k}\end{aligned} \tag{3.6.7}$$

for $k\geq 1$, one finds

$$\frac{d}{dq}\frac{\delta}{\delta\Psi(p)}\,v_0(q)\Big|_{p=q}=-\frac{\delta}{\delta V(p)}\,u_0(p). \tag{3.6.8}$$

This enables us to write down $u_1(p)$,

$$u_1(p)=\frac{1}{8}\,\chi^{(2)}(p)+\frac{1}{8}\,\Psi^{(2)}(p)-\frac{1}{4\,d}\,\chi^{(1)}+\frac{1}{4\,d}\,\Psi^{(1)}(p). \tag{3.6.9}$$

Note that up to the overall factor of two this is identical to the one–loop correlator of the hermitian matrix model [9], as it has to be due to eq. (3.3.11).

[5] Of course this is nothing but $2\,W_0(p,p)$ of the Hermitian matrix model of eq. (1.4.7).

Now we solve eq. (3.6.2) at $g=1$ for $v_1(p)$. It is important to realize that generally eq. (3.6.2) fixes $v_g(p)$ only up to a zero mode contribution $\kappa_g\,\phi^{(0)}(p)$. This comes from the fact that, unlike for the bosonic $u(p)$, we do not know the coefficient of the p^{-1} term for $v(p)$. The zero mode coefficient κ_g will be fixed later on by requiring $v_g(p)$ to be a total derivative of the free energy F_g.

In order to calculate $\delta v_0/\delta V(p)$ we make use of the relation

$$\frac{\delta\,\Xi_k}{\delta V(p)} = (k-\frac{1}{2})\,\Xi_{k+1}\,\frac{1}{M_1}\,\phi_x^{(1)}(p) + \frac{1}{2}\Big[\sum_{r=2}^{k}\frac{\Xi_r}{(-d)^{1+k-r}} + \frac{\Xi_1-\Lambda_1}{(-d)^k}\Big]\,\frac{1}{J_1}\,\phi_y^{(1)}(p), \tag{3.6.10}$$

as well as $\delta\,\Lambda_k/\delta V(p)$ obtained from the above by the replacements $x\leftrightarrow y$, $M_k\leftrightarrow J_k$, $\Xi_k\leftrightarrow\Lambda_k$ and $d\to -d$. The derivatives $\delta\,M_k/\delta V(p)$ and $\delta\,J_k/\delta V(p)$ were calculated in ref. [9]

$$\begin{aligned}\frac{\delta\,M_k}{\delta V(p)} &= -\frac{1}{2}\,(p-x)^{-k-1/2}(p-y)^{-3/2} - (k+1/2)\,\phi_x^{(k+1)}(p)\\ &+\frac{1}{2}\Big[\frac{1}{(-d)^k} - \sum_{i=1}^{k}\frac{1}{(-d)^{k-i+1}}\,\frac{M_i}{J_1}\Big]\,\phi_y^{(1)}(p)\\ &+(k+1/2)\,\frac{M_{k+1}}{M_1}\,\phi_x^{(1)}(p),\end{aligned} \tag{3.6.11}$$

and $\delta\,J_k/\delta V(p)$ is obtained by the usual replacements. Using these and the earlier results one has

$$\begin{aligned}\frac{\delta\,v_0}{\delta V(p)} = W_0(p\mid p) &= \Big[-\frac{(\Xi_1-\Lambda_1)}{4\,d\,M_1}\Big]\,\frac{1}{(p-x)^3} + \Big[-\frac{(\Xi_1-\Lambda_1)}{4\,d\,J_1}\Big]\,\frac{1}{(p-y)^3}\\ &+\Big[\frac{\Xi_2}{4\,d\,M_1}\Big]\,\frac{1}{(p-x)^2} + \Big[-\frac{\Lambda_2}{4\,d\,J_1}\Big]\,\frac{1}{(p-y)^2}\\ &+\Big[\frac{\Lambda_2}{4\,d^2\,J_1} - \frac{\Xi_2}{4\,d^2\,M_1}\Big]\,\frac{1}{(p-x)}\\ &+\Big[\frac{\Xi_2}{4\,d^2\,M_1} - \frac{\Lambda_2}{4\,d^2\,J_1}\Big]\,\frac{1}{(p-y)}.\end{aligned} \tag{3.6.12}$$

For the evaluation of the right hand side of eq. (3.6.2) at genus g we also need to know how the operator $\widehat{\Psi}$ acts on the functions $\phi_x^{(n)}(p)$ and $\phi_y^{(n)}(p)$, in terms of which $u_g(p)$ is given. A straightforward calculation yields

$$\begin{aligned}\widehat{\Psi}\circ\phi_x^{(n)}(p) &= \sum_{k=1}^{n+1}\frac{1}{(p-x)^k}\Big[-\frac{(\Xi_1-\Lambda_1)}{2\,(-d)^{n+2-k}} - \sum_{l=2}^{n+2-k}\frac{\Xi_l}{(-d)^{n+3-k-l}}\Big]\\ &+\frac{1}{(p-y)}\Big[-\frac{(\Xi_1-\Lambda_1)}{2\,(-d)^{n+1}}\Big]\end{aligned} \tag{3.6.13}$$

as well as the analogous expression for $\widehat{\Psi}\circ\phi_y^{(n)}(p)$ obtained from eq. (3.6.13) by the replacements $x\leftrightarrow y$, $M_k\leftrightarrow J_k$, $\Xi_k\leftrightarrow\Lambda_k$ and $d\to -d$.

We now have collected all the ingredients needed to evaluate the right hand side of eq. (3.6.2). After a partial fraction decomposition we may read off the poles at x and y, and therefore obtain the coefficients of the linear combination in the basis functions. We arrived at the result for $g=1$ with the aid of *Maple*, namely

$$v_1(p) = \sum_{i=1}^{3}\Big(B_1^{(i)}\,\chi^{(i)}(p) + E_1^{(i)}\,\Psi^{(i)}(p)\Big) + \kappa_1\,\phi^{(0)}(p), \tag{3.6.14}$$

where the coefficients $B_1^{(i)}$ and $E_1^{(i)}$ are given by

$$\begin{aligned}B_1^{(1)} &= -\frac{1}{8}\,\frac{\Xi_3}{d\,M_1} + \frac{1}{8}\,\frac{\Xi_2}{d^2\,M_1} + \frac{1}{4}\,\frac{\Lambda_2}{d^2\,J_1}\\ &+\frac{1}{8}\,\frac{M_2\,\Xi_2}{d\,{M_1}^2} - \frac{1}{16}\,\frac{M_2\,(\Xi_1-\Lambda_1)}{d^2\,{M_1}^2} + \frac{1}{16}\,\frac{J_2\,(\Xi_1-\Lambda_1)}{d^2\,{J_1}^2}\\ &-\frac{3}{16}\,\frac{(\Xi_1-\Lambda_1)}{d^3\,M_1} - \frac{3}{16}\,\frac{(\Xi_1-\Lambda_1)}{d^3\,J_1},\\ B_1^{(2)} &= \frac{1}{8}\,\frac{\Xi_2}{d\,M_1} + \frac{1}{16}\,\frac{M_2\,(\Xi_1-\Lambda_1)}{d\,{M_1}^2} + \frac{3}{16}\,\frac{(\Xi_1-\Lambda_1)}{d^2\,M_1},\\ B_1^{(3)} &= -\frac{5}{16}\,\frac{(\Xi_1-\Lambda_1)}{d\,M_1},\end{aligned} \tag{3.6.15}$$

and $E_1^{(i)} = B_1^{(i)}(M\leftrightarrow J, \Xi\leftrightarrow\Lambda, d\to -d)$.

Yet κ_1 is still undetermined. To compute it and the remaining doubly fermionic part $\widehat{u}_g(p)$ of the loop correlator $W_g(\,\mid p)$ one can employ the order 2 and order 3 eqs. (3.4.17) and (3.4.18) at genus g. It is, however, much easier to construct the free energy F_g at this stage from our knowledge of $u_g(p)$ and $v_g(p)$.

The Computation of F_g and κ_g

As mentioned earlier, the free energy of the supereigenvalue model depends at most quadratically on the fermionic coupling constants. In this subsection we present an algorithm which allows us to determine F_g and κ_g as soon as the results for $u_g(p)$ and $v_g(p)$ (up to the zero mode coefficient κ_g) are known.

The purely bosonic part of the free energy F_g is just twice the free energy of the hermitian matrix model. By using the results of Ambjørn et al. [9] one may then compute the bosonic part of F_g from $u_g(p)$. Here one rewrites $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$ as derivatives with respect to $V(p)$. One easily verifies the following relations for two lowest basis functions:

$$\begin{aligned}
\chi^{(1)}(p) &= \frac{\delta\, x}{\delta V(p)}, \\
\Psi^{(1)}(p) &= \frac{\delta\, y}{\delta V(p)}, \\
\chi^{(2)}(p) &= \frac{\delta}{\delta V(p)}(-\frac{2}{3}\ln M_1 - \frac{1}{3}\ln d\,), \\
\Psi^{(2)}(p) &= \frac{\delta}{\delta V(p)}(-\frac{2}{3}\ln J_1 - \frac{1}{3}\ln d\,).
\end{aligned} \tag{3.6.16}$$

Combining this with the result for $u_1(p)$ of eq. (3.6.9) yields the purely bosonic piece of F_1

$$F_1^{\mathrm{bos}} = -\frac{1}{12}\ln M_1 - \frac{1}{12}\ln J_1 - \frac{1}{3}\ln d, \tag{3.6.17}$$

which as expected is just twice the result for the Hermitian matrix model (cf. [9]).

For the higher basis functions the situation is not as simple. However, a rewriting of the basis functions allows one to identify $u_g(p)$ as a total derivative. For $\chi^{(n)}(p)$ one uses the recursive form [9]

$$\begin{aligned}
\chi^{(n)}(p) = \frac{1}{M_1}\Big[& \frac{1}{2n-1}\sum_{i=1}^{n-1}(-d)^{i-n}\left(\phi_x^{(i)}(p) - M_i\,\frac{\delta\, y}{\delta V(p)}\right) \\
& -\frac{2}{2n-1}\,\frac{\delta\, M_{n-1}}{\delta V(p)} - \sum_{k=2}^{n-1}\chi^{(k)}(p)\, M_{n-k+1}\Big] \qquad n\geq 2,
\end{aligned} \tag{3.6.18}$$

where $\phi_x^{(n)}$ should be rewritten as

$$\begin{aligned}
\phi_x^{(n)}(p) = & \frac{1}{2n-1}\sum_{i=1}^{n-1}(-d)^{i-n}\left(\phi_x^{(i)}(p) - M_i\,\frac{\delta\, y}{\delta V(p)}\right) \\
& + M_n\,\frac{\delta\, x}{\delta V(p)} - \frac{2}{2n-1}\,\frac{\delta\, M_{n-1}}{\delta V(p)} \qquad n\geq 2
\end{aligned}$$

$$\phi_x^{(1)}(p) = M_1\,\frac{\delta\, x}{\delta V(p)}. \tag{3.6.19}$$

The rewriting of $\Psi^{(n)}(p)$ follows from the above by performing the usual replacements $J\leftrightarrow M$ and $d\to -d$.

The strategy for the part of F_g quadratic in fermionic couplings consists in rewriting $v_g(p)$ as a total derivative in the fermionic potential $\Psi(p)$. We know that eq. (3.6.7) implies

$$\frac{\delta\,(\Xi_1-\Lambda_1)}{\delta\Psi(p)} = -d\,\phi^{(0)}(p) \tag{3.6.20}$$

and

$$\frac{\delta\,\Xi_k}{\delta\Psi(p)} = -\,\vartheta_x^{(k)}(p), \qquad \frac{\delta\,\Lambda_k}{\delta\Psi(p)} = -\vartheta_y^{(k)}(p), \quad k\geq 2 \tag{3.6.21}$$

where

$$\begin{aligned}
\vartheta_x^{(k)}(p) &= (p-x)^{-k}\,[\,(p-x)(p-y)\,]^{1/2} \quad k\geq 1, \\
\vartheta_y^{(k)}(p) &= (p-y)^{-k}\,[\,(p-x)(p-y)\,]^{1/2} \quad k\geq 1.
\end{aligned} \tag{3.6.22}$$

Let us again specialize to $g=1$. Using the above we can reexpress eq. (3.6.14) as

$$\begin{aligned}
\frac{\delta}{\delta\Psi(p)}F_1 + \kappa_1\,\frac{1}{d}\,\frac{\delta}{\delta\Psi(p)}(\Xi_1-\Lambda_1) &= \sum_{i=1}^{3}\Big(B_1^{(i)}\,\chi^{(i)}(p) + E_1^{(i)}\,\Psi^{(i)}(p)\Big) \\
&= \sum_{r=2}^{4}\Big(\beta_1^{(r)}\,\vartheta_x^{(r)}(p) + \epsilon_1^{(r)}\,\vartheta_x^{(r)}(p)\Big) + \gamma_1\,[\,\vartheta_x^{(1)}(p) - \vartheta_y^{(1)}(p)\,],
\end{aligned} \tag{3.6.23}$$

with the new coefficients $\beta_1^{(r)}$, $\epsilon_1^{(r)}$ and γ_1 completely determined by the known coefficients $B_1^{(i)}$ and $E_1^{(i)}$. As the new functions $\vartheta_x^{(r)}(p)$ and $\vartheta_y^{(r)}(p)$ are total derivatives in $\Psi(p)$, this equation allows us to calculate κ_1 and F_1. It is obvious that this method works for higher g as well.

With the help of *Maple* the zero mode coefficient κ_1 of $v_1(p)$ becomes

$$\begin{aligned}
\kappa_1 = & \frac{11\,\Xi_2}{16\,d^3{M_1}^2} - \frac{11\,\Lambda_2}{16\,d^3{J_1}^2} + \frac{5\,\Lambda_2 J_2}{8\,d^2{J_1}^3} + \frac{5\,\Xi_2 M_2}{8\,d^2{M_1}^3} - \frac{5\,\Xi_2 M_3}{16\,d{M_1}^3} \\
& + \frac{5\,\Lambda_2 J_3}{16\,d{J_1}^3} - \frac{\Xi_2}{16\,d^3 J_1 M_1} + \frac{\Lambda_2}{16\,d^3 J_1 M_1} - \frac{\Xi_2 J_2}{16\,d^2{J_1}^2 M_1} - \frac{\Lambda_2 M_2}{16\,d^2{M_1}^2 J_1} \\
& + \frac{3\,\Xi_2{M_2}^2}{8\,d{M_1}^4} - \frac{3\,\Lambda_2{J_2}^2}{8\,d{J_1}^4} + \frac{3\,\Lambda_3 J_2}{8\,d{J_1}^3} - \frac{3\,\Xi_3 M_2}{8\,d{M_1}^3} + \frac{5\,\Xi_4}{16\,d{M_1}^2}
\end{aligned}$$

$$
-\frac{5\,\Lambda_4}{16\,dJ_1{}^2} - \frac{5\,\Xi_3}{8\,d^2M_1{}^2} - \frac{5\,\Lambda_3}{8\,d^2J_1{}^2} + (\Xi_1-\Lambda_1)\Big\{\frac{3\,J_2{}^2}{8\,d^2J_1{}^4}
-\frac{3\,M_2{}^2}{8\,d^2M_1{}^4} - \frac{5\,M_2}{8\,d^3M_1{}^3} - \frac{5\,J_2}{8\,d^3J_1{}^3} + \frac{5\,M_3}{16\,d^2M_1{}^3} - \frac{5\,J_3}{16\,d^2J_1{}^3}
+\frac{11}{16\,d^4J_1{}^2} - \frac{11}{16\,d^4M_1{}^2} + \frac{M_2}{16\,d^3J_1M_1{}^2} + \frac{J_2}{16\,d^3J_1{}^2M_1}\Big\} \tag{3.6.24}
$$

and the doubly fermionic part of F_1 is constructed as well. The result for the full free energy at genus 1 then reads

$$
\begin{aligned}
F_1 = &-\frac{1}{12}\ln M_1 - \frac{1}{12}\ln J_1 - \frac{1}{3}\ln d \\
&-(\Xi_1-\Lambda_1)\Big\{\frac{11\,\Xi_2}{16\,d^4\,M_1{}^2} - \frac{11\,\Lambda_2}{16\,d^4\,J_1{}^2} + \frac{5\,\Lambda_2\,J_2}{8\,d^3\,J_1{}^3} + \frac{5\,\Xi_2\,M_2}{8\,d^3\,M_1{}^3} \\
&-\frac{5\,\Xi_2\,M_3}{16\,d^2\,M_1{}^3} + \frac{5\,\Lambda_2\,J_3}{16\,d^2\,J_1{}^3} - \frac{\Xi_2\,J_2}{16\,d^3\,J_1{}^2M_1} - \frac{\Lambda_2\,M_2}{16\,d^3\,M_1{}^2J_1} \\
&+\frac{3\,\Xi_2\,M_2{}^2}{8\,d^2\,M_1{}^4} - \frac{3\,\Lambda_2\,J_2{}^2}{8\,d^2\,J_1{}^4} + \frac{3\,\Lambda_3\,J_2}{8\,d^2\,J_1{}^3} - \frac{3\,\Xi_3\,M_2}{8\,d^2\,M_1{}^3} + \frac{5\,\Xi_4}{16\,d^2\,M_1{}^2} \\
&-\frac{5\,\Lambda_4}{16\,d^2\,J_1{}^2} - \frac{\Xi_3}{2\,d^3\,M_1{}^2} - \frac{\Lambda_3}{2\,d^3\,J_1{}^2} + \frac{3\,\Xi_2}{16\,d^4\,J_1M_1} - \frac{3\,\Lambda_2}{16\,d^4\,J_1M_1}\Big\} \\
&+\frac{\Xi_2\,\Xi_3}{8\,d^2\,M_1{}^2} + \frac{\Lambda_2\,\Lambda_3}{8\,d^2\,J_1{}^2} - \frac{\Xi_2\,\Lambda_2}{4\,d^3\,J_1M_1}.
\end{aligned} \tag{3.6.25}
$$

The above results hold true for generic bosonic and fermionic potentials. Note that for symmetric bosonic potentials, i.e. $x=-y$, and generic fermionic potentials one has

$$
M_k = (-)^{k+1}\,J_k, \tag{3.6.26}
$$

easily seen from the definitions of eq. (3.5.7). If one takes the fermionic potential to be symmetric as well one finds

$$
\Xi_k = (-)^k\,\Lambda_k \qquad k\geq 2, \tag{3.6.27}
$$

from the defining equation (3.5.8). Then one checks $F_1^{\mathrm{ferm}}=0$ and this continues to hold true for higher genera as well, seen from an analysis of the odd superloop equation in the totally symmetric case.

The Iteration for $\widehat{u}_g(p)$

The remaining part of the loop correlator $W(\,|\,p)$ is now easily derived from F_g by applying the loop insertion operator $\delta/\delta V(p)$ to its doubly fermionic part. For genus 1 the result is

$$
\widehat{u}_1(p) = \sum_{i=1}^{4}\Big(\widehat{A}_1^{(i)}\,\chi^{(i)}(p) + \widehat{D}_1^{(i)}\,\Psi^{(i)}(p)\Big), \tag{3.6.28}
$$

where

$$
\widehat{A}_1^{(4)} = -\frac{35\,(\Xi_1-\Lambda_1)\,\Xi_2}{32\,d^2\,M_1{}^2}
$$

$$
\widehat{A}_1^{(3)} = (\Xi_1-\Lambda_1)\Big\{-\frac{15\,\Xi_3}{16\,d^2\,M_1{}^2} + \frac{45\,\Xi_2}{32\,d^3\,M_1{}^2} + \frac{25\,\Xi_2\,M_2}{32\,d^2\,M_1{}^3} - \frac{5\,\Lambda_2}{32\,d^3\,J_1\,M_1}\Big\}
$$

$$
\begin{aligned}
\widehat{A}_1^{(2)} = &\frac{21\,(\Xi_1-\Lambda_1)\,\Xi_3}{16\,d^3M_1{}^2} - \frac{3\,(\Xi_1-\Lambda_1)\,\Xi_2M_2{}^2}{8\,d^2M_1{}^4} + \frac{3\,(\Xi_1-\Lambda_1)\,\Xi_2J_2}{32\,d^3J_1{}^2M_1} + \frac{\Xi_2\,\Xi_3}{8\,d^2M_1{}^2} \\
&+\frac{3\,\Lambda_2\,\Xi_2}{8\,J_1M_1d^3} + \frac{3\,(\Xi_1-\Lambda_1)\,\Xi_3M_2}{4\,d^2M_1{}^3} - \frac{15\,(\Xi_1-\Lambda_1)\,\Xi_4}{16\,d^2M_1{}^2} - \frac{\Xi_3\,\Xi_2}{4\,d^2M_1{}^2} \\
&-\frac{51\,(\Xi_1-\Lambda_1)\,\Xi_2}{32\,d^4M_1{}^2} - \frac{\Lambda_2\,(\Xi_1-\Lambda_1)\,M_2}{32\,J_1d^3M_1{}^2} - \frac{\Lambda_2\,(\Xi_1-\Lambda_1)}{4\,J_1d^4M_1} \\
&-\frac{33\,(\Xi_1-\Lambda_1)\,\Xi_2M_2}{32\,d^3M_1{}^3} + \frac{5\,(\Xi_1-\Lambda_1)\,\Xi_2M_3}{16\,d^2M_1{}^3} - \frac{9\,(\Xi_1-\Lambda_1)\,\Xi_2}{32\,J_1d^4M_1}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{A}_1^{(1)} = &\frac{33\,(\Xi_1-\Lambda_1)\,\Xi_2M_2}{32\,d^4M_1{}^3} - \frac{(\Xi_1-\Lambda_1)\,\Xi_2M_2}{32\,d^4J_1M_1{}^2} - \frac{35\,(\Xi_1-\Lambda_1)\,\Xi_5}{32\,d^2M_1{}^2} \\
&-\frac{3\,(\Xi_1-\Lambda_1)\,\Xi_2J_2{}^2}{16\,d^3J_1{}^4} + \frac{15\,(\Xi_1-\Lambda_1)\,\Xi_4M_2}{16\,d^2M_1{}^3} + \frac{45\,(\Xi_1-\Lambda_1)\,\Xi_4}{32\,d^3M_1{}^2} \\
&+\frac{5\,(\Xi_1-\Lambda_1)\,\Xi_2}{16\,d^5J_1M_1} - \frac{7\,(\Xi_1-\Lambda_1)\,\Lambda_3}{32\,d^4J_1{}^2} - \frac{\Xi_2\,\Lambda_3}{16\,d^3J_1{}^2} + \frac{\Xi_3\,\Xi_2}{16\,d^3M_1{}^2} \\
&+\frac{51\,(\Xi_1-\Lambda_1)\,\Xi_2}{32\,d^5M_1{}^2} - \frac{51\,(\Xi_1-\Lambda_1)\,\Xi_3}{32\,d^4M_1{}^2} - \frac{\Xi_2\,\Xi_3}{8\,d^3M_1{}^2} + \frac{5\,\Xi_2\,\Xi_4}{16\,d^2M_1{}^2} \\
&-\frac{\Lambda_2\,\Xi_2}{2\,J_1M_1d^4} - \frac{7\,\Lambda_2\,(\Xi_1-\Lambda_1)\,J_2}{32\,J_1{}^3d^4} + \frac{3\,\Lambda_2\,\Xi_3}{8\,d^3J_1M_1} + \frac{7\,\Lambda_2\,(\Xi_1-\Lambda_1)}{16\,J_1{}^2d^5} \\
&+\frac{5\,(\Xi_1-\Lambda_1)\,\Xi_2\,J_2}{16\,J_1{}^3d^4} - \frac{11\,(\Xi_1-\Lambda_1)\,\Xi_2}{32\,J_1{}^2d^5} - \frac{5\,(\Xi_1-\Lambda_1)\,\Xi_2M_3}{16\,d^3M_1{}^3} \\
&-\frac{9\,(\Xi_1-\Lambda_1)\,\Xi_3M_2{}^2}{16\,d^2M_1{}^4} + \frac{3\,(\Xi_1-\Lambda_1)\,\Xi_3J_2}{32\,d^3J_1{}^2M_1} - \frac{9\,(\Xi_1-\Lambda_1)\,\Xi_3}{32\,J_1M_1d^4}
\end{aligned}
$$

$$+\frac{15\,(\Xi_1-\Lambda_1)\,\Xi_3 M_3}{32\,d^2 {M_1}^3}-\frac{9\,(\Xi_1-\Lambda_1)\,\Xi_3 M_2}{8\,d^3 {M_1}^3}+\frac{5\,(\Xi_1-\Lambda_1)\,\Xi_2 J_3}{32\,d^3 {J_1}^3}$$
$$-\frac{\Xi_2}{2\,d}\left(-\frac{11\,\Lambda_2}{16\,d^3 {J_1}^2}+\frac{5\,\Lambda_2 J_2}{8\,d^2 {J_1}^3}+\frac{5\,\Lambda_2 J_3}{16\,d {J_1}^3}+\frac{\Lambda_2}{16\,d^3 J_1 M_1}\right.$$
$$-\frac{\Lambda_2 M_2}{16\,d^2 {M_1}^2 J_1}-\frac{3\,\Lambda_2 {J_2}^2}{8\,d {J_1}^4}+\frac{3\,\Lambda_3 J_2}{8\,d {J_1}^3}-\frac{3\,\Xi_3 M_2}{8\,d {M_1}^3}+\frac{5\,\Xi_4}{16\,d {M_1}^2}$$
$$\left.-\frac{5\,\Lambda_4}{16\,d {J_1}^2}-\frac{5\,\Xi_3}{8\,d^2 {M_1}^2}-\frac{5\,\Lambda_3}{8\,d^2 {J_1}^2}\right)+\frac{3\,(\Xi_1-\Lambda_1)\,\Xi_2 {M_2}^2}{8\,d^3 {M_1}^4}$$
$$-\frac{(\Xi_1-\Lambda_1)\,\Xi_2 J_2}{8\,d^4 {J_1}^2 M_1}+\frac{\Lambda_2\,(\Xi_1-\Lambda_1)\,M_2}{16\,d^4 J_1 {M_1}^2}+\frac{11\,\Lambda_2\,(\Xi_1-\Lambda_1)}{32\,J_1 d^5 M_1}$$
$$+(\Xi_1-\Lambda_1)\left\{\frac{3\,{J_2}^2}{8\,d^2 {J_1}^4}-\frac{3\,{M_2}^2}{8\,d^2 {M_1}^4}-\frac{5\,M_2}{8\,d^3 {M_1}^3}-\frac{5\,J_2}{8\,d^3 {J_1}^3}\right.$$
$$+\frac{5\,M_3}{16\,d^2 {M_1}^3}-\frac{5\,J_3}{16\,d^2 {J_1}^3}+\frac{11}{16\,d^4 {J_1}^2}-\frac{11}{16\,d^4 {M_1}^2}$$
$$\left.+\frac{M_2}{16\,d^3 J_1 {M_1}^2}+\frac{J_2}{16\,d^3 {J_1}^2 M_1}\right\}, \tag{3.6.29}$$

and the analogous expressions for the $\widehat{D}_1^{(i)}$ obtained from the above by replacing $M \leftrightarrow J$, $\Xi \leftrightarrow \Lambda$ and $d \to -d$.

General Structure of u_g, v_g, $\widehat{u}_g$ and F_g

In the following subsection we deduce the number of moments and basis functions the free energy and the superloop correlators at genus g depend on.

Ambjørn et al. [9, 40] have shown that the free energy of the hermitian matrix model depends on $2(3g-2)$ moments. This directly translates to $F_g^{\rm bos}$. Similarly as $u_g = \delta\, F_g^{\rm bos}/\delta V(p)$ and with eq. (3.6.11) we see that u_g contains $2(3g-1)$ bosonic moments and basis functions up to order $(3g-1)$, i.e.

$$u_g(p) = \sum_{k=1}^{3g-1} A_g^{(k)}\,\chi^{(k)}(p) \;+\; D_g^{(k)}\,\Psi^{(k)}(p). \tag{3.6.30}$$

For the structure of v_g consider the leading–order poles on the right hand side of eq. (3.6.2). Label this order by n_g, then with eqs. (3.6.10), (3.6.11) and (3.6.13) the three terms on the right hand side of eq. (3.6.2) give rise to the following poles of leading order

$$\widehat{\Psi}\circ u_g(p) : (3g-1)+1$$
$$v_{g'}\,u_{g-g'} : n_{g'} + \Big(3\,(g-g')-1\Big) + 1$$
$$\frac{\delta}{\delta V(p)}\,v_{g-1} : n_{g-1} \;+\; 3. \tag{3.6.31}$$

From the above we deduce that $n_g = 3\,g$, and therefore

$$v_g(p) = \sum_{k=1}^{3g} B_g^{(k)}\,\chi^{(k)}(p) \;+\; E_g^{(k)}\,\Psi^{(k)}(p) \;+\; \kappa_g\,\phi^{(0)}(p). \tag{3.6.32}$$

As the highest bosonic moments in v_g come from the highest–order basis functions, we see that v_g depends on $2(3\,g)$ bosonic moments. To find the dependence on the number of fermionic moments recall eq. (3.6.7). In order for v_g to have a leading contribution of $\chi^{(3g)}(p)$ the fermionic part of the free energy $F_g^{\rm ferm}$ must contain Ξ_{3g+1}. We are thus led to the conclusion that $F_g^{\rm ferm}$ and v_g both depend on $2(3\,g+1)$ fermionic moments [6]. As the application of the loop insertion operator $\delta/\delta\Psi(p)$ does not change the number of bosonic moments, $F_g^{\rm ferm}$ must contain $2(3\,g)$ bosonic moments.

Knowing the structure of $F_g^{\rm ferm}$ then tells us with eqs. (3.6.10) and (3.6.11) that $\widehat{u}_g$ depends on $2(3\,g+1)$ bosonic and $2(3\,g+2)$ fermionic moments. For genus g it reads

$$\widehat{u}_g(p) = \sum_{k=1}^{3g+1} \widehat{A}_g^{(k)}\,\chi^{(k)}(p) \;+\; \widehat{D}_g^{(k)}\,\Psi^{(k)}(p). \tag{3.6.33}$$

The genus g contribution to the $(\,|s)$–superloop correlator $W_g(\,|p,\ldots,p)$ will then depend on $2(3\,g+s)$ bosonic and $2(3\,g+s+1)$ fermionic moments. Similarly the genus g $(1|s)$–superloop correlator $W_g(p|p,\ldots,p)$ is a function of $2(3\,g+s)$ bosonic and $2(3\,g+s+1)$ fermionic moments.

This concludes our analysis of the iterative process away from the double scaling limit.

7. The Double Scaling Limit

In view of our lack of understanding the supereigenvalue model on a geometrical basis as a discretization of super–Riemann surfaces it appears quite ambitious to speak of a continuum limit. However, due to the striking similarities to the Hermitian matrix model

[6] In some sense this is counterintuitive: In ref. [9] it was shown that there is a relation of the dependence on the number of moments to the dimension of the moduli space of a Riemann surface of genus g, which is $3g-3$. So one could have speculated that something similar holds true for the fermionic moments and the number of fermionic moduli of a super–Riemann surface, which is $2g-2$. However, the discovered slope of 3 in g seems to contradict such an interpretation.

with its well understood continuum limit (discussed in chapter I section 2) it is natural to export these techniques to the supersymmetric case. We shall see that again one finds critical values for the coupling constants g_k and $\xi_{k+1/2}$ allowing one to perform a double scaling limit.

Similar to the situation in the Hermitian matrix model the “naive” $N \to \infty$ limit of the supereigenvalue model is unsatisfactory as it leaves us only with the planar contributions, easily seen from eqs. (3.4.8) and (3.4.9). Crucial for the double scaling limit is the observation that there exists a subspace in the space of couplings $\{g_k, \xi_{k+1/2}\}$ where all higher genus contributions to the free energy F_g diverge. This enables us to take the double scaling limit, where one simultaneously approaches the critical subspace of couplings *and* takes $N \to \infty$, giving contributions to the free energy from all genera. Just as in the Hermitian matrix model Kazakov [13] multicritical points appear, related to extra zeros of the eigenvalue densities accumulating at one endpoint of their support.

For the formulation of this limit the moment technique developed in the above turns out to be extremely useful. One can then determine which terms in the explicit solutions for F_g, $W(p|\,)$ and $W(\,|p)$ calculated in the previous section contribute to the double scaling limit. However, the calculation of these quantities away from the double scaling limit is rather time consuming, especially as a large number of terms turn out to be irrelevant in this limit. In this section we will therefore develop a procedure which directly produces only the terms relevant in the double scaling limit.

The first thing to do is to study the scaling of the moments and basis functions by approaching the critical point.

Multicritical Points

The analysis of the scaling behaviour for the bosonic quantities was carried out by Ambjørn et al. [9, 41] in the framework of the hermitian matrix model. Consider the case of generic, i.e. non–symmetric, potentials $V(p)$ and $\Psi(p)$. The m'th multicritical point is reached when the eigenvalue density $(u_0(p) - V'(p))$ of eq. (3.5.22), which under normal circumstances vanishes as a square root on both ends of its support, acquires $(m-1)$ additional zeros at one end of the cut, say x. Alternatively one would define the critical point by a diverging free energy. A glance at our genus one result of eq. (3.6.25) tells us that F_1 diverges for vanishing M_1 (or J_1). The only moments appearing in the denominators of higher genus F_g's will be M_1 and J_1 as well. Hence the condition for being at an m'th multicritical point is taken to be

$$M_1^c = M_2^c = \cdots M_{m-1}^c = 0, \quad M_k^c \neq 0, \quad k \geq m, \qquad J_l^c \neq 0, \quad l \geq 1, \tag{3.7.1}$$

defining a critical subspace in the space of bosonic couplings g_k. Denote by g_k^c a particular point in this subspace for which the eigenvalue distribution is confined to the interval $[x_c, y_c]$. If we now move away from this point the conditions of eq. (3.7.1) will no longer be fulfilled and the cut will move to the interval $[x, y]$. Assume we control this movement by the parameter a and set [9]

$$\begin{aligned} x &= x_c - a\,\Lambda^{1/m} \\ p &= x_c + a\,\pi \end{aligned} \tag{3.7.2}$$

The scaling of p and the introduction of its scaling variable π is necessary in order to speak of the double scaling limit of the superloop correlators. Λ plays the role of the cosmological constant. We will now deduce the scaling of y by further assuming that the critical subspace $\{g_k^c\}$ is reached as

$$g_k = g \cdot g_k^c, \tag{3.7.3}$$

where g is a function of a to be determined. Imposing the boundary conditions (3.5.3) yields

$$y - y_c \sim a^m, \quad \text{and} \quad g - 1 \sim a^m. \tag{3.7.4}$$

This may be seen as follows. Consider the first boundary condition (3.5.3) using $y = y_c + \Delta y$, eqs. (3.7.2) and (3.7.3) and the fact that the M_i^c for $i \in [1, m-1]$ vanish lead to

$$\begin{aligned} 0 &= \oint_C \frac{d\omega}{2\pi} \frac{g\,V_c'(\omega)}{(\omega - x_c + a\,\Lambda^{1/m})^{1/2}\,(\omega - y_c - \Delta y)^{1/2}} \\ &= g \oint_C \frac{d\omega}{2\pi} \frac{g\,V_c'(\omega)}{(\omega - x_c)^{1/2}\,(\omega - y_c)^{1/2}} [\sum_{i=0}^{\infty} c_i\,\frac{(a\Lambda^{1/m})^i}{(\omega - x_c)^i}][\sum_{i=0}^{\infty} c_i\,\frac{(\Delta y)^i}{(\omega - y_c)^i}] \\ &= g \sum_{i=m}^{\infty} c_i\,M_i^c\,(a\Lambda^{1/m})^i + g\,c_1\,J_1^c\,\Delta y \\ &\quad + \mathcal{O}(a \cdot \Delta y), \end{aligned}$$

from which one deduces $\Delta y \sim a^m$. The computation of $g - 1 \sim a^m$ goes along the same lines starting from the second boundary condition (3.5.3).

Knowing this one easily computes the m'th multicritical scaling behaviour of the bosonic moments

$$M_k \sim a^{m-k}, \quad k \in [1, m-1], \tag{3.7.5}$$

while the higher M–moments and the J–moments do not scale.

See this by direct computation:

$$M_k = \oint_C \frac{d\omega}{2\pi} \frac{g\, V_c'(\omega)}{(\omega - x_c + a\,\Lambda^{1/m})^{k+1/2}\, (\omega - y_c - a^m\,\tilde{\Lambda})^{1/2}}$$

$$= g \oint_C \frac{d\omega}{2\pi} \frac{V_c'(\omega)}{(\omega - x_c)^{k+1/2}\, (\omega - y_c)^{1/2}} [\sum_{i=0}^{\infty} c_i^{(k)}\, \frac{(a\,\Lambda^{1/m})^i}{(\omega - x_c)^i}] [\sum_{j=0}^{\infty} c_j^{(0)}\, \frac{(a^m\,\tilde{\Lambda})^j}{(\omega - y_c)^j}],$$

where for $k \in [1, m-1]$ the leading nonvanishing term is a^{m-k} (from $i = m-k$ and $j = 0$ in the sums).

Moreover the functions $\phi_x^{(n)}(p)$ and $\phi_y^{(n)}(p)$ are found to behave like

$$\phi_x^{(n)}(p) \sim a^{-n-1/2}, \qquad \phi_y^{(n)}(p) \sim a^{-1/2}, \tag{3.7.6}$$

from which one proves the scaling behaviour of the basis functions

$$\chi^{(n)}(p) \sim a^{-m-n+1/2}, \qquad \Psi^{(n)}(p) \sim a^{-1/2}, \tag{3.7.7}$$

by induction.

Let us now turn to the scaling of the fermionic moments Ξ_k and Λ_k. Similar to the bosonic case the function $(v_0(p) - \Psi(p))$ of eq. (3.5.24) usually vanishes at the endpoints of the cut like a square root. We will fine tune the coupling constants $\xi_{k+1/2}$ in such a manner that $(n-1)$ extra zeros accumulate at x, i.e.

$$\Xi_2^c = \cdots \Xi_{n-1}^c = 0, \qquad \Xi_k^c \neq 0, \quad k \geq n, \qquad \Lambda_l \neq 0, \quad l \geq 2, \tag{3.7.8}$$

where $\Xi_k^c \equiv \Xi_k[x_c, y_c, \xi_{k+1/2}^c]$. In addition the analysis of the solution away from the scaling limit tells us that the moments Ξ_1 and Λ_1 will always appear in the combination $(\Xi_1 - \Lambda_1)$ [7]. This suggests to impose the constraint $\Xi_1^c - \Lambda_1^c = 0$ on these moments. As there is no analogue to the boundary conditions (3.5.3) for the fermionic quantities we are free to choose the scaling of the coupling constants $\xi_{k+1/2}$. We set

$$\xi_{k+1/2} = a^{1/2}\, \xi_{k+1/2}^c, \tag{3.7.9}$$

and will comment on this choice later on. From this one derives $(\Xi_1 - \Lambda_1) \sim a^{n-1/2}$ and $\Xi_k \sim a^{n-k+1/2}$ for $k \in [2, n-1]$. All other fermionic moments scale uniformly with $a^{1/2}$. So far the fermionic scaling is completely independent of the bosonic scaling, governed by the integer n. We shall, however, introduce the requirement that the scaling part of the bosonic one–superloop correlator $W_0(\,|\,p)$ of eq. (3.5.21) scales uniformly, i.e. we require $(u_0(p) - V'(p))$ and $\widehat{u}_0(p)$ to scale in the same way [32]. As $(u_0(p) - V'(p)) \sim a^{m-1/2}$ we arrive at the following condition on n:

$$n = m + 1. \tag{3.7.10}$$

And therefore

$$\Xi_1 - \Lambda_1 \sim a^{m+1/2}, \qquad \Xi_k \sim a^{m-k+3/2}, \quad k \in [2, m]$$

$$\Xi_k \sim a^{1/2}, \quad k > m \qquad \Lambda_l \sim a^{1/2}, \quad l > 1. \tag{3.7.11}$$

The double scaling limit is now defined by letting $N \to \infty$ and $a \to 0$ but keeping the string coupling constant $\alpha = a^{-2m-1}\, N^{-2}$ fixed. [8]

The Iteration for $u_g(p)$ and $v_g(p)$

By making use of the above scaling properties of the moments and basis functions we may now develop the iterative procedure which allows us to calculate directly the double scaling relevant versions of $u_g(p)$ and $v_g(p)$. Recall the iterative scheme of the previous sections: By eq. (3.5.13) every u_g and v_g may be written as a linear combination of basis functions $\chi^{(n)}$ and $\Psi^{(n)}$, where the coefficients of this expansion are simply read off the poles at x and y of the right hand sides of the superloop equations (3.6.1) and (3.6.2) after a partial fraction decomposition. To optimize the procedure to only produce terms which are relevant in the double scaling limit we have to analyze the operators appearing on the right hand sides of the superloop equations, i.e. the superloop insertion operators $\delta/\delta V(p)$ and $\delta/\delta\Psi(p)$ as well as the operator $\widehat{\Psi}$.

From the point of view of the $a \to 0$ limit the effect of a given operator in $\delta/\delta V(p)$ acting on an expression which scales with a to some power is to lower this power by a certain amount. Carefully examining each term in $\delta/\delta V(p)$ shows that a is maximally lowered by a power of $(m + 3/2)$. All operators which do not lower a by this amount are subdominant in the scaling limit and may be neglected. The outcome of this analysis for $\delta/\delta V(p)$ is

$$\frac{\delta}{\delta V(p)_x} = \sum_{k=1}^{\infty} \frac{\delta M_k}{\delta V(p)_x} \frac{\delta}{\delta M_k} + \frac{\delta x}{\delta V(p)} \frac{\delta}{\delta x}$$

[7] The only exception is the planar zero mode χ of eq. (3.5.20), however, as seen from eq. (3.5.24) $v_0(p)$ effectively does not depend on this combination.

[8] Let us now comment on the choice of eq. (3.7.9). At first sight one might have expected a scaling like $\xi_{k+1/2} = [1 + o(a)]\, \xi_{k+1/2}^c$. This turns out to be inconsistent because then the condition (3.7.10) demands n to be half–integer which it can not be. If one takes the more general ansatz $\xi_{k+1/2} = a^l\, \xi_{k+1/2}^c$ the condition of uniform scaling of $W(\,|\,p)$ yields the allowed sequence $\{n, l\} = \{m+1, 1/2\}, \{m, 3/2\}, \{m-1, 5/2\}, \ldots$. The scaling of the lowest moments in eq. (3.7.11) remains unchanged, the uniform scaling however already starts with Ξ_n scaling like a^l and thus simply reduces the number of double scaling relevant terms.

$$+ \frac{\delta(\Xi_1 - \Lambda_1)}{\delta V(p)_x} \frac{\delta}{\delta(\Xi_1 - \Lambda_1)} + \sum_{k=2}^{\infty} \frac{\delta \Xi_k}{\delta V(p)_x} \frac{\delta}{\delta \Xi_k} \tag{3.7.12}$$

where [9]

$$\frac{\delta M_k}{\delta V(p)_x} = -(k+\frac{1}{2})\,\phi_x^{(k+1)}(p) \,+\, (k+\frac{1}{2})\,\frac{M_{k+1}}{M_1}\,\phi_x^{(1)}(p) \tag{3.7.13}$$

$$\frac{\delta x}{\delta V(p)} = \frac{1}{M_1}\,\phi_x^{(1)}(p)$$

$$\frac{\delta(\Xi_1 - \Lambda_1)}{\delta V(p)_x} = \frac{1}{2}\,\frac{\Xi_2}{M_1}\,\phi_x^{(1)}(p)$$

$$\frac{\delta \Xi_k}{\delta V(p)_x} = (k-\frac{1}{2})\,\frac{\Xi_{k+1}}{M_1}\,\phi_x^{(1)}(p) \tag{3.7.14}$$

and

$$\phi_x^{(n)}(p) = (p-x)^{-n-1/2}\,{d_c}^{-1/2}, \tag{3.7.15}$$

with $d_c = x_c - y_c$. Repeating this analysis for the fermionic superloop insertion operator $\delta/\delta\Psi(p)$ shows that here a is maximally lowered by a power of $(m+1)$ and the relevant contributions are

$$\frac{\delta}{\delta \Psi(p)_x} = \frac{\delta(\Xi_1 - \Lambda_1)}{\delta \Psi(p)_x} \frac{\delta}{\delta(\Xi_1 - \Lambda_1)} + \sum_{k=2}^{\infty} \frac{\delta \Xi_k}{\delta \Psi(p)_x} \frac{\delta}{\delta \Xi_k} \tag{3.7.16}$$

with

$$\frac{\delta(\Xi_1 - \Lambda_1)}{\delta \Psi(p)_x} = -d_c\,\phi_x^{(0)}(p)$$

$$\frac{\delta \Xi_k}{\delta \Psi(p)_x} = -d_c\,\phi_x^{(k-1)}(p). \tag{3.7.17}$$

Finally we state the double scaling version of the operator $\widehat{\Psi}$ acting on the function $\phi_x^{(n)}(p)$

$$\widehat{\Psi}_x \circ \phi_x^{(n)}(p) = \sum_{k=1}^{n} \frac{\Xi_{n+2-k}}{d_c} \frac{1}{(p-x)^k} + \frac{(\Xi_1 - \Lambda_1)}{2\,d_c} \frac{1}{(p-x)^{n+1}}. \tag{3.7.18}$$

[9] Here the subscript x indicates that the critical behaviour is associated with the endpoint x.

Here the operator $\widehat{\Psi}_x$ is seen to increase the power of a of the expression it acts on by a power of m.

We are now in a position to calculate the double scaling limit of the right hand sides of the loop equations (3.6.1) and (3.6.2) for $u_g(p)$ and $v_g(p)$ provided we know the double scaled versions of $u_1(p), \ldots, u_{g-1}(p)$ and $v_1(p), \ldots, v_{g-1}(p)$. As all the y dependence has disappeared we do not have to perform a partial fraction decomposition of the result. Moreover no J_k and Λ_k dependent terms will contribute if we do not start out with any and we do not.

The starting point of the iterative procedure are of course the genus 0 correlators. Keeping only the double scaling relevant parts of eqs. (3.6.5) and (3.6.8) one has

$$-\frac{d}{dq}\frac{\delta v_0(q)}{\delta \Psi(p)_x}\big|_{p=q} = \frac{\delta u_0(p)}{\delta V(p)_x} = \frac{1}{8}\,\frac{1}{(p-x)^2} \tag{3.7.19}$$

Similarly eq. (3.6.12) turns into

$$\frac{\delta v_0(p)}{\delta V(p)_x} = \Big[-\frac{(\Xi_1 - \Lambda_1)}{4\,d_c\,M_1}\Big]\,\frac{1}{(p-x)^3} + \Big[\frac{\Xi_2}{4\,d_c\,M_1}\Big]\,\frac{1}{(p-x)^2}. \tag{3.7.20}$$

The higher genus correlators $u_g(p)$ and $v_g(p)$ are expressed as linear combinations of the basis function $\chi^{(n)}(p)$ and take the general form

$$u_g(p) = \sum_{k=1}^{3g-1} A_g^{(k)}\,\chi^{(k)}(p) \quad \text{and} \quad v_g(p) = \sum_{k=1}^{3g} B_g^{(k)}\,\chi^{(k)}(p) \,+\, \kappa_g\,\phi^{(0)}(p), \tag{3.7.21}$$

where κ_g is the zero mode coefficient not determined by the first two superloop equations (3.6.1) and (3.6.2). Note that the $A_g^{(k)}$ coefficients should up to a factor of two be identical to those of the double scaled Hermitian matrix model obtained in ref. [9]. We have calculated the $A_g^{(k)}$ and $B_g^{(k)}$ coefficients in the double–scaling limit for $g = 1, 2$ and 3 with the aid of *Maple*. The results of Ambjørn et al. [9] for the $A_g^{(k)}$ coefficients could be reproduced.

Before we state our explicit results let us turn to the general scaling behaviour of the one–superloop correlators

$$W_g(\,|\,p) \sim a^{(1-2g)\,(m+1/2)\,-\,1}, \quad \text{and} \quad W_g(p\,|\,) \sim a^{(1-2g)\,(m+1/2)\,-\,1/2}, \tag{3.7.22}$$

which one proves by induction. As shown in ref. [9] the coefficients $A_g^{(k)}$ of eq. (3.7.21) take the form

$$A_g^{(k)} = \sum_{\alpha_j} \langle\,\alpha_1, \ldots, \alpha_s \mid \alpha\,\rangle_{g,k}\,\frac{M_{\alpha_1}\ldots M_{\alpha_s}}{{M_1}^{\alpha}\,{d_c}^{g-1}}, \tag{3.7.23}$$

where the brackets stand for rational numbers and where α_j, α and s are subject to the constraints

$$\alpha = 2g + s - 2 \qquad \text{and} \qquad \sum_{j=1}^{s} (\alpha_j - 1) = 3g - k - 1 \tag{3.7.24}$$

with $\alpha_j \in [2, 3g-1]$. Explicitly up to genus two one finds

$$A_1^{(1)} = 0 \qquad A_1^{(2)} = \frac{1}{8}$$

$$A_2^{(1)} = 0 \qquad A_2^{(2)} = 0 \qquad A_2^{(3)} = \frac{49\, {M_2}^2}{128\, d_c\, {M_1}^4} - \frac{5\, M_3}{16\, d_c\, {M_1}^3}$$

$$A_2^{(4)} = -\frac{49\, M_2}{64\, d_c\, {M_1}^3} \qquad \frac{105}{128\, d_c\, {M_1}^2}. \tag{3.7.25}$$

The terms above are only potentially relevant, as for the m'th multicritical point all terms containing M_k with $k > m$ are subleading in the double scaling limit.

Similarly the structure of the coefficients $B_g^{(k)}$ may be determined from the iterative procedure and is seen to be

$$B_g^{(k)} = \sum_{\alpha_j, \beta} \langle \alpha_1, \ldots, \alpha_s, \beta \mid \alpha \rangle_{g,k} \frac{M_{\alpha_1} \ldots M_{\alpha_s}\, \Xi_\beta}{{M_1}^\alpha\, {d_c}^g}, \tag{3.7.26}$$

where the brackets denote rational numbers and where we write Ξ_1 for $(\Xi_1 - \Lambda_1)$. One shows that the α, α_j, β and s obey the conditions

$$\alpha = 2g + s - 1, \qquad \text{and} \qquad \sum_{j=1}^{s} (\alpha_j - 1) = 3g + 1 - \beta - k \tag{3.7.27}$$

with $\alpha_j \in [2, 3g]$ and $\beta \in [1, 3g]$. For the zero mode coefficient κ_g the general structure is given by the same expansion as eq. (3.7.26) with $k = 0$. The conditions on α, α_j, β and s then read

$$\alpha = 2g + s, \qquad \text{and} \qquad \sum_{j=1}^{s} (\alpha_j - 1) = 3g + 1 - \beta \tag{3.7.28}$$

where $\alpha_j \in [2, 3g]$ and $\beta \in [1, 3g+1]$.

The explicit results for the $B_g^{(k)}$ coefficients for $g = 1$ and $g = 2$ are given by

$$B_1^{(1)} = -\frac{\Xi_3}{8\, M_1\, d_c} + \frac{M_2 \Xi_2}{8\, {M_1}^2\, d_c},$$

$$B_1^{(2)} = \frac{M_2\, (\Xi_1 - \Lambda_1)}{16\, {M_1}^2 d_c} + \frac{\Xi_2}{8\, M_1\, d_c},$$

$$B_1^{(3)} = -\frac{5\, (\Xi_1 - \Lambda_1)}{16\, M_1\, d_c}.$$

$$\begin{aligned}
B_2^{(1)} = {} & \frac{203\, M_2 \Xi_5}{128\, {d_c}^2 {M_1}^4} - \frac{145\, {M_3}^2 \Xi_2}{128\, {M_1}^5 {d_c}^2} - \frac{105\, \Xi_6}{128\, {d_c}^2 {M_1}^3} + \frac{63\, {M_2}^3 \Xi_3}{32\, {M_1}^6 {d_c}^2} \\
& + \frac{105\, M_4 \Xi_3}{128\, {d_c}^2 {M_1}^4} + \frac{145\, M_3 \Xi_4}{128\, {d_c}^2 {M_1}^4} + \frac{105\, M_5 \Xi_2}{128\, {d_c}^2 {M_1}^4} - \frac{77\, M_4 M_2 \Xi_2}{32\, {M_1}^5 {d_c}^2} \\
& - \frac{87\, M_3 M_2 \Xi_3}{32\, {M_1}^5 {d_c}^2} + \frac{75\, M_3 {M_2}^2 \Xi_2}{16\, {M_1}^6 {d_c}^2} - \frac{63\, {M_2}^4 \Xi_2}{32\, {M_1}^7 {d_c}^2} - \frac{63\, {M_2}^2 \Xi_4}{32\, {M_1}^5 {d_c}^2}, \\
B_2^{(2)} = {} & -\frac{21\, {M_2}^3 \Xi_2}{64\, {M_1}^6 {d_c}^2} - \frac{21\, M_2 \Xi_4}{64\, {d_c}^2 {M_1}^4} + \frac{77\, M_3 M_2 \Xi_2}{128\, {M_1}^5 {d_c}^2} - \frac{35\, M_4 \Xi_2}{128\, {d_c}^2 {M_1}^4} \\
& + \frac{75\, M_3 {M_2}^2 (\Xi_1 - \Lambda_1)}{32\, {M_1}^6 {d_c}^2} + \frac{105\, M_5 (\Xi_1 - \Lambda_1)}{256\, {d_c}^2 {M_1}^4} - \frac{63\, {M_2}^4 (\Xi_1 - \Lambda_1)}{64\, {M_1}^7 {d_c}^2} \\
& + \frac{35\, \Xi_5}{128\, {M_1}^3 {d_c}^2} - \frac{145\, {M_3}^2 (\Xi_1 - \Lambda_1)}{256\, {M_1}^5 {d_c}^2} + \frac{21\, {M_2}^2 \Xi_3}{64\, {M_1}^5 {d_c}^2} - \frac{35\, M_3 \Xi_3}{128\, {d_c}^2 {M_1}^4}, \\
& - \frac{77\, M_4 M_2 (\Xi_1 - \Lambda_1)}{64\, {M_1}^5 {d_c}^2} \\
B_2^{(3)} = {} & -\frac{599\, M_3 M_2 (\Xi_1 - \Lambda_1)}{128\, {M_1}^5 {d_c}^2} + \frac{105\, {M_2}^3 (\Xi_1 - \Lambda_1)}{32\, {M_1}^6 {d_c}^2} - \frac{5\, M_3 \Xi_2}{16\, {d_c}^2 {M_1}^4} \\
& + \frac{385\, M_4 (\Xi_1 - \Lambda_1)}{256\, {d_c}^2 {M_1}^4} + \frac{21\, {M_2}^2 \Xi_2}{64\, {M_1}^5 {d_c}^2} + \frac{7\, M_2 \Xi_3}{128\, {d_c}^2 {M_1}^4}, \\
B_2^{(4)} = {} & -\frac{357\, {M_2}^2 (\Xi_1 - \Lambda_1)}{64\, {M_1}^5 {d_c}^2} + \frac{875\, M_3 (\Xi_1 - \Lambda_1)}{256\, {d_c}^2 {M_1}^4} - \frac{35\, \Xi_3}{128\, {M_1}^3 {d_c}^2} - \frac{63\, M_2 \Xi_2}{128\, {d_c}^2 {M_1}^4}, \\
B_2^{(5)} = {} & \frac{105\, \Xi_2}{128\, {M_1}^3 {d_c}^2} + \frac{1617\, M_2 (\Xi_1 - \Lambda_1)}{256\, {d_c}^2 {M_1}^4}, \\
B_2^{(6)} = {} & -\frac{1155\, (\Xi_1 - \Lambda_1)}{256\, {M_1}^3 {d_c}^2}.
\end{aligned} \tag{3.7.29}$$

Note that the terms listed above are only potentially relevant, depending on which multi–critical model one wishes to consider. For an m'th multi–critical model all terms containing M_k, $k > m$, or Ξ_l, $l > m+1$, vanish in the double–scaling limit. We remind the reader that we assumed to have a non–symmetric potential and that the critical behaviour was associated with the endpoint x. In the case where the critical behaviour is associated with the endpoint y all formulas in this section still hold provided d_c is replaced by $-d_c$, M_k by J_k, Ξ_k by Λ_k and x by y.

The Iteration for F_g, κ_g and $\widehat{u}_g(p)$

Having computed $u_g(p)$ and $v_g(p)$ (up to the zero mode) we may now proceed to calculate the free energy F_g and the zero mode coefficient κ_g. This is done by rewriting u_g and v_g as total derivatives in the superloop insertion operators $\delta/\delta V(p)$ and $\delta/\delta\Psi(p)$ respectively, yielding the bosonic and doubly fermionic parts of F_g as well as κ_g. The procedure to compute the bosonic part of the free energy $F_g^{\rm bos}$ works just as in the Hermitian matrix model described in ref. [9]. We have $F_g^{\rm bos} = 2\,F_g^{\rm herm}$.

From eq. (3.7.22) we see that F_g scales as

$$F_g = F_g^{\rm bos} + F_g^{\rm ferm} \sim a^{(2-2g)(m+1/2)}, \tag{3.7.30}$$

just as the hermitian matrix model at its m'th multicritical point.

To find the bosonic piece of the free energy F_g one uses the following recursive rewriting of $\chi^{(n)}(p)$ as a total derivative

$$\chi^{(n)}(p) = -\frac{1}{M_1}\Big(\frac{2}{2n-1}\,\frac{\delta M_{n-1}}{\delta V(p)} - \sum_{k=2}^{n-1}\chi^{(k)}(p)\,M_{n-k+1}\Big) \qquad n\geq 2, \tag{3.7.31}$$

One does not need any expression for $\chi^{(1)}(p)$ as $A_g^{(1)} = 0$ in eq. (3.7.21) for all $g \geq 2$ [9].

To obtain the doubly fermionic part $F_g^{\rm ferm}$ simply rewrite the basis functions $\chi^{(n)}(p)$ of $v_g(p)$ appearing in (3.7.21) in terms of the functions $\phi_x^{(n)}(p)$ which by eq. (3.7.17) are nothing but total derivatives in $\delta/\delta\Psi(p)$. Doing this for $v_g(p)$ allows one to directly deduce the form of $F_g^{\rm ferm}$ and κ_g. [10]

The explicit results for genus one are

$$F_1^{\rm bos} = -\frac{1}{12}\ln M_1,$$

$$F_1^{\rm ferm} = (\Xi_1-\Lambda_1)\Big(-\frac{5\,\Xi_4}{16\,{d_c}^2{M_1}^2} + \frac{3\,M_2\,\Xi_3}{8\,{d_c}^2{M_1}^3} + \frac{5\,M_3\,\Xi_2}{16\,{d_c}^2{M_1}^3} - \frac{3\,{M_2}^2\,\Xi_2}{8\,{d_c}^2{M_1}^4}\Big) + \frac{\Xi_2\,\Xi_3}{8\,{d_c}^2{M_1}^2} \tag{3.7.32}$$

and

$$\kappa_1 = \frac{5\,\Xi_4}{16\,{M_1}^2 d_c} - \frac{3\,M_2\,\Xi_3}{8\,{M_1}^3 d_c} - \frac{5\,M_3\,\Xi_2}{16\,{M_1}^3 d_c} + \frac{3\,{M_2}^2\,\Xi_2}{8\,{M_1}^4 d_c}. \tag{3.7.33}$$

These expressions may of course be alternatively obtained by taking the double scaling limit of the genus one results computed away from the scaling limit, except for the scaling violating term $F_1^{\rm bos}$ at genus one. This term in eq. (3.7.32) was selected by producing the double scaling version of $u_1(p)$.

Before presenting explicit results for genus two, let us describe the general structure of the free energy at genus g. For the bosonic piece we have [9]

$$F_g^{\rm bos} = \sum_{\alpha_j>1}\langle\,\alpha_1,\ldots,\alpha_s\mid\alpha\,\rangle_g\,\frac{M_{\alpha_1}\ldots M_{\alpha_s}}{M_1^{\alpha}\,{d_c}^{g-1}} \qquad g\geq 2, \tag{3.7.34}$$

where the brackets denote rational numbers. The α_j, α and s obey

$$\alpha = 2g-2+s \qquad\text{and}\qquad \sum_{j=1}^{s}(\alpha_j-1) = 3g-3, \tag{3.7.35}$$

with $\alpha_j\in[2,3g-2]$.

Similarly the general form of $F_g^{\rm ferm}$ for $g\geq 1$ reads

$$F_g^{\rm ferm} = \sum_{\alpha_j,\beta_i}\langle\alpha_1,\ldots,\alpha_s,\beta_1,\beta_2\mid\alpha\rangle_g\,\frac{\Xi_{\beta_1}\,\Xi_{\beta_2}\,M_{\alpha_1}\ldots M_{\alpha_s}}{{M_1}^{\alpha}\,{d_c}^{g+1}}, \tag{3.7.36}$$

where we write Ξ_1 for $(\Xi_1-\Lambda_1)$ and where the brackets denote rational numbers as before. The α_j, β_i, α and s are subject to the constraints

$$\alpha = 2g+s, \quad\text{and}\quad \sum_{j=1}^{s}(\alpha_j-1) = 3g+2-\beta_1-\beta_2, \tag{3.7.37}$$

where $\alpha_j\in[2,3g]$ and $\beta_i\in[1,3g+1]$.

The results for genus two read

$$F_2^{\rm bos} = \frac{29\,M_2\,M_3}{64\,{M_1}^5\,d_c} - \frac{21\,{M_2}^3}{80\,{M_1}^5\,d_c} - \frac{35\,M_4}{192\,{M_1}^3\,d_c}$$

[10] Actually the presentation here is somewhat misleading. In order to compute the $u_g(p)$ and $v_g(p)$ by iteration in the way outlined in the previous subsection one needs to know the *full* $v_1(p),\ldots,v_{g-1}(p)$ including the zero mode coefficients $\kappa_1,\ldots,\kappa_{g-1}$. In practice one hence computes all quantities at genus g, i.e. u_g, v_g, κ_g and F_g, before proceeding to genus $(g+1)$.

$$F_2^{\text{ferm}} = (\Xi_1 - \Lambda_1)\Big[\frac{1015\,\Xi_5\,M_3}{128\,M_1{}^5 d_c{}^3} - \frac{375\,\Xi_4\,M_2\,M_3}{16\,M_1{}^6 d_c{}^3}$$
$$+\frac{315\,\Xi_4\,M_2{}^3}{16\,M_1{}^7 d_c{}^3} + \frac{385\,\Xi_4\,M_4}{64\,M_1{}^5 d_c{}^3} + \frac{1323\,\Xi_2\,M_2{}^5}{64\,M_1{}^9 d_c{}^3} + \frac{693\,\Xi_6\,M_2}{64\,M_1{}^5 d_c{}^3}$$
$$-\frac{1155\,\Xi_7}{256\,M_1{}^4 d_c{}^3} - \frac{63\,\Xi_2\,M_2\,M_5}{4\,M_1{}^6 d_c{}^3} - \frac{525\,\Xi_5\,M_2{}^2}{32\,M_1{}^6 d_c{}^3} - \frac{1155\,\Xi_3\,M_2\,M_4}{64\,M_1{}^6 d_c{}^3}$$
$$+\frac{315\,\Xi_3\,M_5}{64\,M_1{}^5 d_c{}^3} - \frac{2175\,\Xi_3\,M_3{}^2}{256\,M_1{}^6 d_c{}^3} + \frac{675\,\Xi_3\,M_2{}^2\,M_3}{16\,M_1{}^7 d_c{}^3} - \frac{1785\,\Xi_2\,M_3\,M_4}{128\,M_1{}^6 d_c{}^3}$$
$$-\frac{495\,\Xi_2\,M_2{}^3\,M_3}{8\,M_1{}^8 d_c{}^3} + \frac{2205\,\Xi_2\,M_2{}^2\,M_4}{64\,M_1{}^7 d_c{}^3} - \frac{1323\,\Xi_3\,M_2{}^4}{64\,M_1{}^8 d_c{}^3} + \frac{8175\,\Xi_2\,M_3{}^2 M_2}{256\,M_1{}^7 d_c{}^3}$$
$$+\frac{1155\,\Xi_2\,M_6}{256\,M_1{}^5 d_c{}^3}\Big] - \frac{21\,\Xi_2\,\Xi_5\,M_2}{16\,M_1{}^5 d_c{}^3} + \frac{105\,\Xi_2\,\Xi_6}{128\,M_1^4 d_c{}^3} - \frac{145\,\Xi_2\,\Xi_4\,M_3}{128\,M_1{}^5 d_c{}^3}$$
$$+\frac{105\,\Xi_2\,\Xi_4 M_2{}^2}{64\,M_1{}^6 d_c{}^3} + \frac{21\,\Xi_3\,\Xi_4\,M_2}{64\,M_1{}^5 d_c{}^3} - \frac{35\,\Xi_3\,\Xi_5}{128\,M_1{}^4 d_c{}^3} - \frac{63\,\Xi_2\,\Xi_3\,M_2{}^3}{32\,M_1{}^7 d_c{}^3}$$
$$-\frac{35\,\Xi_2\,\Xi_3\,M_4}{32\,M_1{}^5 d_c{}^3} + \frac{195\,\Xi_2\,\Xi_3\,M_2\,M_3}{64\,M_1{}^6 d_c{}^3}. \tag{3.7.38}$$

Of course so far we have determined only the coefficients F_g of the genus expansion of the free energy (cf. eq. (3.4.9)). For an m'th multicritical model the relevant expansion parameter in the double scaling limit is the string coupling constant $\alpha = a^{-2m-1}\,N^{-2}$. If we introduce the bosonic and fermionic scaling moments μ_k and τ_k by (cf. eqs. (3.7.5) and (3.7.11))

$$M_k = a^{m-k}\,\mu_k, \quad k \in [1, m],$$
$$(\Xi_1 - \Lambda_1) = a^{m+1/2}\,\tau_1, \qquad \Xi_l = a^{m-l+3/2}\,\tau_l, \quad l \in [2, m+1], \tag{3.7.39}$$

we get by replacing M_k by μ_k and setting M_k equal to zero for $k > m$ as well as replacing Ξ_l by τ_l and setting Ξ_l to zero for $l > m+1$ in the formulas above exactly the coefficients of the expansion in the string coupling constant. Needless to say that these results apply for non–symmetric potentials as well where the critical behaviour is associated with the endpoint y by performing the usual replacements.

For the zero mode coefficient at genus two we find

$$\kappa_2 = \frac{495\,\Xi_2 M_3 M_2{}^3}{8\,d_c{}^2 M_1{}^8} + \frac{63\,\Xi_2 M_5 M_2}{4\,d_c{}^2 M_1{}^6} - \frac{315\,\Xi_3 M_5}{64\,d_c{}^2 M_1{}^5} + \frac{1323\,\Xi_3 M_2{}^4}{64\,d_c{}^2 M_1{}^8}$$
$$-\frac{675\,\Xi_3 M_3 M_2{}^2}{16\,d_c{}^2 M_1{}^7} + \frac{2175\,\Xi_3 M_3{}^2}{256\,d_c{}^2 M_1{}^6} - \frac{8175\,\Xi_2 M_3{}^2 M_2}{256\,d_c{}^2 M_1{}^7} - \frac{385\,\Xi_4 M_4}{64\,d_c{}^2 M_1{}^5}$$
$$-\frac{2205\,\Xi_2 M_4 M_2{}^2}{64\,d_c{}^2 M_1{}^7} + \frac{375\,\Xi_4 M_3 M_2}{16\,d_c{}^2 M_1{}^6} - \frac{1155\,\Xi_2 M_6}{256\,d_c{}^2 M_1{}^5} + \frac{1785\,\Xi_2 M_4 M_3}{128\,d_c{}^2 M_1{}^6}$$
$$-\frac{693\,\Xi_6\,M_2}{64\,d_c{}^2 M_1{}^5} - \frac{315\,\Xi_4 M_2{}^3}{16\,d_c{}^2 M_1{}^7} + \frac{525\,\Xi_5 M_2{}^2}{32\,d_c{}^2 M_1{}^6} - \frac{1015\,\Xi_5 M_3}{128\,d_c{}^2 M_1{}^5}$$
$$-\frac{1323\,\Xi_2 M_2{}^5}{64\,d_c{}^2 M_1{}^9} + \frac{1155\,\Xi_3 M_4 M_2}{64\,d_c{}^2 M_1{}^6} + \frac{1155\,\Xi_7}{256\,d_c{}^2 M_1{}^4}. \tag{3.7.40}$$

We obtained these results with the aid of a *Maple* program which performs the iteration up to arbitrary genus. In practice the expressions become quite lengthy, e.g. F_3^{ferm} consists of 114 terms.

For the knowledge of the full $W_g(\,|\,p)$ it remains to compute $\widehat{u}_g(p)$. This is of course done by applying $\delta/\delta V(p)$ to F_g^{ferm}. The general structure of $\widehat{u}_g(p)$ turns out to be

$$\widehat{u}_g(p) = \sum_{k=1}^{3g+1} \widehat{A}_g^{(k)}\,\chi^{(k)}(p), \tag{3.7.41}$$

where

$$\widehat{A}_g^{(k)} = \sum_{\alpha_j,\beta_1,\beta_2} \langle \alpha_1, \ldots, \alpha_s, \beta_1, \beta_2 \mid \alpha \rangle_{g,k}\, \frac{M_{\alpha_1}\ldots M_{\alpha_s}\,\Xi_{\beta_1}\,\Xi_{\beta_2}}{M_1{}^{\alpha}\, d_c{}^{g+1}} \tag{3.7.42}$$

underlying the conditions

$$\alpha = s + 3g, \quad \text{and} \quad \sum_{j=1}^{s}(\alpha_j - 1) = 4 + 3g - k - \beta_1 - \beta_2, \tag{3.7.43}$$

with $\alpha_j \in [2, 3g+1]$ and $\beta_i \in [1, 3g+2]$. Due to space let us only state the genus one results

$$\widehat{A}_1^{(1)} = \frac{5\,\Xi_2\,\Xi_4}{32\,d_c{}^2 M_1{}^2} + \frac{3\,M_2 \Xi_2 \Xi_3}{16\,d_c{}^2 M_1{}^3} - \frac{35\,(\Xi_1 - \Lambda_1)\Xi_5}{32\,d_c{}^2 M_1{}^2}$$
$$+\frac{15\,M_3(\Xi_1 - \Lambda_1)\,\Xi_3}{32\,d_c{}^2 M_1{}^3} + \frac{15\,M_2(\Xi_1 - \Lambda_1)\,\Xi_4}{16\,d_c{}^2 M_1{}^3} - \frac{9\,M_2{}^2(\Xi_1 - \Lambda_1)\,\Xi_3}{16\,d_c{}^2 M_1{}^4},$$
$$\widehat{A}_1^{(2)} = -\frac{15\,(\Xi_1 - \Lambda_1)\,\Xi_4}{16\,d_c{}^2 M_1{}^2} + \frac{3\,\Xi_2\,\Xi_3}{8\,d_c{}^2 M_1{}^2} + \frac{3\,M_2(\Xi_1 - \Lambda_1)\,\Xi_3}{4\,d_c{}^2 M_1{}^3}$$
$$+\frac{5\,M_3(\Xi_1 - \Lambda_1)\Xi_2}{16\,d_c{}^2 M_1{}^3} - \frac{3\,M_2{}^2(\Xi_1 - \Lambda_1)\,\Xi_2}{8\,d_c{}^2 M_1{}^4},$$
$$\widehat{A}_1^{(3)} = -\frac{15\,(\Xi_1 - \Lambda_1)\Xi_3}{16\,d_c{}^2 M_1{}^2} + \frac{25\,M_2\,(\Xi_1 - \Lambda_1)\,\Xi_2}{32\,d_c{}^2 M_1{}^3},$$

$$\widehat{A}_1^{(4)} = -\frac{35\,(\Xi_1 - \Lambda_1)\,\Xi_2}{32\,{d_c}^2\,{M_1}^2}. \tag{3.7.44}$$

This concludes our analysis of the double–scaling limit for generic potentials.

The case of symmetric bosonic and generic fermionic potentials was considered in ref. [31] where a doubling of degrees of freedom was observed for genus zero. We then have the independent set of moments $\{M_k, \Xi_k, \Lambda_k\}$. With the methods presented in this paper one can see that this holds for higher genera as well, i.e. the free energy here takes the form

$$F_g^{\text{symm}} = 2\,F_g^{\text{bos}} \;+\; F_g^{\text{ferm,x}} \;+\; F_g^{\text{ferm,y=-x}}, \tag{3.7.45}$$

where F_g^{bos} denotes the bosonic part of the free energy for generic potentials, $F_g^{\text{ferm,x}}$ and $F_g^{\text{ferm,y=-x}}$ denote the doubly fermionic parts in the generic case where the critical behaviour is associated with the endpoints x and $y=-x$ respectively (cf. eq. (3.7.36)). If one chooses to take the fermionic potential to be symmetric as well the doubly fermionic part of the free energy will vanish, as we have already seen on the discrete level in section 6.

8. Identification of the Model

In order to identify the continuum theory described by the supereigenvalue model we will have to discuss some results obtained in super–Liouville theory. This subsection will by no means give a review on this subject, we simply highlight some formulas relevant for the interpretation of our results. For more details see refs. [42, 43, 5].

In the following we consider $N=1$ superconformal field theories with $\widehat{c}=d$ coupled to 2d supergravity. For a base manifold of fixed genus g the partition function is given by

$$Z_g = \sum_s \int [\mathcal{D}E_M^A]\,[\mathcal{D}\mathcal{X}^I]\,e^{-S[\mathcal{X},E]}, \tag{3.8.1}$$

where the sum runs over the spin structures. Here E_M^A denotes the super–zweibein, $\mathcal{X}^I$ the free, conformal superfield and the matter action reads

$$S[\mathcal{X},E] = \frac{1}{2}\int_{M_s} d^2z\,d^2\theta D_\alpha \mathcal{X}^I\,D^\alpha \mathcal{X}^I, \tag{3.8.2}$$

using the superspace notation of ref. [42]. Here D_α are superdifferentials and $I=1,\ldots,d$. The action and the measures $\mathcal{D}E_M^A$ and $\mathcal{D}\mathcal{X}^I$ are invariant under superdiffeomorphisms and local Lorentz transformations. These have to be divided out of the path integral (3.8.1) which is symbolized by square brackets around the volume elements. The action (3.8.2) is also invariant under the group of local superconformal transformations while the measure is not. Thus a superconformal gauge fixing to the reference frame $E = e^{a\,\phi}\,\widehat{E}$ will yield a Jacobian which is assumed to be [42]

$$J(\phi,\widehat{E}) = \exp(-S_{\text{SL}}[\phi,\widehat{E}]\,). \tag{3.8.3}$$

The action functional in the exponent is the super–Liouville action

$$S_{\text{SL}}[\phi,\widehat{E}] = \frac{1}{4\pi}\int d^2z\,d^2\theta\,\widehat{E}\,(\frac{1}{2}\,\widehat{D}_\alpha\phi\,\widehat{D}^\alpha\phi - \sqrt{\frac{9-\widehat{c}}{2}}\,\widehat{Y}\,\phi + \kappa\,\mathcal{O}_{\text{min}}\,e^{a\,\phi}\,) \tag{3.8.4}$$

where ϕ is the Liouville superfield and $\widehat{Y}$ the curvature superfield. The cosmological constant κ couples to the operator of lowest–dimension in the Neveu–Schwarz sector of the matter theory denoted by $\mathcal{O}_{\text{min}}$ [44]. For unitary theories $\mathcal{O}_{\text{min}}$ is unity. Moreover a is a $\widehat{c}$ dependent constant. The precise form of the matter action (3.8.2) is not of importance, all that is needed is that one has some superconformal field theory with central charge $\widehat{c}$. One can now study the partition function (3.8.1) in the form

$$Z_g = \int_0^\infty dL\,Z_g(L), \tag{3.8.5}$$

where L is a characteristic length scale of the super–Riemann surface [11]. The partition function for the super–random surface of fixed length is defined to be

$$Z_g(L) = \sum_s \int \mathcal{D}\mathcal{X}\,\mathcal{D}\phi\,e^{-S_{\text{SL}}-S_{\text{M}}}\,\delta\Big(\int d^2z\,d^2\theta\,\widehat{E} - L\Big), \tag{3.8.6}$$

where the symbol $\mathcal{D}\mathcal{X}$ denotes the integration over all the super–matter, super–ghost and super–moduli contributions, contained in the matter action S_{M}. Similarly to the bosonic case the scaling behaviour of $Z_g(L)$ may be computed [42]

$$Z_g(L) = e^{(\kappa_c-\kappa)\,L}\,L^{(1-g)\,(\gamma_{\text{str.}}-2)-1}\,f_g, \tag{3.8.7}$$

where κ_c and f_g are undetermined and L independent. Here the string susceptibility $\gamma_{\text{str.}}$ of a unitary superconformal field theory takes the form

$$\gamma_{\text{str.}} = 2 + \frac{1}{4}\,(\,\widehat{c} - 9 - \sqrt{(9-\widehat{c}\,)\,(1-\widehat{c}\,)}\,). \tag{3.8.8}$$

[11] Note that L has the dimension of a length. In contrast to the bosonic random surface which has an intrinsic area defined by $\int d^2z\,\sqrt{g}$, the analogous object for a super–random surface is given by $\int d^2z\,d^2\theta\,E = L$. Because of the Grassmann integrations the dimension of L is lowered by 1 compared to the bosonic case.

The minimal $N=1$ superconformal field theories with $\widehat{c} \leq 1$ are classified by a pair of integers (p,q) and have central charge

$$\widehat{c} = 1 - \frac{2\,(p-q)^2}{p\,q}. \tag{3.8.9}$$

The unitary theories (with $\widehat{c} \geq 0$) are given by the pair $(p,q) = (m+2,m)$ with $m \geq 2$. For a non–unitary theory formula (3.8.8) is no longer valid and instead one has [44, 5]

$$\gamma_{\text{str.}} = 2 - \frac{2\,(p+q)}{p+q-2}. \tag{3.8.10}$$

In any case the integral over L in eq. (3.8.5) with the $Z_g(L)$ of eq. (3.8.7) may be performed to give

$$Z_g = (\kappa-\kappa_c)^{(2-\gamma_{\text{str.}})\,(1-g)}\,\Gamma[(g-1)(2-\gamma_{\text{str.}})]\,f_g, \tag{3.8.11}$$

note the analogy to the bosonic result of eq. (1.2.11). Guided by the approach in the Hermitian matrix model, one now identifies the free energy of the supereigenvalue model with the partition function of 2d supergravity coupled to superconformal matter, i.e.

$$N^2\,F_g[g_k,\xi_{k+1/2}] \,\widehat{=}\, Z_g, \tag{3.8.12}$$

and sets $(g_k - g_k^c) \sim (\kappa-\kappa_c)$. Using our result of eq. (3.7.30) and the scaling behaviour of the bosonic couplings in eq. (3.7.4) one finds

$$N^2\,F_g \sim (g_k - g_k^c)^{(2+1/m)\,(1-g)}, \tag{3.8.13}$$

hence we see that $\gamma_{\text{str.}} = -1/m$. The computation of a string susceptibility exponent is of course not enough to identify the continuum model described by the double scaled supereigenvalue model. One has to establish a connection between Liouville operators to supereigenvalue correlators. The Neveu–Schwarz and Ramond sectors of the super–Liouville theory are reflected in the even and odd correlators of the supereigenvalue model $\langle \sum_i {\lambda_i}^k \rangle$ and $\langle \sum_i \theta\,{\lambda_i}^k \rangle$ respectively. Adequate linear combinations of these correlators in the scaling limit can be identified with super–Liouville amplitudes, as was shown by Abdalla and Zadra in ref. [37] by using the planar supereigenvalue results of ref. [31]. The outcome is that the double scaled supereigenvalue model at its m'th critical point describes the coupling of a minimal superconformal field theory of type $(p,q) = (4m,2)$ to 2d supergravity having the central charge

$$\widehat{c} = 1 - \frac{(2m-1)^2}{m} = 0, -\frac{7}{2}, -\frac{22}{3}, \ldots \qquad m \geq 1. \tag{3.8.14}$$

The only unitary model in this series is pure supergravity with $m=1$. Note, however, that this case must be treated separately, as in our approach $m=1$ corresponds to no restrictions on the moments (cf. eq. (3.7.1)), and thus leads to no critical behaviour. This is completely analogous to the bosonic case, where a string susceptibility of $\gamma_{\text{str.}} = -1$ corresponding to $c=-2$ may not be reached by the Hermitian one matrix model, but rather through the model studied in chapter II. For the higher non–unitary superconformal theories one easily convinces oneself that $(p,q) = (4m,2)$ yields $\gamma_{\text{str.}} = -1/m$ by using eq. (3.8.10).

9. Conclusions

We have constructed the supereigenvalue model by imposing the super–Virasoro constraints on its partition function. By integrating out its fermionic degrees of freedom we saw that the bosonic part of the free energy of the supereigenvalue model is simply twice the free energy of the corresponding hermitian matrix model. First the model was solved away from the double scaling limit. The superloop correlators obey a set of integral equations, the superloop equations. These two equations could be split up into a set of four equations, sorted by their order in fermionic coupling constants. By a change of variables from coupling constants to moments we were able to present the planar solution of the superloop equations for general potentials in a very compact form. The remarkable structure of the superloop equations enabled us to develop an iterative procedure for the calculation of higher–genera contributions to the free energy and the superloop correlators. Here it proved sufficient to solve the two lowest–order equations at genus g for the purely bosonic $u_g(p)$ and the fermionic $v_g(p)$ (up to a zero mode contribution). The zero mode as well as the doubly fermionic part of the free energy could then be found by rewriting $v_g(p)$ as a total derivative in the fermionic potential. The purely bosonic part of the free energy can be calculated with the methods of ref. [9]. In principle the application of loop insertion operators to the free energy then yields arbitrary multi–superloop correlators. As we demonstrated for genus one, in practice these expressions become quite lengthy.

We then turned to the double scaling limit of the model in the moment description. The m'th multi–critical point was identified and the scaling properties of the moments and basis functions were derived. The iterative scheme for the calculation of higher genus contributions could be optimized to produce only terms relevant in the double scaling limit. The general form of the free energy and the one–superloop correlators at genus g were stated. We presented explicit result up to genus two. Finally we commented on the identification of the supereigenvalue model at its m'th multi–critical point with 2d supergravity coupled to $(4m,2)$ minimal superconformal matter.

The analogy of structures in the Hermitian one matrix model and in the supereigenvalue model is rather impressive. This lends hope to find an answer to further interesting questions one should address, such as the supersymmetric generalization of two– and multi–matrix models or the generalization of matrix models in external fields, which we shall be discussing in the next chapter.

Chapter IV: The External Field Problem

So far the partition functions of the Hermitian matrix and the supereigenvalue model acted as generating functionals at the same time due to the presence of arbitrary polynomial potentials. This is why solving for the free energy of these models really meant completely determining all possible correlators as well. As we know from field theory an alternative way to build generating functionals is to include external fields as sources coupled linearly to the fields of the theory. In the language of Hermitian matrix models this leads to partition functions depending on the eigenvalues of an external matrix. It is thus not surprising that interrelations between ordinary and external matrix models may be established. The key point here is to find a translation scheme from the coupling constants of the ordinary theory to the eigenvalues of the external matrices known as Miwa transformations. The study of matrix models in external fields has received much attention following the work of Kontsevich [45] whose external field model was shown to be equivalent to the Hermitian matrix model in the double scaling limit. No limiting procedure is necessary as the model directly produces only the scaling relevant terms.

Moreover external field models are closely related to Hermitian two matrix models, which in their scaling limit are capable of describing *all* the (p, q) minimal conformal field theories coupled to 2d gravity. Although the form of the two matrix model is obvious on the matrix level, it is not on the eigenvalue level as two arbitrary Hermitian matrices may not be simultaneously diagonalized by one $U(n)$ transformation.

In regard of these results it would be quite desirable to find the correct supersymmetric generalizations of external Hermitian matrix models. However, in that respect this chapter has the form of an outlook, as we have not been able to perform this program completely. Nevertheless some encouraging results may be reported.

Due to simplicity we study the discrete case, i.e. away from the double scaling limit. After a review of the bosonic case we turn to the supersymmetrization of the observed structures.

1. The Bosonic Case

Let us consider a $N \times N$ Hermitian matrix model with a potential $V_0(\boldsymbol{X})$ and an external Hermitian matrix field $\boldsymbol{L}$ as source term

$$Z_N[\boldsymbol{L}] = \int_{N\times N} \mathcal{D}\boldsymbol{X}\, \exp[\,\mathrm{Tr}\,\boldsymbol{L}\,\boldsymbol{X} - \mathrm{Tr}\,V_0(\boldsymbol{X})\,]. \tag{4.1.1}$$

By taking matrix derivatives of $Z_N[\boldsymbol{L}]$ with respect to the source and defined by

$$(\partial_{\boldsymbol{L}})_{ab} \equiv \frac{\partial}{\partial \boldsymbol{L}_{ba}} \qquad \Rightarrow \qquad (\partial_{\boldsymbol{L}})_{ab}\, Z_N[\boldsymbol{L}] = \langle\, \boldsymbol{X}_{ab}\,\rangle\, Z_N[\boldsymbol{L}] \tag{4.1.2}$$

one may generate arbitrary correlators. In this sense the source $\boldsymbol{L}$ replaces the infinite set of coupling constants g_k of the Hermitian matrix model of eq. (1.1.1).

External field models may be solved in the planar limit and in their genus expansions through the method of Schwinger–Dyson equations [46, 47]. These equations are easily derived. Consider the shift of the matrix $\boldsymbol{X}$ in eq. (4.1.1)

$$\boldsymbol{X} \to \boldsymbol{X} + \epsilon_n\, \boldsymbol{X}^{n+1} \qquad n \geq -1, \tag{4.1.3}$$

with ϵ_n infinitesimal. Then the invariance of the integral under a renaming of integration variables yields

$$\int_{N\times N} \mathcal{D}\boldsymbol{X}\, \mathrm{Tr}\left[\boldsymbol{L}\,\boldsymbol{X}^{n+1} - \boldsymbol{X}^{n+1}\, V_0'(\boldsymbol{X}) + \sum_{k=0}^{n} \mathrm{Tr}\,[\,\boldsymbol{X}^k\,]\,\boldsymbol{X}^{n-k}\right]$$
$$\cdot\, \exp[\,\mathrm{Tr}\,\boldsymbol{L}\,\boldsymbol{X} - \mathrm{Tr}\,V_0(\boldsymbol{X})\,] = 0. \tag{4.1.4}$$

in first order ϵ_n.

See this by calculating the Jacobian of the transformation (4.1.3) to first order ϵ_n:

$$\mathrm{Det}\,\frac{\partial\,\boldsymbol{X}'}{\partial \boldsymbol{X}^T} = e^{\mathrm{Tr}\ln(1+\epsilon_n\,\frac{\partial\,\boldsymbol{X}^{n+1}}{\partial \boldsymbol{X}^T})} = e^{\mathrm{Tr}\,\epsilon_n\,\frac{\partial\,\boldsymbol{X}^{n+1}}{\partial \boldsymbol{X}^T}} = 1 + \epsilon_n\,\mathrm{Tr}\,\frac{\partial \boldsymbol{X}^{n+1}}{\partial \boldsymbol{X}^T}$$

Now

$$\mathrm{Tr}\,\frac{\partial\,\boldsymbol{X}^{n+1}}{\partial \boldsymbol{X}^T} = \frac{\partial}{\partial \boldsymbol{X}_{ba}}\,\boldsymbol{X}_{b\,i_1}\,\boldsymbol{X}_{i_1 i_2}\ldots\boldsymbol{X}_{i_n\,a} = \sum_{k=1}^{n}\mathrm{Tr}\,\boldsymbol{X}^k\,\mathrm{Tr}\,\boldsymbol{X}^{n-k},$$

and combining this with the variation of the potential and the source term yields eq. (4.1.4)

Using derivatives with respect to the source eq. (4.1.4) may be rewritten as

$$\mathrm{Tr}\left[\boldsymbol{L}\,(\partial_{\boldsymbol{L}})^{n+1} - (\partial_{\boldsymbol{L}})^{n+1}\,V_0'(\partial_{\boldsymbol{L}}) + \sum_{k=0}^{n}(\partial_{\boldsymbol{L}})^k\,\mathrm{Tr}\,(\partial_{\boldsymbol{L}})^{n-k}\right] Z_N[\boldsymbol{L}] = 0, \tag{4.1.5}$$

and by pulling out all matrix derivatives we have

$$\mathrm{Tr}\,(\partial_{\boldsymbol{L}})^{n+1}\left[\boldsymbol{L} - V_0'(\partial_{\boldsymbol{L}})\right] Z_N[\boldsymbol{L}] = 0. \tag{4.1.6}$$

This system is in fact equivalent to a single matrix valued equation

$$\left[V_0'(\partial_{\boldsymbol{L}}) - \boldsymbol{L}\right] Z_N[\boldsymbol{L}] = 0, \tag{4.1.7}$$

known as the Gross–Newman equation [46].

Interestingly enough the external field model of eq. (4.1.1) with the potential

$$V_0(\boldsymbol{X}) = \frac{1}{2}\,\boldsymbol{X}^2 - n\,\ln\boldsymbol{X} \tag{4.1.8}$$

is closely related to the $n\times n$ Hermitian one matrix model with general polynomial potential

$$\mathcal{Z}_n[\,g_k] = \int_{n\times n}\mathcal{D}\boldsymbol{X}\;\exp[-\mathrm{Tr}\sum_{k=0}^{\infty} g_k\,\boldsymbol{X}^k\,]. \tag{4.1.9}$$

The exact relation between the partition functions (4.1.1) and (4.1.9) was noted by Chekhov and Makeenko [48] and reads

$$\mathcal{Z}_n[\,g_k] = \exp[-\frac{1}{2}\,\mathrm{Tr}\,\boldsymbol{L}^2\,]\,Z_N[\boldsymbol{L}], \tag{4.1.10}$$

provided $\boldsymbol{L}$ and g_k are related by the Miwa–transformation

$$\begin{aligned} g_k &= \frac{1}{k}\,\mathrm{Tr}\,\boldsymbol{L}^{-k} - \frac{1}{2}\,\delta_{k,2} \qquad \text{for } k\geq 1\\ g_0 &= -\mathrm{Tr}\,\ln\boldsymbol{L} \end{aligned} \tag{4.1.11}$$

in the limit $N\to\infty$ which turns the g_k's into independent variables. Note that in eq. (4.1.10) the $N\times N$ external field model is related to a $n\times n$ Hermitian one matrix model. The proof of eq. (4.1.10) is based on the fact that the Gross–Newman equation (4.1.7) for the potential (4.1.8) is equivalent to the Virasoro constraints of eq. (1.3.2) for the Hermitian one matrix model (4.1.9).

Let us prove the relation (4.1.10). Taking the Gross–Newman equation (4.1.7) and applying one more matrix derivative $\partial_{\boldsymbol{L}}$ to get rid of the singular $\partial_{\boldsymbol{L}}^{-1}$ term coming from the derivative of the logarithm we find

$$\Big(\,\partial_{\boldsymbol{L}}^2 - (n+1) - \boldsymbol{L}\,\partial_{\boldsymbol{L}}\Big)\,Z_N[\boldsymbol{L}] = 0, \tag{4.1.12}$$

which may be restated as

$$\Big(\,\partial_{\boldsymbol{L}}^2 + L\,\partial_{\boldsymbol{L}} - n\Big)\,\exp[-\frac{1}{2}\,\mathrm{Tr}\,\boldsymbol{L}^2\,]\,Z_N[\boldsymbol{L}] = 0. \tag{4.1.13}$$

Now insert the Miwa transformation into the matrix differential operator of this equation. By using the chain rule we have

$$\partial_{\boldsymbol{L}} = \sum_{k=0}^{\infty}\frac{\partial g_k}{\partial\boldsymbol{L}^T}\,\partial_{g_k} = -\sum_{k=0}^{\infty}\boldsymbol{L}^{-k-1}\,\partial_{g_k} \tag{4.1.14}$$

and

$$\begin{aligned} (\partial_{\boldsymbol{L}})^2 &= -\sum_{k=0}^{\infty}(\,\partial_{\boldsymbol{L}}\,\boldsymbol{L}^{-k-1}\,)\,\partial_{g_k} + \sum_{k,l}\boldsymbol{L}^{-k-l-2}\,\partial_{g_k}\,\partial_{g_l}\\ &= \sum_{k=0}^{\infty}\Big\{\sum_{a=1}^{k+1}\frac{1}{\boldsymbol{L}^{k+2-a}}\,\mathrm{Tr}\left(\frac{1}{\boldsymbol{L}^a}\right)\partial_{g_k} + \frac{1}{\boldsymbol{L}^{k+2}}\sum_{n=0}^{k}\partial_{g_n}\,\partial_{g_{k-n}}\Big\}. \end{aligned} \tag{4.1.15}$$

Rewriting the double sum in the first term of eq. (4.1.15) leads us to

$$(\partial_{\boldsymbol{L}})^2 = \sum_{m=-1}^{\infty}\frac{1}{\boldsymbol{L}^{m+2}}\sum_{a=1}^{\infty}\mathrm{Tr}\left(\frac{1}{\boldsymbol{L}^a}\right)\partial_{g_{m+a}} + \sum_{m=0}^{\infty}\frac{1}{\boldsymbol{L}^{m+2}}\sum_{k=0}^{m}\partial_{g_k}\,\partial_{g_{m-k}}, \tag{4.1.16}$$

which is already strongly reminiscent to the Virasoro constraints if one would ignore the delta function addition in eq. (4.1.11). Putting all this together we see that

$$\begin{aligned} (\partial_{\boldsymbol{L}}^2 + \boldsymbol{L}\,\partial_{\boldsymbol{L}} - n) = \sum_{m=-1}^{\infty}\frac{1}{\boldsymbol{L}^{m+2}}\Big\{\sum_{k=1}^{\infty}\mathrm{Tr}\,(\boldsymbol{L}^{-k})\,\partial_{g_{m+k}} - \partial_{g_{m+2}} + \sum_{k=0}^{m}\partial_{g_k}\,\partial_{g_{m-k}}\Big\}\\ -(\,n + \partial_{g_0}\,), \end{aligned} \tag{4.1.17}$$

so by identifying $k\,g_k = \mathrm{Tr}\,(\boldsymbol{L}^{-k}) - \delta_{k,2}$ in the above we recover the Virasoro generator $\mathcal{L}_m$

$$\begin{aligned} (\partial_{\boldsymbol{L}}^2 + \boldsymbol{L}\,\partial_{\boldsymbol{L}} - n) = \sum_{m=-1}^{\infty}\frac{1}{\boldsymbol{L}^{m+2}}\underbrace{\Big\{\sum_{k=0}^{\infty}k\,g_k\,\partial_{g_{m+k}} + \sum_{k=0}^{m}\partial_{g_k}\,\partial_{g_{m-k}}\Big\}}_{=\mathcal{L}_m}\\ -(\,n + \partial_{g_0}\,), \end{aligned} \tag{4.1.18}$$

of eq. (1.3.3). Therefore the Virasoro constraints obeyed by $\mathcal{Z}_n[\,g_k]$ are equivalent to the Gross–Newman equation for $Z_N[\boldsymbol{L}]$, provided that

$$\partial_{g_0}\,\mathcal{Z}_n[g_k] = -n\,\mathcal{Z}_n[\,g_k] \tag{4.1.19}$$

is valid. But this is obviously true for an $n\times n$ Hermitian matrix model, as $\partial_{g_0}\,\mathcal{Z}_n[g_k] = -\langle \mathrm{Tr}\,1\rangle\,\mathcal{Z}_n[g_k]$ seen from the definition in eq. (4.1.9). Hence we have proven the relation (4.1.10). Amazingly enough this identity does not depend on the dimension N of the external matrix field.

Of course these identities may also be stated in the eigenvalue language. As the partition function of the external field model (4.1.1) only depends on the eigenvalues of the matrix source $\boldsymbol{L}$, the integral over the angular variables may be performed with the help of the Itzykson–Zuber formula [49] to give the external eigenvalue model

$$Z_N[l_i] = \Big(\int\prod_{i=1}^{N}dx_i\Big)\,\frac{\prod_{i<j}(x_i-x_j)}{\prod_{i<j}(l_i-l_j)}\,\exp\Big[\sum_{i=1}^{N}(-V_0(x_i)+l_i\,x_i\,)\Big], \tag{4.1.20}$$

modulo an irrelevant multiplicative constant. The x_i denote the eigenvalues of $\boldsymbol{X}$ and l_i those of $\boldsymbol{L}$. Integrating $Z_N[\,l_i\,]$ with the measure $\prod_{i<j}(l_i-l_j)^2\,\exp[-\sum_i V_1(l_i)\,]$ over l_i yields the eigenvalue formulation of the Hermitian two matrix model.

In terms of eigenvalues the Schwinger–Dyson equation (4.1.13) now takes the form

$$\sum_{i=1}^{N}\Big[\,\frac{\partial^2}{\partial {l_i}^2}+\sum_{i\neq j}\frac{1}{l_i-l_j}\,\Big(\frac{\partial}{\partial l_i}-\frac{\partial}{\partial l_j}\Big)+l_i\,\frac{\partial}{\partial l_i}-n\,\Big]\;\exp[-\frac{1}{2}\sum_{k=1}^{\infty}{l_k}^2\,]\,Z_N[l_i]=0, \tag{4.1.21}$$

which may be derived by considering the variation $\delta l_i = \epsilon\,l_i$ in eq. (4.1.20) or by studying the eigenvalue dependence of the matrix derivatives $\partial_{\boldsymbol{L}}$ and $\partial_{\boldsymbol{L}}^2$.

The identity (4.1.10) may also be proven directly on the level of the eigenvalue models, by inserting the Miwa–transformations into eq. (4.1.9) and using integral identities for Hermite polynomials [50].

The equivalence of external field and Hermitian matrix models continues to hold true in the continuum limit. Here the Kontsevich model [45], which is built from a cubic $V_0(\boldsymbol{L})$, is the matrix model describing the Hermitian matrix model in the double scaling limit, as shown in [51]. Let me just mention, that in the framework of generalized Kontsevich models a unification of all (p,q) minimal models coupled to 2d gravity could be achieved [52].

2. The Super–Miwa Transformations

Our analysis of the supereigenvalue model in chapter III revealed a striking number of analogies to the Hermitian matrix model. It is thus rather tempting to ask whether there exists an external field formulation of the supereigenvalue model as well, generalizing the results of the previous section. The partition function of such an external supereigenvalue model should depend on a set of even and odd external "supereigenvalues" l_i and θ_i respectively. Similar to the Gross–Newman equation the Schwinger–Dyson equation of this model would then be represented as a differential operator in the external fields l_i and θ_i annihilating the partition function. After performing a super–Miwa transformation of this differential operator from the l_i and θ_i to the coupling constants g_k and $\xi_{k+1/2}$ one should recover the super–Virasoro generators, thus proving the equivalence of these two models.

Unfortunately we have not succeeded in performing this program so far. However, we can report on some preliminary results. The correct differential operator in external fields could be identified, leading to the super–Virasoro constraints through a set of super–Miwa transformations. This operator may even be formulated on the matrix level, thus giving hope to find a true supersymmetric external *matrix* model.

Let us consider a pair of external $N\times N$ matrices, the Grassmann even matrix $\boldsymbol{L}$ accompanied by the matrix $\boldsymbol{\Lambda}$ with Grassmann odd entries. These shall be related to the coupling constants g_k and $\xi_{k+1/2}$ of the supereigenvalue model via the super–Miwa transformations

$$\begin{aligned} g_k &= \frac{1}{k}\,\mathrm{Tr}\,\boldsymbol{L}^{-k} \qquad k\geq 1 \qquad\qquad \xi_{k+1/2} = -\mathrm{Tr}\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-k-1} \qquad k\geq 0\\ g_0 &= -\mathrm{Tr}\,\ln\boldsymbol{L}, \end{aligned} \tag{4.2.1}$$

generalizing eq. (4.1.11). Note that at this stage we have not included delta function additions as in the bosonic case of eq. (4.1.11). The matrix derivatives

$$\partial_{\boldsymbol{L}} = \frac{\partial}{\partial\boldsymbol{L}^T} \qquad\text{and}\qquad \partial_{\boldsymbol{\Lambda}} = \frac{\partial}{\partial\boldsymbol{\Lambda}^T} \tag{4.2.2}$$

may then be reexpressed as differential operators in the coupling constants by using the chain rule as

$$\partial_{\boldsymbol{\Lambda}} = -\sum_{k=0}^{\infty}\frac{\partial\,\mathrm{Tr}\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-k-1}}{\partial\boldsymbol{\Lambda}^T}\,\partial_{\xi_{k+1/2}} = -\sum_{k=0}^{\infty}\boldsymbol{L}^{-k-1}\,\partial_{\xi_{k+1/2}} \tag{4.2.3}$$

and

$$\begin{aligned} \partial_{\boldsymbol{L}} &= \sum_{k=0}^{\infty}\Big(\frac{1}{k}\,\frac{\partial\,\mathrm{Tr}\,\boldsymbol{L}^{-k}}{\partial\boldsymbol{L}^T}\,\partial_{g_k} - \frac{\partial\,\mathrm{Tr}\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-k-1}}{\partial\boldsymbol{L}^T}\,\partial_{\xi_{k+1/2}}\Big)\\ &= \sum_{k=0}^{\infty}\Big(-\boldsymbol{L}^{-k-1}\,\partial_{g_k} + \sum_{a=1}^{k+1}\boldsymbol{L}^{-a}\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-k-2+a}\,\partial_{\xi_{k+1/2}}\Big). \end{aligned} \tag{4.2.4}$$

As the super–Virasoro generators $\mathcal{G}_{n+1/2}$ and $\mathcal{L}_n$ are maximally quadratic in the ∂_{g_k} and $\partial_{\xi_{k+1/2}}$ it is obvious that we should study quadratic matrix differentials in $\boldsymbol{L}$ and $\boldsymbol{\Lambda}$. The first candidate is $\partial_{\boldsymbol{\Lambda}}\,\partial_{\boldsymbol{L}}$ which after some calculation may be found to be

$$\mathrm{Tr}\,(\partial_{\boldsymbol{\Lambda}}\,\partial_{\boldsymbol{L}}) = \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{\infty} k\,g_k\,\partial_{\xi_{k+m+1/2}} + \sum_{k=0}^{m} \partial_{g_{m-k}}\,\partial_{\xi_{k+1/2}}\Big] + \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{m-1} k\,\partial_{\xi_{m-k-1/2}}\,\partial_{\xi_{k+1/2}}\Big]. \tag{4.2.5}$$

Note that already a number of factors present in the super–Virasoro generators of eq. (3.1.1) appear. The next candidate to consider is the expression $\boldsymbol{\Lambda}\,\partial^2_{\boldsymbol{L}}$. A rather tedious computation reveals that

$$\mathrm{Tr}\,(\boldsymbol{\Lambda}\,\partial^2_{\boldsymbol{L}}) = \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{\infty} k\,g_k\,\partial_{g_{k+m}} + \sum_{k=0}^{m} \partial_{g_k}\,\partial g_{m-k} + \sum_{k=0}^{\infty} k\,\xi_{k+1/2}\,\partial_{\xi_{m+k+1/2}}\Big], \tag{4.2.6}$$

where all terms of order 3 in $\boldsymbol{\Lambda}$ cancel due to their fermionic structure. Additionally one proves the following identities

$$0 = \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{\infty}(k+m+1)\,\xi_{k+1/2}\,\partial_{\xi_{k+m+1/2}}\Big]$$

$$0 = \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{\infty}\xi_{k+1/2}\,\partial_{g_{m+k+1}}\Big] + \sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\Big[\sum_{k=0}^{\infty} k\,g_k\,\partial_{g_{k+m}}\Big]. \tag{4.2.7}$$

So by putting eq. (4.2.5), (4.2.6) and (4.2.7) together we see that the correct differential operator yielding the super–Virasoro constraints is as simple as

$$\begin{aligned}\mathrm{Tr}\,[\,\partial_{\boldsymbol{\Lambda}}\,\partial_{\boldsymbol{L}} + \boldsymbol{\Lambda}\,\partial^2_{\boldsymbol{L}}\,] = {}&\sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{L}^{-m-2}\,)\,\Big\{\sum_{k=0}^{\infty} k\,g_k\,\partial_{\xi_{k+m+1/2}} + \sum_{k=0}^{\infty}\xi_{k+1/2}\,\partial_{g_{m+k+1}} + \sum_{k=0}^{m}\partial_{g_{m-k}}\,\partial_{\xi_{k+1/2}}\Big\}\\ &+ \sum_{m=-1}^{\infty} 2\mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\Big\{\sum_{k=0}^{\infty} k\,g_k\,\partial_{g_{k+m}} + \frac{1}{2}\sum_{k=0}^{m}\partial_{g_k}\,\partial_{g_{m-k}}\\ &\qquad + \sum_{k=0}^{\infty}(k+\frac{m+1}{2})\,\xi_{k+1/2}\,\partial_{\xi_{k+m+1/2}} + \frac{1}{2}\sum_{k=0}^{m-1} k\,\partial_{m-k-1/2}\,\partial_{k+1/2}\Big\}\\ = {}&\sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{L}^{-m-2}\,)\,\mathcal{G}_{m+1/2} + 2\sum_{m=-1}^{\infty} \mathrm{Tr}\,(\,\boldsymbol{\Lambda}\,\boldsymbol{L}^{-m-2}\,)\,\mathcal{L}_m,\end{aligned} \tag{4.2.8}$$

with the super–Virasoro generators $\mathcal{G}_{m+1/2}$ and $\mathcal{L}_m$ defined in eq. (3.1.1) [1].

The relation (4.2.8) may also be stated on the level of “supereigenvalues” l_i and θ_i, despite the fact that we do not know how the θ_i should be related to the matrix $\boldsymbol{\Lambda}$. With the obvious supereigenvalue version of the super–Miwa transformations

$$\begin{aligned} g_k &= \frac{1}{k}\sum_i {l_i}^{-k} \qquad k\ge 1 \qquad\qquad \xi_{k+1/2} = -\sum_i \theta_i\,{l_i}^{-k-1} \qquad k\ge 0\\ g_0 &= -\sum_i \ln l_i ,\end{aligned} \tag{4.2.9}$$

one has the following identity

$$\frac{\partial}{\partial\theta_i}\,\frac{\partial}{\partial l_i} + \theta_i\,\frac{\partial^2}{\partial {l_i}^2} + \sum_{j\ne i}\frac{1}{l_i-l_j-\theta_i\,\theta_j}\,\Big[\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_j} + (\,2\,\theta_i-\theta_j\,)\,(\,\frac{\partial}{\partial l_i} - \frac{\partial}{\partial l_j}\,)\Big] = \sum_{m=-1}^{\infty} {l_i}^{-m-2}\,[\,\mathcal{G}_{m+1/2} + 2\,\theta_i\,\mathcal{L}_m\,], \tag{4.2.10}$$

as already noted by Huang and Zhang [53] in a different context. So we find ourselves in the situation where we know the matrix and eigenvalue based formulation of the Schwinger–Dyson equations of a model, but do not know the model they belong to. Including delta function additions in the super–Miwa transformations of eqs. (4.2.1) and (4.2.9) as they appear in the bosonic transformations of eq. (4.1.11) leads to terms linear in $\partial_{\boldsymbol{\Lambda}}$ and $\partial_{\boldsymbol{L}}$. Unfortunately we have not succeeded in formulating the correct external supersymmetric model so far.

3. Properties of External Eigenvalue Models

In section 1 we saw how the Virasoro constraints emerged out of the Schwinger–Dyson equation associated with the shift $\delta\boldsymbol{X} = \epsilon\,\boldsymbol{X}$ for the external field model of eq. (4.1.1) with

[1] Note that the factors of N^2 are recovered through the rescalings $g_k \to N\,g_k$ and $\xi_{k+1/2} \to N\,\xi_{k+1/2}$.

the special potential (4.1.8). There is yet another way in which the Virasoro structure is hidden in the Schwinger–Dyson equations (4.1.6) of the external field models with arbitrary potentials as noted by Makeenko and Semenoff in ref. [47]. It turns out that this structure may be generalized to an external supereigenvalue model.

Let us first study the bosonic case in the eigenvalue description. The external field model of eq. (4.1.20) reads

$$Z_N[\,l\,] = \Big(\int \prod_{i=1}^{N} dx_i\Big)\,\frac{\Delta[\,x\,]}{\Delta[\,l\,]}\,\exp\Big[\sum_{i=1}^{N}(-V_0(x_i) + l_i\,x_i\,)\Big], \tag{4.3.1}$$

where $\Delta[\,l\,] = \prod_{i<j}(l_i - l_j)$ is the van der Monde determinant. The Schwinger–Dyson equations associated to the shift $x_i \to x_i + \epsilon_n\,{x_i}^{n+1}$ for $n \geq -1$ may be written in the form

$$L_n\,\Delta[\,l\,]\,Z_N[\,l\,] = 0 \qquad \text{for} \quad n \geq -1 \tag{4.3.2}$$

with

$$L_n = -\sum_i V_0'\Big(\frac{\partial}{\partial l_i}\Big)\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} + \sum_i \Big(\frac{\partial}{\partial l_i}\Big)^{n+1} l_i + \frac{1}{2}\sum_{k=0}^{n}\sum_{j\neq i}\Big(\frac{\partial}{\partial l_i}\Big)^{k}\Big(\frac{\partial}{\partial l_j}\Big)^{n-k}. \tag{4.3.3}$$

The differential operators L_n obey the Virasoro algebra without central extension, as one verifies by direct computation

$$[\,L_n\,, L_m\,] = (n-m)\,L_{n+m}. \tag{4.3.4}$$

Note that the generators L_n annihilate $\Delta[\,l\,]\,Z_N[\,l\,]$. One may easily construct generators $\mathcal{L}_n$ which annihilate the partition function $Z_N[\,l\,]$ itself. For this purpose introduce the "long derivatives"

$$\nabla_{l_i} \equiv \Delta^{-1}[\,l\,]\,\frac{\partial}{\partial l_i}\,\Delta[\,l\,] = \frac{\partial}{\partial l_i} + \sum_{i\neq j}\frac{1}{l_i - l_j}, \tag{4.3.5}$$

which commute with each other. The Virasoro constraints of eqs. (4.3.2) and (4.3.3) now read

$$\mathcal{L}_n\,Z_N[\,l\,] = 0 \qquad \text{for} \quad n \geq -1 \tag{4.3.6}$$

and

$$\mathcal{L}_n = \sum_i\Big(-V_0'(\nabla_{l_i})\,(\nabla_{l_i})^{n+1} + (\nabla_{l_i})^{n+1}\,l_i + \frac{1}{2}\sum_{k=0}^{n}\sum_{i\neq j}(\nabla_{l_i})^{k}\,(\nabla_{l_j})^{n-k}\Big). \tag{4.3.7}$$

Due to the commutativity of the ∇_{l_i}'s the generators of eq. (4.3.7) obey the Virasoro algebra (4.3.4) as well.

For the supersymmetric case let us take the following ansatz for an external supereigenvalue model

$$Z_N[\,l,\mu\,] = \int\Big(\prod_i d\lambda_i\,d\theta_i\Big)\,\frac{\Delta^{\alpha}[\,\lambda,\theta\,]}{\Delta^{\alpha}[\,l,\mu\,]}\,\exp\Big[\sum_{i=1}^{N}(-V(\lambda_i) + \lambda_i\,l_i + \theta_i\,\mu_i\,)\Big], \tag{4.3.8}$$

with the measure $\Delta^{\alpha}[\,\lambda,\theta\,] = \prod_{i<j}(\lambda_i - \lambda_j - \theta_i\,\theta_j)^{\alpha}$ and α undetermined. Note that the fermionic variables θ_i enter only through the source term and the measure. One can now study the Schwinger–Dyson equations of this model associated to the shift in integration variables

$$\delta\lambda_i = -\epsilon_n\,\theta_i\,{\lambda_i}^{n+1} \qquad \text{and} \qquad \delta\theta_i = \epsilon_n\,{\lambda_i}^{n+1} \tag{4.3.9}$$

with $n \geq -1$ and ϵ_n an Grassmann odd parameter. The resulting correlator may be rewritten in the form

$$G_{n+1/2}\,\Delta^{\alpha}[\,l,\mu\,]\,Z_N[\,l,\mu\,] = 0 \qquad \text{for} \quad n \geq -1 \tag{4.3.10}$$

where

$$\begin{aligned} G_{n+1/2} = &-\sum_i \frac{\partial}{\partial\mu_i}\,V'\Big(\frac{\partial}{\partial l_i}\Big)\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} + \sum_i \frac{\partial}{\partial\mu_i}\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} l_i \\ &+ \sum_i \mu_i\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} + \alpha\sum_{k=0}^{n}\sum_{i\neq j}\frac{\partial}{\partial\mu_i}\Big(\frac{\partial}{\partial l_i}\Big)^{k}\Big(\frac{\partial}{\partial l_i}\Big)^{n-k}. \end{aligned} \tag{4.3.11}$$

And in fact these generators obey the super–Virasoro algebra which one may directly check

$$\begin{aligned} \{\,G_{n+1/2}, G_{m+1/2}\,\} &= 2\,L_{n+m+1} \\ [\,L_n\,, L_m\,] &= \Big(n-m\Big)\,L_{n+m} \\ [\,L_n\,, G_{m+1/2}\,] &= \Big(\frac{n-1}{2} - m\Big)\,G_{n+m+1/2}, \end{aligned} \tag{4.3.12}$$

and where the generators L_n take the more complicated form

$$\begin{aligned} L_n = &-\sum_i V'\Big(\frac{\partial}{\partial l_i}\Big)\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} + \sum_i l_i\Big(\frac{\partial}{\partial l_i}\Big)^{n+1} + \frac{1}{2}\sum_i (n+1)\,\mu_i\,\frac{\partial}{\partial\mu_i}\Big(\frac{\partial}{\partial l_i}\Big)^{n} \\ &+\frac{1}{2}\sum_i (n+1)\Big(\frac{\partial}{\partial l_i}\Big)^{n} + \frac{\alpha}{2}\sum_{k=0}^{n}\sum_{i\neq j}\Big(\frac{\partial}{\partial l_i}\Big)^{k}\Big(\frac{\partial}{\partial l_j}\Big)^{n-k} \\ &+\frac{\alpha}{2}\sum_{k=0}^{n-1}\sum_{i\neq j} k\,\frac{\partial}{\partial\mu_i}\Big(\frac{\partial}{\partial l_i}\Big)^{n-k-1}\frac{\partial}{\partial\mu_j}\Big(\frac{\partial}{\partial l_j}\Big)^{k}. \end{aligned} \tag{4.3.13}$$

In fact no constraints on the constant α in eq. (4.3.8) arise. In order to obtain super–Virasoro generators directly annihilating the partition function $Z_N[l,\mu]$ introduce the "long derivatives"

$$\nabla_{l_i} \equiv \Delta^{-\alpha}[l,\mu]\,\frac{\partial}{\partial l_i}\,\Delta^{\alpha}[l,\mu] = \frac{\partial}{\partial l_i} + \sum_{i\neq j}\frac{\alpha}{l_i - l_j - \mu_i\,\mu_j}$$
$$\nabla_{\mu_i} \equiv \Delta^{-\alpha}[l,\mu]\,\frac{\partial}{\partial \mu_i}\,\Delta^{\alpha}[l,\mu] = \frac{\partial}{\partial \mu_i} - \sum_{i\neq j}\frac{\alpha\,\mu_j}{l_i - l_j}. \tag{4.3.14}$$

Note that the ∇_{l_i} commute and the ∇_{μ_i} anticommute with each other. Replacing the derivatives ∂_{l_i} by ∇_{l_i} and the ∂_{μ_i} by ∇_{μ_i} in eqs. (4.3.11) and (4.3.13) then yields the super–Virasoro generators $\mathcal{G}_{n+1/2}$ and $\mathcal{L}_n$ which annihilate the partition function $Z_N[l,\mu]$.

So we have seen that the "naive" ansatz of eq. (4.3.8) is capable of producing a generalization of the properties found in the external Hermitian matrix model. However all efforts to use this ansatz as the model generating the Schwinger–Dyson equation (4.2.10) associated to the super–Miwa transformations have failed.

This concludes our short outlook on external supereigenvalue and supermatrix models.

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