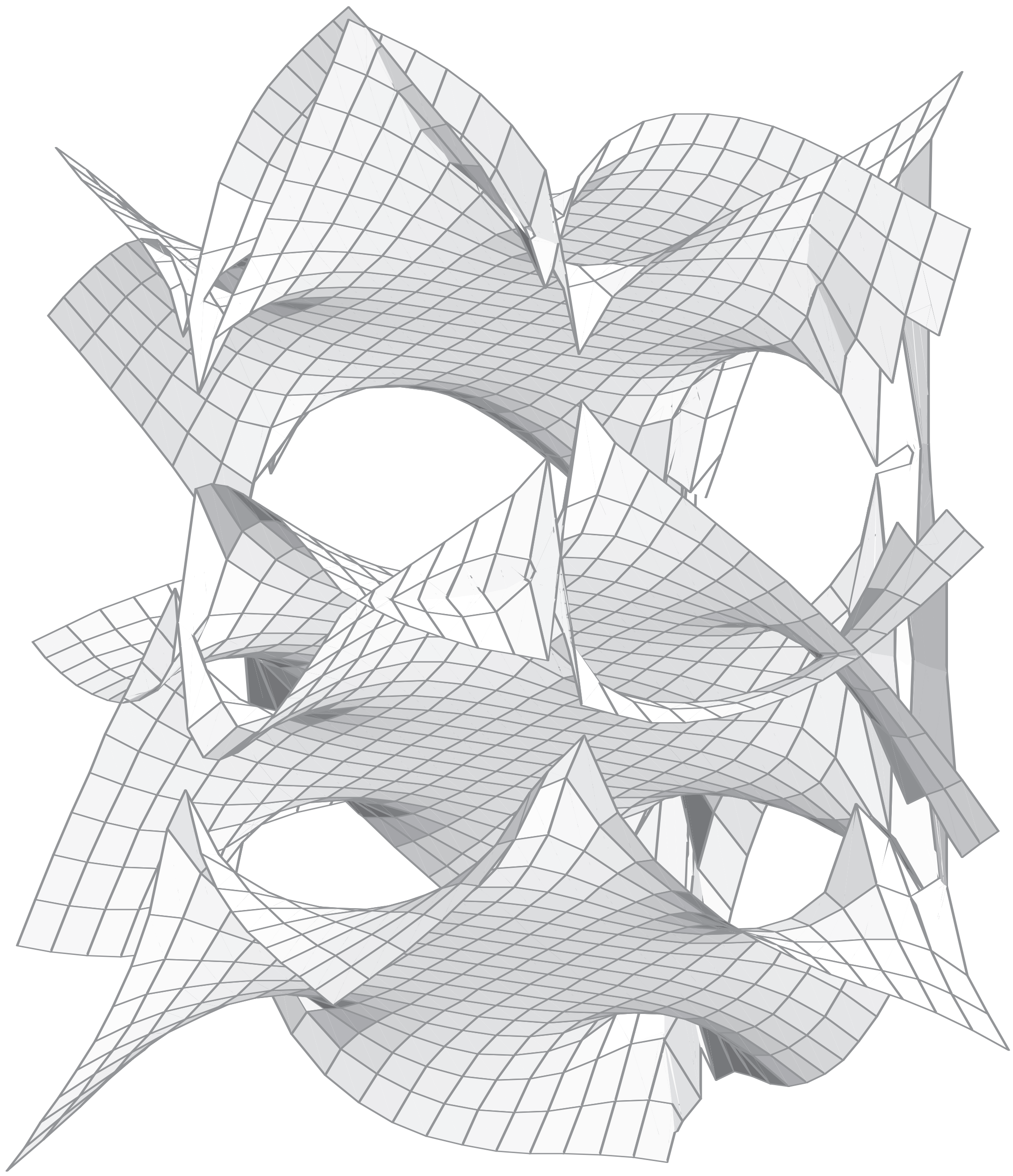
Aspects of Twistor Geometry and Supersymmetric Field Theories within Superstring Theory



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For there is nothing hidden, except that it should be made known;
neither was anything made secret, but that it should come to light.
Mark 4,22

Wir müssen wissen, wir werden wissen.
David Hilbert

To those who taught me

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Zusammenfassung

Die Resultate, die in dieser Arbeit vorgestellt werden, lassen sich im Wesentlichen zwei Forschungsrichtungen in der Stringtheorie zuordnen: Nichtantikommutative Feldtheorie sowie Twistorstringtheorie.

Nichtantikommutative Deformationen von Superräumen entstehen auf natürliche Weise bei Typ II Superstringtheorie in einem nichttrivialen Graviphoton-Hintergrund, und solchen Deformationen wurde in den letzten zwei Jahren viel Beachtung geschenkt. Zunächst konzentrieren wir uns auf die Definition der nichtantikommutativen Deformation von $\mathcal{N}=4$ super Yang-Mills-Theorie. Da es für die Wirkung dieser Theorie keine Superraumformulierung gibt, weichen wir statt dessen auf die äquivalenten *constraint equations* aus. Während der Herleitung der deformierten Feldgleichungen schlagen wir ein nichtantikommutatives Analogon zu der Seiberg-Witten-Abbildung vor.

Eine nachteilige Eigenschaft nichantikommutativer Deformationen ist, dass sie Supersymmetrie teilweise brechen (in den einfachsten Fällen halbieren sie die Zahl der erhaltenen Superladungen). Wir stellen in dieser Arbeit eine sog. Drinfeld-Twist-Technik vor, mit deren Hilfe man supersymmetrische Feldtheorien derart reformulieren kann, dass die gebrochenen Supersymmetrien wieder manifest werden, wenn auch in einem *getwisteten* Sinn. Diese Reformulierung ermöglicht es, bestimmte chirale Ringe zu definieren und ergibt supersymmetrische Ward-Takahashi-Identitäten, welche von gewöhnlichen supersymmetrischen Feldtheorien bekannt sind. Wenn man Seibergs *naturalness argument*, welches die Symmetrien von Niederenergie-Wirkungen betrifft, auch im nichtantikommutativen Fall zustimmt, so erhält man Nichtrenormierungstheoreme selbst für nichtantikommutative Feldtheorien.

Im zweiten und umfassenderen Teil dieser Arbeit untersuchen wir detailliert geometrische Aspekte von Supertwistorräumen, die gleichzeitig Calabi-Yau-Supermannigfaltigkeiten sind und dadurch als *target space* für topologische Stringtheorien geeignet sind. Zunächst stellen wir die Geometrie des bekanntesten Beispiels für einen solchen Supertwistorraum, $\mathbb{C}P^{3|4}$, vor und führen die Penrose-Ward-Transformation, die bestimmte holomorphe Vektorbündel über dem Supertwistorraum mit Lösungen zu den $\mathcal{N}=4$ supersymmetrischen selbstdualen Yang-Mills-Gleichungen verbindet, explizit aus. Anschließend diskutieren wir mehrere dimensionale Reduktionen des Supertwistorraumes $\mathbb{C}P^{3|4}$ und die implizierten Veränderungen an der Penrose-Ward-Transformation.

Fermionische dimensionale Reduktionen bringen uns dazu, exotische Supermannigfaltigkeiten, d.h. Supermannigfaltigkeiten mit zusätzlichen (bosonischen) nilpotenten Dimensionen, zu studieren. Einige dieser Räume können als *target space* für topologische Strings dienen und zumindest bezüglich des Satzes von Yau fügen diese sich gut in das Bild der Calabi-Yau-Supermannigfaltigkeiten ein.

Bosonische dimensionale Reduktionen ergeben die Bogomolny-Gleichungen sowie Matrixmodelle, die in Zusammenhang mit den ADHM- und Nahm-Gleichungen stehen. (Tatsächlich betrachten wir die Supererweiterungen dieser Gleichungen.) Indem wir bestimmte Terme zu der Wirkung dieser Matrixmodelle hinzufügen, können wir eine komplette Äquivalenz zu den ADHM- und Nahm-Gleichungen erreichen. Schließlich kann die natürliche Interpretation dieser zwei Arten von BPS-Gleichungen als spezielle D-Branekonfigurationen in Typ IIB Superstringtheorie vollständig auf die Seite der topologischen Stringtheorie übertragen werden. Dies führt zu einer Korrespondenz zwischen topologischen und physikalischen D-Branesystemen und eröffnet die interessante Perspektive, Resultate von beiden Seiten auf die jeweils andere übertragen zu können.

Abstract

There are two major topics within string theory to which the results presented in this thesis are related: non-anticommutative field theory on the one hand and twistor string theory on the other hand.

Non-anticommutative deformations of superspaces arise naturally in type II superstring theory in a non-trivial graviphoton background and they have received much attention over the last two years. First, we focus on the definition of a non-anticommutative deformation of $\mathcal{N}=4$ super Yang-Mills theory. Since there is no superspace formulation of the action of this theory, we have to resort to a set of constraint equations defined on the superspace $\mathbb{R}^{4|16}_{\hbar}$, which are equivalent to the $\mathcal{N}=4$ super Yang-Mills equations. In deriving the deformed field equations, we propose a non-anticommutative analogue of the Seiberg-Witten map.

A mischievous property of non-anticommutative deformations is that they partially break supersymmetry (in the simplest case, they halve the number of preserved supercharges). In this thesis, we present a so-called Drinfeld-twisting technique, which allows for a reformulation of supersymmetric field theories on non-anticommutative superspaces in such a way that the broken supersymmetries become manifest even though in some sense twisted. This reformulation enables us to define certain chiral rings and it yields supersymmetric Ward-Takahashi-identities, well-known from ordinary supersymmetric field theories. If one agrees with Seiberg's naturalness arguments concerning symmetries of low-energy effective actions also in the non-anticommutative situation, one even arrives at non-renormalization theorems for non-anticommutative field theories.

In the second and major part of this thesis, we study in detail geometric aspects of supertwistor spaces which are simultaneously Calabi-Yau supermanifolds and which are thus suited as target spaces for topological string theories. We first present the geometry of the most prominent example of such a supertwistor space, $\mathbb{C}P^{3|4}$, and make explicit the Penrose-Ward transform which relates certain holomorphic vector bundles over the supertwistor space to solutions to the $\mathcal{N}=4$ supersymmetric self-dual Yang-Mills equations. Subsequently, we discuss several dimensional reductions of the supertwistor space $\mathbb{C}P^{3|4}$ and the implied modifications to the Penrose-Ward transform.

Fermionic dimensional reductions lead us to study exotic supermanifolds, which are supermanifolds with additional even (bosonic) nilpotent dimensions. Certain such spaces can be used as target spaces for topological strings, and at least with respect to Yau's theorem, they fit nicely into the picture of Calabi-Yau supermanifolds.

Bosonic dimensional reductions yield the Bogomolny equations describing static monopole configurations as well as matrix models related to the ADHM- and the Nahm equations. (In fact, we describe the superextensions of these equations.) By adding certain terms to the action of these matrix models, we can render them completely equivalent to the ADHM and the Nahm equations. Eventually, the natural interpretation of these two kinds of BPS equations by certain systems of D-branes within type IIB superstring theory can completely be carried over to the topological string side via a Penrose-Ward transform. This leads to a correspondence between topological and physical D-brane systems and opens interesting perspectives for carrying over results from either sides to the respective other one.

Contents

Chapter I

Introduction

I.1 High-energy physics and string theory

Today, there are essentially two well-established approaches to describing fundamental physics, both operating in different regimes: Einstein's theory of General Relativity[1], which governs the dynamics of gravitational effects on a large scale from a few millimeters to cosmological distances and the framework called quantum field theory, which incorporates the theory of special relativity into quantum mechanics and captures phenomena at scales from a fraction of a millimeter to 10^{-19}m. In particular, there is the quantum field theory called the standard model of elementary particles, which is a quantum gauge theory with gauge group $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ and describes the electromagnetic, the weak and the strong interactions on equal footing. Although this theory has already been developed between 1970 and 1973, it still proves to be overwhelmingly consistent with the available experimental data today.

Unfortunately, a fundamental difference between these two approaches is disturbing the beauty of the picture. While General Relativity is a classical description of spacetime dynamics in terms of the differential geometry of smooth manifolds, the standard model has all the features of a quantum theory as e.g. uncertainty and probabilistic predictions. One might therefore wonder whether it is possible or even necessary to quantize gravity.

The first question for the possibility of quantizing gravity is already not easy to answer. Although promoting supersymmetry to a local symmetry almost immediately yields a classical theory containing gravity, the corresponding quantum field theory is non-renormalizable. That is, an infinite number of renormalization conditions is needed at the very high energies near the Planck scale and the theory thus looses all its predictive power[2]. Two remedies to this problem are conceivable: either to assume that there are additional degrees of freedom between the standard model energy scale and the Planck scale or to assume some underlying dependence of the infinite number of renormalization conditions on a finite subset[3].

Today, there are essentially two major approaches to quantizing gravity, which are believed to overcome the above mentioned shortcoming: string theory, which trades the infinite number of renormalization conditions for an infinite tower of higher-spin gauge symmetries, and the so-called loop quantum gravity approach [242]. As of now, it is not even clear whether these two approaches are competitors or merely two aspects of the same underlying theory. Furthermore, there is no help to be expected from experimental input since on the one hand, neither string theory nor loop quantum gravity have yielded any truly verifiable (or better: falsifiable) results so far and on the other hand there is

[1] or more appropriately: General Theory of Relativity

[2] It is an amusing thought to imagine that supergravity was indeed the correct theory and therefore nature was in principle unpredictable.

[3] See also the discussion in `http://golem.ph.utexas.edu/~distler/blog/archives/000639.html`.

simply no quantitative experimental data for any kind of quantum gravity effect up to now.

The second question of the need for quantum gravity is often directly answered positively, due to the argument given in [93] which amounts to a violation of uncertainty if a classical gravitational field is combined with quantum fields[4]. This line of reasoning has, however, been challenged until today, see e.g. [52], and it seems to be much less powerful than generally believed.

There is another reason for quantizing gravity, which is, however, of purely aesthetical value: A quantization of gravity would most likely allow for the unification of all the known forces within one underlying principle. This idea of unification of forces dates back to the electro-magnetic unification by James Clerk Maxwell, was strongly supported by Hermann Weyl and Albert Einstein and found its present climax in the electroweak unification by Abdus Salam and Steven Weinberg. Furthermore, there is a strong argument which suggest that quantizing gravity makes unification or at least simultaneous quantization of all other interactions unavoidable from a phenomenological point of view: Because of the weakness of gravity compared to the other forces there is simply no decoupling regime which is dominated by pure quantum gravity effects and in which all other particle interactions are negligible.

Unification of General Relativity and the standard model is difficult due to the fundamental difference in the ways both theories describe the world. In General Relativity, gravitational interactions deform spacetime, and reciprocally originate from such deformations. In the standard model, interactions arise from the exchange of messenger particles. It is furthermore evident that in order to quantize gravity, we have to substitute spacetime by something more fundamental, which still seems to be completely unknown.

Although the critical superstring theories, which are currently the only candidate for a unified description of nature including a quantum theory of gravity, still do not lead to verifiable results, they may nevertheless be seen as a guiding principle for studying General Relativity and quantum field theories. For this purpose, it is important to find string/gauge field theory dualities, of which the most prominent example is certainly the AdS/CFT correspondence [187]. These dualities provide a dictionary between certain pairs of string theories and gauge theories, which allows to perform field theoretic calculations in the mathematically often more powerful framework of string theory.

The recently proposed twistor string theory [297] gives rise to a second important example of such a duality. It has been in its context that string theoretical methods have led for the first time[5] to field theoretic predictions, which would have been almost impossible to make with state-of-the-art quantum field theoretical[6] technology.

As a large part of this thesis will be devoted to studying certain aspects of this twistor string theory, let us present this theory in more detail. Twistor string theory was introduced in 2003 by Edward Witten [297] and is essentially founded on the marriage

[4]It is argued that if measurement by a gravitational wave causes a quantum mechanical wave function to collapse then the uncertainty relation can only be preserved if momentum conservation is violated. On the other hand, if there is no collapse of the wave function, one could transmit signals faster than with light.

[5]Another string inspired prediction of real-world physics has arisen from the computation of shear viscosity via AdS/CFT-inspired methods in [224].

[6]One might actually wonder about the perfect timing of the progress in high energy physics: These calculations are needed for the interpretation of the results at the new particle accelerator at CERN, which will start collecting data in 2007.

of Calabi-Yau and twistor geometry in the supertwistor space $\mathbb{C}P^{3|4}$. Both of these geometries will therefore accompany most of our discussion.

Calabi-Yau manifolds are complex manifolds which have a trivial first Chern class. They are Ricci-flat and come with a holomorphic volume element. The latter property allows to define a Chern-Simons action on these spaces, which will play a crucial rôle throughout this thesis. Calabi-Yau manifolds naturally emerge in string theory as candidates for internal compactification spaces. In particular, topological strings of B-type – a subsector of the superstrings in type IIB superstring theory – can be consistently defined on spaces with vanishing first Chern number only and their dynamics is then governed by the above-mentioned Chern-Simons theory.

Twistor geometry, on the other hand, is a novel description of spacetime, which was introduced in 1967 by Roger Penrose [216]. Although this approach has found many applications in both General Relativity and quantum theory, it is still rather unknown in the mathematical and physical communities and it has only been recently that new interest was sparked among string theorists by Witten's seminal paper [297]. Interestingly, twistor geometry was originally designed as a unified framework for quantum theory and gravity, but so far, it has not yielded significant progress in this direction. Its value in describing various aspects of field theories, however, keeps growing.

Originally, Witten showed that the topological B-model on the supertwistor space $\mathbb{C}P^{3|4}$ in the presence of n "almost space-filling[7]" D5-superbranes is equivalent to $\mathcal{N}=4$ self-dual Yang-Mills theory. By adding D1-instantons, one can furthermore complete the self-dual sector to the full $\mathcal{N}=4$ super Yang-Mills theory. Following Witten's paper, various further target spaces for twistor string theory have been considered as well [231, 4, 244, 215, 104, 298, 63, 229, 64], which lead, e.g., to certain dimensional reductions of the supersymmetric self-dual Yang-Mills equations. There has been a vast number of publications dedicated to apply twistor string theory to determining scattering amplitudes in ordinary and supersymmetric gauge theories (see e.g. [181] and [234] for an overview), but only half a year after Witten's original paper, disappointing results appeared. In [31], it was discovered that it seems hopeless to decouple conformal supergravity from the part relevant for the description of super Yang-Mills theory in twistor string theory already at one-loop level. Therefore, the results for gauge theory loop amplitudes are mostly obtained today by "gluing together" tree level amplitudes.

Nevertheless, research on twistor string theory continued with a more mathematically based interest. As an important example, the usefulness of Calabi-Yau supermanifolds in twistor string theory suggests an extension of the famous mirror conjecture to supergeometry. This conjecture states that Calabi-Yau manifolds come in pairs of families, which are related by a mirror map. There is, however, a class of such manifolds, the so-called rigid Calabi-Yau manifolds, which cannot allow for an ordinary mirror. A resolution to this conundrum had been proposed in [259], where the mirror of a certain rigid Calabi-Yau manifold was conjectured to be a supermanifold. Several publications in this direction have appeared since, see [167, 4, 24, 238, 3] and references therein.

Returning now to the endeavor of quantizing gravity, we recall that it is still not known what ordinary spacetime should exactly be replaced with. The two most important extensions of spacetime discussed today are certainly supersymmetry and noncommutativity. The former extension is a way to avoid a severe restriction in constructing quantum field theories: An ordinary bosonic symmetry group, which is nontrivially combined with

[7] a restriction on the fermionic worldvolume directions of the D-branes

the Poincaré group of spacetime transformations renders all interactions trivial. Since supersymmetry is a fermionic symmetry, this restriction does not apply and we can extend the set of interesting theories by some particularly beautiful ones. Furthermore, supersymmetry seems to be *the* ingredient to make string theory well-defined. Although, supersymmetry preserves the smooth underlying structure of spacetime and can be nicely incorporated into the quantum field theoretic framework, there is a strong hint that this extension is a first step towards combining quantum field theory with gravity: As stated above, we naturally obtain a theory describing gravity by promoting supersymmetry to a local symmetry. Besides being in some cases the low-energy limit of certain string theories, it is believed that this so-called supergravity is the only consistent theory of an interacting spin $\frac{3}{2}$-particle, the superpartner of the spin 2 graviton.

Nevertheless, everything we know today about a possible quantum theory of gravity seems to tell us that a smooth structure of spacetime described by classical manifolds can not persist to arbitrarily small scales. One rather expects a deformation of the coordinate algebra which should be given by relations like

$$[\hat{x}^\mu, \hat{x}^\nu] \sim \Theta^{\mu\nu} \quad \text{and} \quad \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} \sim C^{\alpha\beta}$$

for the bosonic and fermionic coordinates of spacetime. The idea of bosonic deformations of spacetime coordinates can in fact be traced back to work by H. S. Snyder in 1947 [263]. In the case of fermionic coordinates, a first model using a deformed coordinate algebra appeared in [249]. Later on, it was found that both deformations naturally arise in various settings in string theory.

So far, mostly the simplest possible deformations of ordinary (super)spaces have been considered, i.e. those obtained by constant deformation parameters $\Theta^{\mu\nu}$ and $C^{\alpha\beta}$ on flat spacetimes. The non-(anti)commutative field theories defined on these deformed spaces revealed many interesting features, which are not common to ordinary field theories. Further hopes, as e.g. that noncommutativity could tame field theoretic singularities have been shattered with the discovery of UV/IR mixing in amplitudes within noncommutative field theories.

The fact that such deformations are unavoidable for studying nontrivial string backgrounds have kept the interest in this field alive and deformations have been applied to a variety of theories. For $\mathcal{N} = 4$ super Yang-Mills theory, the straightforward superspace approach broke down, but by considering so-called constraint equations, which live on an easily deformable superspace, also this theory can be rendered non-anticommutative, and we will discuss this procedure in this thesis.

Among the most prominent recent discoveries[8] in noncommutative geometry is certainly the fact that via a so-called Drinfeld twist, one can in some sense undo the deformation. More explicitly, Lorentz invariance is broken to some subgroup by introducing a nontrivial deformation tensor $\Theta^{\mu\nu}$. The Drinfeld twist, however, allows for a recovering of a twisted Lorentz symmetry. This regained symmetry is important for discussing fundamental aspects of noncommutative field theory as e.g. its particle content and formal questions like the validity of Haag's theorem. In this thesis, we will present the application of a similar twist in the non-anticommutative situation and regain a twisted form of the supersymmetry, which had been broken by non-anticommutativity. This allows us to carry over several useful aspects of supersymmetric field theories to non-anticommutative ones.

[8] or better: "recently recalled discoveries"

I.2 Epistemological remarks

String theory is certainly the physical theory which evokes the strongest emotions among professional scientists. On the one hand, there are the advocates of string theory, never tired of stressing its incredible inherent beauty and the deep mathematical results arising from it. On the other hand, there are strong critics, who point out that so far, string theory had not made any useful predictions[9] and that the whole endeavor had essentially been a waste of money and brain power, which had better been spent on more down-to-earth questions. For this reason, let us briefly comment on string theory from an epistemological point of view.

The epistemological model used implicitly by today's physics community is a mixture of rationalism and empiricism as both doctrines by themselves have proven to be insufficient in the history of natural sciences. The most popular version of such a mixture is certainly Popper's *critical rationalism* [232], which is based on the observation that no finite number of experiments can verify a scientific theory but a single negative outcome can falsify it. For the following discussion we will adopt this point of view.

Thus, we assume that there is a certain pool of theories, which are in an evolutionary competition with each other. A theory is permanently excluded from the pool if one of its predictions contradicts an experimental result. Theories can be added to this pool if they have an equal or better predictive power as any other member of this pool. Note that the way these models are created is – contrary to many other authors – of no interest to Popper. However, we have to restrict the set of possible theories, which we are admitting in the pool: only those, which can be experimentally falsified are empirical and thus of direct scientific value; all other theories are metaphysical[10]. One can therefore state that when Pauli postulated the existence of the neutrino which he thought to be undetectable, he introduced a metaphysical theory to the pool of competitors and he was aware that this was a rather inappropriate thing to do. Luckily, the postulate of the existence of the neutrino became an empirical statement with the discovery of further elementary forces and the particle was finally discovered in 1956. Here, we have therefore the interesting example of a metaphysical theory, which became an empirical one with improved experimental capabilities.

In Popper's epistemological model, there is furthermore the class of self-immunizing theories. These are theories, which constantly modify themselves to fit new experimental results and therefore come with a mechanism for avoiding being falsified. According to Popper, these theories have to be discarded altogether. He applied this reasoning in particular to dogmatic political concepts like e.g. Marxism and Plato's idea of the perfect state. At first sight, one might count supersymmetry to such self-immunizing theories: so far, all predictions for the masses of the superpartners of the particles in the standard model were falsified which resulted in successive shifts of the postulated supersymmetry breaking scales out of the reach of the then up-to-date experiments. Besides self-immunizing, the theory even becomes "temporarily metaphysical" in this way. However, one has to take into account that it is not supersymmetry per se which is falsified, but the symmetry breaking mechanisms it can come with. The variety of such imaginable breaking mechanisms remains, however, a serious problem.

[9]It is doubtful that these critics would accept the exception of twistor string theory, which led to new ways of calculating certain gauge theory amplitudes.

[10]Contrary to the logical positivism, Popper attributes some meaning to such theories in the process of developing new theories.

When trying to put string theory in the context of the above discussed framework, there is clearly the observation that so far, string theory has not made any predictions which would allow for a falsification. At the moment, it is therefore at most a "temporarily metaphysical" theory. Although it is reasonable to expect that with growing knowledge of cosmology and string theory itself, many predictions of string theory will eventually become empirical, we cannot compare its status to the one of the neutrino at the time of its postulation by Pauli, simply for the reason that string theory is not an actually fully developed theory. So far, it appears more or less as a huge collection of related and interwoven ideas[11] which contain strong hints of being capable of explaining both the standard model and General Relativity on equal footing. But without any doubt, there are many pieces still missing for giving a coherent picture; a background independent formulation – the favorite point brought regularly forth by advocates of loop quantum gravity – is only one of the most prominent ones.

The situation string theory is in can therefore be summarized in two points. First, we are clearly just in the process of developing the theory; it should not yet be officially added to our competitive pool of theories. For the development of string theory, it is both necessary and scientifically sound to use metaphysical guidelines as e.g. beauty, consistency, mathematical fertility and effectiveness in describing the physics of the standard model and General Relativity. Second, it is desirable to make string theory vulnerable to falsification by finding essential features of all reasonable string theories. Epistemologically, this is certainly the most important task and, if successful, would finally turn string theory into something worthy of being called a fully physical theory.

Let us end these considerations with an extraordinarily optimistic thought: It could also be possible that there is only one unique theory, which is consistent with all we know so far about the world. If this were true, we could immediately abandon most of the epistemological considerations made so far and turn to a purely rationalistic point of view based on our preliminary results about nature so far. That is, theories in our pool would no longer be excluded from the pool by experimental falsification but by proving their mathematical or logical inconsistency with the need of describing the standard model and General Relativity in certain limits. This point of view is certainly very appealing. However, even if our unreasonably optimistic assumption was true, we might not be able to make any progress without the help of further experimental input.

Moreover, a strong opposition is forming against this idea, which includes surprisingly many well-known senior scientists as e.g. Leonard Susskind [266] and Steven Weinberg [286]. In their approach towards the fundamental principles of physics, which is known as *the landscape*, the universe is divided into a statistical ensemble of sub-universes, each with its own set of string compactification parameters and thus its own low-energy effective field theory. Together with the *anthropic principle*[12], this might explain why our universe actually is as it is. Clearly, the danger of such a concept is that questions which might in fact be answerable by physical principles can easily be discarded as irrelevant due to anthropic reasoning.

[11] For convenience sake, we will label this collection of ideas by *string theory*, even though this nomenclature is clearly sloppy.

[12] Observers exist only in universes which are suitable for creating and sustaining them.

I.3 Outline

In this thesis, the material is presented in groups of subjects, and it has been mostly ordered in such a way that technical terms are not used before a definition is given. This, however, will sometimes lead to a considerable amount of material placed between the introduction of a concept and its first use. By adding as many cross-references as possible, an attempt is made to compensate for this fact.

Definitions and conventions which are not introduced in the body of the text, but might nevertheless prove to be helpful, are collected in appendix A.

The thesis starts with an overview of the necessary concepts in complex geometry. Besides the various examples of certain complex manifolds as e.g. flag manifolds and Calabi-Yau spaces, in particular the discussion of holomorphic vector bundles and their description in terms of Dolbeault and Čech cohomologies is important.

It follows a discussion about basic issues in supergeometry. After briefly reviewing supersymmetry, which is roughly speaking the physicist's name for a $\mathbb{Z}_2$-grading, an overview of the various approaches to superspaces is given. Moreover, the new results obtained in [244] on exotic supermanifolds are presented here. These spaces are supermanifolds endowed with additional even nilpotent directions. We review the existing approaches for describing such manifolds and introduce an integration operation on a certain class of them, the so-called thickened and fattened complex manifolds. We furthermore examine the validity of Yau's theorem for such exotic Calabi-Yau supermanifolds, and we find, after introducing the necessary tools, that the results fit nicely into the picture of ordinary Calabi-Yau supermanifolds which was presented in [239]. We close the chapter with a discussion of spinors in arbitrary dimensions during which we also fix all the necessary reality conditions used throughout this thesis.

The next chapter deals with the various field theories which are vital for the further discussion. It starts by recalling elementary facts on supersymmetric field theories, in particular their quantum aspects as e.g. non-renormalization theorems. It follows a discussion of super Yang-Mills theories in various dimensions and their related theories as chiral or self-dual subsectors and dimensional reductions thereof. The second group of field theories that will appear in the later discussion are Chern-Simons-type theories (holomorphic Chern-Simons theory and holomorphic BF-theories), which are introduced as well. Eventually, a few remarks are made about certain aspects of conformal field theories which will prove useful in what follows.

The aspects of string theory entering into this thesis are introduced in the following chapter. We give a short review on string theory basics and superstring theories before elaborating on topological string theories. One of the latter, the topological B-model, will receive much attention later due to its intimate connection with holomorphic Chern-Simons theory. We will furthermore need some background information on the various types of D-branes which will appear naturally in the models on which we will focus our attention. We close this chapter with a few rather general remarks on several topics in string theory.

Noncommutative deformations of spacetime and the properties of field theories defined on these spaces is the topic of the next chapter. After a short introduction, we present the result of [245], i.e. the non-anticommutative deformation of $\mathcal{N} = 4$ super Yang-Mills equations using an equivalent set of constraint equations on the superspace $\mathbb{R}^{4|16}$. The second half of this chapter is based on the publication [136], in which the analysis of [57] on a Lorentz invariant interpretation of noncommutative spacetime was extended to the non-anticommutative situation. This Drinfeld-twisted supersymmetry allows for carrying over various quantum aspects of supersymmetric field theories to the non-anticommutative situation.

The following chapter on twistor geometry constitutes the main part of this thesis. After a detailed introduction to twistor geometry, integrability and the Penrose-Ward transform, we present in four sections the results of the publications [228, 244, 229, 243].

First, the Penrose-Ward transform using supertwistor spaces is discussed in complete detail, which gives rise to an equivalence between the topological B-model and thus holomorphic Chern-Simons theory on the supertwistor space $\mathbb{C}P^{3|4}$ and $\mathcal{N} = 4$ self-dual Yang-Mills theory. While Witten [297] has motivated this equivalence by looking at the field equations of these two theories on the linearized level, the publication [228] analyzes the complete situation to all orders in the fields. We furthermore scrutinize the effects of the different reality conditions which can be imposed on the supertwistor spaces.

This discussion is then carried over to certain exotic supermanifolds, which are simultaneously Calabi-Yau supermanifolds. We report here on the results of [244], where the possibility of using exotic supermanifolds as a target space for the topological B-model was examined. After restricting the structure sheaf of $\mathbb{C}P^{3|4}$ by combining an even number of Graßmann-odd coordinates into Graßmann-even but nilpotent ones, we arrive at Calabi-Yau supermanifolds, which allow for a twistor correspondence with further spaces having $\mathbb{R}^4$ as their bodies. Also a Penrose-Ward transform is found, which relates holomorphic vector bundles over the exotic Calabi-Yau supermanifolds to solutions of bosonic subsectors of $\mathcal{N} = 4$ self-dual Yang-Mills theory.

Subsequently, the twistor correspondence as well as the Penrose-Ward transform are presented for the case of the mini-supertwistor space, a dimensional reduction of the $\mathcal{N} = 4$ supertwistor space discussed previously. This variant of the supertwistor space $\mathbb{C}P^{3|4}$ has been introduced in [63], where it has been shown that twistor string theory with the mini-supertwistor space as a target space is equivalent to $\mathcal{N} = 8$ super Yang-Mills theory in three dimensions. Following Witten [297], D1-instantons were added here to the topological B-model in order to complete the arising BPS equations to the full super Yang-Mills theory. Here, we consider the geometric and field theoretic aspects of the same situation *without* the D1-branes as done in [229]. We identify the arising dimensional reduction of holomorphic Chern-Simons theory with a holomorphic BF-type theory and describe a twistor correspondence between the mini-supertwistor space and its moduli space of sections. Furthermore, we establish a Penrose-Ward transform between this holomorphic BF-theory and a super Bogomolny model on $\mathbb{R}^3$. The connecting link in this correspondence is a partially holomorphic Chern-Simons theory on a Cauchy-Riemann supermanifold which is a real one-dimensional fibration over the mini-supertwistor space.

While the supertwistor spaces examined so far naturally yield Penrose-Ward transforms for certain self-dual subsectors of super Yang-Mills theories, the superambitwistor space $\mathcal{L}^{5|6}$ introduced in the following section as a quadric in $\mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$ yields an analogue equivalence between holomorphic Chern-Simons theory on $\mathcal{L}^{5|6}$ and full $\mathcal{N} = 4$ super Yang-Mills theory. After developing this picture to its full extend as given in [228], we moreover discuss in detail the geometry of the corresponding dimensional reduction yielding the mini-superambitwistor space $\mathcal{L}^{4|6}$.

The Penrose-Ward transform built upon the space $\mathcal{L}^{4|6}$ yields solutions to the $\mathcal{N} = 8$ super Yang-Mills equations in three dimensions as was shown in [243]. We review the construction of this new supertwistor space by dimensional reduction of the superambitwistor space $\mathcal{L}^{5|6}$ and note that the geometry of the mini-superambitwistor space comes with some surprises. First, this space is not a manifold, but only a fibration. Nevertheless, it satisfies an analogue to the Calabi-Yau condition and therefore might be suited as a target space for the topological B-model. We conjecture that this space is the mirror to a certain mini-supertwistor space. Despite the strange geometry of the mini-superambitwistor space, one can translate all ingredients of the Penrose-Ward transform to this situation and establish a one-to-one correspondence between generalized holomorphic bundles over the mini-superambitwistor space and solutions to the $\mathcal{N} = 8$ super Yang-Mills equations in three dimensions. Also the truncation to the Yang-Mills-Higgs subsector can be conveniently described by generalized holomorphic bundles over formal *sub-neighborhoods* of the mini-ambitwistor space.

We close this chapter with a presentation of the ADHM and the Nahm constructions, which are intimately related to twistor geometry and which will allow us to identify certain field theories with D-brane configurations in the following.

The next to last chapter is devoted to matrix models. We briefly recall basic aspects of the most prominent matrix models and introduce the new models, which were studied in [176]. In this paper, we construct two matrix models from twistor string theory: one by dimensional reduction onto a rational curve and another one by introducing noncommutative coordinates on the fibres of the supertwistor space $\mathcal{P}^{3|4} \to \mathbb{C}P^1$. Examining the resulting actions, we note that we can relate our matrix models to a recently proposed string field theory. Furthermore, we comment on their physical interpretation in terms of D-branes of type IIB, critical $\mathcal{N} = 2$ and topological string theory. By extending one of the models, we can carry over all the ingredients of the super ADHM construction to a D-brane configuration in the supertwistor space $\mathcal{P}^{3|4}$ and establish a correspondence between a D-brane system in ten dimensional string theory and a topological D-brane system. The analogous correspondence for the Nahm construction is also established.

After concluding in the last chapter, we elaborate on the remaining open questions raised by the results presented in this thesis and mention several directions for future research.

I.4 Publications

During my PhD-studies, I was involved in the following publications:

1. C. Sämann and M. Wolf, *Constraint and super Yang-Mills equations on the deformed superspace* $\mathbb{R}_\hbar^{(4|16)}$, JHEP **0403** (2004) 048 [hep-th/0401147].
2. A. D. Popov and C. Sämann, *On supertwistors, the Penrose-Ward transform and N = 4 super Yang-Mills theory,* Adv. Theor. Math. Phys. **9** (2005) 931 [hep-th/0405123].
3. C. Sämann, *The topological B-model on fattened complex manifolds and subsectors of* $\mathcal{N} = 4$ *self-dual Yang-Mills theory,* JHEP **0501** (2005) 042 [hep-th/0410292].
4. A. D. Popov, C. Sämann and M. Wolf, *The topological B-model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations,* JHEP **0510** (2005) 058 [hep-th/0505161].
5. M. Ihl and C. Sämann, *Drinfeld-twisted supersymmetry and non-anticommutative superspace,* JHEP **0601** (2006) 065 [hep-th/0506057].

6. C. Sämann, *On the mini-superambitwistor space and $\mathcal{N} = 8$ super Yang-Mills theory,* hep-th/0508137.
7. O. Lechtenfeld and C. Sämann, *Matrix models and D-branes in twistor string theory*, JHEP **0603** (2006) 002 [hep-th/0511130].

Chapter II

Complex Geometry

In this chapter, we review the basic notions of complex geometry, which will be heavily used throughout this thesis due to the intimate connection of this subject with supersymmetry and the topological B-model. The following literature has proven to be useful for studying this subject: [201, 135] (complex geometry), [145, 111, 246] (Calabi-Yau geometry), [225, 142] (Dolbeault- and Čech-description of holomorphic vector bundles), [50, 188] (deformation theory), [113, 121] (algebraic geometry).

II.1 Complex manifolds

II.1.1 Manifolds

Similarly to the structural richness one gains when turning from real analysis to complex analysis, there are many new features arising when turning from real (and smooth) to complex manifolds. For this, the requirement of having smooth transition functions between patches will have to be replaced by demanding that the transition functions are holomorphic.

§1 Holomorphic maps. A map $f:\mathbb{C}^m\to\mathbb{C}^n:(z^1,\dots,z^m)\mapsto(w^1,\dots,w^n)$ is called *holomorphic* if all the w^i are holomorphic in each of the coordinates z^j, where $1\leq i\leq n$ and $1\leq j\leq m$.

§2 Complex manifolds. Let M be a topological space with an open covering $\mathfrak{U}$. Then M is called a *complex manifold* of *dimension* n if for every $U\in\mathfrak{U}$ there is a homeomorphism[1] $\phi_U:U\to\mathbb{C}^n$ such that for each $U\cap V\neq\varnothing$ the *transition function* $\phi_{UV}:=\phi_U\phi_V^{-1}$, which maps open subsets of $\mathbb{C}^n$ to $\mathbb{C}^n$, is holomorphic. A pair (U,ϕ_U) is called a chart and the collection of all charts form a *holomorphic structure*.

§3 Graßmannian manifolds. An ubiquitous example of complex manifolds are *Graßmannian manifolds*. Such manifolds $G_{k,n}(\mathbb{C})$ are defined as the space of k-dimensional vector subspaces in $\mathbb{C}^n$. The most common example is $G_{1,n}$ which is the *complex projective space* $\mathbb{C}P^n$. This space is globally described by *homogeneous coordinates* $(\omega^1,\dots,\omega^{n+1})\in\mathbb{C}^n\backslash\{(0,\dots,0)\}$ together with the identification $(\omega^1,\dots,\omega^{n+1})\sim(t\omega^1,\dots,t\omega^{n+1})$ for all $t\in\mathbb{C}^\times$. An open covering of $\mathbb{C}P^n$ is given by the collection of open patches U_j for which $\omega^j\neq 0$. On such a patch U_j, we can introduce n *inhomogeneous coordinates* $(z^1,\dots,\hat{z}^j,\dots,z^{n+1})$ with $z^i=\frac{\omega^i}{\omega^j}$, where the hat indicates an omission. For convenience, we will always shift the indices on the right of the omission to fill the gap, i.e. $z^i\to z^{i-1}$ for $i>j$.

§4 Theorem. (Chow) Since we will often use complex projective spaces and their subspaces, let us recall the following theorem by Chow: Any submanifold of $\mathbb{C}P^m$ can be defined by the zero locus of a finite number of homogeneous polynomials. Note that the

[1] i.e. ϕ_U is bijective and ϕ_U and ϕ_U^{-1} are continuous

zero locus of a set of polynomials is in general not a manifold (due to singularities), but an *algebraic variety*.

§5 Flag manifolds. Complex flag manifolds are a major tool in the context of twistor geometry and the Penrose-Ward correspondence, cf. chapter VII. They can be considered as generalizations of projective spaces and Graßmann manifolds. An r-tuple $(L_1, \dots, L_r)$ of vector spaces of dimensions $\dim_{\mathbb{C}} L_i = d_i$ with $L_1 \subset \dots \subset L_r \subset \mathbb{C}^n$ and $0 < d_1 < \dots < d_r < n$ is called a *flag* in $\mathbb{C}^n$. The *(complex) flag manifold* $F_{d_1 \dots d_r, n}$ is the compact space

$$F_{d_1 \dots d_r, n} \; := \; \{\text{all flags } (L_1, \dots, L_r) \text{ with } \dim_{\mathbb{C}} L_i \; = \; d_i, \; i \; = \; 1, \dots, r\} \; . \qquad \text{(II.1)}$$

Simple examples of flag manifolds are $F_{1,n} = \mathbb{C}P^{n-1}$ and $F_{k,n} = G_{k,n}(\mathbb{C})$. The flag manifold $F_{d_1 \dots d_r, n}$ can also be written as the coset space

$$F_{d_1 \dots d_r, n} \; = \; \frac{\mathsf{U}(n)}{\mathsf{U}(n-d_r) \times \dots \times \mathsf{U}(d_2 - d_1) \times \mathsf{U}(d_1)} \; , \qquad \text{(II.2)}$$

and therefore its dimension is

$$\dim_{\mathbb{C}} F_{d_1 \dots d_r, n} \; = \; d_1(n - d_1) + (d_2 - d_1)(n - d_2) + \dots + (d_r - d_{r-1})(n - d_r) \; . \qquad \text{(II.3)}$$

§6 Weighted projective spaces. A further generalization of complex projective spaces are spaces which are obtained from $(\mathbb{C}^{m+1}) \backslash \{0\}$ with coordinates (z^i) by the identification $(z^1, z^2, \dots, z^{m+1}) \sim (t^{q_1} z^1, t^{q_2} z^2, \dots, t^{q_{m+1}} z^{m+1})$ with $t \in \mathbb{C}^\times$. These spaces are called *weighted projective spaces* and denoted by $W\mathbb{C}P^m(q_1, \dots, q_{m+1})$. Note that $W\mathbb{C}P^m(1, \dots, 1) = \mathbb{C}P^m$.

A subtlety when working with weighted projective spaces is the fact that they may not be smooth but can have non-trivial fixed points under the coordinate identification, which lead to singularities. Therefore, these spaces are mostly used as embedding spaces for smooth manifolds.

§7 Stein manifolds. A complex manifold that can be embedded as a closed submanifold into a complex Euclidean space is called a *Stein manifold*. Such manifolds play an important rôle in making Čech cohomology sets on a manifold independent of the covering, see section II.2.3, §32.

§8 Equivalence of manifolds. Two complex manifolds M and N are *biholomorphic* if there is a biholomorphic map[2] $m : M \to N$. This is equivalent to the fact that there is an identical cover $\mathfrak{U}$ of M and N and that there are biholomorphic functions h_a on each patch $U_a \in \mathfrak{U}$ such that we have the following relation between the transition functions: $f^M_{ab} = h_a^{-1} \circ f^N_{ab} \circ h_b$ on $U_a \cap U_b \neq \varnothing$. Two complex manifolds are called *diffeomorphic* if their underlying smooth manifolds are diffeomorphic. The transition functions of two diffeomorphic manifolds on an identical cover $\mathfrak{U}$ are related by $f^M_{ab} = s_a^{-1} \circ f^N_{ab} \circ s_b$ on nonempty intersections $U_a \cap U_b \neq \varnothing$, where the s_a are smooth functions on the patches U_a.

We call complex manifolds *smoothly equivalent* if they are diffeomorphic and *holomorphically equivalent* if they are biholomorphic. In one dimension, holomorphic equivalence implies conformal equivalence, cf. section IV.4.1.

§9 Functions on manifolds. Given a manifold M, we will denote the set of functions $\{f : M \to \mathbb{C}\}$ on M by $\mathscr{F}(M)$. Smooth functions will be denoted by $C^\infty(M)$ and holomorphic functions by $\mathcal{O}(M)$.

[2] a holomorphic map with a holomorphic inverse

II.1.2 Complex structures

It is quite obvious that many real manifolds of even dimension might also be considered as complex manifolds after a change of variables. The tool for making this statement exact is a complex structure.

§10 Modules and vector spaces. A *left module* over a ring Λ (or an Λ-left-module) is an Abelian group G together with an operation $(\lambda \in \Lambda, a \in G) \mapsto \lambda a \in G$, which is linear in both components. Furthermore, we demand that this operation is associative, i.e. $(\lambda\mu)a = \lambda(\mu a)$ and normalized according to $\mathbb{1}_\Lambda a = a$.

Analogously, one defines a *right module* with right multiplication and that of a *bimodule* with simultaneously defined, commuting left and right multiplication.

A *vector space* is a module over a field and in particular, a complex vector space is a module over $\mathbb{C}$. Later on, we will encounter supervector spaces which are modules over $\mathbb{Z}_2$-graded rings, cf. III.2.3, §20.

§11 Complex structures. Given a real vector space V, a *complex structure* on V is a map $I : V \to V$ with $I^2 = -\mathbb{1}_V$. This requires the vector space to have even dimensions and is furthermore to be seen as a generalization of $\mathrm{i}^2 = -1$. After defining the scalar multiplication of a complex number $(a + \mathrm{i}b) \in \mathbb{C}$ with a vector $v \in V$ as $(a + \mathrm{i}b)v := av + bIv$, V is a complex vector space. On the other hand, each complex vector space has a complex structure given by $Iv = \mathrm{i}v$.

§12 Canonical complex structure. The obvious identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is obtained by equating $z^i = x^i + \mathrm{i}y^i$, which induces the *canonical complex structure*

$$\begin{aligned} I(x^1, \ldots, x^n, y^1, \ldots, y^n) &= (-y^1, \ldots, -y^n, x^1, \ldots, x^n) , \\ \text{and thus} \quad I &= \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} . \end{aligned} \tag{II.4}$$

§13 Almost complex structure. Given a real differentiable manifold M of dimension $2n$, an *almost complex structure* is a smooth tensor field I of type (1,1) on each patch of M, such that at each point $x \in M$, I_x is a complex structure on T_xM. The pair (M, I) is called an *almost complex manifold.* Note that each real manifold with even dimension locally admits such a tensor field, and the equations $I_b{}^a \frac{\partial}{\partial x^a} f = \mathrm{i}\frac{\partial}{\partial x^b} f$ are just the Cauchy-Riemann equations. Thus, holomorphic maps $f : \mathbb{C}^n \supset U \to \mathbb{C}^m$ are exactly those which preserve the almost complex structure.

§14 Complexification. Given a real space S with a real scalar multiplication $\cdot : \mathbb{R}\times S \to S$, we define its *complexification* as the tensor product $S^c = S \otimes_{\mathbb{R}} \mathbb{C}$. We will encounter an example in the following paragraph.

§15 Holomorphic vector fields. Consider the complexification of the tangent space $TM^c = TM \otimes_{\mathbb{R}} \mathbb{C}$. This space decomposes at each point x into the direct sum of eigenvectors of I with eigenvalues $+\mathrm{i}$ and $-\mathrm{i}$, which we denote by $T^{1,0}_x M$ and $T^{0,1}_x M$, respectively, and therefore we have $TM^c = T^{1,0}M \oplus T^{0,1}M$. Sections of $T^{1,0}M$ and $T^{0,1}M$ are called *vector fields of type* $(1, 0)$ and $(0, 1)$, respectively. Vector fields of type $(1, 0)$ whose action on arbitrary functions will be holomorphic will be called *holomorphic vector fields* and *antiholomorphic vector fields* are defined analogously. This means in particular that a vector field X given locally by $X = \xi^i \frac{\partial}{\partial z^i}$, where $(\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n})$ is a local basis of $T^{1,0}M$, is a holomorphic vector field if the ξ^i are holomorphic functions. We will denote the space of vector fields on M by $\mathscr{X}(M)$. The above basis is complemented by the basis $(\frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n})$ of $T^{0,1}M$ to a full local basis of TM^c.

§16 Integrable complex structures. If an almost complex structure is induced from a holomorphic structure, cf. §2, one calls this almost complex structure *integrable*. Thus, an almost complex manifold with an integrable complex structure is a complex manifold.

§17 Newlander-Nirenberg theorem. Let (M, I) be an almost complex manifold. Then the following statements are equivalent:

(i) The almost complex structure I is integrable.

(ii) The *Nijenhuis tensor* $N(X,Y) = \frac{1}{4}([X,Y] + I[X,IY] + I[IX,Y] - [IX,IY])$ (the torsion) vanishes for arbitrary vector fields $X, Y \in \mathscr{X}(M)$.

(iii) The Lie bracket $[X,Y]$ closes in $T^{1,0}M$, i.e. for $X, Y \in T^{1,0}M$, $[X,Y] \in T^{1,0}M$.

§18 Complex differential forms. Analogously to complex tangent spaces, we introduce the space of complex differential forms on a complex manifold M as the complexification of the space of real differential forms: $\Omega^q(M)^c := \Omega^q(M) \otimes_{\mathbb{R}} \mathbb{C}$. Consider now a q-form $\omega \in \Omega^q(M)^c$. If $\omega(V_1, \ldots, V_q) = 0$ unless r of the V_i are elements of $T^{1,0}M$ and $s = q - r$ of them are elements of $T^{0,1}M$, we call ω a form of *bidegree* (r, s). We will denote the space of forms of bidegree (r, s) on M by $\Omega^{r,s}(M)$. It is now quite obvious that Ω^q (uniquely) splits into $\bigoplus_{r+s=q} \Omega^{r,s}(M)$.

Clearly, elements of $\Omega^{1,0}$ and $\Omega^{0,1}$ are dual to elements of $T^{1,0}M$ and $T^{0,1}M$, respectively. Local bases for $\Omega^{1,0}$ and $\Omega^{0,1}$ dual to the ones given in §15 are then given by $(\mathrm{d}z^1, \ldots, \mathrm{d}z^n)$ and $(\mathrm{d}\bar{z}^1, \ldots, \mathrm{d}\bar{z}^n)$ and satisfy the orthogonality relations $\langle \mathrm{d}z^i, \frac{\partial}{\partial z^j} \rangle = \delta^i_j$, $\langle \mathrm{d}\bar{z}^{\bar{\imath}}, \frac{\partial}{\partial z^j} \rangle = \langle \mathrm{d}z^i, \frac{\partial}{\partial \bar{z}^{\bar{\jmath}}} \rangle = 0$ and $\langle \mathrm{d}\bar{z}^{\bar{\imath}}, \frac{\partial}{\partial \bar{z}^{\bar{\jmath}}} \rangle = \delta^{\bar{\imath}}_{\bar{\jmath}}$.

§19 The exterior derivative. The *exterior derivative* d maps a form of bidegree (r, s) to a form which is the sum of an $(r+1, s)$-form and an $(r, s+1)$-form: Given an (r, s) form ω on a complex manifold M by

$$\omega \;=\; \frac{1}{r!s!}\, \omega_{i_1 \ldots i_r \bar{\imath}_{r+1} \ldots \bar{\imath}_{r+s}} \mathrm{d}z^{i_1} \wedge \ldots \mathrm{d}z^{i_r} \wedge \ldots \mathrm{d}\bar{z}^{\bar{\imath}_{r+s}} \ , \tag{II.5}$$

we define

$$\mathrm{d}\omega \;=\; \frac{1}{r!s!} \left(\partial_k \omega_{i_1 \ldots \bar{\imath}_{r+s}} \mathrm{d}z^k \wedge \mathrm{d}z^{i_1} \wedge \ldots \mathrm{d}\bar{z}^{\bar{\imath}_{r+s}} + \bar{\partial}_{\bar{k}} \omega_{i_1 \ldots \bar{\imath}_{r+s}} \mathrm{d}z^{i_1} \wedge \ldots \mathrm{d}\bar{z}^{\bar{k}} \wedge \ldots \mathrm{d}\bar{z}^{\bar{\imath}_{r+s}} \right) \ ,$$

which agrees with the definition of d on M interpreted as a real manifolds. One therefore splits $\mathrm{d} = \partial + \bar{\partial}$, where $\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$ and $\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$. The operators ∂ and $\bar{\partial}$ are called the *Dolbeault operators*. A holomorphic r-form is given by an $\omega \in \Omega^{r,0}(M)$ satisfying $\bar{\partial}\omega = 0$ and holomorphic 0-forms are holomorphic functions. The Dolbeault operators are nilpotent, i.e. $\partial^2 = \bar{\partial}^2 = 0$, and therefore one can construct the Dolbeault cohomology groups, see section II.2.3.

§20 Real structure. A *real structure* τ on a complex vector space V is an antilinear involution $\tau : V \to V$. This implies that $\tau^2(v) = v$ and $\tau(\lambda v) = \bar{\lambda} v$ for all $\lambda \in \mathbb{C}$ and $v \in V$. Therefore, a real structure maps a complex structure I to $-I$. One can use such a real structure to reduce a complex vector space to a real vector subspace. A real structure on a complex manifold is a complex manifold with a real structure on its tangent spaces. For an example, see the discussion in section VII.3.1, §4.

II.1.3 Hermitian structures

§21 Hermitian inner product. Given a complex vector space (V, I), a *Hermitian inner product* is an inner product g satisfying $g(X,Y) = g(IX, IY)$ for all vectors $X, Y \in V$

("I is g-orthogonal"). Note that every inner product $\tilde{g}$ can be turned into a Hermitian one by defining $g = \frac{1}{2}(\tilde{g}(X,Y) + \tilde{g}(IX,IY))$. To have an *almost Hermitian inner product* on an almost complex manifold M, one smoothly defines a g_x on T_xM for every $x \in M$.

§22 Hermitian structure. Every Hermitian inner product g can be uniquely extended to a *Hermitian structure* h, which is a map $h : V \times V \to \mathbb{C}$ satisfying

(*i*) $h(u,v)$ is $\mathbb{C}$-linear in v for every $u \in V$

(*ii*) $\overline{h(u,v)} = h(v,u)$ for all vectors $u, v \in V$.

(*iii*) $h(u,u) \geq 0$ for all vectors $u \in V$ and $h(u,u) = 0 \Leftrightarrow u = 0$.

For Hermitian structures on an almost complex manifold M, we demand additionally that h understood as a map $h : \Gamma(TM) \times \Gamma(TM) \to \mathscr{F}(M)$ maps every pair of smooth sections to smooth functions on M.

§23 Hermitian metric. When interpreting a smooth manifold M as a complex manifold via an integrable almost complex structure, one can extend the Riemannian metric g to a map $\tilde{g}_x : T_xM^c \times T_xM^c \to \mathbb{C}$ by

$$\tilde{g}_x : (X + \mathrm{i}Y, U + iV) \;\mapsto\; g_x(X,U) - g_x(Y,V) + \mathrm{i}(g_x(X,V) + g_x(Y,U)) \; . \tag{II.6}$$

A metric obtained in this way and satisfying $\tilde{g}_x(I_xX, I_xY) = \tilde{g}_x(X,Y)$ is called a *Hermitian metric*. Given again bases $(\frac{\partial}{\partial z^i})$ and $(\frac{\partial}{\partial \bar{z}^i})$ spanning locally $T^{1,0}M$ and $T^{0,1}M$, respectively, we have

$$g_{ij} \;=\; g_{\bar{\imath}\bar{\jmath}} \;=\; 0 \quad \text{and} \quad g \;=\; g_{i\bar{\jmath}} \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{\jmath}} + g_{\bar{\imath}j} \mathrm{d}\bar{z}^{\bar{\imath}} \otimes \mathrm{d}z^j \tag{II.7}$$

for a Hermitian metric g. A complex manifold with a Hermitian metric is called a *Hermitian manifold*.

§24 Theorem. A complex manifold always admits a Hermitian metric. Given a Riemannian metric on a complex manifold, one obtains a Hermitian metric e.g. by the construction described in §21.

§25 Kähler form. Given a Hermitian manifold (M,g), we define a tensor field J of type $(1,1)$ by $J(X,Y) = g(IX,Y)$ for every pair of sections (X,Y) of TM. As $J(X,Y) = g(IX,Y) = g(IIX,IY) = -g(IY,X) = -J(Y,X)$, the tensor field is antisymmetric and defines a two-form, the *Kähler form* of the Hermitian metric g. As easily seen, J is invariant under the action of I. Let m be the complex dimension of M. One can show that $\wedge^m J$ is a nowhere vanishing, real $2m$-form, which can serve as a volume element and thus every Hermitian manifold (and so also every complex manifold) is orientable.

§26 Kähler manifold. A *Kähler manifold* is a Hermitian manifold (M,g) on which one of the following three equivalent conditions holds:

(*i*) The Kähler form J of g satisfies $\mathrm{d}J = 0$.

(*ii*) The Kähler form J of g satisfies $\nabla J = 0$.

(*iii*) The almost complex structure satisfies $\nabla I = 0$,

where ∇ is the Levi-Civita connection of g. The metric g of a Kähler manifold is called a *Kähler metric.*

§27 Kähler potential. Given a Kähler manifold (M,g) with Kähler form J, it follows from $\mathrm{d}J = (\partial + \bar\partial)\mathrm{i}g_{i\bar\jmath}\mathrm{d}z^i \wedge \mathrm{d}\bar z^{\bar\jmath} = 0$ that

$$\frac{\partial g_{i\bar\jmath}}{\partial z^l} = \frac{\partial g_{l\bar\jmath}}{\partial z^i} \quad\text{and}\quad \frac{\partial g_{i\bar\jmath}}{\partial \bar z^l} = \frac{\partial g_{l\bar\jmath}}{\partial \bar z^i} \ . \tag{II.8}$$

Thus, we can define a local real function $\mathscr{K}$ such that $g = \partial\bar\partial\mathscr{K}$ and $J = \mathrm{i}\partial\bar\partial\mathscr{K}$. This function is called the *Kähler potential* of g. Conversely, if a metric is derived from a Kähler potential, it automatically satisfies (II.8).

§28 Examples. A simple example is the Kähler metric on $\mathbb{C}^m$ obtained from the Kähler potential $\mathscr{K} = \frac{1}{2}\sum z^i \bar z^{\bar\imath}$, which is the complex analog of $(\mathbb{R}^{2m}, \delta)$. Also easily seen is the fact that any orientable complex manifold M with $\dim_{\mathbb{C}} M = 1$ is Kähler: since J is a real two-form, $\mathrm{d}J$ has to vanish on M. These manifolds are called *Riemann surfaces.* Furthermore, any complex submanifold of a Kähler manifold is Kähler.

An important example is the complex projective space $\mathbb{C}P^n$, which is also a Kähler manifold. In homogeneous coordinates (ω^i) and inhomogeneous coordinates (z^i) (see §3), one can introduce a positive definite function

$$\mathscr{K}_i = \sum_{j=1}^{n+1} \left|\frac{\omega^j}{\omega^i}\right|^2 = \sum_{j=1}^{n} |z_i^j|^2 + 1 \tag{II.9}$$

on the patch U_i, which globally defines a closed two-form J by $J := \mathrm{i}\partial\bar\partial \ln \mathscr{K}_j$, as one easily checks. From this form, we obtain a metric by $g(X,Y) := J(X, IY)$, the *Fubini-Study metric* of $\mathbb{C}P^n$. In components, it reads on the patch U_i

$$g_i(X,Y) = 2\sum_{j,\bar\jmath} \frac{\delta_{j\bar\jmath}\mathscr{K}_i - z_i^j \bar z_i^{\bar\jmath}}{\mathscr{K}_i^2} X^j \bar Y^{\bar\jmath} \ . \tag{II.10}$$

Note that $S^2 \cong \mathbb{C}P^1$ is the only sphere which admits a complex structure. Above we saw that it is also a Kähler manifold.

§29 Kähler differential geometry. On a Kähler manifold (M,g) with Kähler potential $\mathscr{K}$, the components of the Levi-Civita connection simplify considerably. We introduce the Christoffel symbols as in Riemannian geometry by

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) \ . \tag{II.11}$$

Upon turning to complex coordinates and using the identity (II.8), we see that

$$\Gamma^l_{jk} = g^{l\bar s}\frac{\partial g_{k\bar s}}{\partial z^j} \quad\text{and}\quad \Gamma^{\bar l}_{\bar\jmath\bar k} = g^{\bar l s}\frac{\partial g_{\bar k s}}{\partial \bar z^{\bar\jmath}} \ , \tag{II.12}$$

and all other components vanish. Connections of this type, which are metric compatible ($\nabla_k g_{i\bar\jmath} = \nabla_{\bar k} g_{i\bar\jmath} = 0$) are called *Hermitian connections.*

The torsion and curvature tensors are again defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \ , \tag{II.13}$$
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ , \tag{II.14}$$

and the only non-vanishing components of the Riemann tensor and the Ricci tensor are

$$R_{i\bar\jmath k\bar l} = g_{i\bar s}\frac{\partial \Gamma^{\bar s}_{\bar\jmath\bar l}}{\partial z^k} \quad\text{and}\quad \mathrm{Ric}_{\bar\imath j} := R^{\bar k}_{\bar\imath\bar k j} = -\frac{\partial\Gamma^{\bar k}_{\bar\imath\bar k}}{\partial z^j} \ , \tag{II.15}$$

respectively.

§30 The Ricci form. Given the Ricci tensor Ric on a Kähler manifold M, we define the *Ricci form* $\mathscr{R}$ by

$$\mathscr{R}(X,Y) \ := \ \mathrm{Ric}(IX,Y). \tag{II.16}$$

Thus we have in components $\mathscr{R} = \mathrm{i}R_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}$. Note that on a Kähler manifold with metric $g_{\mu\nu}$, the Ricci form is closed and can locally be expressed as $\mathscr{R} = \mathrm{i}\partial\bar{\partial}\ln G$, where $G = \det(g_{\mu\nu}) = \sqrt{g}$. Furthermore, its cohomology class is (up to a real multiple) equal to the Chern class of the canonical bundle on M.

A manifold with vanishing Ricci form is called *Ricci-flat*. Kähler manifolds with this property are called *Calabi-Yau manifolds* and will be discussed in section II.3.

§31 Monge-Ampère equation. A differential equation of the type

$$(\partial_x\partial_x u)(\partial_y\partial_y u) - (\partial_x\partial_y u)^2 \ = \ f(x,y,u,\partial_x u,\partial_y u) \tag{II.17}$$

is called a *Monge-Ampère equation*. The condition of vanishing Ricci form obviously yields such an equation. We will explicitly discuss a related example in section III.3.4.

§32 Hyper-Kähler manifold. A *hyper-Kähler manifold* is a Riemannian manifold with three Kähler structures I, J and K which satisfy $IJK = -1$. Equivalently, one can define a hyper-Kähler manifold as a Riemannian manifold with holonomy group contained in $\mathsf{Sp}(m)$, which is the group of $m \times m$ quaternionic unitary matrices with m being half the complex dimension of the manifold.

§33 't Hooft tensors. The *'t Hooft tensors* (or eta-symbols) are given by

$$\eta^{i(\pm)}_{\mu\nu} \ := \ \varepsilon_{i\mu\nu 4} \pm \delta_{i\mu}\delta_{\nu 4} \mp \delta_{i\nu}\delta_{\mu 4} \tag{II.18}$$

and satisfy the relation $\eta^{i(\pm)}_{\mu\nu} = \pm * \eta^{i(\pm)}_{\mu\nu}$, where $*$ is the Hodge star operator. They form three Kähler structures, which give rise to a hyper-Kähler structure on the Euclidean spacetime $\mathbb{R}^4$. Note furthermore that any space of the form $\mathbb{R}^{4m}$ with $m \in \mathbb{N}$ is evidently a hyper-Kähler manifold.

II.2 Vector bundles and sheaves

II.2.1 Vector bundles

§1 Homotopy lifting property. Let E, B, and X be topological spaces. A map $\pi : E \to B$ is said to have the *homotopy lifting property* with respect to the space X if, given the commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{h} & E \\ \downarrow p & & \downarrow \pi \\ X \times [0,1] & \xrightarrow{h_t} & B \end{array} \tag{II.19}$$

there is a map $G : X \times [0,1] \to E$, which gives rise to two commutative triangles. That is, $G(x,0) = h(x)$ and $\pi \circ G(x,t) = h_t(x)$. Note that we assumed that all the maps are continuous.

§2 Fibration. A *fibration* is a continuous map $\pi : E \to B$ between topological spaces E and B, which satisfies the homotopy lifting property for all topological spaces X.

§3 Complex vector bundles. A *complex vector bundle* E over a complex manifold M is a vector bundle $\pi : E \to M$, and for each $x \in M$, $\pi^{-1}(x)$ is a complex vector space. As we will see in the following paragraph, holomorphic vector bundles are complex vector bundles which allow for a trivialization with holomorphic transition functions.

Every vector bundle is furthermore a fibration. The prove for this can be found e.g. in [125].

In the following, we will denote the space of smooth sections of the vector bundle $\pi : E \to M$ by $\Gamma(M, E)$.

§4 Holomorphic vector bundle. A *holomorphic vector bundle* E of rank k over a manifold M with $\dim_{\mathbb{C}} M = n$ is a $(k+n)$-dimensional complex manifold E endowed with a holomorphic projection $\pi : E \to M$ satisfying the conditions

(i) $\pi^{-1}(p)$ is a k-dimensional complex vector space for all $p \in M$.

(ii) For each point $p \in M$, there is a neighborhood U and a biholomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$. The maps ϕ_U are called *local trivializations.*

(iii) The *transition functions* f_{UV} are holomorphic maps $U \cap V \to \mathsf{GL}(k, \mathbb{C})$.

Holomorphic vector bundles of dimension $k = 1$ are called *line bundles.*

§5 Examples. Let M be a complex manifold of dimension m. The holomorphic tangent bundle $T^{1,0}M$, its dual, the holomorphic cotangent bundle $T^{1,0\vee}M$, in fact all the bundles $\Lambda^{p,0}M$ with $0 \leq p \leq m$ are holomorphic vector bundles. The complex line bundle $K_M := \Lambda^{m,0}M$ is called the *canonical bundle*; K_M is also a holomorphic bundle.

On the spaces $\mathbb{C}P^m$, one defines the *tautological line bundle* as: $\mathbb{C}^{m+1} \to \mathbb{C}P^m$. One can proof that the canonical bundle over $\mathbb{C}P^m$ is isomorphic to the $(m+1)$th exterior power of the tautological line bundle. For more details on these line bundles, see also the remarks in §28.

§6 Holomorphic structures. Given a complex vector bundle E over M, we define the bundle of E-valued forms on M by $\Lambda^{p,q}E := \Lambda^{p,q}M \otimes E$. An operator $\bar{\partial} : \Gamma(M, \Lambda^{p,q}E) \to \Gamma(M, \Lambda^{p,q+1}E)$ is called a *holomorphic structure* if it satisfies $\bar{\partial}^2 = 0$. It is obvious that the action of $\bar{\partial}$ is independent of the chosen trivialization, as the transition functions are holomorphic and $\bar{\partial}$ does not act on them. Note furthermore that the operator $\bar{\partial}$ satisfies a graded Leibniz rule when acting on the wedge product of a (p,q)-form ω and an arbitrary form σ:

$$\bar{\partial}(\omega \wedge \sigma) \;=\; (\bar{\partial}\omega) \wedge \sigma + (-1)^{p+q}\omega \wedge (\bar{\partial}\sigma) \ . \tag{II.20}$$

§7 Theorem. A complex vector bundle E is holomorphic if and only if there exists a holomorphic structure $\bar{\partial}$ on E. For more details on this statement, see section II.2.3.

§8 Connections and curvature. Given a complex vector bundle $E \to M$, a *connection* is a $\mathbb{C}$-linear map $\nabla : \Gamma(M, E) \to \Gamma(M, \Lambda^1 E)$ which satisfies the Leibniz rule

$$\nabla(f\sigma) \;=\; \mathrm{d}f \otimes \sigma + f\nabla\sigma \ , \tag{II.21}$$

where $f \in C^\infty(M)$ and $\sigma \in \Gamma(M, E)$. A connection gives a means of transporting frames of E along a path in M. Given a smooth path $\gamma : [0,1] \to M$ and a frame $\mathbf{e}_0$ over $\gamma(0)$, there is a unique frame $\mathbf{e}_t$ consisting of sections of $\gamma^* E$ such that

$$\nabla_{\dot{\gamma}(t)}\mathbf{e}_t \;=\; 0 \tag{II.22}$$

for all $t \in [0,1]$. This frame is called the *parallel transport* of $\mathbf{e}_0$ along γ. As we can parallel transport frames, we can certainly do the same with vector fields.

The *curvature* associated to ∇ is defined as the two-form $\mathcal{F}_\nabla = \nabla^2$. Given locally constant sections $(\sigma_1, \ldots, \sigma_k)$ over U defining a basis for each fibre over U, we can represent a connection by a collection of one-forms $\omega_{ij} \in \Gamma(U, \Lambda^1 U)$: $\nabla\sigma_i = \omega_{ij} \otimes \sigma_j$. The components of the corresponding curvature $\mathcal{F}_\nabla\sigma_i = \mathcal{F}_{ij} \otimes \sigma_j$ are easily calculated to be $\mathcal{F}_{ij} = \mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$. Roughly speaking, the curvature measures the difference between the parallel transport along a loop and the identity.

Identifying $\nabla^{0,1}$ with the holomorphic structure $\bar{\partial}$, one immediately sees from the theorem §7 that the $(0, 2)$-part of the curvature of a holomorphic vector bundle has to vanish.

§9 Chern connection. Conversely, given a Hermitian structure on a holomorphic vector bundle with holomorphic structure $\bar{\partial}$, there is a unique connection ∇, the *Chern connection*, for which $\nabla^{0,1} = \bar{\partial}$.

§10 Connections on Hermitian manifolds. On a Hermitian manifold (M, I, h), there are two natural connections: the Levi-Civita connection and the Chern connection. They both coincide if and only if h is Kähler.

§11 Holonomy groups. Let M be a manifold of dimension d endowed with a connection ∇. A vector $V \in T_pM$ will be transformed to another vector $V' \in T_pM$ when parallel transported along a closed curve through p. The group of all such transformations is called the *holonomy group* of the manifold M. Using the Levi-Civita connection which will not affect the length of the vector V during the parallel transport, the holonomy group will be a subgroup of $\mathsf{SO}(d)$ on real manifolds and a subgroup of $\mathsf{U}(d)$ for Kähler manifolds. Flat manifolds will clearly have the trivial group consisting only of the element $\mathbb{1}$ as their holonomy groups. Complex manifolds, whose holonomy groups are $\mathsf{SU}(d)$ are called *Calabi-Yau manifolds* and will be discussed in section II.3.

§12 Characteristic classes. *Characteristic classes* are subsets of cohomology classes and are used to characterize topological properties of manifolds and bundles. Usually they are defined by polynomials in the curvature two-form $\mathcal{F}_\nabla$. Therefore, every trivial bundle has a trivial characteristic class, and thus these classes indicate the nontriviality of a bundle. In the following, we will restrict our discussion mainly to Chern classes, as they play a key rôle in the definition of Calabi-Yau manifolds.

§13 Chern class. Given a complex vector bundle $E \to M$ with fibres $\mathbb{C}^k$ endowed with a connection ω defining a field strength $\mathcal{F}$, we define the *total Chern class*[3] by

$$c(\mathcal{F}) \;:=\; \det\left(\mathbb{1} + \frac{\mathrm{i}\mathcal{F}}{2\pi}\right) \;. \tag{II.23}$$

One can split $c(\mathcal{F})$ into the direct sums of forms of even degrees:

$$c(\mathcal{F}) \;=\; 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \ldots \;. \tag{II.24}$$

The $2j$-form $c_j(\mathcal{F})$ is called the *j-th Chern class*. Note that when talking about the Chern class of a manifold, one means the Chern class of its tangent bundle calculated from the curvature of the Levi-Civita connection.

§14 Chern number. If M is compact and of real dimension $2d$, one can pair any product of Chern classes of total degree $2d$ with oriented homology classes of M which results in integers called the *Chern numbers* of E. As a special example, consider the possible first Chern classes of a line bundle L over the Riemann sphere $\mathbb{C}P^1 \cong S^2$. It is $H^2(S^2) \cong \mathbb{Z}$ and the number corresponding to the first Chern class of the line bundle L is called the *first Chern number*.

[3] named after Shiing-shen Chern, who introduced it in the 1940s

§15 Properties of Chern classes. The zeroth Chern class is always equal to 1. For a manifold M with dimension d, we clearly have $c_n(\mathcal{F}) = 0$ for $n > d$.

§16 Calculating Chern classes. A simple method for calculating Chern classes is available if one can diagonalize $\mathcal{F}$ by an element $g \in \mathsf{GL}(k, \mathbb{C})$ such that $g^{-1}\mathcal{F}g = \mathrm{diag}(x_1, \ldots, x_n) =: D$. One then easily derives that

$$\begin{aligned} \det(\mathbb{1} + D) &= \det(\mathrm{diag}(1 + x_1, \ldots, 1 + x_n)) \\ &= 1 + \mathrm{tr}\, D + \tfrac{1}{2}((\mathrm{tr}\, D)^2 - \mathrm{tr}\, D^2) + \ldots + \det D\ . \end{aligned} \tag{II.25}$$

§17 Theorem. Consider two complex vector bundles $E \to M$ and $F \to M$ with total Chern classes $c(E)$ and $c(F)$. Then the total Chern class of a Whitney sum bundle[4] $(E \oplus F) \to M$ is given by $c(E \oplus F) = c(E) \wedge c(F)$. In particular, the first Chern classes add: $c_1(E \oplus F) = c_1(E) + c_1(F)$.

§18 Whitney product formula. Given a short exact sequence of vector bundles A, B and C as

$$0 \ \to\ A\ \to\ B\ \to\ C\ \to\ 0\ , \tag{II.26}$$

we have a splitting $B = A \oplus C$ and together with the above theorem, we obtain the formula $c(A) \wedge c(C) = c(B)$. This formula will be particularly useful for calculating the Chern classes of the superambitwistor space $\mathcal{L}^{5|6}$, see the short exact sequence (VII.323).

§19 Further rules for calculations. Given two vector bundles E and F over a complex manifold M, we have the following formulæ:

$$c_1(E \otimes F) \ =\ \mathrm{rk}(F)c_1(E) + \mathrm{rk}(E)c_1(F)\ , \tag{II.27}$$

$$c_1\left(E^\vee\right) \ =\ -c_1(E)\ . \tag{II.28}$$

§20 Chern classes from degeneracy loci. Chern classes essentially make statements about the degeneracy of sets of sections of vector bundles via a Gauß-Bonnet formula. More precisely, given a vector bundle E of rank e over M, the c_{e+1-i}th Chern class is Poincaré-dual to the degeneracy cycle of i generic global sections. This degeneracy locus is obtained by arranging the i generic sections in an $e \times i$-dimensional matrix C and calculating the locus in M, where C has rank less than i. We will present an example in paragraph §28. For more details, see e.g. [113].

§21 Chern character. Let us also briefly introduce the characteristic classes called Chern characters, which play an important rôle in the Atiyah-Singer index theorem. We will need them for instanton configurations, in which the number of instantons is given by an integral over the second Chern character. One defines the *total Chern character* of a curvature two-form $\mathcal{F}$ as

$$\mathrm{ch}(\mathcal{F}) \ =\ \mathrm{tr}\ \exp\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) \tag{II.29}$$

and the *j-th Chern character* as a part of the corresponding Taylor expansion

$$\mathrm{ch}_j(\mathcal{F}) \ =\ \frac{1}{j!}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^j\ . \tag{II.30}$$

Note that $\mathrm{ch}(\mathcal{F})$ is a polynomial of finite order on a finite-dimensional manifold. Furthermore, one can express Chern characters in terms of Chern classes, e.g.

$$\mathrm{ch}_1(\mathcal{F}) \ =\ c_1(\mathcal{F}) \quad \text{and} \quad \mathrm{ch}_2(\mathcal{F}) \ =\ \tfrac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}))\ . \tag{II.31}$$

[4]A *Whitney sum* of two vector bundles over a manifold M yields the vector bundle whose fibres are the direct sums of the fibres of the original two bundles.

The zeroth Chern character $\mathrm{ch}_0(\mathcal{F})$ is simply the dimension of the vector bundle associated to the curvature two-form $\mathcal{F}$.

II.2.2 Sheaves and line bundles

§22 Sheaf. A *presheaf* $\mathfrak{S}$ on a topological space X is an association of a group[5] $\mathfrak{S}(U)$ to every open set $U \subset X$ together with a restriction map $\rho_{UV} : \mathfrak{S}(V) \to \mathfrak{S}(U)$ for $U \subset V \subset X$, which satisfies $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$ for $U \subset V \subset W \subset X$ and $\rho_{WW} = \mathrm{id}$. A presheaf becomes a *sheaf* under two additional conditions:

(i) Sections are determined by local data: Given two sections $\sigma, \tau \in \mathfrak{S}(V)$ with $\rho_{UV}(\sigma) = \rho_{UV}(\tau)$ for every open set $U \subset V$, we demand that $\sigma = \tau$ on V.

(ii) Compatible local data can be patched together: If $\sigma \in \mathfrak{S}(U)$ and $\tau \in \mathfrak{S}(V)$ such that $\rho_{(U\cap V)U}(\sigma) = \rho_{(U\cap V)V}(\tau)$ then there exists an $\chi \in \mathfrak{S}(U \cup V)$ such that $\rho_{U(U\cup V)}(\chi) = \sigma$ and $\rho_{V(U\cup V)}(\chi) = \tau$.

§23 Turning a presheaf into a sheaf. One can associate a sheaf $\mathfrak{S}$ to a presheaf $\mathfrak{S}_0$ on a topological space X by the following construction: Consider two local sections s and $s' \in \mathfrak{S}_0(U)$ for an open set $U \subset X$. We call s and s' equivalent at the point $x \in X$ if there is a neighborhood $V_x \subset U$, such that $\rho_{V_xU}(s) = \rho_{V_xU}(s')$. The corresponding equivalence classes are called *germs* of sections in the point x and the space of germs at x is denoted by $\mathfrak{S}_x$. We can now define the sheaf $\mathfrak{S}$ as the union of the spaces of germs $\mathfrak{S} := \bigcup_{x\in X} \mathfrak{S}_x$, as this union clearly has the required properties.

§24 Subsheaf. A *subsheaf* of a sheaf $\mathfrak{S}$ over a topological space X is a sheaf $\mathfrak{S}'$ over X such that $\mathfrak{S}'(U)$ is a subgroup of $\mathfrak{S}(U)$ for any open set $U \subset X$. The restriction maps on $\mathfrak{S}'$ are inherited from the ones on $\mathfrak{S}$.

§25 Examples. Examples for sheaves are the sheaf of holomorphic functions $\mathscr{O}(U)$, the sheaves of continuous and smooth functions[6] $C^0(U)$ and $C^\infty(U)$ and the sheaves of smooth (r,s)-forms $\Omega^{r,s}(U)$, where U is a topological space (a complex manifold in the latter example).

§26 Structure sheaf. One can interpret a manifold M as a *locally ringed space*, which[7] is a topological space M together with a sheaf F of commutative rings on M. This sheaf F is called the structure sheaf of the locally ringed space and one usually denotes it by $\mathscr{O}_M$. In the case that $(M, \mathscr{O}_M)$ is a complex manifold, F is the sheaf of holomorphic functions on M.

§27 Locally free sheaf. A sheaf $\mathfrak{E}$ is *locally free* and *of rank r* if there is an open covering $\{U_j\}$ such that $\mathfrak{E}|_{U_j} \cong \mathscr{O}_{U_j}^{\oplus r}$. One can show that (isomorphism classes of) locally free sheaves of rank r over a manifold M are in one-to-one correspondence with (isomorphism classes of) vector bundles of rank r over M. The sheaf $\mathfrak{E}$ corresponding to a certain vector bundle E is given by the sheaf dual to the sheaf of sections of E. For this reason, the terms vector bundle and (locally free) sheaf are often used sloppily for the same object.

We will denote by $\mathscr{O}(U)$ the sheaf of holomorphic functions, and the holomorphic vector bundle over U, whose sections correspond to elements of $\mathscr{O}(U)$, by $\mathcal{O}(U)$.

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[5] Usually, the definition of a sheaf involves only Abelian groups, but extensions to non-Abelian groups are possible, see e.g. the discussion in [226].

[6] Note that $C^0(U, \mathfrak{S})$ will denote the set of Čech 0-cochains taking values in the sheaf $\mathfrak{S}$.

[7] A special case of locally ringed spaces are the better-known *schemes*.

§28 Holomorphic line bundles. A *holomorphic line bundle* is a holomorphic vector bundle of rank 1. Over the Riemann sphere $\mathbb{C}P^1 \cong S^2$, these line bundles can be completely characterized by an integer $d \in \mathbb{Z}$, cf. §14.

Given the standard patches U_+ and U_- on the Riemann sphere $\mathbb{C}P^1$ with the inhomogeneous coordinates $\lambda_\pm$ glued via $\lambda_\pm = 1/\lambda_\mp$ on the intersection $U_+ \cap U_-$ of the patches, the *holomorphic line bundle* $\mathcal{O}(d)$ is defined by its transition function $f_{+-} = \lambda_+^d$ and thus we have $z_+ = \lambda_+^d z_-$, where $z_\pm$ are complex coordinates on the fibres over $U_\pm$.

For $d \geq 0$, global sections of the bundle $\mathcal{O}(d)$ are polynomials of degree d in the inhomogeneous coordinates $\lambda_\pm$ and homogeneous polynomials of degree d in homogeneous coordinates. The $\mathcal{O}(d)$ line bundle has first Chern number d, since – according to the Gauß-Bonnet formula of paragraph §20 – the first Chern class is Poincaré dual to the degeneracy loci of one generic global section. These loci are exactly the d points given by the zeros of a degree d polynomial. Furthermore, the first Chern class is indeed sufficient to characterize a complex line bundle up to topological (smooth) equivalence, and therefore it also suffices to characterize a holomorphic line bundle up to holomorphic equivalence.

The complex conjugate bundle to $\mathcal{O}(d)$ is denoted by $\bar{\mathcal{O}}(d)$. Its sections have transition functions $\bar{\lambda}_+^d$: $\bar{z}_+ = \bar{\lambda}_+^d \bar{z}_-$.

This construction can be generalized to higher-dimensional complex projective spaces $\mathbb{C}P^n$. Recall that these spaces are covered by $n+1$ patches. In terms of the homogeneous coordinates λ_i, $i = 0, \ldots, n$, the line bundle $\mathcal{O}(d) \rightarrow \mathbb{C}P^n$ is defined by the transition function $f_{ij} = (\lambda_j/\lambda_i)^d$.

We will sometimes use the notation $\mathcal{O}_{\mathbb{C}P^n}(d)$, to label the line bundle of degree d over $\mathbb{C}P^n$. Furthermore, $\mathcal{O}_{\mathbb{C}P^n}$ denotes the trivial line bundle over $\mathbb{C}P^n$, and $\mathcal{O}^k(d)$ is defined as the direct sum of k line bundles of rank d.

Note that bases of the (1,0)-parts of the tangent and the cotangent bundles of the Riemann sphere $\mathbb{C}P^1$ are sections of $\mathcal{O}(2)$ and $\mathcal{O}(-2)$, respectively. Furthermore, the canonical bundle of $\mathbb{C}P^n$ is $\mathcal{O}(-n-1)$ and its tautological line bundle is $\mathcal{O}(-1)$.

§29 Theorem. (Grothendieck) Any holomorphic bundle E over $\mathbb{C}P^1$ can be decomposed into a direct sum of holomorphic line bundles. This decomposition is unique up to permutations of holomorphically equivalent line bundles. The Chern numbers of the line bundles are holomorphic invariants of E, but only their sum is also a topological invariant.

II.2.3 Dolbeault and Čech cohomology

There are two convenient descriptions of holomorphic vector bundles: the Dolbeault and the Čech description. Since the Penrose-Ward transform (see chapter VII) heavily relies on both of them, we recollect here the main aspects of these descriptions and comment on their equivalence.

§30 Dolbeault cohomology groups. As the Dolbeault operator $\bar{\partial}$ is nilpotent, one can introduce the Dolbeault complex

$$\Omega^{r,0}(M) \xrightarrow{\bar{\partial}} \Omega^{r,1}(M) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{r,m}(M) \tag{II.32}$$

on a complex manifold M together with the (r,s)th $\bar{\partial}$*-cohomology group*

$$H_{\bar{\partial}}^{r,s}(M) = \frac{\text{cocycles}}{\text{coboundaries}} = \frac{Z_{\bar{\partial}}^{r,s}(M)}{B_{\bar{\partial}}^{r,s}(M)} . \tag{II.33}$$

Here, the *cocycles* $Z_{\bar{\partial}}^{r,s}(M)$ are the elements ω of $\Omega^{r,s}(M)$ which are closed, i.e. $\bar{\partial}\omega = 0$ and the *coboundaries* are those elements ω which are exact, i.e. for $s > 0$ there is a form $\tau \in \Omega^{r,s-1}(M)$ such that $\bar{\partial}\tau = \omega$.

The *Hodge number* $h^{r,s}$ is the complex dimension of $H_{\bar{\partial}}^{r,s}(M)$. The corresponding *Betti number* of the de Rham cohomology of the underlying real manifold is given by $b_k = \sum_{p=0}^{k} h^{p,k-p}$ and the *Euler number* of a d-dimensional real manifold is defined as $\chi = \sum_{p=0}^{d}(-1)^p b_p$.

The Poincaré lemma can be directly translated to the complex situation and thus every $\bar{\partial}$-closed form is locally $\bar{\partial}$-exact.

§31 Holomorphic vector bundles and Dolbeault cohomology. Assume that G is a group having a representation in terms of $n \times n$ matrices. We will denote by $\mathfrak{S}$ the sheaf of smooth G-valued functions on a complex manifold M and by $\mathfrak{A}$ the sheaf of flat (0,1)-connections on a principal G-bundle $P \to M$, i.e. germs of solutions to

$$\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \;=\; 0 \;. \tag{II.34}$$

Note that elements $\mathcal{A}^{0,1}$ of $\Gamma(M, \mathfrak{A})$ define a holomorphic structure $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + \mathcal{A}^{0,1}$ on a trivial rank n complex vector bundle over M. The moduli space $\mathcal{M}$ of such holomorphic structures is obtained by factorizing $\Gamma(M, \mathfrak{A})$ by the group of gauge transformations, which is the set of elements g of $\Gamma(M, \mathfrak{S})$ acting on elements $\mathcal{A}^{0,1}$ of $\Gamma(M, \mathfrak{A})$ as

$$\mathcal{A}^{0,1} \;\mapsto\; g\mathcal{A}^{0,1}g^{-1} + g\bar{\partial}g^{-1} \;. \tag{II.35}$$

Thus, we have $\mathcal{M} \cong \Gamma(M, \mathfrak{A})/\Gamma(M, \mathfrak{S})$ and this is the description of holomorphic vector bundles in terms of Dolbeault cohomology.

§32 Čech cohomology sets. Consider a trivial principal G-bundle P over a complex manifold M covered by a collection of patches $\mathfrak{U} = \{U_a\}$ and let G have a representation in terms of $n \times n$ matrices. Let $\mathfrak{G}$ be an arbitrary sheaf of G-valued functions on M. The set of *Čech q-cochains* $C^q(\mathfrak{U}, \mathfrak{G})$ is the collection $\psi = \{\psi_{a_0 \dots a_q}\}$ of sections of $\mathfrak{G}$ defined on nonempty intersections $U_{a_0} \cap \dots \cap U_{a_q}$. Furthermore, we define the sets of Čech 0- and 1-cocycles by

$$Z^0(\mathfrak{U}, \mathfrak{G}) \;:=\; \{\, \psi \in C^0(\mathfrak{U}, \mathfrak{G}) \mid \psi_a \;=\; \psi_b \text{ on } U_a \cap U_b \;\neq\; \varnothing\} \;=\; \Gamma(\mathfrak{U}, \mathfrak{G}) \;, \tag{II.36}$$

$$\begin{aligned} Z^1(\mathfrak{U}, \mathfrak{S}) \;:=\; \{\, \chi \in C^1(\mathfrak{U}, \mathfrak{G}) \mid \chi_{ab} \;&=\; \chi_{ba}^{-1} \text{ on } U_a \cap U_b \;\neq\; \varnothing, \\ \chi_{ab}\chi_{bc}\chi_{ca} \;&=\; \mathbb{1} \text{ on } U_a \cap U_b \cap U_c \;\neq\; \varnothing\} \;. \end{aligned} \tag{II.37}$$

This definition implies that the Čech 0-cocycles are independent of the covering: it is $Z^0(\mathfrak{U}, \mathfrak{G}) = Z^0(M, \mathfrak{G})$, and we define the *zeroth Čech cohomology set* by $\check{H}^0(M, \mathfrak{G}) := Z^0(M, \mathfrak{G})$. Two 1-cocycles χ and $\tilde{\chi}$ are called *equivalent* if there is a 0-cochain $\psi \in C^0(\mathfrak{U}, \mathfrak{G})$ such that $\tilde{\chi}_{ab} = \psi_a \chi_{ab} \psi_b^{-1}$ on all $U_a \cap U_b \neq \varnothing$. Factorizing $Z^1(\mathfrak{U}, \mathfrak{G})$ by this equivalence relation gives the *first Čech cohomology set* $\check{H}^1(\mathfrak{U}, \mathfrak{G}) \cong Z^1(\mathfrak{U}, \mathfrak{G})/C^0(\mathfrak{U}, \mathfrak{G})$.

If the patches U_a of the covering $\mathfrak{U}$ are Stein manifolds, one can show that the first Čech cohomology sets are independent of the covering and depend only on the manifold M, e.g. $\check{H}^1(\mathfrak{U}, \mathfrak{S}) = \check{H}^1(M, \mathfrak{S})$. This is well known to be the case in the situations we will consider later on, i.e. for purely bosonic twistor spaces. Let us therefore imply that all the coverings in the following have patches which are Stein manifolds unless otherwise stated.

Note that in the terms introduced above, we have $\mathcal{M} \cong \check{H}^0(M, \mathfrak{A})/\check{H}^0(M, \mathfrak{S})$.

§33 Abelian Čech cohomology. If the structure group G of the bundle P defined in the previous paragraph is Abelian, one usually replaces in the notation of the group action the multiplication by addition to stress commutativity. Furthermore, one can then define a full *Abelian Čech complex* from the operator[8] $\check{\mathrm{d}} : C^q(M,\mathfrak{S}) \to C^{q+1}(M,\mathfrak{S})$ whose action on Čech q-cochains ψ is given by

$$(\check{\mathrm{d}}\psi)_{a_0,a_1,\ldots,a_{q+1}} \;:=\; \sum_{\nu=0}^{q+1}(-1)^\nu \psi_{a_0,a_1,\ldots,\hat{a}_\nu,\ldots,a_{q+1}}\ , \tag{II.38}$$

where the hat $\hat{\cdot}$ denotes an omission. The nilpotency of $\check{\mathrm{d}}$ is easily verified, and the *Abelian Čech cohomology* $\check{H}^q(M,\mathfrak{S})$ is the cohomology of the Čech complex.

More explicitly, we will encounter the following three Abelian Čech cohomology groups: $\check{H}^0(M,\mathfrak{S})$, which is the space of global sections of $\mathfrak{S}$ on M, $\check{H}^1(M,\mathfrak{S})$, for which the cocycle and coboundary conditions read

$$\chi_{ac} \;=\; \chi_{ab}+\chi_{bc} \quad \text{and} \quad \chi_{ab} \;=\; \psi_a - \psi_b\ , \tag{II.39}$$

respectively, where $\chi \in C^1(M,\mathfrak{S})$ and $\psi \in C^0(M,\mathfrak{S})$, and $\check{H}^2(M,\mathfrak{S})$, for which the cocycle and coboundary conditions read

$$\varphi_{abc}-\varphi_{bcd}+\varphi_{cda}-\varphi_{dab} \;=\; 0 \quad \text{and} \quad \varphi_{abc} \;=\; \chi_{ab}-\chi_{ac}+\chi_{bc}\ , \tag{II.40}$$

where $\varphi \in C^2(M,\mathfrak{S})$, as one easily derives from (II.38).

§34 Holomorphic vector bundles and Čech cohomology. Given a complex manifold M, let us again denote the sheaf of smooth G-valued functions on M by $\mathfrak{S}$. We introduce additionally its subsheaf of holomorphic functions and denote it by $\mathfrak{H}$.

Contrary to the connections used in the Dolbeault description, the Čech description of holomorphic vector bundles uses transition functions to define vector bundles. Clearly, such a collection of transition functions has to belong to the first Čech cocycle set of a suitable sheaf $\mathfrak{G}$. Furthermore, we want to call two vector bundles equivalent if there exists an element h of $C^0(M,\mathfrak{G})$ such that

$$f_{ab} \;=\; h_a^{-1}\tilde{f}_{ab}h_b \quad \text{on all} \;\; U_a\cap U_b \;\neq\; \varnothing\ . \tag{II.41}$$

Thus, we observe that holomorphic and smooth vector bundles have transition functions which are elements of the Čech cohomology sets $\check{H}^1(M,\mathfrak{H})$ and $\check{H}^1(M,\mathfrak{S})$, respectively.

§35 Equivalence of the Dolbeault and Čech descriptions. For simplicity, let us restrict our considerations to topologically trivial bundles, which will prove to be sufficient. To connect both descriptions, let us first introduce the subset $\mathfrak{X}$ of $C^0(M,\mathfrak{S})$ given by a collection of G-valued functions $\psi = \{\psi_a\}$, which fulfill

$$\psi_a\bar{\partial}\psi_a^{-1} \;=\; \psi_b\bar{\partial}\psi_b^{-1} \tag{II.42}$$

on any two arbitrary patches U_a, U_b from the covering $\mathfrak{U}$ of M. Due to (II.34), elements of $\check{H}^0(M,\mathfrak{A})$ can be written as $\psi\bar{\partial}\psi^{-1}$ with $\psi \in \mathfrak{X}$. Thus, for every $\mathcal{A}^{0,1} \in \check{H}^0(M,\mathfrak{A})$ we have corresponding elements $\psi \in \mathfrak{X}$. One of these ψ can now be used to define the transition functions of a topologically trivial rank n holomorphic vector bundle E over M by the formula

$$f_{ab} \;=\; \psi_a^{-1}\psi_b \quad \text{on} \;\; U_a\cap U_b \;\neq\; \varnothing\ . \tag{II.43}$$

[8]The corresponding picture in the non-Abelian situation has still not been constructed in a satisfactory manner.

It is easily checked that the f_{ab} constructed in this way are holomorphic. Furthermore, they define holomorphic vector bundles which are topologically trivial, but not holomorphically trivial. Thus, they belong to the kernel of a map $\rho : \check{H}^1(M,\mathfrak{H}) \to \check{H}^1(M,\mathfrak{S})$.

Conversely, given a transition function f_{ab} of a topologically trivial vector bundle on the intersection $U_a \cap U_b$, we have

$$0 = \bar{\partial} f_{ab} \;=\; \bar{\partial}(\psi_a^{-1}\psi_b) \;=\; \psi_a^{-a}(\psi_a\bar{\partial}\psi_a^{-1} - \psi_b\bar{\partial}\psi_b^{-1})\psi_b \;=\; \psi_a^{-1}(\mathcal{A}_a - \mathcal{A}_b)\psi_b \; . \tag{II.44}$$

Hence on $U_a \cap U_b$, we have $\mathcal{A}_a = \mathcal{A}_b$ and we have defined a global $(0,1)$-form $\mathcal{A}^{0,1} := \psi\bar{\partial}\psi^{-1}$.

The bijection between the moduli spaces of both descriptions is easily found. We have the short exact sequence

$$0 \to \mathfrak{H} \;\xrightarrow{i}\; \mathfrak{S} \;\xrightarrow{\delta^0}\; \mathfrak{A} \;\xrightarrow{\delta^1}\; 0 \; , \tag{II.45}$$

where i denotes the embedding of $\mathfrak{H}$ in $\mathfrak{S}$, δ^0 is the map $\mathfrak{S} \ni \psi \mapsto \psi\bar{\partial}\psi^{-1} \in \mathfrak{A}$ and δ^1 is the map $\mathfrak{A} \ni \mathcal{A}^{0,1} \mapsto \bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1}$. This short exact sequence induces a long exact sequence of cohomology groups

$$0 \;\to\; \check{H}^0(M,\mathfrak{H}) \;\xrightarrow{i_*}\; \check{H}^0(M,\mathfrak{S}) \;\xrightarrow{\delta^0_*}\; \check{H}^0(M,\mathfrak{A}) \;\xrightarrow{\delta^1_*}\; \check{H}^1(M,\mathfrak{H}) \;\xrightarrow{\rho}\; \check{H}^1(M,\mathfrak{S}) \;\to\; \dots \; ,$$

and from this we see that $\ker\rho \cong \check{H}^0(M,\mathfrak{A})/\check{H}^0(M,\mathfrak{S}) \cong \mathcal{M}$. Thus, the moduli spaces of both descriptions are bijective and we have the equivalence

$$(E, f_{+-} = \mathbb{1}_n, A^{0,1}) \;\sim\; (\tilde{E}, \tilde{f}_{+-}, \tilde{A}^{0,1} = 0) \; . \tag{II.46}$$

This fact is at the heart of the Penrose-Ward transform, see chapter VII.

§36 Remark concerning supermanifolds. In the later discussion, we will need to extend these results to supermanifolds and exotic supermanifolds, see chapter III. Note that this is not a problem, as our above discussion was sufficiently abstract. Furthermore, we can assume that the patches of a supermanifold are Stein manifolds if and only if the patches of the corresponding body are Stein manifolds since infinitesimal neighborhoods cannot be covered partially. Recall that having patches which are Stein manifolds render the Čech cohomology sets independent of the covering.

II.2.4 Integrable distributions and Cauchy-Riemann structures

Cauchy-Riemann structures are a generalization of the concept of complex structures to real manifolds of arbitrary dimension, which we will need in discussing aspects of the mini-twistor geometry in section VII.6.

§37 Integrable distribution. Let M be a smooth manifold of real dimension d and $T_\mathbb{C}M$ its complexified tangent bundle. A subbundle $\mathcal{T} \subset T_\mathbb{C}M$ is said to be *integrable* if

(i) $\mathcal{T} \cap \bar{\mathcal{T}}$ has constant rank k,

(ii) $\mathcal{T}$ and[9] $\mathcal{T} \cap \bar{\mathcal{T}}$ are closed under the Lie bracket.

Given an integrable distribution $\mathcal{T}$, we can choose local coordinates $u^1, \dots, u^l, v^1, \dots, v^k$, $x^1, \dots, x^m, y^1, \dots, y^m$ on any patch U of the covering of M such that $\mathcal{T}$ is locally spanned by the vector fields

$$\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^k}, \frac{\partial}{\partial \bar{w}^1}, \dots, \frac{\partial}{\partial \bar{w}^m} \; , \tag{II.47}$$

where $\bar{w}^1 = x^1 - \mathrm{i}y^1, \dots, \bar{w}^m = x^m - \mathrm{i}y^m$ [208].

[9]We use the same letter for the bundle $\mathcal{T}$ and a distribution generated by its sections.

§38 $\mathcal{T}$-differential. For any smooth function f on M, let $\mathrm{d}_\mathcal{T} f$ denote the restriction of $\mathrm{d}f$ to $\mathcal{T}$, i.e. $\mathrm{d}_\mathcal{T}$ is the composition

$$C^\infty(M) \stackrel{\mathrm{d}}{\longrightarrow} \Omega^1(M) \longrightarrow \Gamma(M, \mathcal{T}^*) , \tag{II.48}$$

where $\Omega^1(M) := \Gamma(M, T^*M)$ and $\mathcal{T}^*$ denotes the sheaf of (smooth) one-forms dual to $\mathcal{T}$ [237]. The operator $\mathrm{d}_\mathcal{T}$ can be extended to act on relative q-forms from the space $\Omega^q_\mathcal{T}(M) := \Gamma(M, \Lambda^q \mathcal{T}^*)$,

$$\mathrm{d}_\mathcal{T} \ : \ \Omega^q_\mathcal{T}(M) \ \to \ \Omega^{q+1}_\mathcal{T}(M) \ , \quad \text{for} \quad q \ \geq \ 0 \ . \tag{II.49}$$

This operator is called a *$\mathcal{T}$-differential*.

§39 $\mathcal{T}$-connection. Let E be a smooth complex vector bundle over M. A covariant differential (or connection) on E along the distribution $\mathcal{T}$ – a *$\mathcal{T}$-connection* [237] – is a $\mathbb{C}$-linear mapping

$$\nabla_\mathcal{T} \ : \ \Gamma(M, E) \ \to \ \Gamma(M, \mathcal{T}^* \otimes E) \tag{II.50}$$

satisfying the Leibniz formula

$$\nabla_\mathcal{T}(f\sigma) \ = \ f \nabla_\mathcal{T} \sigma + \mathrm{d}_\mathcal{T} f \otimes \sigma \ , \tag{II.51}$$

for a local section $\sigma \in \Gamma(M, E)$ and a local smooth function f. This $\mathcal{T}$-connection extends to a map

$$\nabla_\mathcal{T} \ : \ \Omega^q_\mathcal{T}(M, E) \ \to \ \Omega^{q+1}_\mathcal{T}(M, E) \ , \tag{II.52}$$

where $\Omega^q_\mathcal{T}(M, E) := \Gamma(M, \Lambda^q \mathcal{T}^* \otimes E)$. Locally, $\nabla_\mathcal{T}$ has the form

$$\nabla_\mathcal{T} \ = \ \mathrm{d}_\mathcal{T} + A_\mathcal{T} \ , \tag{II.53}$$

where the standard $\mathsf{End}E$-valued $\mathcal{T}$-connection one-form $A_\mathcal{T}$ has components only along the distribution $\mathcal{T}$.

§40 $\mathcal{T}$-flat vector bundles. As usual, $\nabla^2_\mathcal{T}$ naturally induces a relative 2-form

$$\mathcal{F}_\mathcal{T} \in \Gamma(M, \Lambda^2 \mathcal{T}^* \otimes \mathsf{End}E) \tag{II.54}$$

which is the curvature of $A_\mathcal{T}$. We say that $\nabla_\mathcal{T}$ (or $A_\mathcal{T}$) is flat if $\mathcal{F}_\mathcal{T} = 0$. For a flat $\nabla_\mathcal{T}$, the pair $(E, \nabla_\mathcal{T})$ is called a $\mathcal{T}$-flat vector bundle [237].

Note that the complete machinery of Dolbeault and Čech descriptions of vector bundles naturally generalizes to $\mathcal{T}$-flat vector bundles. Consider a manifold M covered by the patches $\mathfrak{U} := \{U_{(a)}\}$ and a topologically trivial vector bundle $(E, f_{+-} = \mathbb{1}, \nabla_\mathcal{T})$ over M, with an expression

$$A_\mathcal{T}|_{U_{(a)}} \ = \ \psi_a \mathrm{d}_\mathcal{T} \psi_a^{-1} \tag{II.55}$$

of the flat $\mathcal{T}$-connection, where the ψ_a are smooth $\mathsf{GL}(n, \mathbb{C})$-valued superfunctions on every patch $U_{(a)}$, we deduce from the triviality of E that $\psi_a \mathrm{d}_\mathcal{T} \psi_a^{-1} = \psi_b \mathrm{d}_\mathcal{T} \psi_b^{-1}$ on the intersections $U_{(a)} \cap U_{(b)}$. Therefore, it is $\mathrm{d}_\mathcal{T}(\psi_+^{-1}\psi_-) = 0$ and we can define a $\mathcal{T}$-flat complex vector bundle $\tilde{E}$ with the canonical flat $\mathcal{T}$-connection $\mathrm{d}_\mathcal{T}$ and the transition function $\tilde{f}_{ab} := \psi_a^{-1}\psi_b$. The bundles E and $\tilde{E}$ are equivalent as smooth bundles but not as $\mathcal{T}$-flat bundles. However, we have an equivalence of the following data:

$$(E, f_{+-} = \mathbb{1}_n, A_\mathcal{T}) \ \sim \ (\tilde{E}, \tilde{f}_{+-}, \tilde{A}_\mathcal{T} = 0) \ , \tag{II.56}$$

similarly to the holomorphic vector bundles discussed in the previous section.

§41 Cauchy-Riemann structures. A *Cauchy-Riemann structure* on a smooth manifold M of real dimension d is an integrable distribution, which is a complex subbundle $\bar{\mathscr{D}}$ of rank m of the complexified tangent bundle $T_{\mathbb{C}}M$. The pair $(M, \bar{\mathscr{D}})$ is called a *Cauchy-Riemann manifold of dimension* $d = \dim_{\mathbb{R}} M$*, of rank* $m = \dim_{\mathbb{C}} \bar{\mathscr{D}}$ *and of codimension* $d - 2m$. In particular, a Cauchy-Riemann structure on M in the special case $d = 2m$ is a complex structure on M. Thus, the notion of Cauchy-Riemann manifolds generalizes the one of complex manifolds. Furthermore, given a vector bundle E over M, the pair $(E, \nabla_{\bar{\mathscr{D}}})$, where $\nabla_{\bar{\mathscr{D}}}$ is a $\bar{\mathscr{D}}$-connection, is a *Cauchy-Riemann vector bundle.*

II.3 Calabi-Yau manifolds

Calabi-Yau manifolds are compact d-dimensional Kähler manifolds with holonomy group $\mathsf{SU}(d)$. E. Calabi conjectured in 1954 that such manifolds should admit a Ricci-flat metric in every Kähler class. In 1971, this conjecture was proven by S. T. Yau.

II.3.1 Definition and Yau's theorem

§1 Calabi-Yau manifolds. A *local Calabi-Yau manifold* is a complex Kähler manifold with vanishing first Chern class. A *Calabi-Yau manifold* is a compact local Calabi-Yau manifold.

The notion of a local Calabi-Yau manifold stems from physicists and using it has essentially two advantages: First, one can consider sources of fluxes on these spaces without worrying about the corresponding "drains". Second, one can easily write down metrics on many local Calabi-Yau manifolds, as e.g. on the conifold [214]. We will sometimes drop the word "local" if the context determines the situation.

§2 Theorem. (Yau) Yau has proven that for every complex Kähler manifold M with vanishing first Chern class $c_1 = 0$ and Kähler form J, there exists a unique Ricci-flat metric on M in the same Kähler class as J.

This theorem is particularly useful, as it links the relatively easily accessible first Chern class to the existence of a Ricci-flat metric. The latter property is hard to check explicitly in most cases, in particular, because no Ricci-flat metric is known on any (compact) Calabi-Yau manifold. Contrary to that, the first Chern class is easily calculated, and we will check the Calabi-Yau property of our manifolds in this way.

§3 Holonomy of a Calabi-Yau manifold. Ricci-flatness of a d-dimensional complex manifold M implies the vanishing of the trace part of the Levi-Civita connection, which in turn restricts the holonomy group of M to $\mathsf{SU}(d)$. In fact, having holonomy group $\mathsf{SU}(d)$ is equivalent for a d-dimensional compact complex manifold to being Calabi-Yau. For such manifolds with holonomy group $\mathsf{SU}(d)$, it can furthermore be shown that $h^{0,d} = h^{d,0} = 1$ and $h^{0,i} = h^{i,0} = 0$ for $1 < i < d$. The nontrivial element of $H^{d,0}(M)$ defines the *holomorphic volume form* $\Omega^{d,0}$, one of the key properties of a Calabi-Yau manifold, which we will exploit to define the action of holomorphic Chern-Simons theory, see section IV.3.2. Arranging the Hodge numbers similarly as in Pascal's triangle, one obtains the

Hodge diamond, which looks, e.g. for $d = 2$ as

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{II.57}$$

§4 Equivalent definitions of Calabi-Yau manifolds. Let us summarize all equivalent conditions on a compact complex manifold M of dimension n for being a Calabi-Yau manifold:

- ▷ M is a Kähler manifold with vanishing first Chern class.
- ▷ M admits a Levi-Civita connection with $\mathsf{SU}(n)$ holonomy.
- ▷ M admits a nowhere vanishing holomorphic $(n,0)$-form $\Omega^{n,0}$.
- ▷ M admits a Ricci-flat Kähler metric.
- ▷ M has a trivial canonical bundle.

§5 Deformations of Calabi-Yau manifolds. Let us briefly comment on the moduli space parameterizing deformations of Calabi-Yau manifolds, which preserve Ricci flatness. For a more general discussion of deformation theory, see section II.4.

Consider a Calabi-Yau manifold M with Ricci-flat metric g in the Kähler class J. Deformations of the metric consist of pure index type ones and such of mixed type ones $\delta g = \delta g_{ij} \mathrm{d}z^i \mathrm{d}z^j + \delta g_{i\bar{\jmath}} \mathrm{d}z^i \mathrm{d}z^{\bar{\jmath}} + c.c.$. The deformation of mixed type are given by elements of $H^{1,1}(M)$ and are associated to deformations of the Kähler class J which – roughly speaking – determines the size of the Calabi-Yau manifold. Deformations of pure type are associated with elements of $H^{1,2}(M)$ and demand a redefinition of coordinates to yield a Hermitian metric. Therefore the complex structure is deformed, which determines the shape of the Calabi-Yau manifold.

§6 Comments on the moduli spaces. Above we saw that the moduli space of deformations of a Calabi-Yau manifold apparently decomposes into a Kähler moduli space and a complex structure moduli space. In fact, the situation is unfortunately more subtle, and we want to briefly comment on this.

Given a Kähler form $J = \mathrm{i} g_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}$ on a Calabi-Yau manifold M, there is the positivity constraint for volumes $\int_{S_r} J^{\wedge r} > 0$ for submanifolds $S_r \subset M$ with complex dimension r. For any allowed Kähler structure J, sJ is also allowed for $s \in \mathbb{R}^{>0}$. Thus, the moduli space of Kähler forms is a cone. Nevertheless, it is well known from string-theoretic arguments that all elements of $H^2(M, \mathbb{R})$ should be admitted. The solution is, to allow neighboring Kähler cones to exist, sharing a common wall and interpreting them as belonging to a Calabi-Yau manifold with different topology, which solves the positivity problem. Passing through the wall of a Kähler cone changes the topology but preserves the Hodge numbers.

The complex structure moduli space has similar singularities: If the Calabi-Yau manifold is defined by a homogeneous polynomial P in a projective space, points z_0 where $P(z_0) = \partial_{z^i} P(z)|_{z=z_0} = 0$ are called the discriminant locus in the moduli space. The Calabi-Yau manifold fails to be a complex manifold there, as the tangent space is not well defined, but collapses to a point. Note, however, that in string theory, such geometric transitions do not cause any problems.

§7 K3 manifolds. A *K3 manifold* is a complex Kähler manifold M of complex dimension 2 with $\mathsf{SU}(2)$ holonomy and thus it is a Calabi-Yau manifold. All K3 manifolds can be shown to be smoothly equivalent. They have Euler number $\chi(M) = 24$ and Pontryagin classes $p_q(M) = 1$. Their only nontrivial Hodge number (i.e. the Hodge number not fixed in the Hodge diamond by the Calabi-Yau property) is $h^{1,1} = 20$. K3 manifolds play an important rôle in string theory compactifications. The K3 manifold's name stems from the three mathematicians Kummer, Kähler and Kodaira who named it in the 1950s shortly after the K2 mountain was climbed for the first time.

§8 Rigid Calabi-Yau manifolds. There is a class of so-called *rigid Calabi-Yau manifolds*, which do not allow for deformations of the complex structure. This fact causes problems for the mirror conjecture, see section V.3.5, as it follows that the mirrors of these rigid Calabi-Yau manifolds have no Kähler moduli, which is inconsistent with them being Kähler manifolds.

II.3.2 Calabi-Yau 3-folds

Calabi-Yau 3-folds play a central rôle in the context of string compactification, see section V.2.2, §13. A ten-dimensional string theory is usually split into a four-dimensional theory and a six-dimensional $\mathcal{N} = 2$ superconformal theory. For a theory to preserve $\mathcal{N} = 2$ supersymmetry, the manifold has to be Kähler, conformal invariance demands Ricci-flatness. Altogether, the six-dimensional theory has to live on a Calabi-Yau 3-fold.

§9 Triple intersection form. On a Calabi-Yau 3-fold M, one can define a topological invariant called the *triple intersection form*:

$$I^{1,1} : \left(H^{1,1}_{\bar\partial}(M)\right)^{\wedge 3} \rightarrow \mathbb{R}, \quad I^{1,1}(A, B, C) := \int_M A \wedge B \wedge C \ . \tag{II.58}$$

§10 Calabi-Yau manifolds in weighted projective spaces. Calabi-Yau manifolds can be described by the zero locus of polynomials in weighted projective spaces, which is the foundation of toric geometry. For example, a well-known group of Calabi-Yau manifolds are the *quintics* in $\mathbb{C}P^4$ defined by a homogeneous[10] quintic polynomial $q(z_0, \ldots z_4)$:

$$M_q = \{(z_0, \ldots, z_4) \in \mathbb{C}P^4 : q(z_0, \ldots, z_4) = 0\} \ . \tag{II.59}$$

Another example is the complete intersection of two cubics in $\mathbb{C}P^5$:

$$M_c = \{(z_0, \ldots, z_4) \in \mathbb{C}P^5 : c_1(z_0, \ldots, z_5) = c_2(z_0, \ldots, z_5) = 0\} \ , \tag{II.60}$$

where c_1 and c_2 are homogeneous cubic polynomials.

§11 Calabi-Yau manifolds from vector bundles over $\mathbb{C}P^1$. A very prominent class of local Calabi-Yau manifolds can be obtained from the vector bundles $\mathcal{O}(a) \oplus \mathcal{O}(b) \rightarrow \mathbb{C}P^1$, where the Calabi-Yau condition of vanishing first Chern class amounts to $a+b = -2$.

To describe these bundles, we will always choose the standard inhomogeneous coordinates $\lambda_\pm$ on the patches $U_\pm$ covering the base $\mathbb{C}P^1$, together with the coordinates $z^1_\pm$ and $z^2_\pm$ in the fibres over the patches $U_\pm$. The transition functions on the overlap are implicitly given by

$$z^1_+ = \lambda^a_+ z^1_- \ , \quad z^2_+ = \lambda^b_+ z^2_- \ , \quad \lambda_+ = \frac{1}{\lambda_-} \ . \tag{II.61}$$

[10] Homogeneity follows from the fact that $p(\lambda z_0, \ldots, \lambda z_5)$ has to vanish for all λ.

The holomorphic volume form on these spaces, whose existence is granted by vanishing of the first Chern class, can be defined to be $\Omega^{3,0}_{\pm} = \pm \mathrm{d}z^1_{\pm} \wedge \mathrm{d}z^2_{\pm} \wedge \mathrm{d}\lambda_{\pm}$.

In more physical terms, this setup corresponds to a (β,γ)-system of weight $a/2$ (and $b/2$), where the two bosonic fields describe the sections of the $\mathcal{O}(a) \oplus \mathcal{O}(b)$ vector bundle over $\mathbb{C}P^1$.

One of the most common examples is the bundle $\mathcal{O}(0) \oplus \mathcal{O}(2) \to \mathbb{C}P^1$, which is, e.g., the starting point in the discussion of Dijkgraaf and Vafa relating matrix model computations to effective superpotential terms in supersymmetric gauge theories [78]. Switching to the coordinates $x = z^1_+$, $u = 2z^2_-$, $v = 2z^2_+$, $y = 2\lambda z^2_-$, we can describe the above Calabi-Yau as $\mathbb{C} \times A_1$, where the A_1 singularity is given by[11] $uv - y^2 = 0$. Note that A_1 is a local K3 manifold.

Another example is the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$, which we will discuss in the following section. Note that the (projective) twistor space of $\mathbb{C}^4$, $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$, is not a Calabi-Yau manifold, however, it can be extended by fermionic coordinates to a Calabi-Yau supermanifold, see section VII.4.

II.3.3 The conifold

§12 The conifold. The *conifold* is the algebraic variety $\mathscr{C}$ defined by the equation

$$f(w) \;=\; w_1^2 + w_2^2 + w_3^2 + w_4^2 \;=\; 0 \tag{II.62}$$

in $\mathbb{C}^4$ as a complex three-dimensional subspace of codimension 1. One immediately notes that in the point $\vec{w} = (w_i) = 0$, the tangent space of $\mathscr{C}$ collapses to a point which is indicated by a simultaneous vanishing of the defining equation $f(w)$ and all of its derivatives. Such points on a algebraic variety are called *double points* and are points at which varieties fail to be smooth. Thus, $\mathscr{C}$ is only a manifold for $w \neq 0$. As we will see later on, there are two possible ways of repairing this singularity: a resolution and a deformation (or smoothing). Away from the origin, the shape is most efficiently determined by intersecting $\mathscr{C}$ with a seven sphere in $\mathbb{R}^8$:

$$|w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 \;=\; r^2 \;. \tag{II.63}$$

Splitting w_i in real and imaginary components x_i and y_i, one obtains the equations $\vec{x}^2 - 2\mathrm{i}\vec{x}\cdot\vec{y} - \vec{y}^{\,2} = 0$ for the conifold and obviously $\vec{x}^2 + \vec{y}^{\,2} = r^2$ for the sphere. The intersection is given by the three equations

$$\vec{x}\cdot\vec{y} \;=\; 0, \quad \vec{x}^2 \;=\; \vec{y}^{\,2} \;=\; \frac{r^2}{2} \;. \tag{II.64}$$

The last two equations define 3-spheres, the first one reduces one 3-sphere to a 2-sphere, since fixing $\vec{x}$ requires $\vec{y}$ to be orthogonal, leaving an S^2. The radius of the spheres is $r/\sqrt{2}$, so that the conifold is indeed a cone over the base $B = S^2 \times S^3$. This base space is also known as the space $T^{1,1}$,

$$B \;=\; T^{1,1} \;=\; \mathsf{SU}(2) \times \frac{\mathsf{SU}(2)}{\mathsf{U}(1)} \;\cong\; \frac{\mathsf{SO}(4)}{\mathsf{U}(1)} \;. \tag{II.65}$$

By changing coordinates to $z_{1,3} = w_3 \pm \mathrm{i}w_4$ and $z_{2,4} = \mathrm{i}w_1 \mp w_2$, one obtains another defining equation for $\mathscr{C}$:

$$z_1 z_3 - z_2 z_4 \;=\; 0 \;. \tag{II.66}$$

[11] In general, A_k is the space $\mathbb{C}^2/\mathbb{Z}_{k+1}$.

This equation leads to a definition using the determinant of a matrix, which will be quite useful later on:

$$\mathcal{W} \;=\; \begin{pmatrix} z_1 & z_2 \\ z_4 & z_3 \end{pmatrix}, \qquad \det \mathcal{W} \;=\; 0 \;. \tag{II.67}$$

From this matrix, one can introduce the radial coordinate of the conifold by

$$r = \operatorname{tr}(\mathcal{W}^\dagger \mathcal{W}) \in \mathbb{R} \tag{II.68}$$

which parameterizes the distance from the origin in $\mathbb{C}^4$: $\vec{x}^2 + \vec{y}^2 = r^2$. We see that the geometry is invariant under $r \to \lambda r$, so that ∂_r is a Killing vector.

Let the angles of the S^3 be denoted by (θ_1, ϕ_1, ψ) and the ones of the S^2 by (θ_2, ϕ_2). Then by taking ψ and combining it with the radius, to $v = r\mathrm{e}^{\mathrm{i}\psi}$, one gets a complex cone over $\mathbb{C}P^1 \times \mathbb{C}P^1$.

§13 The deformed conifold. The deformation $\mathscr{C}_{\mathrm{def}}$ of the conifold is obtained by deforming the defining equation (II.62) to

$$f \;=\; w_1^2 + w_2^2 + w_3^2 + w_4^2 \;=\; z_1 z_3 - z_2 z_4 \;=\; \varepsilon \;. \tag{II.69}$$

Due to $\vec{x}^2 - \vec{y}^2 = \varepsilon$, the range of the radial coordinate $r^2 = \vec{x}^2 + \vec{y}^2$ is $\varepsilon \leq r < \infty$. Thus, the tip of the conifold was pushed away from the origin to the point $r^2 = \varepsilon^2$, which corresponds to $\vec{x}^2 = \varepsilon^2$, $\vec{y}^2 = 0$. The deformed conifold can be identified with T^*S^3. In the case of the singular conifold, the base $S^3 \times S^2$ completely shrank to a point. Here, we note that the S^3 at the tip, given by $\vec{x}^2 = \varepsilon$, has finite radius $r = \varepsilon$ and only the S^2 given by $\vec{x} \cdot \vec{y} = 0$, $\vec{y}^2 = 0$ shrinks to a point. This is depicted in figure II.1.

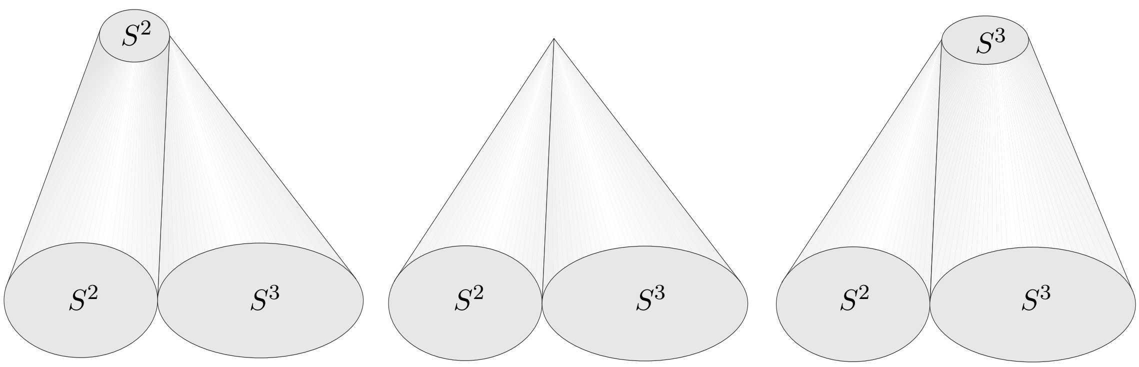

Figure II.1. *The resolved, the singular and the deformed conifolds.*

§14 The resolved conifold. The *resolved conifold* $\mathscr{C}_{\mathrm{res}}$ is defined by replacing the defining equation of the conifold $\det \mathcal{W} = 0$ with

$$\mathcal{W}\begin{pmatrix} \lambda_{\dot 1} \\ \lambda_{\dot 2} \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_4 & z_3 \end{pmatrix}\begin{pmatrix} \lambda_{\dot 1} \\ \lambda_{\dot 2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for} \;\; (\lambda_{\dot\alpha}) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{II.70}$$

Here, $(\lambda_{\dot\alpha}) \neq 0$ is a homogeneous coordinate on the Riemann sphere $\mathbb{C}P^1 \cong S^2$. Switching to the inhomogeneous coordinates $\lambda_+ := \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}}$ and $\lambda_- := \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}}$, we note that solutions to (II.70) are of the form

$$\mathcal{W} \;=\; \begin{pmatrix} -z_2\lambda_+ & z_2 \\ -z_3\lambda_+ & z_3 \end{pmatrix} \;=\; \begin{pmatrix} z_1 & z_1\lambda_- \\ z_4 & z_4\lambda_- \end{pmatrix} , \tag{II.71}$$

and thus the coordinates (z_2, z_3, λ_+) and (z_1, z_4, λ_-) describe $\mathscr{C}_{\rm res}$ on two patches $U_\pm$ with transition functions $z_1 = -\lambda_+ z_2$ and $z_4 = -\lambda_+ z_3$. Up to a sign, which can easily be absorbed by a redefinition of the coordinates, this is the rank two vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$. Contrary to the case of the deformed conifold, the S^3 at the tip vanishes while the S^2 keeps its finite size.

§15 Metric on the conifold. Recall that the metric for a *real* d-dimensional cone takes the form

$$g_{mn} \mathrm{d}x^m \mathrm{d}x^n \;=\; \mathrm{d}\rho^2 + \rho^2 h_{ij} \mathrm{d}x^i \mathrm{d}x^j , \qquad \text{(II.72)}$$

where h_{ij} is the metric on the $(d-1)$-dimensional base space. If this base is not the space S^{d-1}, there is a singularity at $\rho = 0$. In the case of the conifold, the base manifold is $S^2 \times S^3$, and thus the singularity at $\rho = 0$ is the one already present in the discussion above. A detailed discussion of the explicit form of the natural metric on the conifold is found in [214] and the references therein.

§16 The conifold transition. The transition from a deformed conifold through a singular conifold to a resolved conifold is an allowed process in string theory which amounts to a topology change. An application of this transition is found in the famous large N duality in [276]: In type IIA string theory compactified on the deformed conifold, i.e. on T^*S^3, wrapping N $D6$-branes around the S^3 produces $U(N)$ Yang-Mills theory in the remaining four noncompact directions filling spacetime. In the large N limit, this is equivalent to type IIA string theory on the small resolution, i.e. on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$. The inverse process is found in the mirror picture of this situation: N $D5$-branes wrapped around the sphere of the small resolution give rise to $U(N)$ Yang-Mills theory, the large N limit corresponds to type IIB string theory compactified on the deformed conifold T^*S^3.

II.4 Deformation theory

Deformation theory is an important tool in twistor theory as well as in the Kodaira-Spencer theory of gravity [32], the closed string theory corresponding to the topological B-model. In the former theory, one considers deformations of a $\mathbb{C}P^1$ which is holomorphically embedded in an open subspace of $\mathbb{C}P^3$. These deformations are called *relative deformations*. The Kodaira-Spencer theory of gravity, on the other hand, is a theory which describes the deformation of the total complex structure of a Calabi-Yau manifold as a result of closed string interactions.

II.4.1 Deformation of compact complex manifolds

§1 Deformation of complex structures. Consider a complex manifold M covered by patches U_a on which there are coordinates $z_a = (z_a^i)$ together with transition functions f_{ab} on nonempty intersections $U_a \cap U_b \neq \varnothing$ satisfying the compatibility condition (cf. section II.2.3, §34) $f_{ac} = f_{ab} \circ f_{bc}$ on all $U_a \cap U_b \cap U_c \neq \varnothing$. A *deformation of the complex structure* is obtained by making the coordinates z_a and the transition functions f_{ab} depend on an additional set of parameters $t = (t^1, t^2, \ldots)$ such that

$$z_a(t) \;=\; f_{ab}(t, z_b(t)) \quad \text{and} \quad f_{ac}(t, z_c(t)) \;=\; f_{ab}(t, f_{bc}(t, z_c(t))) \;, \qquad \text{(II.73)}$$

and $z_a(0)$ and $f_{ab}(0)$ are the coordinates and transition functions we started from.

§2 Infinitesimal deformations. One linearizes the second equation in (II.73) by differentiating it with respect to the parameter t and considering $Z_{ac} := \frac{\partial f_{ac}}{\partial t}|_{t=0}$. This leads to the *linearized cocycle condition*

$$Z_{ac} \ = \ Z_{ab} + Z_{bc} \ , \tag{II.74}$$

and thus the vector field Z_{ac} is an element of the Čech 1-cocycles on M with values in the sheaf of germs of holomorphic vector fields $\mathfrak{h}$ (see section II.2.3, §33). Trivial deformations, on the other hand, are those satisfying $f_{ab}(t, h_b(t, z_b(t))) = h_a(t, z_a(t))$, where the h_a are holomorphic functions for fixed t, as then the manifolds for two arbitrary parameters of t are biholomorphic and thus equivalent. Infinitesimally, this amounts to $Z_{ab} = Z_a - Z_b$, where $Z_a = \frac{\partial h_a}{\partial t}$. The latter equation is the Abelian coboundary condition, and thus we conclude[12] that nontrivial infinitesimal deformations of the complex structure of a complex manifold M are given by the first Čech cohomology group $\check{H}(M, \mathfrak{h})$.

These considerations motivate the following theorem:

§3 Theorem. (Kodaira-Spencer-Nirenberg) If $\check{H}^1(M, \mathfrak{h}) = 0$, any small deformation of M is trivial. If $\check{H}^1(M, \mathfrak{h}) \neq 0$ and $\check{H}^2(M, \mathfrak{h}) = 0$ then there exists a complex manifold $\mathcal{M}$ parameterizing a family of complex structures on M such that the tangent space to $\mathcal{M}$ is isomorphic to $\check{H}^1(M, \mathfrak{h})$.

Thus, the dimension of the Čech cohomology group $\check{H}^1(M, \mathfrak{h})$ gives the number of parameters of inequivalent complex structures on M, while $\check{H}^2(M, \mathfrak{h})$ gives the *obstructions* to the construction of deformations.

Note that this theorem can also be adapted to be suited for deformations of complex vector bundles over a fixed manifold M.

§4 Beltrami differential. Given a complex manifold M with a Dolbeault operator $\bar{\partial}$, one can describe perturbations of the complex structure by adding a $T^{1,0}M$-valued (0,1)-form A, which would read in local coordinates (z^i) as

$$\tilde{\bar{\partial}} \ := \ \mathrm{d}\bar{z}^{\bar{\imath}} \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} + \mathrm{d}\bar{z}^{\bar{\imath}} A^j_{\bar{\imath}} \frac{\partial}{\partial z^j} \ . \tag{II.75}$$

For such an operator $\tilde{\bar{\partial}}$ to define a complex structure, it has to satisfy $\tilde{\bar{\partial}}^2 = 0$. One can show that this integrability condition amounts to demanding that $\check{H}^2(M, \mathfrak{h}) = 0$.

§5 Rigid manifolds. A complex manifold M with vanishing first Čech cohomology group is called *rigid*.

§6 Example. Consider the complex line bundles $\mathcal{O}(n)$ over the Riemann sphere $\mathbb{C}P^1$. The Čech cohomology group $\check{H}^0(\mathbb{C}P^1, \mathcal{O}(n))$ vanishes for $n < 0$ and amounts to global sections of $\mathcal{O}(n)$ otherwise. As $\mathfrak{h} = \mathcal{O}(2)$, we conclude by using Serre duality[13] that $\dim \check{H}^1(\mathbb{C}P^1, \mathcal{O}(2)) = \dim \check{H}^0(\mathbb{C}P^1, \mathcal{O}(-4)) = 0$ and thus $\mathcal{O}_{\mathbb{C}P^1}(n)$ is rigid.

II.4.2 Relative deformation theory

§7 Normal bundle. Given a manifold X and a submanifold $Y \subset X$, we define the *normal bundle* $\mathcal{N}$ to Y in X via the short exact sequence

$$1 \ \rightarrow \ TY \ \rightarrow \ TX|_Y \ \rightarrow \ \mathcal{N} \ \rightarrow \ 1 \ . \tag{II.76}$$

[12] being slightly sloppy

[13] The spaces $\check{H}^i(M, \mathcal{O}(E))$ and $\check{H}^{n-i}(M, \mathcal{O}(E^\vee \otimes \Lambda^{n,0}))$ are dual, where M is a compact complex manifold of dimension n, E a holomorphic vector bundle and $\Lambda^{n,0}$ a $(n, 0)$-form. One can then pair elements of these spaces and integrate over M.

Therefore, $\mathcal{N} = \frac{TX|_Y}{TY}$. This space can roughly be seen as the local orthogonal complement to Y in X. For complex manifolds, it is understood that one considers the holomorphic tangent spaces $T^{1,0}X$ and $T^{1,0}Y$.

§8 Infinitesimal motions. Deformations of Y in X will obviously be described by elements of $\check{H}^0(Y, \mathcal{O}(\mathcal{N}))$ at the infinitesimal level, but let us be more explicit.

Assume that Y is covered by patches U_a with coordinates $z_a(t)$ and transition functions $f_{ab}(t, z_b(t))$ satisfying the linearized cocycle condition (II.74). Furthermore, let $h_a(t, z_a(t))$ be a family of embeddings of U_a into X, which is holomorphic for each t in the coordinates $z_a(t)$ and satisfies the conditions

$$h_a(0, z_a(0)) = \text{id} \quad \text{and} \quad h_b(t, z_b(t)) = h_a(t, f_{ab}(t, z_b(t))) \quad \text{on} \quad U_a \cap U_b \ . \tag{II.77}$$

One can again linearize the latter condition and consider the vectors only modulo tangent vectors to Y (which would correspond to moving Y tangent to itself in X, leaving Y invariant). One obtains $\frac{\partial h_a}{\partial t} = \frac{\partial h_b}{\partial t}$ and therefore these vector fields define global sections of the normal bundle.

Obstructions to these deformations can be analyzed by considering the second order expansion of (II.77), which leads to the condition that the first Čech cohomology group $\check{H}^1(Y, \mathcal{O}(\mathcal{N}))$ must be trivial.

Altogether, we can state that

§9 Theorem. (Kodaira) If $\check{H}^1(Y, \mathcal{O}(\mathcal{N})) = 0$ then there exists a $d = \dim_{\mathbb{C}} \check{H}^0(Y, \mathcal{O}(\mathcal{N}))$ parameter family of deformations of Y inside X.

§10 Examples. Consider a projective line $Y = \mathbb{C}P^1$ embedded in the complex projective space $\mathbb{C}P^3$ and let X be a neighborhood of Y in $\mathbb{C}P^3$. The normal bundle is just $\mathcal{N} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and we have furthermore

$$\check{H}^0(Y, \mathcal{O}(\mathcal{N})) \cong \check{H}^0(Y, \mathcal{O}(1)) \oplus \check{H}^0(Y, \mathcal{O}(1)) \cong \mathbb{C}^4 \quad \text{and} \quad \check{H}^1(Y, \mathcal{O}(\mathcal{N})) = 0 \ . \tag{II.78}$$

We will make extensive use of this example later in the context of the twistor correspondence.

As another example, consider the resolved conifold $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$, which we discussed above. There are no deformations of the base space $Y = \mathbb{C}P^1$ inside this vector bundle, as $\check{H}^0(Y, \mathcal{O}(\mathcal{N})) = \varnothing$, and thus Y is rigid in X. For example, if one would wrap D-branes around this $\mathbb{C}P^1$, they are fixed and cannot fluctuate.

Chapter III

Supergeometry

The intention of this chapter is to give a concise review of the geometric constructions motivated by supersymmetry and fix the relevant conventions. Furthermore, we discuss so-called exotic supermanifolds, which are supermanifolds with additional even nilpotent directions, reporting some novel results.

Physicists use the prefix "super" to denote objects which come with a $\mathbb{Z}_2$-grading. With this grading, each superobject can be decomposed into an *even* or *bosonic* part and an *odd* or *fermionic* part, the latter being nilquadratic. It can thus capture the properties of the two fundamental species of elementary particles: bosons (e.g. photons) and fermions (e.g. electrons).

The relevant material to this chapter is found in the following references: [8, 35, 186, 101] (general supersymmetry), [72, 27, 189, 56] (superspaces and supermanifolds), [92, 91, 244] (exotic supermanifolds and thickenings).

III.1 Supersymmetry

§1 Need for supersymmetry. The Coleman-Mandula theorem [65] states that if the S-matrix of a quantum field theory in more than 1+1 dimensions possesses a symmetry which is not a direct product with the Poincaré group, the S-matrix is trivial. The only loophole to this theorem [117] is to consider an additional $\mathbb{Z}_2$-graded symmetry which we call *supersymmetry* (SUSY). Although SUSY was introduced in the early 1970s and led to a number of aesthetically highly valuable theories, it is still unknown if it actually plays any rôle in nature. The reason for this is mainly that supersymmetry – as we have not detected any superpartners to the particle spectrum of the standard model – is broken by some yet unknown mechanism. However, there are some phenomenological hints for the existence of supersymmetry from problems in the current non-supersymmetric standard model of elementary particles. Among those are the following:

▷ The gauge couplings of the standard model seem to unify at $M_U \sim 2 \cdot 10^{16}$GeV in the minimal supersymmetric standard model (MSSM), while there is no unification in non-SUSY theories.

▷ The *hierarchy problem*, i.e. the mystery of the unnaturally big ratio of the Planck mass to the energy scale of electroweak symmetry breaking ($\sim$ 300GeV), which comes with problematic radiative corrections of the Higgs mass. In the MSSM, these corrections are absent.

▷ Dark matter paradox: the *neutralino*, one of the extra particles in the supersymmetric standard model, might help to explain the missing dark matter in the universe. This dark matter is not observed but needed for correctly explaining the dynamics in our galaxy and accounts for 25% of the total matter[1] in our universe.

[1] another 70% stem from so-called *dark energy*

Other nice features of supersymmetry are seriously less radiative corrections and the emergence of gravity if supersymmetry is promoted to a local symmetry. Furthermore, all reasonable string theories appear to be supersymmetric. Since gravity has eventually to be reconciled with the standard model (and – as mentioned in the introduction – string theories are candidates with very few competitors) this may also be considered as a hint.

There might even be a mathematical reason for considering at least supergeometry: mirror symmetry (see section V.3.5) essentially postulates that every family of Calabi-Yau manifolds comes with a mirror family, which has a rotated Hodge diamond. For some so-called "rigid" Calabi-Yau manifolds this is impossible, but there are proposals that the corresponding mirror partners might be Calabi-Yau supermanifolds [259].

Be it as it may, we will soon know more about the phenomenological value of supersymmetry from the experimental results that will be found at the new large hadron collider (LHC) at CERN.

III.1.1 The supersymmetry algebra

§2 The supersymmetry algebra. The supersymmetric extension of the Poincaré algebra on a pseudo-Euclidean four-dimensional space is given by

$$\begin{aligned}
[P_\rho, M_{\mu\nu}] &= \mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\ ,\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho})\ ,\\
[P_\mu, Q_{\alpha i}] &= 0\ , \quad [P_\mu, \bar{Q}^i_{\dot\alpha}] = 0\ ,\\
[M_{\mu\nu}, Q_{i\alpha}] &= \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}\ , \quad [M_{\mu\nu}, \bar{Q}^{i\dot\alpha}] = \mathrm{i}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{i\dot\beta}\ .\\
\{Q_{\alpha i}, \bar{Q}^j_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^j_i\ , \quad \{Q_{\alpha i}, Q_{\beta j}\} = \varepsilon_{\alpha\beta}Z_{ij}\ , \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}\bar{Z}^{ij}\ ,
\end{aligned} \tag{III.1}$$

where $\eta_{\mu\nu}$ is either the Euclidean metric, the Minkowskian one or the Kleinian one with $(\eta_{\mu\nu}) = \mathrm{diag}(-1,-1,+1,+1)$. The generators P, M, Q and $\bar{Q}$ correspond to translations, Lorentz transformations and supersymmetry transformations (translations in chiral directions in superspace), respectively. The terms $Z^{ij} = -Z^{ji}$ are allowed central extensions of the algebra, i.e. $[Z^{ij}, \cdot] = 0$. We will almost always put them to zero in the following.

The indices i, j run from 1 up to the number of supersymmetries, usually denoted by $\mathcal{N}$. In four dimensions, the indices α and $\dot\alpha$ take values $1, 2$. In particular, we have therefore $4\mathcal{N}$ supercharges $Q_{\alpha i}$ and $\bar{Q}^i_{\dot\alpha}$.

§3 Sigma matrix convention. On four-dimensional Minkowski space, we use

$$\sigma_0 := \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \quad \sigma_1 := \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$$

and thus $\sigma_\mu = (\mathbb{1}, \sigma_i)$ together with the definition $\bar\sigma_\mu = (\mathbb{1}, -\sigma_i)$, where σ_i denotes the three Pauli matrices. On Euclidean spacetime, we define

$$\sigma_1 := \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \quad \sigma_4 := \begin{pmatrix} \mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}$$

with $\sigma_\mu = (\sigma_i, \mathrm{i}\mathbb{1})$, $\bar\sigma_\mu = (-\sigma_i, -\mathrm{i}\mathbb{1})$ and on Kleinian space $\mathbb{R}^{2,2}$ we choose

$$\sigma_1 := \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \quad \sigma_4 := \begin{pmatrix} -\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}\ .$$

In the above supersymmetry algebra, we also made use of the symbols

$$\sigma^{\nu\mu}{}_\alpha{}^\beta := \tfrac{1}{4}(\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta} - \sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta}) \quad \text{and} \quad \bar\sigma^{\nu\mu\dot\alpha}{}_{\dot\beta} := \tfrac{1}{4}(\bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta} - \bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta})\ . \tag{III.2}$$

§4 Immediate consequences of supersymmetry. While every irreducible representation of the Poincaré algebra corresponds to a particle, every irreducible representation of the supersymmetry algebra corresponds to several particles, which form a supermultiplet. As P^2 commutes with all generators of the supersymmetry algebra, all particles in a supermultiplet have the same mass. Furthermore, the energy P_0 of any state is always positive, as

$$2\sigma^{\mu}_{\alpha\dot{\alpha}}\langle\psi|P_\mu|\psi\rangle \;=\; \langle\psi|\{Q_{\alpha i},\bar{Q}^i_{\dot{\alpha}}\}|\psi\rangle \;=\; ||Q_{\alpha i}|\psi\rangle||^2+||\bar{Q}^i_{\dot{\alpha}}|\psi\rangle||^2 \;\geq\; 0\;. \tag{III.3}$$

And finally, we can deduce that a supermultiplet contains an equal number of bosonic and fermionic degrees of freedom (i.e. physical states with positive norm). To see this, introduce a parity operator $\mathcal{P}$ which gives -1 and 1 on a bosonic and fermionic state, respectively. Consider then

$$2\sigma^{\mu}_{\alpha\dot{\alpha}}\,\mathrm{tr}\,(\mathcal{P}P_\mu) \;=\; \mathrm{tr}\,(\mathcal{P}\{Q_{\alpha i},\bar{Q}^i_{\dot{\alpha}}\}) \;=\; \mathrm{tr}\,(-Q_{\alpha i}\mathcal{P}\bar{Q}^i_{\dot{\alpha}}+\mathcal{P}\bar{Q}^i_{\dot{\alpha}}Q_{\alpha i}) \;=\; 0\;, \tag{III.4}$$

where we used the facts that $\mathcal{P}$ anticommutes with $Q_{\alpha i}$ and that the trace is cyclic. Any non-vanishing P_μ then proves the above statement.

III.1.2 Representations of the supersymmetry algebra

§5 Casimir operators of the supersymmetry algebra. To characterize all irreducible representations of the supersymmetry algebra, we need to know its Casimir operators. Recall that the Casimirs of the Poincaré algebra are the mass operator $P^2=P_\mu P^\mu$ with eigenvalues m^2 together with the square of the Pauli-Ljubanski vector $W_\mu=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu W^{\rho\sigma}$ with eigenvalues $-m^2s(s+1)$ for massive and $W_\mu=\lambda P_\mu$ for massless states, where s and λ are the spin and the helicity, respectively.

In the super Poincaré algebra, P^2 is still a Casimir, while W^2 has to be replaced by the *superspin operator* given by $C^2=C_{\mu\nu}C^{\mu\nu}$ with

$$C_{\mu\nu} \;=\; (W_\mu-\tfrac{1}{4}\bar{Q}^i_{\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta}_\mu Q_{i\beta})P_\nu-(W_\nu-\tfrac{1}{4}\bar{Q}^i_{\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\beta}_\nu Q_{i\beta})P_\mu\;. \tag{III.5}$$

Thus $C^2=P^2W^2-\frac{1}{4}(P\cdot W)^2$, and in the massive case, this operator has eigenvalues $-m^4s(s+1)$, where s is called the *superspin*.[2]

§6 Massless representations. First, let us consider massless representations in a frame with $P_\mu=(E,0,0,E)$ which leads to

$$\sigma^\mu P_\mu \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}\;. \tag{III.6}$$

From the relation $\{Q_{\alpha i},\bar{Q}^j_{\dot{\beta}}\}=2\sigma^\mu_{\alpha\dot{\beta}}P_\mu$ one deduces by a similar argument to (III.3) that $Q_{i1}=\bar{Q}^i_{\dot{1}}=0$. Together with the supersymmetry algebra, this also implies that the central charges Z^{ij} have to vanish. The remaining supercharges Q_{i2} and $\bar{Q}^i_{\dot{2}}$ are proportional to fermionic annihilation and creation operators, respectively. From the commutation relations of these operators with the generator of rotations $J_3=M_{12}$, one sees that they indeed lower respectively rise the helicity of a state by $\frac{1}{2}$. Choosing a highest weight state $|h\rangle$ annihilated by all the Q_{i2}s, we can built a supermultiplet by acting with the $\bar{Q}^i_{\dot{2}}$s on it. Altogether, one obtains $2^{\mathcal{N}}$ states due to the nilpotency of the $\bar{Q}^i_{\dot{2}}$s.

[2]When having additional conformal invariance, one can define the analog of a *superhelicity*.

§7 Massless supermultiplets. For $\mathcal{N}=1$, a supermultiplet consists of fields of helicity $(\lambda_0, \lambda_0+\frac{1}{2})$. Since we do not want to exceed $\lambda = 1$ for physical[3] reasons, we thus find the chiral (scalar) multiplet with helicities $(0,\frac{1}{2})$ together with its CPT conjugate $(-\frac{1}{2},0)$ and the vector (gauge) multiplet with helicities $(\frac{1}{2},1)$ and $(-1,-\frac{1}{2})$.

For $\mathcal{N}=2$, a supermultiplet consists of fields with helicities $(\lambda_0,\lambda_0+\frac{1}{2},\lambda_0+\frac{1}{2},\lambda_0+1)$, and we therefore find the vector multiplet $(0,\frac{1}{2},\frac{1}{2},1)$ and its corresponding CPT conjugate $(-1,-\frac{1}{2},-\frac{1}{2},0)$. Note that this multiplet amounts essentially to the sum of a chiral and a vector multiplet in $\mathcal{N}=1$ language. Furthermore, there is the hypermultiplet, consisting of fields with helicities $(-\frac{1}{2},0,0,\frac{1}{2})$, which can but does not necessarily have to be its own CPT adjoint.

For $\mathcal{N}=4$, there is a single multiplet $(-1,4\times-\frac{1}{2},6\times 0,4\times\frac{1}{2},1)$ with helicities not larger than one: a gauge potential (helicities ±1), four Weyl fermions and their conjugates with helicities $\pm\frac{1}{2}$ and three complex scalars with helicity 0.

§8 Massive representations. For massive representations, we choose the frame $P_\mu = (m,0,0,0)$. By an appropriate $\mathsf{U}(\mathcal{N})$ rotation of the generators, we can bring the matrix of central charges Z^{ij} to a block diagonal form $(Z^{ij}) = \mathrm{diag}(z^k)$, where the z^k are antisymmetric 2×2 matrices. Here, we assumed that $\mathcal{N}$ was even. If $\mathcal{N}$ was odd there would be an additional zero eigenvalue of the matrix (Z^{ij}). The supercharges can be rearranged to fermionic creation and annihilation operators according to

$$a^r_\alpha \;:=\; \tfrac{1}{\sqrt{2}}\left(Q^{(2r-2)+1}_\alpha + \varepsilon_{\alpha\beta}(Q^{2r}_\beta)^\dagger\right)\ ,\quad b^r_\alpha \;:=\; \tfrac{1}{\sqrt{2}}\left(Q^{(2r-2)+1}_\alpha - \varepsilon_{\alpha\beta}(Q^{2r}_\beta)^\dagger\right) \qquad \text{(III.7)}$$

with $r=1,\ldots,\frac{\mathcal{N}}{2}$, for which the only non-vanishing anticommutators are

$$\{a^r_\alpha,(a^s_\beta)^\dagger\} \;=\; (2m-q_r)\delta_{rs}\delta_{\alpha\beta}\ ,\quad \{b^r_\alpha,(b^s_\beta)^\dagger\} \;=\; (2m+q_r)\delta_{rs}\delta_{\alpha\beta}\ , \qquad \text{(III.8)}$$

where q_r is the upper right entry of z^r. The positivity of the Hilbert space requires $2m \geq |q_r|$ for all r. For values of q_r saturating the boundary, the corresponding operators a^r_α and b^r_α have to be put to zero.

Thus, we obtain $2\mathcal{N}-2k$ fermionic oscillators amounting to $2^{2(\mathcal{N}-k)}$ states, where k is the number of q_r for which $2m=|q_r|$. The multiplets for $k>0$ are called *short multiplets* or *BPS multiplets*, in the case $k=\frac{\mathcal{N}}{2}$ one calls them *ultrashort multiplets*.

III.2 Supermanifolds

For supersymmetric quantum field theories, a representation of the super Poincaré algebra on fields is needed. Such representations can be defined by using functions which depend on both commuting and anticommuting coordinates. Note that such a $\mathbb{Z}_2$-grading of the coordinates comes with a $\mathbb{Z}_2$-grading of several other objects, as e.g. derivatives, integral forms, vector fields etc.

There are basically three approaches to $\mathbb{Z}_2$-graded coordinates on spaces:

- ▷ The first one just introduces a set of Graßmann variables, which serve as formal parameters in the calculation and take the rôle of the anticommuting coordinates. This setup is the one most commonly used in physics. Deeper formalizations can be found, and we briefly present the sheaf-theoretic approach, in which a supermanifold is interpreted as an ordinary manifold with a structure sheaf enlarged to a supercommutative ring, cf. the definition of a locally ringed space in II.2.2, §26.

[3] Otherwise, our supermultiplet will necessarily contain gravitini and gravitons.

- ▷ The second one, pioneered by A. Rogers and B. S. DeWitt, allows the coordinates to take values in a Graßmann algebra. This approach, though mathematically in many ways more appealing than the first one, has serious drawbacks, as physics seems to be described in an unnatural manner.
- ▷ A unifying approach has been proposed by A. Schwarz [248] by defining all objects of supermathematics in a categorial language. This approach, however, also comes with some problematic aspects, which we will discuss later.

III.2.1 Supergeneralities

§1 $\mathbb{Z}_2$-grading. A set S is said to posses a $\mathbb{Z}_2$*-grading* if one can associate to each element $s \in S$ a number $\tilde{s} \in \{0, 1\}$, its *parity*. If there is a product structure defined on S, the product has furthermore to respect the grading, i.e.

$$s_1 \cdot s_2 \ = \ s_3 \quad \Rightarrow \quad \tilde{s}_3 \equiv \tilde{s}_1 + \tilde{s}_2 \mod 2 \ . \tag{III.9}$$

In the following, we will sometimes use a tilde over an index to refer to the grading or parity of the object naturally associated to that index. Objects with parity 0 are called *even*, those with parity 1 are called *odd*.

§2 Supervector space. A *supervector space* is a $\mathbb{Z}_2$-graded vector space. In some cases, one considers a *supervector space* as a module over a ring with nilpotent elements. Here, the multiplication with elements of the ring has to respect the grading. A supervector space of dimension $m|n$ is the span of a basis with m even elements and n odd elements.

§3 Sign rule. A heuristic sign rule which can be used as a guideline for operating with $\mathbb{Z}_2$-graded objects is the following: If in a calculation in an ordinary algebra one has to interchange two terms a and b in a monomial then in the corresponding superalgebra, one has to insert a factor of $(-1)^{\tilde{a}\tilde{b}}$.

§4 Supercommutator. The *supercommutator* is the natural generalization of the commutator for $\mathbb{Z}_2$-graded rings reflecting the above sign rule. Depending on the grading of the involved objects, it behaves as a commutator or an anticommutator:

$$[\![a, b]\!] \ := \ a \cdot b - (-1)^{\tilde{a}\tilde{b}} b \cdot a \ . \tag{III.10}$$

From this definition, we immediately conclude that

$$[\![a, b]\!] \ = \ -(-1)^{\tilde{a}\tilde{b}} [\![b, a]\!] \ . \tag{III.11}$$

Note that instead of explicitly writing commutators and anticommutators in the supersymmetry algebra (III.1), we could also have used supercommutators everywhere.

§5 Super Jacobi identity. In an associative $\mathbb{Z}_2$-graded ring A, the supercommutator satisfies the following *super Jacobi identity*:

$$[\![a, [\![b, c]\!]]\!] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} [\![b, [\![c, a]\!]]\!] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})} [\![c, [\![a, b]\!]]\!] \ = \ 0 \tag{III.12}$$

for $a, b, c \in A$, as one easily verifies by direct calculation.

§6 Supercommutative rings. A $\mathbb{Z}_2$-graded ring A is called *supercommutative* if the supercommutator $[\![a, b]\!]$ vanishes for all elements $a, b \in A$.

§7 Superalgebra. One can lift a supervector space V to a *superalgebra* by endowing it with an associative multiplication respecting the grading (i.e. $\widetilde{ab} \equiv \tilde{a} + \tilde{b} \mod 2$) and a unit $\mathbb{1}$ with $\tilde{\mathbb{1}} = 0$. If we have an additional bracket on V which satisfies the super Jacobi identity (III.12), we obtain a corresponding super Lie algebra structure.

§8 Super Poisson structure. A *super Poisson structure* is a super Lie algebra structure which satisfies the super Jacobi identity and the equations

$$\{\!\{f, gh\}\!\} \;=\; \{\!\{f, g\}\!\}h + (-1)^{\tilde{f}\tilde{g}} g\{\!\{f, h\}\!\} \quad \text{and} \quad \{\!\{fg, h\}\!\} \;=\; f\{\!\{g, h\}\!\} + (-1)^{\tilde{g}\tilde{h}}\{\!\{f, h\}\!\}g\;. \tag{III.13}$$

§9 Supermatrices. Linear transformations on a supervector space are described by *supermatrices*. Given a supervector space V of dimension $m|n$, the basis e is a tuple of m even and n odd elements of V, and we will always assume this order of basis vectors in the following. Even supermatrices are those, which preserve the parity of the basis vectors and thus have the block structure

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\;, \tag{III.14}$$

with the elements A and D being even and the elements B and C being odd. The blocks of odd supermatrices have inverse parities. Note that there are furthermore supermatrices which are not $\mathbb{Z}_2$-graded in the above scheme. They are said to have *mixed parity*. Due to their existence, the supermatrices do not form a supervector space.

§10 Supertrace. The *supertrace* of a standard supermatrix which has the form (III.14) is defined by

$$\mathrm{str}(K) \;=\; \mathrm{tr}\, A - \,\mathrm{tr}\, D\;. \tag{III.15}$$

This definition ensures that $\mathrm{str}(KL) = \mathrm{str}(LK)$, and – after a suitable definition – invariance under transposition of the matrix K. Furthermore, we have $\mathrm{str}(\mathbb{1}_{m|n}) = \mathrm{tr}\,(\mathbb{1}_m) - \,\mathrm{tr}\,(\mathbb{1}_n) = m - n$.

§11 Superdeterminant. A *superdeterminant* is easily defined by integrating the classical variational law

$$\delta \ln \det K \;=\; \mathrm{tr}\,(K^{-1}\delta K) \tag{III.16}$$

together with the boundary condition $\mathrm{sdet}(\mathbb{1}_{m|n}) = 1$. This definition yields for a matrix K of the form (III.14)

$$\mathrm{sdet}(K) \;=\; \frac{\det(A - DB^{-1}C)}{\det(B)} \;=\; \frac{\det(A)}{\det(B)\det(\mathbb{1}_m - CA^{-1}DB^{-1})}\;. \tag{III.17}$$

We will present the derivation of this result in an analogous case in §15 of section III.3.4.

Our definition preserves in particular the product rule for ordinary determinants, i.e. we have $\mathrm{sdet}(KL) = \mathrm{sdet}(K)\mathrm{sdet}(L)$. The superdeterminant $\mathrm{sdet}(\cdot)$ is also called the *Berezinian*.

§12 Almost nilpotent algebra. An *almost nilpotent algebra* is an associative, finite-dimensional, unital, $\mathbb{Z}_2$-graded supercommutative algebra in which the ideal of nilpotent elements has codimension 1.

III.2.2 Graßmann variables

§13 Graßmann variables. Define a set of formal variables $\lambda := \{\theta^i\}$ which satisfy the algebra

$$\{\theta^i, \theta^j\} \;=\; \theta^i\theta^j + \theta^j\theta^i \;:=\; 0\;. \tag{III.18}$$

The elements of this set are called *Graßmann variables*. Trivial consequences of the algebra are their anticommutativity: $\theta^i\theta^j = -\theta^j\theta^i$ and their nilquadraticy: $(\theta^i)^2 = 0$. The parity of a Graßmann variables is odd: $\widetilde{\theta^i} = 1$.

§14 Graßmann algebras. The algebra generated by a set of $N \in \mathbb{N} \cup \{\infty\}$ Graßmann variables over $\mathbb{C}$ or $\mathbb{R}$ is called the *Graßmann algebra* Λ_N. A Graßmann algebra is a $\mathbb{Z}_2$-graded algebra, and thus every element $z \in \Lambda_N$ can be decomposed into an even part $z_0 \in \Lambda_0$ with $\tilde{z}_0 = 0$ and an odd part $z_1 \in \Lambda_1$ with $\tilde{z}_1 = 1$ as well as into a *body* $z_B \in \Lambda_B := \Lambda_N \cap \mathbb{C}$ and a *soul* $z_S \in \Lambda_S := \Lambda_N \backslash \mathbb{C}$. Note that the soul is nilpotent, and an element z of the Graßmann algebra Λ_N has the multiplicative inverse $z^{-1} = \frac{1}{z_B} \sum_{i=0}^{N} (-\frac{z_S}{z_B})^i$ if and only if the body is non-vanishing. Elements of a Graßmann algebra are also called *supernumbers*.

Note that a Graßmann algebra is an almost nilpotent algebra.

§15 Derivatives with respect to Graßmann variables. Recall that a derivative is a linear map which annihilates constants and satisfies a Leibniz rule. For Graßmann variables, one easily finds that the most appropriate definition of a derivative is

$$\frac{\partial}{\partial \theta^i}(a + \theta^i b) \;:=\; b \;, \tag{III.19}$$

where a and b are arbitrary constants in θ^i. Due to the nilpotency of Graßmann variables, this definition fixes the derivative completely, and it gives rise to the following super Leibniz rule:

$$\frac{\partial}{\partial \theta^i}(ab) \;=\; \left(\frac{\partial}{\partial \theta^i} a\right) b + (-1)^{\tilde{a}} a \left(\frac{\partial}{\partial \theta^i} b\right) \;. \tag{III.20}$$

Note that in our conventions, all the derivatives with respect to Graßmann variables act from the left.

§16 Integration over Graßmann variables. The corresponding rule for an integration $\int \mathrm{d}\theta^i$ is fixed by demanding that $\int \mathrm{d}\theta^i$ is a linear functional and that[4] $\frac{\partial}{\partial \theta^i} \int \mathrm{d}\theta^i f = \int \mathrm{d}\theta^i \frac{\partial}{\partial \theta^i} f = 0$, where f is an arbitrary function of θ^i. The latter condition is the foundation of integration by parts and Stokes' formula. Thus we have to define

$$\int \mathrm{d}\theta^i \; (a + \theta^i b) \;:=\; b \;, \tag{III.21}$$

and integration over a Graßmann variable is equivalent to differentiating with respect to it. This integration prescription was first introduced by F. A. Berezin, one of the pioneers of Graßmann calculus, and is therefore called *Berezin integration.*

When performing a change of coordinates, the Jacobian is replaced by the Berezinian, i.e. the usual determinant is replaced by the superdeterminant, and we will encounter several examples for this later on.

§17 Complex conjugation of Graßmann variables. After defining a complex conjugation on $* : \lambda \to \lambda$, we call the elements of λ *complex* Graßmann variables and those elements $\xi \in \mathrm{span}(\lambda) = \Lambda$ for which $\xi^* = \xi$ *real.*

We will have to introduce different explicit antilinear involutions defining reality conditions for Graßmann variables in our discussion later on. However, we can already fix two conventions: First, our reality conditions will always be compatible with

$$\overline{\frac{\partial}{\partial \theta}} \;=\; \frac{\partial}{\partial \bar{\theta}} \;. \tag{III.22}$$

Furthermore, we adopt the following convention for the conjugation of products of Graßmann variables and supernumbers in general:

$$\tau(\theta^1 \theta^2) \;=\; \tau(\theta^2)\tau(\theta^1) \quad \text{and} \quad \tau(z^1 z^2) \;=\; \tau(z^2)\tau(z^1) \;. \tag{III.23}$$

[4] no sum over i implied

This choice is almost dictated by the fact that we need the relation $(AB)^\dagger = B^\dagger A^\dagger$ for matrix-valued superfunctions. A slight drawback here is that the product of two real objects will be imaginary. This is furthermore the most common convention used for supersymmetry in Minkowski space, and the difference to the convention $\tau(\theta^1\theta^2) = \tau(\theta^1)\tau(\theta^2)$ is just a factor of i in the Graßmann generators. A more detailed discussion can be found in [56].

III.2.3 Superspaces

§18 Superspace from an enlarged structure sheaf. A *superspace* is a pair $(M, \mathcal{O}_M)$, where M is a topological space and $\mathcal{O}_M$ is a sheaf of supercommutative rings such that the stalk $\mathcal{O}_{M,x}$ at any point $x \in M$ is a local ring. Thus, $(M, \mathcal{O}_M)$ is a locally ringed space with a structure sheaf which is supercommutative.

§19 Superspace according to A. Schwarz. We will describe the categorial approach of A. Schwarz in more detail in section III.3.1.

§20 The space $\mathbb{R}^{m|n}$. Given a set of n Graßmann variables $\{\theta^i\}$, the space $\mathbb{R}^{0|n}$ is the set of points denoted by the formal coordinates θ^i. The space $\mathbb{R}^{m|n}$ is the cartesian product $\mathbb{R}^m \times \mathbb{R}^{0|n}$, and we say that $\mathbb{R}^{m|n}$ is of dimension $m|n$. This construction straightforwardly generalizes to the complex case $\mathbb{C}^{m|n}$. Besides being the simplest superspace, $\mathbb{R}^{m|n}$ will serve as a local model (i.e. a patch) for supermanifolds. In the formulation of Manin, we can put $\mathbb{R}^{m|n} = (\mathbb{R}^m, \Lambda_n)$.

§21 Maps on $\mathbb{R}^{m|n}$. A function $f : \mathbb{R}^{m|n} \to \mathbb{R} \otimes \Lambda_n$ is an element of $\mathscr{F}(\mathbb{R}^m) \otimes \Lambda_n$ and we will denote this set by $\mathscr{F}(\mathbb{R}^{m|n})$. Smooth functions will correspondingly be denoted by $C^\infty(\mathbb{R}^{m|n})$. Choosing coordinates (x^i, θ^j), where $1 \leq i \leq m$ and $1 \leq j \leq n$, we can write f as

$$f(x,\theta) \;=\; f_0(x) + f_j(x)\theta^j + f_{k_1k_2}(x)\theta^{k_1}\theta^{k_2} + \ldots + f_{l_1\ldots l_n}(x)\theta^{l_1}\ldots\theta^{l_n} \;. \tag{III.24}$$

We will often also use the notation $f(x,\theta) = f_I(x)\theta^I$, where I is a multiindex. Note that functions on $\mathbb{R}^{0|n}$ are just the supernumbers defined in §14, and if f_0 is nowhere vanishing then there is a function $f^{-1}(x)$, which is given by the inverse of the supernumber $f(x)$, such that $f(x)f^{-1}(x)$ is the constant function with value 1.

The formula for the inverse of a supernumber can be generalized to matrix valued supernumbers (and therefore to matrix valued superfunctions) $\psi \in \mathsf{GL}(n,\mathbb{R}) \otimes \Lambda_n$:

$$\psi^{-1} \;=\; \psi_B^{-1} - \psi_B^{-1}\psi_S\psi_B^{-1} + \psi_B^{-1}\psi_S\psi_B^{-1}\psi_S\psi_B^{-1} - \psi_B^{-1}\psi_S\psi_B^{-1}\psi_S\psi_B^{-1}\psi_S\psi_B^{-1} + \ldots \;,$$

where $\psi = \psi_B + \psi_S$ is the usual decomposition into body and soul.

§22 Superspace for $\mathcal{N}$-extended supersymmetry. The *superspace for $\mathcal{N}$-extended supersymmetry* in four dimensions is the space $\mathbb{R}^{4|4\mathcal{N}}$ (or $\mathbb{C}^{4|4\mathcal{N}}$ as the complex analogue), i.e. a real four-dimensional space $\mathbb{R}^4$ with arbitrary signature endowed additionally with $4\mathcal{N}$ Graßmann coordinates. These coordinates are grouped into Weyl spinors $\theta^{\alpha i}$ and $\bar{\theta}^i_{\dot{\alpha}}$, where $\alpha, \dot{\alpha} = 1, 2$ and $i = 1, \ldots, \mathcal{N}$. The spinor indices are raised and lowered with the antisymmetric ε-symbol defined by $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = -1$.

For simplicity, let us introduce the following shorthand notation: we can drop undotted contracted spinor indices if the left one is the upper index and dotted contracted spinor indices if the right one is the upper one, i.e. $\theta\theta = \theta^\alpha\theta_\alpha$ and $\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$.

§23 Spinorial notation. It will often be convenient to rewrite the spacetime coordinates x^μ in spinor notation, using (local) isomorphisms as e.g. $\mathsf{SO}(4) \simeq \mathsf{SU}(2) \times \mathsf{SU}(2)$ for a Euclidean spacetime[5] by $x^{\alpha\dot\alpha} = -\mathrm{i}\sigma_\mu^{\alpha\dot\alpha} x^\mu$. The sigma matrices are determined by the signature of the metric under consideration (see section III.1.1, §3). This will simplify considerably the discussion at many points later on, however, it requires some care when comparing results from different sources.

As shorthand notations for the derivatives, we will use in the following

$$\partial_{\alpha\dot\alpha} := \frac{\partial}{\partial x^{\alpha\dot\alpha}} , \quad \partial_{\alpha i} := \frac{\partial}{\partial \theta^{\alpha i}} \quad \text{and} \quad \bar\partial^i_{\dot\alpha} := \frac{\partial}{\partial \bar\theta_i^{\dot\alpha}} . \tag{III.25}$$

§24 Representation of the supersymmetry algebra. On the superspace $\mathbb{R}^{4|4\mathcal{N}}$ described by the coordinates $(x^\mu, \theta^{\alpha i}, \bar\theta^i_{\dot\alpha})$, one can define a representation of the supersymmetry algebra by introducing the action of the superderivatives on functions[6] $f \in \mathscr{F}(\mathbb{R}^{4|4\mathcal{N}})$

$$D_{\alpha i} f := \partial_{\alpha i} f + \bar\theta_i^{\dot\alpha} \partial_{\alpha\dot\alpha} f \quad \text{and} \quad \bar D^i_{\dot\alpha} f := -\bar\partial^i_{\dot\alpha} f - \theta^{\alpha i}\partial_{\alpha\dot\alpha} f , \tag{III.26}$$

as well as the action of the supercharges

$$Q_{\alpha i} f := \partial_{\alpha i} f - \bar\theta_i^{\dot\alpha} \partial_{\alpha\dot\alpha} f \quad \text{and} \quad \bar Q^i_{\dot\alpha} f := -\bar\partial^i_{\dot\alpha} f + \theta^{\alpha i}\partial_{\alpha\dot\alpha} f . \tag{III.27}$$

The corresponding transformations induced by the supercharges $Q_{\alpha i}$ and $\bar Q^i_{\dot\alpha}$ in superspace read

$$\begin{aligned} \delta x^{\alpha\dot\alpha} &= \theta^{\alpha i}\xi_i^{\dot\alpha} , & \delta\theta^{\alpha i} &= 0 , & \delta\bar\theta_i^{\dot\alpha} &= \xi_i^{\dot\alpha} , \\ \delta x^{\alpha\dot\alpha} &= \xi^{\alpha i}\bar\theta_i^{\dot\alpha} , & \delta\theta^{\alpha i} &= \xi^{\alpha i} , & \delta\bar\theta_i^{\dot\alpha} &= 0 , \end{aligned} \tag{III.28}$$

respectively, where $(\xi^{\alpha i}, \xi_i^{\dot\alpha})$ are odd parameters.

§25 Chiral superspaces and chiral coordinates. The superspace $\mathbb{R}^{4|4\mathcal{N}}$ splits into the two *chiral superspaces* $\mathbb{R}^{4|2\mathcal{N}}_L$ and $\mathbb{R}^{4|2\mathcal{N}}_R$ where the subscripts L and R stand for left-handed (chiral) and right-handed (anti-chiral). The theories under consideration often simplify significantly when choosing the appropriate coordinate system for the chiral superspaces. In the left-handed case, we choose

$$(y_L^{\alpha\dot\alpha} := x^{\alpha\dot\alpha} + \theta^{\alpha i}\bar\theta_i^{\dot\alpha},\ \theta^{\alpha i},\ \bar\theta_i^{\dot\alpha}) . \tag{III.29}$$

The representations of the superderivatives and the supercharges read in these chiral coordinates as

$$\begin{aligned} D_{\alpha i} f &= \partial_{\alpha i} f + 2\bar\theta_i^{\dot\alpha}\partial^L_{\alpha\dot\alpha} f , & \bar D^i_{\dot\alpha} f &= -\bar\partial^i_{\dot\alpha} f , \\ Q_{\alpha i} f &= \partial_{\alpha i} f , & \bar Q^i_{\dot\alpha} f &= -\bar\partial^i_{\dot\alpha} f + 2\theta^{\alpha i}\partial^L_{\alpha\dot\alpha} f , \end{aligned} \tag{III.30}$$

where $\partial^L_{\alpha\dot\alpha}$ denotes a derivative with respect to $y^{\alpha\dot\alpha}$. Due to $\partial^L_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha}$, we can safely drop the superscript "L" in the following.

One defines the anti-chiral coordinates accordingly as

$$(y_R^{\alpha\dot\alpha} := x^{\alpha\dot\alpha} - \theta^{\alpha i}\bar\theta_i^{\dot\alpha},\ \theta^{\alpha i},\ \bar\theta_i^{\dot\alpha}) . \tag{III.31}$$

Note that we will also work with superspaces of Euclidean signature and the complexified superspaces, in which θ and $\bar\theta$ are not related via complex conjugation. In these cases, we will often denote $\bar\theta$ by η.

[5]Similar isomorphisms also exist for Minkowski and Kleinian signature.

[6]One should stress that the convention presented here is suited for Minkowski space and differs from one used later when discussing supertwistor spaces for Euclidean superspaces.

III.2.4 Supermanifolds

§26 Supermanifolds. Roughly speaking, a *supermanifold* is defined to be a topological space which is locally diffeomorphic to $\mathbb{R}^{m|n}$ or $\mathbb{C}^{m|n}$. In general, a supermanifold contains a purely bosonic part, the *body*, which is parameterized in terms of the supermanifold's bosonic coordinates. The body of a supermanifold is a real or complex manifold by itself. The $\mathbb{Z}_2$-grading of the superspace used for parameterizing the supermanifold induces a grading on the ring of functions on the supermanifold. For objects like subspaces, forms etc. which come with a dimension, a degree etc., we use the notation $i|j$, where i and j denote the bosonic and fermionic part, respectively.

Due to the different approaches to supergeometry, we recall the most basic definitions used in the literature. For a more extensive discussion of supermanifolds, see [56] and references therein as well as the works [28, 162, 178].

§27 The parity inverting operator Π. Given a vector bundle $E \to M$, the parity inverting operator Π acts by reversing the parity of the fiber coordinates.

§28 Examples of simple supermanifolds. Consider the tangent bundle TM over a manifold M of dimension n. The dimension of TM is $2n$, and a point in TM can be locally described by n coordinates on the base space (x^i) and n coordinates in the fibres (y^i). The parity inverted tangent bundle ΠTM is of dimension $n|n$ and locally described by the n coordinates (x^i) on the base space together with the n Graßmann coordinates (θ^i) in the fibres. More explicitly, we have e.g. $\Pi T\mathbb{R}^4 = \mathbb{R}^{4|4}$, the superspace for $\mathcal{N} = 1$ supersymmetry.

Another example which we will often encounter is the space $\Pi\mathcal{O}(n) \to \mathbb{C}P^1$ which is described by complex variables $\lambda_\pm$ and Graßmann variables $\theta_\pm$ on the two standard patches $U_\pm$ of $\mathbb{C}P^1$ with $\theta_+ = \lambda_+^n \theta_-$ on $U_+ \cap U_-$. This bundle has first Chern number $-n$, as in fermionic integration, the Jacobian is replaced by an inverse of the Jacobian (the Berezinian).

§29 Supermanifolds in the sheaf-theoretic approach. We do not want to repeat the formal discussion of [189] at this point, but merely make some remarks. It is clear that a supermanifold will be a superspace as defined above with some additional restrictions. These restrictions basically state that it is possible to decompose a supermanifold globally into its *body*, which is a (in some sense maximal) ordinary real or complex manifold, and into its *soul*, which is the "infinitesimal cloud" surrounding the body and complementing it to the full supermanifold.

Let us consider as an example the chiral superspace $\mathbb{R}^{4|4\mathcal{N}}$ and the complex projective superspace $\mathbb{C}P^{3|4}$. Their bodies are the spaces $\mathbb{R}^{4|0} = \mathbb{R}^4$ and $\mathbb{C}P^{3|0} = \mathbb{C}P^3$, respectively.

§30 Supermanifolds according to B. S. DeWitt. This construction of a supermanifold will not be used in this thesis and is only given for completeness sake.

First, we define the *superdomain* $\mathbb{R}^m_c \times \mathbb{R}^n_a$ to be an open superspace described by $m + n$ real coordinates $u^i \in \Lambda_0$ and $v^j \in \Lambda_1$, with $i = 1, \dots m$, $j = 1, \dots n$. Note that $\mathbb{R}^m_c \times \mathbb{R}^n_a$ is *not* a supervector space in this approach to supermathematics.

Furthermore, a topology on this space can be obtained from the topology of the embedded real space $\mathbb{R}^m$ via the canonical projection

$$\pi : \mathbb{R}^m_c \times \mathbb{R}^n_a \;\to\; \mathbb{R}^m \; . \tag{III.32}$$

That is, a subset $Y \subset \mathbb{R}^m_c \times \mathbb{R}^n_a$ is open if its projection $\pi(Y)$ onto $\mathbb{R}^m$ is open. Therefore, a superdomain is not Hausdorff, but only projectively Hausdorff.

A *supermanifold* of dimension $m|n$ is then a topological space which is locally diffeomorphic to $\mathbb{R}^m_c \times \mathbb{R}^n_a$.

The definition of the body of such a supermanifold is a little more subtle, as one expects the body to be invariant under coordinate transformations. This implies that we introduce equivalence classes of points on such supermanifolds, and only then we can define a body as the real manifold which consists of all these equivalence classes. For further details, see [72] or [56].

III.2.5 Calabi-Yau supermanifolds and Yau's theorem

§31 Calabi-Yau supermanifolds. A *Calabi-Yau supermanifold* is a supermanifold which has vanishing first Chern class. Thus, Calabi-Yau supermanifolds come with a nowhere vanishing holomorphic measure Ω. Note, however, that Ω is not a differential form in the Graßmann coordinates, since Graßmann differential forms are dual to Graßmann vector fields and thus transform contragrediently to them. Berezin integration, however is equivalent to differentiation, and thus a volume element has to transform as a product of Graßmann vector fields, i.e. with the inverse of the Jacobi determinant. Such forms are called integral forms and for short, we will call Ω a *holomorphic volume form*, similarly to the usual nomenclature for Calabi-Yau manifolds.

§32 Comments on the definition. This definition has become common usage, even if not all such spaces admit a Ricci-flat metric. Counterexamples to Yau's theorem for Calabi-Yau supermanifolds can be found in [239].

Nevertheless, one should remark that vanishing of the first Chern class – and not Ricci-flatness – is necessary for a consistent definition of the topological B-model on a manifold (see section V.3.3). And, from another viewpoint, it is only with the help of a holomorphic volume form that one can give an action for holomorphic Chern-Simons theory (see section IV.3.2). Thus, the nomenclature is justified from a physicist's point of view.

§33 Examples. The most important example discussed in recent publications is certainly the space $\mathbb{C}P^{3|4}$ and its open subset

$$\mathcal{P}^{3|4} \;=\; \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \;\to\; \mathbb{C}P^1 \;. \tag{III.33}$$

The latter space is clearly a Calabi-Yau supermanifold, since its first Chern class is trivial.[7] The space $\mathcal{P}^{3|4}$ is covered by two patches $\mathcal{U}_\pm$, on which its holomorphic volume form is given by

$$\hat{\Omega}^{3,0|4,0}_\pm \;=\; \pm \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \mathrm{d}\lambda_\pm\, \mathrm{d}\eta^\pm_1 \mathrm{d}\eta^\pm_2 \mathrm{d}\eta^\pm_3 \mathrm{d}\eta^\pm_4 \;, \tag{III.34}$$

where $\lambda_\pm$ is the coordinate on the base space, while $z^\alpha_\pm$ and $\eta^\pm_i$ are coordinates of the bosonic and fermionic line bundles, respectively. Note that the body of a Calabi-Yau supermanifold is not a Calabi-Yau manifold, in general, as also in the case of the above example: the body of $\mathcal{P}^{3|4}$ is $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$ which is not a Calabi-Yau manifold.

A further class of examples for superspaces with the Calabi-Yau property is given by the weighted projective spaces $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with $p+q=4$, which were proposed as target spaces for the topological B-model in [297] and studied in detail in [231].

To complete the list, we will also encounter the superambitwistor space $\mathcal{L}^{5|6}$, which is a quadric in the product of two supertwistor spaces, and the mini-supertwistor space

[7] Recall that $\Pi\mathcal{O}(1)$ contributes -1 to the total first Chern number, see §28.

$\mathcal{P}^{2|4} := \mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4$. The corresponding mini-superambitwistor space $\mathcal{L}^{4|6}$ is not a supermanifold, see section VII.7.3.

§34 Yau's theorem on supermanifolds. In [239], it was shown that Yau's theorem is not valid for all supermanifolds. That is, even if the first Chern class is vanishing on a supermanifold with Kähler form J, this does not imply that this supermanifold admits a super Ricci-flat metric in the same Kähler class as J. To construct a counterexample, one can start from a Kähler manifold with vanishing first Chern class and one fermionic and an arbitrary number of bosonic dimensions. One finds that such a supermanifold admits a Ricci-flat metric if and only if its scalar curvature is vanishing [239]. As the weighted projective spaces $W\mathbb{C}P^{m,1}(1,\ldots,1|m)$ provide examples, for which this condition is not met, we find that the naïve form of Yau's theorem is not valid for supermanifolds.

In a following paper [302], it was conjectured that this was an artifact of supermanifolds with one fermionic dimensions, but in the paper [240] published only shortly afterwards, counterexamples to the naïve form of Yau's theorem with two fermionic dimensions were presented.

III.3 Exotic supermanifolds

In this section, we want to give a brief review of the existing extensions or generalizations of supermanifolds, having additional dimensions described by even nilpotent coordinates. Furthermore, we will present a discussion of Yau's theorem on exotic supermanifolds. In the following, we shall call every (in a well-defined way generalized) manifold which is locally described by k even, l even and nilpotent and q odd and nilpotent coordinates an *exotic supermanifold* of dimension $(k \oplus l|q)$. In section VII.5, some of the exotic supermanifolds defined in the following will serve as target spaces for a topological B-model.

III.3.1 Partially formal supermanifolds

§1 Supermathematics via functors. The objects of supermathematics, as e.g. supermanifolds or supergroups, are naturally described as covariant functors from the category of Graßmann algebras to corresponding categories of ordinary mathematical objects, as manifolds or groups, [248]. A generalization of this setting is to consider covariant functors with the category of almost nilpotent (AN) algebras as domain [160, 159]. Recall that an AN algebra Ξ can be decomposed into an even part Ξ_0 and an odd part Ξ_1 as well as in the canonically embedded ground field (i.e. $\mathbb{R}$ or $\mathbb{C}$), Ξ_B, and the nilpotent part Ξ_S. The parts of elements $\xi \in \Xi$ belonging to Ξ_B and Ξ_S are called the *body* and the *soul* of ξ, respectively.

§2 Superspaces and superdomains. A *superspace* is a covariant functor from the category of AN algebras to the category of sets. Furthermore, a *topological superspace* is a functor from the category of AN algebras to the category of topological spaces.

Consider now a tuple $(x^1,\ldots,x^k,y^1,\ldots,y^l,\zeta^1,\ldots,\zeta^q)$ of k even, l even and nilpotent and q odd and nilpotent elements of an AN algebra Ξ, i.e. $x^i \in \Xi_0$, $y^i \in \Xi_0 \cap \Xi_S$ and $\zeta^i \in \Xi_1$. The functor from the category of AN algebras to such tuples is a superspace denoted by $\mathbb{R}^{k\oplus l|q}$. An open subset $U^{k\oplus l|q}$ of $\mathbb{R}^{k\oplus l|q}$, which is obtained by restricting the fixed ground field Ξ_B of the category of AN algebras to an open subset, is called a *superdomain* of dimension $(k \oplus l|q)$. After defining a graded basis $(e_1,\ldots,e_k,f_1,\ldots,f_l,\varepsilon_1,\ldots,\varepsilon_q)$

consisting of $k+l$ even and q odd vectors, one can consider the set of linear combinations $\{x^i e_i + y^j f_j + \zeta^\alpha \varepsilon_\alpha\}$ which forms a *supervector space* [160, 159].

Roughly speaking, one defines a partially formal supermanifold[8] of dimensions $(k\oplus l|q)$ as a topological superspace smoothly glued together from superdomains $U^{k\oplus l|q}$. Although we will not need the exact definition in the subsequent discussion, we will nevertheless give it here for completeness sake.

§3 Maps between superspaces. We define a *map between two superspaces* as a natural transformation of functors. More explicitly, consider two superspaces $\mathcal{M}$ and $\mathcal{N}$. Then a map $F : \mathcal{M} \to \mathcal{N}$ is a map between superspaces if F is compatible with the morphisms of AN algebras $\alpha : \Xi \to \Xi'$. We call a smooth map $\kappa : \mathbb{R}_\Xi^{k\oplus l|q} \to \mathbb{R}_\Xi^{k'\oplus l'|q'}$ between two superdomains Ξ_0-*smooth* if for every $x \in \mathbb{R}_\Xi^{k\oplus l|q}$ the tangent map $(\kappa_\Xi)_* : T_x \to T_{\kappa_\Xi(x)}$ is a homomorphism of Ξ_0-modules. Furthermore, we call a map $\kappa : \mathbb{R}^{k\oplus l|q} \to \mathbb{R}^{k'\oplus l'|q'}$ *smooth* if for all AN algebras Ξ the maps κ_Ξ are Ξ_0-smooth.

§4 Partially formal supermanifolds. Now we can be more precise: A *partially formal supermanifold of dimension* $(k \oplus l|q)$ is a superspace locally equivalent to superdomains of dimension $(k\oplus l|q)$ with smooth transition functions on the overlaps. Thus, a partially formal supermanifold is also an exotic supermanifold.

However, not every exotic supermanifold is partially formal. We will shortly encounter examples of such cases: exotic supermanifolds, which are constructed using a particular AN algebra instead of working with the category of AN algebras.

The definitions used in this section stem from [160, 159], where one also finds examples of applications.

Unfortunately, it is not clear how to define a general integration over the even nilpotent part of such spaces; even the existence of such an integral is questionable. We will comment on this point later on. This renders partially formal supermanifolds useless as target spaces for a topological string theory, as we need an integration to define an action. Therefore, we have to turn to other generalizations.

III.3.2 Thick complex manifolds

§5 Formal neighborhoods. Extensions to m-th formal neighborhoods of a submanifold X in a manifold $Y \supset X$ and the more general thickening procedure have been proposed and considered long ago[9] in the context of twistor theory, in particular for ambitwistor spaces, e.g. in [290, 90, 170, 92]. We will ignore this motivation and only recollect the definitions needed for our discussion in chapter VII.

§6 Thickening of complex manifolds. Given a complex manifold X with structure sheaf $\mathcal{O}_X$, we consider a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{(m)}$ on X with a homomorphism $\alpha : \mathcal{O}_{(m)} \to \mathcal{O}_X$, such that locally $\mathcal{O}_{(m)}$ is isomorphic to $\mathcal{O}[y]/(y^{m+1})$ where y is a formal (complex) variable and α is the obvious projection. The resulting ringed space or scheme $X_{(m)} := (X, \mathcal{O}_{(m)})$ is called a *thick complex manifold*. Similarly to the nomenclature of supermanifolds, we call the complex manifold X the *body* of $X_{(m)}$.

§7 Example. As a simple example, let X be a closed submanifold of the complex manifold Y with codimension one. Let $\mathcal{I}$ be the ideal of functions vanishing on X. Then

[8]This term was introduced in [161].

[9]In fact, the study of infinitesimal neighborhoods goes back to [107] and [112]. For a recent review, see [54].

$\mathcal{O}_{(m)} = \mathcal{O}_Y/\mathcal{I}^{m+1}$ is called an *infinitesimal neighborhood* or the m-th *formal neighborhood* of X. This is a special case of a thick complex manifold. Assuming that X has complex dimension n, $\mathcal{O}_{(m)}$ is also an exotic supermanifold of dimension $(n \oplus 1|0)$. More explicitly, let $(x^1, \ldots, x^n)$ be local coordinates on X and $(x^1, \ldots, x^n, y)$ local coordinates on Y. Then the ideal $\mathcal{I}$ is generated by y and $\mathcal{O}_{(m)}$ is locally a formal polynomial in y with coefficients in $\mathcal{O}_X$ together with the identification $y^{m+1} \sim 0$. Furthermore, one has $\mathcal{O}_{(0)} = \mathcal{O}_X$.

Returning to the local description as a formal polynomial in y, we note that there is no object y^{-1} as it would violate associativity by an argument like $0 = y^{-1}y^{m+1} = y^{-1}yy^m = y^m$. However, the inverse of a formal polynomial in y is defined if (and only if) the zeroth order monomial has an inverse. Suppose that $p = a + \sum_{i=1}^m f_i y^i = a + b$. Then we have $p^{-1} = \frac{1}{a}\sum_{i=0}^m (-\frac{b}{a})^i$, analogously to the inverse of a supernumber.

§8 Vector bundles. A *holomorphic vector bundle* on $(X, \mathcal{O}_{(m)})$ is a locally free sheaf of $\mathcal{O}_{(m)}$-modules.

The *tangent space* of a thick complex manifold is the sheaf of derivations $D : \mathcal{O}_{(m)} \to \mathcal{O}_{(m)}$. Let us consider again our above example $X_{(m)} = (X, \mathcal{O}_{(m)})$. Locally, an element of $TX_{(m)}$ will take the form $D = f\frac{\partial}{\partial y} + \sum_j g^j \frac{\partial}{\partial x^j}$ together with the differentiation rules

$$\frac{\partial}{\partial y} y \;=\; 1 \;, \quad \frac{\partial}{\partial y} x^i \;=\; \frac{\partial}{\partial x^i} y \;=\; 0 \;, \quad \frac{\partial}{\partial x^i} x^j \;=\; \delta_i^j \;. \tag{III.35}$$

All this and the introduction of cotangent spaces for thick complex manifolds is found in [92].

§9 Integration on thick complex manifolds. In defining a (definite) integral over the nilpotent formal variable y, which is needed for formulating hCS theory by giving an action, one faces the same difficulty as in the case of Berezin integration: the integral should not be taken over a specific range as we integrate over an infinitesimal neighborhood which would give rise to infinitesimal intervals. Furthermore, this neighborhood is purely formal and so has to be the integration. Recall that a suitable integration I should satisfy the rule[10] $DI = ID = 0$, where D is a derivative with respect to a variable over which I integrates. The first requirement $DI = 0$ states that the result of definite integration does not depend on the variables integrated over. The requirement $ID = 0$ for integration domains with vanishing boundary (or functions vanishing on the boundary) is the foundation of Stokes' formula and integration by parts. It is easy to see that the condition $DI = ID = 0$ demands that

$$I \;=\; c \cdot \frac{\partial^m}{\partial y^m} \;, \tag{III.36}$$

where y is the local formal variable from the definition of $X_{(m)}$ and c is an arbitrary normalization constant, e.g. $c = 1/m!$ would be most convenient. Thus, we define

$$\int \mathrm{d}y \; f \;:=\; \frac{1}{m!}\frac{\partial^m}{\partial y^m} f \;. \tag{III.37}$$

This definition only relies on an already well-defined operation and thus is well-defined itself.[11] Additionally, it also agrees with the intuitive picture that the integral of a con-

[10] This rule can also be used to fix Berezin integration, cf. section III.2, §16.

[11] From this definition, we see the problem arising for partially formal supermanifolds: The integration process on thick complex manifolds returns the coefficient of the monomial with highest possible power in y. For partially formal supermanifolds, where one works with the category of AN algebras, such a highest power does not exist as it is different for each individual AN algebra.

stant over an infinitesimal neighborhood should vanish. Integration over a thick complex manifold is an integro-differential operation.

§10 Change of coordinates. Consider now a change of coordinates $(x^1,\dots,x^n,y) \to (\tilde{x}^1,\dots,\tilde{x}^n,\tilde{y})$ which leaves invariant the structure of the thick complex manifold. That is, $\tilde{x}^i$ is independent of y, and $\tilde{y}$ is a polynomial only in y with vanishing zeroth order coefficient and non-vanishing first order coefficient. Because of $\partial_{\tilde{y}} = \frac{\partial y}{\partial \tilde{y}}\partial_y$, we have the following transformation of a volume element under such a coordinate change:

$$\mathrm{d}\tilde{x}^1\dots\mathrm{d}\tilde{x}^n\mathrm{d}\tilde{y} \;=\; \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)\mathrm{d}x^1\dots\mathrm{d}x^n\left(\frac{\partial y}{\partial \tilde{y}}\right)^m \mathrm{d}y\ . \tag{III.38}$$

The theorems in [92] concerning obstructions to finding $X_{(m+1)}$ given $X_{(m)}$ will not be needed in the following, as we will mainly work with order one thickenings (or fattenings) and in the remaining cases, the existence directly follows by construction.

III.3.3 Fattened complex manifolds

§11 Fattening of complex manifolds. Fattened complex manifolds [91] are straightforward generalizations of thick complex manifolds. Consider again a complex manifold X with structure sheaf $\mathcal{O}_X$. The *m-th order fattening with codimension k* of X is the ringed space $X_{(m,k)} = (X,\mathcal{O}_{(m,k)})$ where $\mathcal{O}_{(m,k)}$ is locally isomorphic to

$$\mathcal{O}[y^1,\dots,y^k]/(y^1,\dots,y^k)^{m+1}\ . \tag{III.39}$$

Here, the y^i are again formal complex variables. We also demand the existence of the (obvious) homomorphism $\alpha : \mathcal{O}_{(m,k)} \to \mathcal{O}_X$. It follows immediately that a fattening with codimension 1 is a thickening. Furthermore, an (m,k)-fattening of an n-dimensional complex manifold X is an exotic supermanifold of dimension $(n\oplus k|0)$ and we call X the *body* of $X_{(m,k)}$.

As in the case of thick complex manifolds, there are no inverses for the y^i, but the inverse of a formal polynomial p in the y^i decomposed into $p = a + b$, where b is the nilpotent part of p, exists again if and only if $a \neq 0$ and it is then given by $p^{-1} = \frac{1}{a}\sum_{i=0}^m(-\frac{b}{a})^i$. A *holomorphic vector bundle* on $\mathcal{O}_{(m,k)}$ is a locally free sheaf of $\mathcal{O}_{(m,k)}$-modules. The *tangent space* of a thick complex manifold is also generalized in an obvious manner.

§12 Integration on fattened complex manifolds. We define the integral analogously to thick complex manifolds as

$$\int \mathrm{d}y^1\dots\mathrm{d}y^k\ f \;:=\; \frac{1}{m!}\frac{\partial^m}{\partial(y^1)^m}\ \cdots\ \frac{1}{m!}\frac{\partial^m}{\partial(y^k)^m}f\ . \tag{III.40}$$

A change of coordinates $(x^1,\dots,x^n,y^1,\dots,y^k) \to (\tilde{x}^1,\dots,\tilde{x}^n,\tilde{y}^1,\dots,\tilde{y}^k)$ must again preserve the structure of the fat complex manifold: $\tilde{x}^i$ is independent of the y^i and the $\tilde{y}^i$ are nilpotent polynomials in the y^i with vanishing monomial of order 0 and at least one non-vanishing monomial of order 1. Evidently, all the $\tilde{y}^i$ have to be linearly independent. Such a coordinate transformation results in a more complicated transformation law for the volume element:

$$\mathrm{d}\tilde{x}^1\dots\mathrm{d}\tilde{x}^n\mathrm{d}\tilde{y}^1\dots\mathrm{d}\tilde{y}^k \;=\; \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right)\mathrm{d}x^1\dots\mathrm{d}x^n\left(\frac{\partial y^{i_1}}{\partial \tilde{y}^1}\cdots\frac{\partial y^{i_k}}{\partial \tilde{y}^k}\right)^m \mathrm{d}y^{i_1}\dots\mathrm{d}y^{i_k}\ , \tag{III.41}$$

where a sum over the indices $(i_1, \ldots, i_k)$ is implied. In this case, the coefficient for the transformation of the nilpotent formal variables cannot be simplified. Recall that in the case of ordinary differential forms, the wedge product provides the antisymmetry needed to form the determinant of the Jacobi matrix. In the case of Berezin integration, the anticommutativity of the derivatives with respect to Graßmann variables does the same for the inverse of the Jacobi matrix. Here, we have neither of these and therefore no determinant appears.

§13 Thick and fattened supermanifolds. After thickening or fattening a complex manifold, one can readily add fermionic dimensions. Given a thickening of an n-dimensional complex manifold of order m, the simplest example is possibly $\Pi TX_{(m)}$, an $(n \oplus 1|n+1)$ dimensional exotic supermanifold. However, we will not study such objects in the following.

III.3.4 Exotic Calabi-Yau supermanifolds and Yau's theorem

§14 Exotic Calabi-Yau supermanifolds. Following the convention for supermanifolds (cf. section III.2, §31), we shall call an exotic supermanifold Calabi-Yau if its first Chern class vanishes and it therefore comes with a holomorphic volume form. For exotic supermanifolds, too, the Calabi-Yau property is not sufficient for the existence of a Ricci-flat metric, as we will derive in the following.

§15 Exotic trace and exotic determinant. We start from a $(k \oplus l|q)$-dimensional exotic supermanifold with local coordinate vector $(x^1, \ldots, x^k, y^1, \ldots, y^l, \zeta^1, \ldots, \zeta^q)^T$. An element of the tangent space is described by a vector $(X^1, \ldots, X^k, Y^1, \ldots, Y^l, Z^1, \ldots, Z^q)^T$. Both the metric and linear coordinate transformations on this space are defined by non-singular matrices

$$K = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix} , \qquad \text{(III.42)}$$

where the elements A, B, D, E, J are of even and G, H, C, F are of odd parity. As a definition for the *extended supertrace* of such matrices, we choose

$$\mathrm{etr}(K) := \mathrm{tr}\,(A) + \mathrm{tr}\,(E) - \mathrm{tr}\,(J) , \qquad \text{(III.43)}$$

which is closely related to the supertrace and which is the appropriate choice to preserve cyclicity: $\mathrm{etr}(KM) = \mathrm{etr}(MK)$. Similarly to [72], we define the extended superdeterminant by

$$\delta \ln \mathrm{edet}(K) := \mathrm{etr}(K^{-1}\delta K) \quad \text{together with} \quad \mathrm{edet}(\mathbb{1}) := 1 , \qquad \text{(III.44)}$$

which guarantees $\mathrm{edet}(KM) = \mathrm{edet}(K)\mathrm{edet}(M)$. Proceeding analogously to [72], one decomposes K into the product of a lower triangular matrix, a block diagonal matrix and an upper diagonal matrix. The triangular matrices can be chosen to have only 1 as diagonal entries and thus do not contribute to the total determinant. The block diagonal matrix is of the form

$$K' = \begin{pmatrix} A & 0 & 0 \\ 0 & E - DA^{-1}B & 0 \\ 0 & 0 & R \end{pmatrix} , \qquad \text{(III.45)}$$

with $R = J - GA^{-1}C - (H - GA^{-1}B)(E - DA^{-1}B)^{-1}(F - DA^{-1}C)$. The determinant of a block diagonal matrix is easily calculated and in this case we obtain

$$\mathrm{edet}(K) \;=\; \mathrm{edet}(K') \;=\; \frac{\det(A)\det(E - DA^{-1}B)}{\det(R)}\ . \tag{III.46}$$

Note that for the special case of no even nilpotent dimensions, for which one should formally set $B = D = F = H = 0$, one recovers the formulæ for the supertrace ($E = 0$) and the superdeterminant ($E = \mathbb{1}$ to drop the additional determinant).

§16 Yau's theorem on exotic supermanifolds. In [239], the authors found that Kähler supermanifolds with one fermionic dimension admit Ricci-flat supermetrics if and only if the body of the Kähler supermanifold admits a metric with vanishing scalar curvature,[12] and thus Yau's theorem (see section II.3.1) is only valid under additional assumptions. Let us investigate the same issue for the case of a $(p \oplus 1|0)$-dimensional exotic supermanifold Y with one even nilpotent coordinate y. We denote the ordinary p-dimensional complex manifold embedded in Y by X. The extended Kähler potential on Y is given by a real-valued function $\mathscr{K} = f^0 + f^1 y\bar{y}$, such that the metric takes the form

$$g := \left(\partial_i \bar{\partial}_{\bar{\jmath}} \mathscr{K}\right) \;=\; \begin{pmatrix} f^0_{,i\bar{\jmath}} + f^1_{,i\bar{\jmath}}\, y\bar{y} & f^1_{,i}\, y \\ f^1_{,\bar{\jmath}}\, \bar{y} & f^1 \end{pmatrix}\ . \tag{III.47}$$

For the extended Ricci-tensor to vanish, the extended Kähler potential has to satisfy the Monge-Ampère equation $\mathrm{edet}(g) := \mathrm{edet}(\partial_i \bar{\partial}_{\bar{\jmath}} \mathscr{K}) = 1$. In fact, we find

$$\begin{aligned}
\mathrm{edet}(g) \;&=\; \det\left(f^0_{,i\bar{\jmath}} + f^1_{,i\bar{\jmath}} y\bar{y}\right)\left(f^1 - f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n} y\bar{y}\right)\\
&=\; \det\left[\left(f^0_{,i\bar{\jmath}} + f^1_{,i\bar{\jmath}} y\bar{y}\right)\left(\sqrt[p]{f^1} - \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n} y\bar{y}}{p (f^1)^{\frac{p-1}{p}}}\right)\right]\\
&=\; \det\left[f^0_{,i\bar{\jmath}} \sqrt[p]{f^1} + \left(f^1_{,i\bar{\jmath}} \sqrt[p]{f^1} - f^0_{,i\bar{\jmath}} \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{p (f^1)^{\frac{p-1}{p}}}\right) y\bar{y}\right]\\
&=\; \det\left[f^0_{,l\bar{\jmath}} \sqrt[p]{f^1}\right] \det\left[\delta_i^k + \left(g^{\bar{m}k} f^1_{,i\bar{m}} - \delta_i^k \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{p f^1}\right) y\bar{y}\right]\ ,
\end{aligned}$$

where $g^{\bar{m}n}$ is the inverse of $f^0_{,n\bar{m}}$. Using the relation $\ln\det(A) = \mathrm{tr}\,\ln(A)$, we obtain

$$\mathrm{edet}(g) \;=\; \det\left[f^0_{,l\bar{\jmath}} \sqrt[p]{f^1}\right]\left(1 + \left(g^{\bar{m}i} f^1_{,i\bar{m}} - \frac{f^1_{\bar{m}} g^{\bar{m}n} f^1_{,n}}{f^1}\right) y\bar{y}\right)\ . \tag{III.48}$$

From demanding extended Ricci-flatness, it follows that

$$f^1 \;=\; \frac{1}{\det\left(f^0_{,l\bar{\jmath}}\right)} \quad \text{and} \quad \left(g^{\bar{\jmath}i} f^1_{,i\bar{\jmath}} - \frac{f^1_{\bar{\jmath}} g^{\bar{\jmath}i} f^1_{,i}}{f^1}\right) \;=\; 0\ . \tag{III.49}$$

The second equation can be simplified to

$$g^{\bar{\jmath}i}\left(f^1_{,i\bar{\jmath}} - \frac{f^1_{\bar{\jmath}} f^1_{,i}}{f^1}\right) \;=\; f^1 g^{\bar{\jmath}i}\left(\ln(f^1)\right)_{,i\bar{\jmath}} \;=\; 0\ , \tag{III.50}$$

[12]For related work, see [302, 240, 179].

and together with the first equation in (III.49), it yields

$$g^{\bar{\jmath}i}\left(\ln\frac{1}{\det\left(f^0_{,k\bar{\jmath}}\right)}\right)_{,i\bar{\jmath}} = -g^{\bar{\jmath}i}\left(\ln\det\left(f^0_{,k\bar{\jmath}}\right)\right)_{,i\bar{\jmath}} = -g^{\bar{\jmath}i}R_{,i\bar{\jmath}} = 0\ . \tag{III.51}$$

This equation states that an exotic supermanifold Y of dimension $(p\oplus 1|0)$ admits an extended Ricci-flat metric if and only if the embedded ordinary manifold X has vanishing scalar curvature. A class of examples for which this additional condition is not satisfied are the weighted projective spaces $W\mathbb{C}P^{m-1\oplus 1|0}(1,\ldots,1\oplus m|\cdot)$, which have vanishing first Chern class but do not admit a Kähler metric with vanishing Ricci scalar.

Thus, we obtained exactly the same result as in [239], which is somewhat surprising as the definition of the extended determinant involved in our calculation strongly differs from the definition of the superdeterminant. However, this agreement might be an indication that fattened complex manifolds – together with the definitions made above – fit nicely in the whole picture of extended Calabi-Yau spaces.

III.4 Spinors in arbitrary dimensions

The main references for this section are [29, 285] and appendix B of [219].

III.4.1 Spin groups and Clifford algebras

§1 Spin group. The *spin group* $\mathsf{Spin}(p,q)$ is the *double cover* (or *universal cover*) of the Lorentz group $\mathsf{SO}(p,q)$. Explicitly, it is defined by the short exact sequence

$$1\ \to\ \mathbb{Z}_2\ \to\ \mathsf{Spin}(p,q)\ \to\ \mathsf{SO}(p,q)\ \to\ 1\ . \tag{III.52}$$

§2 Clifford algebra. Let V be a $(p+q)$-dimensional vector space V with a pseudo-Euclidean scalar product g_{AB} invariant under the group $\mathsf{O}(p,q)$. Consider furthermore $p+q$ symbols γ_A with a product satisfying

$$\gamma_A\gamma_B+\gamma_B\gamma_A\ =\ -2g_{AB}\mathbb{1}\ . \tag{III.53}$$

The *Clifford algebra* $\mathscr{C}(p,q)$ is then a 2^{p+q} dimensional vector space spanned by the basis

$$(\mathbb{1},\gamma_A,\gamma_A\gamma_B,\ldots,\gamma_1\ldots\gamma_{p+q})\ . \tag{III.54}$$

Note that this is a $\mathbb{Z}_2$-graded algebra, $\mathscr{C}(p,q)=\mathscr{C}_+(p,q)\oplus\mathscr{C}_-(p,q)$, where $\mathscr{C}_+(p,q)$ and $\mathscr{C}_-(p,q)$ denote the elements consisting of an even and odd number of symbols γ_A, respectively.

§3 Representation of the Clifford algebra. A faithful representation of the Clifford algebra for $d=2k+2$ can be found by recombining the generators γ_A as follows:

$$\gamma_{0\pm}\ =\ \tfrac{1}{2}(\pm\gamma_0+\gamma_1)\quad\text{and}\quad\gamma_{a\pm}\ =\ \tfrac{1}{2}(\gamma_{2a}\pm \mathrm{i}\gamma_{2a+1})\quad\text{for}\quad a\ \neq\ 0\ . \tag{III.55}$$

This yields the fermionic oscillator algebra

$$\{\gamma_{a+},\gamma_{b-}\}\ =\ \delta^{ab}\ ,\quad \{\gamma_{a+},\gamma_{b+}\}\ =\ \{\gamma_{a-},\gamma_{b-}\}\ =\ 0\ , \tag{III.56}$$

and by the usual highest weight construction, one obtains a 2^{k+1}-dimensional representation. That is, starting from a state $|h\rangle$ with $\gamma_{a-}|h\rangle=0$ for all a, we obtain all the states by acting with arbitrary combinations of the γ_{a+} on $|h\rangle$. As every γ_{a+} can appear at most once, this leads to 2^{k+1} states, which can be constructed iteratively, see [219]. Given such a representation for $d = 2k+2$, one can construct a representation for $d = 2k+3$ by adding the generator $\gamma_d = \mathrm{i}^{-k}\gamma_0 \dots \gamma_{d-2}$. Thus, the faithful representations of the Clifford algebra on a space with dimension d are $2^{[\frac{d}{2}]}$-dimensional.

§4 Embedding a Spin group in a Clifford algebra. Given a Clifford algebra $\mathscr{C}(p,q)$, the generators

$$\Sigma_{AB} \;=\; -\tfrac{\mathrm{i}}{4}[\gamma_A, \gamma_B] \tag{III.57}$$

form a representation of the Lie algebra of $\mathsf{Spin}(p,q)$. This is the *Dirac representation*, which is a reducible representation of the underlying Lorentz algebra. As examples, consider in four dimensions the decomposition of the Dirac representation into two Weyl representations $\mathbf{4}_{\text{Dirac}} = \mathbf{2} + \mathbf{2}'$ as well as the similar decomposition in ten dimensions: $\mathbf{32}_{\text{Dirac}} = \mathbf{16} + \mathbf{16}'$.

§5 Examples. The following table contains those examples of spin groups which are most frequently encountered. For further examples and more details, see [47].

$\mathsf{Spin}(2) \cong \mathsf{U}(1)$	$\mathsf{Spin}(3) \cong \mathsf{SU}(2)$	$\mathsf{Spin}(4) \cong \mathsf{SU}(2) \times \mathsf{SU}(2)$
$\mathsf{Spin}(1,1) \cong \mathbb{R}^\times$	$\mathsf{Spin}(2,1) \cong \mathsf{SL}(2,\mathbb{R})$	$\mathsf{Spin}(3,1) \cong \mathsf{SL}(2,\mathbb{C})$
$\mathsf{Spin}(5) \cong \mathsf{Sp}(2)$	$\mathsf{Spin}(6) \cong \mathsf{SU}(4)$	$\mathsf{Spin}(2,2) \cong \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$
$\mathsf{Spin}(4,1) \cong \mathsf{Sp}(1,1)$	$\mathsf{Spin}(5,1) \cong \mathsf{SL}(2,\mathbb{H})$	

III.4.2 Spinors

§6 Spinors. A *spinor* on a spacetime with Lorentz group $\mathsf{SO}(p,q)$ is an element of the representation space of the group $\mathsf{Spin}(p,q)$. Generically, a (Dirac) spinor is thus of complex dimension $2^{[(p+q)/2]}$.

§7 Minkowski space. On d-dimensional Minkowski space, the $2^{[\frac{d}{2}]}$-dimensional Dirac representation splits into two *Weyl representations*, which are the two sets of eigenstates of the chirality operator

$$\gamma \;=\; \mathrm{i}^{-k}\gamma_0\gamma_1 \dots \gamma_{d-1} \;, \tag{III.58}$$

where γ has eigenvalues ± 1. This operator can be used to define a projector onto the two Weyl representation:

$$P_\pm \;:=\; \frac{\mathbb{1} \pm \gamma}{2} \;. \tag{III.59}$$

In dimensions $d = 0,1,2,3,4 \mod 8$, one can also impose a *Majorana condition* on a Dirac spinor, which demands that a so-called Majorana spinor ψ is its own charge conjugate:

$$\psi^c \;=\; \psi \quad \text{with} \quad \psi^c \;:=\; C\gamma_0\psi^* \;. \tag{III.60}$$

Here, C is the charge conjugation operator, satisfying

$$C\gamma_\mu C^{-1} \;=\; -\gamma_\mu^T \quad \text{and} \quad C\gamma_0(C\gamma_0)^* \;=\; \mathbb{1} \;. \tag{III.61}$$

The latter equation implies $(\psi^c)^c = \psi$. (Note that in the remaining cases $d = 5,6,7 \mod 8$, one can group the spinors into doublets and impose a symplectic Majorana condition. We will encounter such a condition in the case of Graßmann variables on Euclidean spacetime in §13.)

The Majorana condition is essentially equivalent to the Weyl condition in dimensions $d = 0, 4 \mod 8$. In dimensions $d = 2 \mod 8$, one can impose both the Weyl and the Majorana conditions simultaneously, which yields *Majorana-Weyl spinors.* The latter will appear when discussing ten-dimensional super Yang-Mills theory in section IV.2.1.

Two-spinors, in particular the commuting ones needed in twistor theory, will be discussed in §2 of section VII.1.1 in more detail.

§8 Euclidean space. The discussion of Euclidean spinors is quite parallel, and one basically identifies the properties of representations of $\mathsf{Spin}(p)$ with those of $\mathsf{Spin}(p+1,1)$. The Dirac representation decomposes again into two Weyl representations, and one can impose a Majorana condition for $d = 0,1,2,6,7 \mod 8$. In the cases $d = 3,4,5 \mod 8$, one has to switch to a pseudoreal representation.

§9 Vectors from spinors. The generators of the Clifford algebra can be interpreted as linear maps on the spinor space. Thus they (and their reduced versions) can be used to convert vector indices into two spinor indices and vice versa. We already used this fact in introducing the notation $x^{\alpha\dot\alpha} := -\mathrm{i}\sigma_\mu^{\alpha\dot\alpha} x^\mu$. In particular, this example together with conventions for commuting two-spinors are given in section VII.1.1, §2. For more details in general dimensions, see [217].

§10 Reality conditions. A real structure is an antilinear involution τ, which gives rise to a reality condition by demanding that $\tau(\cdot) = \cdot$. The real structures which we will define live on superspaces with four- or three-dimensional bodies. In the four-dimensional case, there are two such involutions for Kleinian signature[13] on the body, and each one for bodies with Euclidean and Minkowski signature. In the three-dimensional case, there is evidently just a Euclidean and a Minkowski signature possible on the body. We want to stress in advance that contrary to the Minkowskian signature $(3,1)$, the variables $\theta^{\alpha i}$ and $\eta_i^{\dot\alpha} = \bar\theta_i^{\dot\alpha}$ are independent for both signatures (4,0) and (2,2).

In the following, we will consider the superspaces $\mathbb{R}^{4|4\mathcal{N}}$ and $\mathbb{R}^{3|4\mathcal{N}}$ with coordinates $(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \theta^{\alpha i})$ and $(y^{\dot\alpha\dot\beta}, \eta_i^{\dot\alpha}, \theta^{\alpha i})$, respectively. The latter coordinates are obtained from dimensional reduction via the formula $y^{\dot\alpha\dot\beta} := -\mathrm{i}x^{(\dot\alpha\dot\beta)}$, see section IV.2.5, §29 for more details.

§11 Kleinian case. For this case, we introduce two real structures τ_1 and τ_0, which act on the bosonic coordinates of our superspace as

$$\begin{aligned} \tau_1(x^{2\dot2}) &:= \bar x^{1\dot1}, \qquad \tau_1(x^{2\dot1}) := \bar x^{1\dot2}, \\ \tau_0(x^{\alpha\dot\alpha}) &:= \bar x^{\alpha\dot\alpha}. \end{aligned} \tag{III.62}$$

For τ_1, we can thus extract the real coordinates $x^\mu \in \mathbb{R}^{2,2}$, $\mu = 1,\dots,4$ by

$$x^{2\dot2} = \bar x^{1\dot1} = -(x^4 + \mathrm{i}x^3) \quad \text{and} \quad x^{2\dot1} = \bar x^{1\dot2} = -(x^2 - \mathrm{i}x^1). \tag{III.63}$$

and the real coordinates $x^a \in \mathbb{R}^{2,1}$, $a = 1,2,3$ by

$$y^{\dot1\dot1} = -\bar y^{\dot2\dot2} = (x^1 + \mathrm{i}x^2) =: y, \quad y^{\dot1\dot2} = \bar y^{\dot1\dot2} = -x^3. \tag{III.64}$$

For the fermionic coordinates, the actions of the two real structures read as

$$\tau_1 \begin{pmatrix} \theta^{1i} \\ \theta^{2i} \end{pmatrix} = \begin{pmatrix} \bar\theta^{2i} \\ \bar\theta^{1i} \end{pmatrix}, \quad \tau_1 \begin{pmatrix} \eta_i^{\dot1} \\ \eta_i^{\dot2} \end{pmatrix} = \begin{pmatrix} \bar\eta_i^{\dot2} \\ \bar\eta_i^{\dot1} \end{pmatrix} \tag{III.65}$$

[13] i.e. signature (2,2)

and

$$\tau_0(\theta^{\alpha i}) \;=\; \bar\theta^{\alpha i} \quad\text{and}\quad \tau_0(\eta_i^{\dot\alpha}) \;=\; \bar\eta_i^{\dot\alpha}\ , \tag{III.66}$$

matching the definition for commuting spinors. The resulting Majorana-condition is then

$$\tau_1(\theta^{\alpha i}) \;=\; \theta^{\alpha i} \quad\text{and}\quad \tau_1(\eta_i^{\dot\alpha}) \;=\; \eta_i^{\dot\alpha} \;\Leftrightarrow\; \theta^{2i} \;=\; \bar\theta^{1i} \quad\text{and}\quad \eta_i^{\dot 2} \;=\; \bar\eta_i^{\dot 1}\ , \tag{III.67}$$

$$\tau_0(\theta^{\alpha i}) \;=\; \theta^{\alpha i} \quad\text{and}\quad \tau_0(\eta_i^{\dot\alpha}) \;=\; \eta_i^{\dot\alpha} \;\Leftrightarrow\; \theta^{\alpha i} \;=\; \bar\theta^{\alpha i} \quad\text{and}\quad \eta_i^{\dot\alpha} \;=\; \bar\eta_i^{\dot\alpha}\ . \tag{III.68}$$

§12 Minkowski case. Here, we define a real structure τ_M by the equations

$$\tau_M(x^{\alpha\dot\beta}) \;=\; -\overline{x^{\beta\dot\alpha}} \quad\text{and}\quad \tau_M(\eta_i^{\dot\alpha}) \;=\; \overline{\theta^{\alpha i}}\ , \tag{III.69}$$

where the indices $\alpha = \dot\alpha$ and $\beta = \dot\beta$ denote the same number.

§13 Euclidean case. In the Euclidean case, the real structure acts on the bosonic coordinates according to

$$\tau_{-1}(x^{2\dot 2}) \;:=\; \bar x^{1\dot 1}\ ,\quad \tau_{-1}(x^{2\dot 1}) \;:=\; -\bar x^{1\dot 2} \tag{III.70}$$

and the prescription for a change to real coordinates $x^\mu \in \mathbb{R}^4$ reads as

$$x^{2\dot 2} \;=\; \bar x^{1\dot 1} \;=\; -(-x^4 + \mathrm{i}x^3) \quad\text{and}\quad x^{2\dot 1} \;=\; -\bar x^{1\dot 2} \;=\; (x^2 - \mathrm{i}x^1) \tag{III.71}$$

in four bosonic dimensions. In the three-dimensional case, we have

$$y^{\dot 1\dot 1} \;=\; -\bar y^{\dot 2\dot 2} \;=\; (x^1 + \mathrm{i}x^2) \;=:\; y\ ,\quad y^{\dot 1\dot 2} \;=\; \bar y^{\dot 1\dot 2} \;=\; -x^3\ . \tag{III.72}$$

Here, we can only fix a real structure on the fermionic coordinates if the number of supersymmetries $\mathcal{N}$ is even (see e.g. [163, 182]). In these cases, one groups together the fermionic coordinates in pairs and defines matrices

$$(\varepsilon_r{}^s) \;:=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ ,\quad r,s \;=\; 1,2 \quad\text{and}\quad (T_i{}^j) \;:=\; \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\ ,\quad i,j \;=\; 1,\dots,4\ .$$

The action of τ_{-1} is then given by

$$\tau_{-1}\begin{pmatrix} \theta^{11} & \theta^{12} \\ \theta^{21} & \theta^{22} \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar\theta^{11} & \bar\theta^{12} \\ \bar\theta^{21} & \bar\theta^{22} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{III.73}$$

for $\mathcal{N}$=2 and by

$$\tau_{-1}\begin{pmatrix} \theta^{11} & \cdots & \theta^{14} \\ \theta^{21} & \cdots & \theta^{24} \end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar\theta^{11} & \cdots & \bar\theta^{14} \\ \bar\theta^{21} & \cdots & \bar\theta^{24} \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $\mathcal{N}$=4. The last equation can also be written in components as

$$\tau_{-1}(\theta^{\alpha i}) \;=\; -\varepsilon^{\alpha\beta}T_j{}^i\bar\theta^{\beta j}\ , \tag{III.74a}$$

where there is a summation over β and j. The same definition applies to $\eta_i^{\dot\alpha}$:

$$\tau_{-1}(\eta_i^{\dot\alpha}) \;=\; \varepsilon^{\dot\alpha\dot\beta}T_i{}^j\bar\eta_j^{\dot\beta}\ . \tag{III.74b}$$

The reality conditions here are symplectic Majorana conditions, which read explicitly

$$\tau_{-1}(\theta^{\alpha i}) \;=\; \theta^{\alpha i} \quad\text{and}\quad \tau_{-1}(\eta_i^{\dot\alpha}) \;=\; \eta_i^{\dot\alpha}\ . \tag{III.75}$$

We have for instance for $\mathcal{N}$=4

$$\tau\begin{pmatrix} \eta_1^{\dot 1} & \eta_2^{\dot 1} & \eta_3^{\dot 1} & \eta_4^{\dot 1} \\ \eta_1^{\dot 2} & \eta_2^{\dot 2} & \eta_3^{\dot 2} & \eta_4^{\dot 2} \end{pmatrix} \;=\; \begin{pmatrix} -\bar\eta_2^{\dot 2} & \bar\eta_1^{\dot 2} & -\bar\eta_4^{\dot 2} & \bar\eta_3^{\dot 2} \\ \bar\eta_2^{\dot 1} & -\bar\eta_1^{\dot 1} & \bar\eta_4^{\dot 1} & -\bar\eta_3^{\dot 1} \end{pmatrix}\ . \tag{III.76}$$

Chapter IV

Field Theories

The purpose of this chapter is to give an overview of the field theories we will encounter in this thesis. Basic facts together with the necessary well-known results are recalled for convenience and in order to fix our notation.

IV.1 Supersymmetric field theories

First, let us briefly recall some elementary facts on supersymmetric field theories which will become useful in the subsequent discussion. In particular, we will discuss the $\mathcal{N}=1$ superfield formalism and present some features of supersymmetric quantum field theories as supersymmetric Ward-Takahashi identities and non-renormalization theorems. The relevant references for this section are [287, 49, 186, 8, 35].

IV.1.1 The $\mathcal{N}=1$ superspace formalism

When discussing the massless representations of the $\mathcal{N}=1$ supersymmetry algebra in III.1.2, §6, we encountered two multiplets: the chiral multiplet with fields having helicities $(0, \frac{1}{2})$ and the vector multiplet consisting of fields with helicities $(\frac{1}{2}, 1)$. There is a nice way of representing both multiplets as superfunctions on the superspace $\mathbb{R}^{4|4}$, which allows us to easily write down supersymmetric actions and often simplifies further examinations of supersymmetric theories significantly. Throughout this section, we will assume a superspace with Minkowski signature.

§1 General superfield. A general superfield on the $\mathcal{N}=1$ superspace $\mathbb{R}^{4|4}$ with coordinates $(x^\mu, \theta^{\alpha i}, \bar{\theta}^i_{\dot{\alpha}})$ can be expanded as a power series in the Graßmann variables with highest monomial $\theta^2\bar{\theta}^2$. However, this representation of the supersymmetry algebra is reducible and by applying different constraints onto the general superfield, we will obtain two irreducible representations: the chiral superfield and the vector superfield.

§2 Chiral superfield. Chiral and anti-chiral superfields Φ and $\bar{\Phi}$ are defined via the condition

$$\bar{D}_{\dot{\alpha}}\Phi \;=\; 0 \quad \text{and} \quad D_\alpha\bar{\Phi} \;=\; 0\;, \tag{IV.1}$$

respectively. These conditions are most generally solved by restricting the functions Φ and $\bar{\Phi}$ to the chiral and anti-chiral subspaces $\mathbb{R}^{4|2}_L$ and $\mathbb{R}^{4|2}_R$ of the superspace $\mathbb{R}^{4|4}$:

$$\Phi \;=\; \Phi(y_L^{\alpha\dot{\alpha}}, \theta^\alpha) \quad \text{and} \quad \bar{\Phi} \;=\; \bar{\Phi}(y_R^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\alpha}})\;. \tag{IV.2}$$

Let us now focus on the chiral superfields, the anti-chiral ones are obtained by complex conjugation. Their component expansion reads as[1]

$$\Phi(y,\theta) \;=\; \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y)\;, \tag{IV.3}$$

[1]Recall our convention for spinor bilinears, e.g. $\theta\theta = \theta^\alpha\theta_\alpha$ and $\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$.

where ϕ is a complex scalar with helicity 0, ψ_α is a Weyl spinor with helicity $\frac{1}{2}$ and F is an auxiliary field which causes the supersymmetry algebra to close off-shell. The field Φ now contains the complete chiral multiplet and the supersymmetry transformations are easily read off to be

$$\delta\phi \;=\; \sqrt{2}\varepsilon\psi\ ,\quad \delta\psi_\alpha \;=\; \sqrt{2}\partial_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha} - \sqrt{2}F\varepsilon_\alpha \quad\text{and}\quad \delta F \;=\; \sqrt{2}\partial_{\alpha\dot\alpha}\psi^\alpha\bar\varepsilon^{\dot\alpha}\ . \tag{IV.4}$$

Note that this superfield contains 4 real bosonic and 4 real fermionic degrees of freedom off-shell. On-shell, the component F becomes an auxiliary field and we are left with 2 real bosonic and 2 real fermionic degrees of freedom.

Correspondingly, the complex conjugate field $\bar\Phi$ is an anti-chiral superfield containing the anti-chiral multiplet with fields of helicity 0 and $-\frac{1}{2}$.

§3 Vector superfield. To represent the vector multiplet containing fields of helicity $\pm\frac{1}{2}$ and ± 1, it is clear that we will need both left- and right-handed Graßmann variables, and the vector superfield will be a function on the full $\mathcal{N}=1$ superspace $\mathbb{R}^{4|4}$. Naïvely, this gives rise to 16 components in the superfield expansion. However, by imposing the so-called *Wess-Zumino gauge*, one can reduce the components and obtain the following field expansion

$$V_{\mathrm{WZ}} \;=\; \mathrm{i}\theta\bar\theta\sigma^\mu A_\mu(x) + \mathrm{i}\theta\theta\bar\theta\bar\lambda(x) - \mathrm{i}\bar\theta\bar\theta\theta\lambda(x) + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta D(x)\ , \tag{IV.5}$$

giving rise to the *real* Lie algebra valued vector superfield $V_{\mathrm{WZ}} = -V_{\mathrm{WZ}}^\dagger$, where we chose the generators of the gauge group to be anti-Hermitian. A disadvantage of this gauge is that it is not invariant under supersymmetry transformations, i.e. any supersymmetry transformation will cause additional terms in the expansion (IV.5) to appear, which, however, can subsequently be gauged away.

A gauge transformation is now generated by a Lie algebra valued chiral superfield Λ and acts on a vector superfield V by

$$\mathrm{e}^V \;\to\; \mathrm{e}^{i\Lambda^\dagger}\mathrm{e}^V\mathrm{e}^{-\mathrm{i}\Lambda}\ . \tag{IV.6}$$

There are two corresponding field strengths defined by

$$W_\alpha \;=\; -\tfrac{1}{4}\bar D\bar D\left(\mathrm{e}^{-2V}D_\alpha\mathrm{e}^{2V}\right) \quad\text{and}\quad \bar W_{\dot\alpha} \;=\; -\tfrac{1}{4}DD\left(\mathrm{e}^{2V}\bar D_{\dot\alpha}\mathrm{e}^{-2V}\right)\ , \tag{IV.7}$$

the first of which is chiral (since $\bar D^3 = 0$), the second anti-chiral. Both field strengths transform covariantly under the gauge transformations (IV.6):

$$W_\alpha \;\to\; \mathrm{e}^{\mathrm{i}\Lambda}W_\alpha\mathrm{e}^{-\mathrm{i}\Lambda} \quad\text{and}\quad \bar W_{\dot\alpha} \;\to\; \mathrm{e}^{\mathrm{i}\Lambda^\dagger}\bar W_{\dot\alpha}\mathrm{e}^{-\mathrm{i}\Lambda^\dagger}\ . \tag{IV.8}$$

Eventually, let us stress that all the above formulæ were given for a non-Abelian gauge group and simplify considerably for Abelian gauge groups.

§4 SUSY invariant actions. Actions which are invariant under supersymmetry are now easily constructed by considering polynomials in superfields and integrating over the appropriate superspace. When constructing such actions, one has however to guarantee that the action is Hermitian and that additional symmetries, as e.g. gauge invariance are manifest. The former is easily achieved by adding complex conjugated terms for chiral expressions. An example for such a gauge invariant action is

$$S \sim \int \mathrm{d}^4x\ \operatorname{tr}\left(\int \mathrm{d}^2\theta\ WW + \int \mathrm{d}^2\bar\theta\ \bar W\bar W\right)\ , \tag{IV.9}$$

corresponding to $\mathcal{N}=1$ super Yang-Mills theory. Note that both terms in (IV.9) are real and equal. Furthermore, a coupling to chiral matter is achieved via an additional term $\sim \int \mathrm{d}^4x\mathrm{d}^4\theta\ \bar\Phi\mathrm{e}^{2V}\Phi$ in the action. In the latter case, gauge transformations act on the chiral superfields according to $\Phi \to \mathrm{e}^{\mathrm{i}\Lambda}\Phi$ and $\bar\Phi \to \bar\Phi\mathrm{e}^{-\mathrm{i}\Lambda^\dagger}$.

§5 On-shell massive representations for superspin 0. By combining two complex massless chiral superfields via the equations

$$-\tfrac{1}{4}\bar{D}^2\bar{\Phi} + m\Phi \;=\; 0 \quad \text{and} \quad -\tfrac{1}{4}D^2\Phi + m\bar{\Phi} \;=\; 0\ , \tag{IV.10}$$

one can find real on-shell representations in terms of $\mathcal{N} = 1$ superfields also for a massive multiplet of superspin 0. This fact is essential in constructing the Wess-Zumino model.

IV.1.2 The Wess-Zumino model

One of the most popular supersymmetric field theories is the Wess-Zumino model. It is well-suited as a toy model to demonstrate features of supersymmetric field theories arising due to their supersymmetry, as e.g. non-renormalization theorems.

§6 Action. This model was proposed by J. Wess and B. Zumino in [288] and is given by the action

$$S_{\text{WZM}} \;:=\; \int \mathrm{d}^4x\mathrm{d}^4\theta\ \bar{\Phi}\Phi + \int \mathrm{d}^4x\mathrm{d}^2\theta\ \mathscr{L}_c(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\ \bar{\mathscr{L}}_c(\bar{\Phi})\ , \tag{IV.11}$$

where $\mathscr{L}_c$ is a holomorphic function of a complex field, the *chiral superpotential.* The kinetic term arises from $\bar{\Phi}\Phi$ after Taylor-expanding the chiral superfields around the non-chiral coordinate $x^{\alpha\dot{\alpha}}$. While $\mathscr{L}_c$ is classically unrestricted, renormalizability demands that it is at most a third-order polynomial, and we will adapt the common notation

$$\mathscr{L}_c \;:=\; \frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3\ . \tag{IV.12}$$

We have dropped the monomial of order 1, as it simply amounts to a constant shift in the superfield Φ.

§7 Equations of motion. The corresponding equations of motion are easily derived to be

$$-\tfrac{1}{4}\bar{D}^2\bar{\Phi} + \mathscr{L}_c'(\Phi) \;=\; 0 \quad \text{and} \quad -\tfrac{1}{4}D^2\Phi + \bar{\mathscr{L}}_c'(\bar{\Phi}) \;=\; 0\ . \tag{IV.13}$$

§8 The Landau-Ginzburg model. A possibility of generalizing the action (IV.11) is to allow for several chiral superfields. Such a model with n massless chiral superfields Φ_a and a polynomial interaction is called a *Landau-Ginzburg model* and its action reads as

$$S = \int \mathrm{d}^4x \left(\int \mathrm{d}^4\theta\ \mathscr{K}(\Phi_a, \bar{\Phi}_a) + \frac{1}{2}\int \mathrm{d}^2\theta\ \mathscr{L}_c(\Phi_a) + \frac{1}{2}\int \mathrm{d}^2\theta\ \bar{\mathscr{L}}_c(\bar{\Phi}_a) \right)\ , \tag{IV.14}$$

where $\mathscr{L}_c(\Phi_a)$ is again called the (chiral) superpotential. The vacua of the theory are the critical points of $\mathscr{L}_c(\Phi_a)$.

The function $\mathscr{K}(\Phi_a, \bar{\Phi}_a)$ can be considered as a *Kähler potential* and defines the *Kähler metric* $g_{i\bar{\jmath}} := \partial_i\partial_{\bar{\jmath}}\mathscr{K}(\Phi_a, \bar{\Phi}_a)$. Note that the component fields in the action (IV.14) couple via the Kähler metric $g_{i\bar{\jmath}}$ and higher derivatives of the Kähler potential. For vanishing $\mathscr{L}_c$, the Landau-Ginzburg is a nonlinear sigma model (cf. section V.3.1), which defines a Kähler geometry via $\mathscr{K}$. One can also obtain a supersymmetric nonlinear sigma model from a Kähler geometry.

It is known that the Landau-Ginzburg models with a single chiral superfield Φ and polynomial interaction $\mathscr{L}_c(\Phi) = \Phi^{P+2}$ has central charge $c_P = \frac{3P}{P+2}$ at its infrared fixed point and can be shown to be essentially the P-th minimal model.

IV.1.3 Quantum aspects

The heavy constraints imposed by supersymmetry on a quantum field theory become manifest at quantum level: the additional symmetry leads to a cancellation of contributions from certain Feynman diagrams and the vacuum energy does not receive any quantum corrections. Furthermore, there is additional structure found in the correlation functions, the so-called chiral rings, which we will discuss momentarily. Certain properties of these rings lead quite directly to non-renormalization theorems, which strongly constrain the allowed quantum corrections and simplify considerably the study of a supersymmetric quantum field theory.

§9 Quantization. Consider a quantum field theory with a set of fields φ and an action functional $S[\varphi]$ which splits into a free and an interaction part $S[\varphi] = S_0[\varphi] + S_{\rm int}[\varphi]$. The generating functional is given by

$$\mathcal{Z}[J] \;:=\; \int \mathscr{D}\varphi \mathrm{e}^{\mathrm{i}S[\varphi]+\int \mathrm{d}^4x\ \varphi J} \tag{IV.15}$$

from which the n-point correlation functions are defined by

$$G_n(x_1,\ldots,x_n) \;:=\; \frac{1}{\mathcal{Z}[J]}\frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} \;. \tag{IV.16}$$

Perturbation theory is done in terms of the power expansion of the following reformulation of the generating functional:

$$\mathcal{Z}[J] \;=\; \mathrm{e}^{\mathrm{i}S_{\rm int}[\frac{\delta}{\delta J}]}\mathcal{Z}_0[J] \quad \text{with} \quad \mathcal{Z}_0[J] \;:=\; \int \mathscr{D}\varphi \mathrm{e}^{\mathrm{i}S_0[\varphi]+\int \mathrm{d}^4x\ \varphi J}\;, \tag{IV.17}$$

which will yield a power series in the coupling constants contained in $S_{\rm int}$.

§10 Low energy effective action. Consider a quantum field theory with fields ϕ and action S. We choose some cutoff Λ and decompose the fields into high- and low-frequency parts:

$$\phi \;=\; \phi_H + \phi_L \quad \text{with} \quad \phi_H : \omega \;>\; \Lambda \quad \phi_L : \omega \;<\; \Lambda\;. \tag{IV.18}$$

In the path integral, we then perform the integration of all high-frequency fields ϕ_H and arrive at

$$\int \mathscr{D}\phi_L \mathscr{D}\phi_H \mathrm{e}^{\mathrm{i}S[\phi_L,\phi_H]} \;=\; \int \mathscr{D}\phi_L \mathrm{e}^{\mathrm{i}S_\Lambda[\phi_L]}\;, \tag{IV.19}$$

where $S_\Lambda[\phi_L]$ is the so-called[2] *low energy effective action* or *Wilsonian effective action*. For more details, see e.g. [220].

§11 Chiral rings and correlation functions. The chiral rings of operators in supersymmetric quantum field theories are cohomology rings of the supercharges $Q_{i\alpha}$ and $\bar{Q}^i_{\dot\alpha}$. Correlation functions which are built out of elements of a single such chiral ring have peculiar properties.

Recall that the vacuum is annihilated by both the supersymmetry operators $Q_{\alpha i}$ and $\bar{Q}^i_{\dot\alpha}$. Using this fact, we see that $Q_{\alpha i}$- and $\bar{Q}^i_{\dot\alpha}$-exact operators cause a correlation function built of $Q_{\alpha i}$ or $\bar{Q}^i_{\dot\alpha}$-closed operators to vanish, e.g.

$$\begin{aligned}\langle\{\!\![Q,A]\!\!\}\bar{\mathcal{O}}_1\ldots\bar{\mathcal{O}}_n\rangle &= \langle\{\!\![Q,A\bar{\mathcal{O}}_1\ldots\bar{\mathcal{O}}_n]\!\!\}\rangle \pm \langle A\{\!\![Q,\bar{\mathcal{O}}_1]\!\!\}\ldots\bar{\mathcal{O}}_n\rangle\\ &\quad\pm\ldots\pm\langle A\bar{\mathcal{O}}_1\ldots\{\!\![Q,\bar{\mathcal{O}}_n]\!\!\}\rangle\\ &= \langle QA\bar{\mathcal{O}}_1\ldots\bar{\mathcal{O}}_n\rangle \pm \langle A\bar{\mathcal{O}}_1\ldots\bar{\mathcal{O}}_nQ\rangle \;=\; 0\;.\end{aligned} \tag{IV.20}$$

The resulting two cohomology rings are called the *chiral* and *anti-chiral ring*.

[2]This effective action is not to be confused with the 1PI effective action related to the one particle irreducible diagrams and calculated from the standard generating functional via a Legendre transform.

§12 Supersymmetric Ward-Takahashi identities. The existence of chiral rings in our theory leads to supersymmetric Ward-Takahashi identities. Since one can write any derivative with respect to bosonic coordinates as an anticommutator of the supercharges due to the supersymmetry algebra, any such derivative of correlation functions built purely out of chiral or anti-chiral operators will vanish:

$$\frac{\partial}{\partial x^\mu}\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \;=\; \frac{\partial}{\partial x^\mu}\langle \bar{\mathcal{O}}_1 \dots \bar{\mathcal{O}}_n \rangle \;=\; 0 \ . \tag{IV.21}$$

The correlation functions do not depend on the bosonic coordinates of the operators, and hence one can move them to a far distance of each other, which causes the correlation function to factorize[3]:

$$\langle \bar{\mathcal{O}}_1(x_1) \dots \bar{\mathcal{O}}_n(x_n) \rangle \;=\; \langle \bar{\mathcal{O}}_1(x_1^\infty) \rangle \dots \langle \bar{\mathcal{O}}_n(x_n^\infty) \rangle \ . \tag{IV.22}$$

Such a correlation function therefore does not contain any contact terms. This phenomenon is called *clustering* in the literature.

Another direct consequence of the existence of chiral rings is the holomorphic dependence of the chiral correlation functions on the coupling constants, i.e.

$$\frac{\partial}{\partial \bar{\lambda}}\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \;=\; 0 \ . \tag{IV.23}$$

As an illustrative example for this, consider the case of a $\mathcal{N}=1$ superpotential "interaction" term added to the Lagrangian,

$$\mathcal{L}_{\rm W} \;=\; \int \mathrm{d}^2\theta\ \lambda\Phi + \int \mathrm{d}^2\bar{\theta}\ \bar{\lambda}\bar{\Phi} \ , \tag{IV.24}$$

where $\Phi = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) - \theta^2 F(y)$ is a chiral superfield and one of the supersymmetry transformations is given by $\{Q_\alpha, \psi_\beta\} \sim \varepsilon_{\alpha\beta}F$. Then we have

$$\begin{aligned} \frac{\partial}{\partial \bar{\lambda}}\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \;&=\; \int \mathrm{d}^4y \mathrm{d}^2\bar{\theta}\langle \mathcal{O}_1 \dots \mathcal{O}_n \bar{\Phi} \rangle \;\sim\; \int \mathrm{d}^4y \langle \mathcal{O}_1 \dots \mathcal{O}_n F \rangle \\ &\sim\; \int \mathrm{d}^4y \langle \mathcal{O}_1 \dots \mathcal{O}_n \{\bar{Q}_{\dot\alpha}, \bar{\psi}^{\dot\alpha}\} \rangle \;=\; 0 \ . \end{aligned} \tag{IV.25}$$

§13 Non-renormalization theorems. It is in the non-renormalization theorems[4] that supersymmetric field theories reveal their full power as quantum field theories.

- ▷ Every term in the effective action of an $\mathcal{N}=1$ supersymmetric quantum field theory can be written as an integral over the full superspace.
- ▷ The general structure of the effective action of the Wess-Zumino model is given by

  $$\Gamma[\Phi,\bar{\Phi}] \;=\; \sum_n \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n \int \mathrm{d}^4\theta\ f(x_1,\dots,x_n) F_1(x_1,\theta) \dots F_n(x_n,\theta) \ ,$$

  where the F_i are local functions of the fields Φ,$\bar{\Phi}$ and their covariant derivatives.
- ▷ The superpotential of the Wess-Zumino model is not renormalized at all. For more details on this point, see section VI.3.3, §15.

[3]This observation has first been made in [209].

[4]For more details and a summary of non-renormalization theorems, see [48].

▷ This is also true for $\mathcal{N}=1$ super Yang-Mills theories. The renormalization of the kinetic term only happens through the gauge coupling and here not beyond one-loop order.

▷ All vacuum diagrams sum up to zero and thus, consistently with our analysis of the supersymmetry algebra, the vacuum energy is indeed zero.

▷ The action of $\mathcal{N}=2$ supersymmetric theories can always be written as

$$\tfrac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\mathrm{d}^2\theta^1\mathrm{d}^2\theta^2\ \mathscr{F}(\Psi)\ , \tag{IV.26}$$

where $\mathscr{F}$ is a holomorphic function of Ψ called the *prepotential*. The field Ψ is the $\mathcal{N}=2$ chiral superfield composed of a $\mathcal{N}=1$ chiral superfield Φ and the super field strength W_α according to

$$\Psi\ =\ \Phi(y,\theta^1)+\sqrt{2}\theta^{2\alpha}W_\alpha(y,\theta^1)+\theta^{2\alpha}\theta^2_\alpha G(y,\theta^1)\ . \tag{IV.27}$$

For $\mathcal{N}=2$ super Yang-Mills theory, the prepotential is $\mathscr{F}\sim\ \mathrm{tr}\,(\Psi^2)$.

▷ The β-function for $\mathcal{N}=4$ super Yang-Mills theory vanishes and hence the coupling constant does not run.

IV.2 Super Yang-Mills theories

In the following section, we describe the maximally supersymmetric Yang-Mills theories one obtains from $\mathcal{N}=1$ in ten dimensions by dimensional reduction. The key references for our discussion are [41, 120, 119] (super Yang-Mills) and [83, 274] (instantons and monopoles). Further references are found in the respective sections.

IV.2.1 Maximally supersymmetric Yang-Mills theories

§1 Preliminaries. We start from d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ with Minkowski metric $\eta_{\mu\nu}=\mathrm{diag}(+1,-1,\dots,-1)$. On this space, consider a vector bundle with a connection, i.e. a one-form A_μ taking values in the Lie algebra of a chosen gauge group G. We will always assume that the corresponding generators are anti-Hermitian. The associated *field strength* is defined by

$$F_{\mu\nu}\ :=\ [\nabla_\mu,\nabla_\nu]\ =\ \partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\ . \tag{IV.28}$$

Consider furthermore a spinor λ transforming in the double cover $\mathsf{Spin}(1,d-1)$ of $\mathsf{SO}(1,d-1)$ and in the adjoint representation of the gauge group G. Its *covariant derivative* is defined by $\nabla_\mu\lambda:=\partial_\mu\lambda+[A_\mu,\lambda]$.

Gauge transformations, which are parameterized by smooth sections g of the trivial bundle $G\times\mathbb{R}^{1,d-1}$, will act on the above fields according to

$$A_\mu\ \mapsto\ g^{-1}A_\mu g+g^{-1}\partial_\mu g\ ,\quad F_{\mu\nu}\ \mapsto\ g^{-1}F_{\mu\nu}g\ ,\quad \lambda\ \mapsto\ g^{-1}\lambda g\ , \tag{IV.29}$$

and thus the terms

$$\mathrm{tr}\,\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\quad\text{and}\quad\mathrm{tr}\,\left(\mathrm{i}\bar\lambda\Gamma^\mu\nabla_\mu\lambda\right) \tag{IV.30}$$

are gauge invariant quantities. Note that Γ^μ is a generator of the Clifford algebra $\mathscr{C}(1,d-1)$.

§2 $\mathcal{N}=1$ super Yang-Mills theory. Recall that a gauge potential in d dimensions has $d-2$ degrees of freedom, while the counting for a Dirac spinor yields $2^{\lfloor \frac{d}{2} \rfloor}$. By additionally imposing a Majorana or a Weyl condition, we can further halve the degrees of freedom of the spinor. Thus the action

$$S = \int \mathrm{d}^d x \;\; \mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\bar{\lambda}\Gamma^\mu\nabla_\mu\lambda\right) \tag{IV.31}$$

can only posses a linear supersymmetry in dimensions four, six and ten. More explicitly, supersymmetry is possible in $d=10$ with both the Majorana and the Weyl condition imposed on the spinor λ, in $d=6$ with the Weyl condition imposed on λ and in $d=4$ with either the Majorana or the Weyl condition[5] imposed on λ. These theories will then have $\mathcal{N}=1$ supersymmetry.

In the following, we will always be interested in maximally supersymmetric Yang-Mills theories and thus start from $\mathcal{N}=1$ in $d=10$ with 16 supercharges. Higher numbers of supersymmetries will lead to a graviton appearing in the supermultiplet, and in supergravity [277], one in fact considers theories with 32 supercharges. On the other hand, as we saw by the above considerations of degrees of freedom, we cannot construct $\mathcal{N}=1$ supersymmetric field theories in higher dimensions. Further theories will then be obtained by dimensional reduction.

§3 $\mathcal{N}=1$ SYM theory in $d=10$. This theory is defined by the action[6] [41]

$$S = \int \mathrm{d}^{10} x \;\; \mathrm{tr}\left(-\tfrac{1}{4}\hat{F}_{MN}\hat{F}^{MN} + \tfrac{\mathrm{i}}{2}\hat{\bar{\lambda}}\Gamma^M\hat{\nabla}_M\hat{\lambda}\right)\;, \tag{IV.32}$$

where $\hat{\lambda}$ is a 16-dimensional Majorana-Weyl spinor and therefore satisfies

$$\hat{\lambda} \;=\; C\hat{\bar{\lambda}}^T \quad \text{and} \quad \hat{\lambda} \;=\; +\Gamma\hat{\lambda}\;. \tag{IV.33}$$

Here, C is the charge conjugation operator and $\Gamma = \mathrm{i}\Gamma_0\ldots\Gamma_9$. The supersymmetry transformations are given by

$$\delta\hat{A}_M \;=\; \mathrm{i}\bar{\alpha}\Gamma_M\hat{\lambda} \quad \text{and} \quad \delta\hat{\lambda} \;=\; \Sigma_{MN}\hat{F}^{MN}\alpha\;. \tag{IV.34}$$

§4 Constraint equations. Instead of deriving the equations of motion of ten-dimensional SYM theory from an action, one can also use so-called *constraint equations*, which are the compatibility conditions of a linear system and thus fit naturally in the setting of integrable systems. These constraint equations are defined on the superspace $\mathbb{R}^{10|16}$ with Minkowski signature on the body. They read

$$\{\hat{\nabla}_A, \hat{\nabla}_B\} \;=\; 2\Gamma^M_{AB}\hat{\nabla}_M\;, \tag{IV.35}$$

where $\hat{\nabla}_M = \partial_M + \hat{\omega}_M$ is the covariant derivative in ten dimensions and

$$\hat{\nabla}_A \;:=\; D_A + \hat{\omega}_A \;:=\; \frac{\partial}{\partial\theta^A} + \Gamma^M_{AB}\theta^B\frac{\partial}{\partial x^M} + \hat{\omega}_A \tag{IV.36}$$

is the covariant superderivative. Note that both the fields $\hat{\omega}_M$ and $\hat{\omega}_A$ are superfields. From these potentials, we construct the spinor superfield and the bosonic curvature

$$\hat{\lambda}^B \;:=\; \tfrac{1}{10}\Gamma^{MAB}[\hat{\nabla}_M, \hat{\nabla}_A] \quad \text{and} \quad \hat{F}_{MN} \;:=\; [\hat{\nabla}_M, \hat{\nabla}_N]\;. \tag{IV.37}$$

[5] Here, actually both are equivalent.

[6] In this section, we will always denote the fields of the ten-dimensional theory by a hat.

Using Bianchi identities and identities for the Dirac matrices in ten dimensions, one obtains the superfield equations

$$\Gamma^M_{AB}\hat{\nabla}_M\hat{\lambda}^B \;=\; 0 \quad \text{and} \quad \hat{\nabla}^M\hat{F}_{MN}+\tfrac{1}{2}\Gamma_{NAB}\{\hat{\lambda}^A,\hat{\lambda}^B\} \;=\; 0\;. \tag{IV.38}$$

One can show that these equations are satisfied if and only if they are satisfied to zeroth order in their θ-expansion [120]. We will present this derivation in more detail for the case of $\mathcal{N}=4$ SYM theory in four dimensions in section IV.2.2.

§5 Dimensional reduction. A *dimensional reduction* of a theory from $\mathbb{R}^d$ to $\mathbb{R}^{d-q}$ is essentially a Kaluza-Klein compactification on the q-torus T^q, cf. also section V.2.3. The fields along the compact directions can be expanded as a discrete Fourier series, where the radii of the cycles spanning T^q appear as inverse masses of higher Fourier modes. Upon taking the size of the cycles to zero, the higher Fourier modes become infinitely massive and thus decouple. In this way, the resulting fields become independent of the compactified directions. The Lorentz group on $\mathbb{R}^d$ splits during this process into the remaining Lorentz group on $\mathbb{R}^{d-q}$ and an internal symmetry group $\mathsf{SO}(q)$. When dimensionally reducing a supersymmetric gauge theory, the latter group will be essentially the R-symmetry of the theory and the number of supercharges will remain the same, see also [253] for more details.

Let us now exemplify this discussion with the dimensional reduction of ten-dimensional $\mathcal{N}=1$ SYM theory to $\mathcal{N}=4$ SYM theory in four dimensions.

§6 $\mathcal{N}=4$ SYM theory in $d=4$. The dimensional reduction from $d=10$ to $d=4$ is easiest understood by replacing each spacetime index M by (μ,ij). This reflects the underlying splittings of $\mathsf{SO}(9,1)\rightarrow\mathsf{SO}(3,1)\times\mathsf{SO}(6)$ and $\mathsf{Spin}(9,1)\rightarrow\mathsf{Spin}(3,1)\times\mathsf{Spin}(6)$, where $\mathsf{Spin}(6)\cong\mathsf{SU}(4)$. The new index μ belongs to the four-dimensional vector representation of $\mathsf{SO}(3,1)$, while the indices ij label the representation of $\mathsf{Spin}(6)$ by antisymmetric tensors of $\mathsf{SU}(4)$. Accordingly, the gauge potential $\hat{A}_M$ is split into (A_μ,ϕ_{ij}) as

$$\hat{A}_\mu \;=\; A_\mu \quad \text{and} \quad \phi_{i4} \;=\; \frac{\hat{A}_{i+3}+\mathrm{i}\hat{A}_{i+6}}{\sqrt{2}} \quad \text{with} \quad \phi^{ij} \;:=\; \tfrac{1}{2}\varepsilon^{ijkl}\phi_{kl}\;. \tag{IV.39}$$

The gamma matrices decompose as

$$\Gamma^\mu \;=\; \gamma^\mu\otimes\mathbb{1} \quad \text{and} \quad \Gamma^{ij} \;=\; \gamma_5\otimes\begin{pmatrix} 0 & \rho^{ij} \\ \rho_{ij} & 0 \end{pmatrix}\;, \tag{IV.40}$$

where Γ^{ij} is antisymmetric in ij and ρ^{ij} is a 4×4 matrix given by

$$(\rho_{ij})_{kl} \;:=\; \varepsilon_{ijkl} \quad \text{and} \quad \left(\rho^{ij}\right)_{kl} \;:=\; \frac{1}{2}\varepsilon^{ijmn}\varepsilon_{mnkl}\;. \tag{IV.41}$$

With these matrices, one finds that

$$\Gamma \;=\; \Gamma_0\cdots\Gamma_9 \;=\; \gamma_5\otimes\mathbb{1}_8 \quad \text{and} \quad C_{10} \;=\; C\otimes\begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix}\;, \tag{IV.42}$$

where C is again the charge conjugation operator. For the spinor λ, we have

$$\lambda \;=\; \begin{pmatrix} L\chi^i \\ R\tilde{\chi}_i \end{pmatrix} \quad \text{with} \quad \tilde{\chi}_i \;=\; C(\bar{\chi}^i)^T\;, \quad L=\frac{\mathbb{1}+\gamma_5}{2}\;, \quad R \;=\; \frac{\mathbb{1}-\gamma_5}{2}\;. \tag{IV.43}$$

The resulting action and further details on the theory are found in section IV.2.2.

§7 Remark on the Euclidean case. Instead of compactifying the ten-dimensional theory on the torus T^6, one can also compactify this theory on the *Minkowski torus* $T^{5,1}$, using an appropriate decomposition of the Clifford algebra [25] and adjusted reality conditions on the fields. This derivation, however, leads to a non-compact R-symmetry group and one of the scalars having a negative kinetic term. As it is not possible to start from an $\mathcal{N}=1$ SYM action in ten Euclidean dimensions containing Majorana-Weyl spinors, it is better to adjust the $\mathcal{N}=4$ SYM action on four-dimensional Minkowski space "by hand" for obtaining the corresponding Euclidean action by using a Wick rotation. This is consistent with the procedure we will use later on: to consider all fields and symmetry groups in the complex domain and apply the desired reality conditions only later on.

§8 Further dimensional reductions. There are further dimensional reductions which are in a similar spirit to the above discussed reduction from ten-dimensional $\mathcal{N}=1$ SYM theory to $\mathcal{N}=4$ SYM theory in four dimensions. Starting from $\mathcal{N}=1$ SYM theory in ten dimensions (six dimensions), one obtains $\mathcal{N}=2$ SYM theory in six dimensions (four dimensions). Equally well one can continue the reduction of $\mathcal{N}=2$ SYM theory in six dimensions to $\mathcal{N}=4$ SYM theory in four dimensions. The reduction of $\mathcal{N}=4$ SYM theory in four dimensions to three dimensions leads to a $\mathcal{N}=8$ SYM theory, where $\mathcal{N}=8$ supersymmetry arises from splitting the complex $\mathsf{Spin}(3,1)\cong\mathsf{SL}(2,\mathbb{C})$ supercharges in four dimensions into real $\mathsf{Spin}(2,1)\cong\mathsf{SL}(2,\mathbb{R})$ ones in three dimensions. We will discuss this case in more detail in section IV.2.5.

IV.2.2 $\mathcal{N}=4$ SYM theory in four dimensions

The maximally supersymmetric Yang-Mills theory in four dimensions is the one with $\mathcal{N}=4$ supersymmetry and thus with 16 supercharges. This theory received much attention, as it is a conformal theory even at quantum level and therefore its β-function vanishes. In fact, both perturbative contributions and instanton corrections are finite, and it is believed that $\mathcal{N}=4$ SYM theory is *finite at quantum level*. In the recent development of string theory, this theory played an important rôle in the context of the AdS/CFT correspondence [187] and twistor string theory [297].

§9 Action and supersymmetry transformations. The field content of four-dimensional $\mathcal{N}=4$ SYM theory obtained by dimensional reduction as presented above consists of a gauge potential A_μ, four chiral and anti-chiral spinors χ^i_α and $\chi^{\dot\alpha}_i$ and three complex scalars arranged in the antisymmetric matrix ϕ_{ij}. These fields are combined in the action

$$\begin{aligned} S=\int \mathrm{d}^4x\ \operatorname{tr}\Big\{&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\nabla_\mu\phi_{ij}\nabla^\mu\phi^{ij}-\tfrac{1}{4}\left[\phi_{ij},\phi_{kl}\right]\left[\phi^{ij},\phi^{kl}\right]\\ &+\mathrm{i}\bar\chi\gamma^\mu\nabla_\mu L\chi-\tfrac{\mathrm{i}}{2}\left(\bar{\tilde\chi}^i[L\chi^j,\phi_{ij}]-\bar\chi_i[R\tilde\chi_j,\phi^{ij}]\right)\Big\}\ , \end{aligned} \tag{IV.44}$$

where we introduced the shorthand notation $\phi^{ij}:=\frac{1}{2}\varepsilon^{ijkl}\phi_{kl}$ which also implies $\phi_{ij}=\frac{1}{2}\varepsilon_{ijkl}\phi^{kl}$. The corresponding supersymmetry transformations are parameterized by four complex spinors α^i which satisfy the same Majorana condition as χ^i. We have

$$\begin{aligned} \delta A_\mu &= \mathrm{i}\left(\bar\alpha_i\gamma_\mu L\chi^i-\bar\chi_i\gamma_\mu L\alpha^i\right)\ ,\\ \delta\phi_{ij} &= \mathrm{i}\left(\bar\alpha_j R\tilde\chi_i-\bar\alpha_i R\tilde\chi_j+\varepsilon_{ijkl}\bar{\tilde\alpha}^k L\chi^l\right)\ ,\\ \delta L\chi^i &= \sigma_{\mu\nu}F^{\mu\nu}L\alpha^i-\gamma^\mu\nabla_\mu\phi^{ij}R\tilde\alpha_j+\tfrac{1}{2}[\phi^{ik},\phi_{kj}]L\alpha^j\ ,\\ \delta R\tilde\chi_i &= \sigma_{\mu\nu}F^{\mu\nu}R\tilde\alpha_i+\gamma^\mu\nabla_\mu\phi_{ij}L\alpha^j+\tfrac{1}{2}[\phi_{ik},\phi^{kj}]R\tilde\alpha_j\ . \end{aligned} \tag{IV.45}$$

One should stress that contrary to the Yang-Mills supermultiplets for $\mathcal{N} \leq 3$, the $\mathcal{N} = 4$ supermultiplet is irreducible.

§10 Underlying symmetry groups. Besides the supersymmetry already discussed in the upper paragraph, the theory is invariant under the Lorentz group $\mathsf{SO}(3,1)$ and the R-symmetry group $\mathsf{Spin}(6) \cong \mathsf{SU}(4)$. Note, however, that the true automorphism group of the $\mathcal{N} = 4$ supersymmetry algebra is the group $\mathsf{U}(4)$. Due to its adjoint action on the fields, the sign of the determinant is not seen and therefore only the subgroup $\mathsf{SU}(4)$ of $\mathsf{U}(4)$ is realized.

As already mentioned, this theory is furthermore conformally invariant and thus we have the conformal symmetry group $\mathsf{SO}(4,2) \cong \mathsf{SU}(2,2)$, which also gives rise to conformal supersymmetry additionally to Poincaré supersymmetry.

Altogether, the underlying symmetry group is the supergroup $\mathsf{SU}(2,2|4)$. As this is also the symmetry of the space $AdS_5 \times S^5$, $\mathcal{N} = 4$ SYM theory is one of the major ingredients of the AdS/CFT correspondence.

§11 $\mathcal{N} = 3$ and $\mathcal{N} = 4$ SYM theories. The automorphism group of the $\mathcal{N} = 3$ supersymmetry algebra is $\mathsf{U}(3)$. However, the R-symmetry group of $\mathcal{N} = 3$ SYM theory is only $\mathsf{SU}(3)$. With respect to the field content and its corresponding action and equations of motion, $\mathcal{N} = 3$ and $\mathcal{N} = 4$ SYM theory are completely equivalent. When considering the complexified theories, one has to impose an additional condition in the case $\mathcal{N} = 4$, which reads [290] $\phi_{ij} = \frac{1}{2}\varepsilon_{ijkl}\bar{\phi}^{kl}$ and makes the fourth supersymmetry linear.

§12 Spinorial notation. Let us switch now to spinorial notation (see also III.2.3, §23), which will be much more appropriate for our purposes. Furthermore, we will choose a different normalization for our fields to match the conventions in the publications we will report on.

In spinorial notation, we essentially substitute indices μ by pairs $\alpha\dot{\alpha}$, i.e. we use the spinor representation $(\mathbf{\frac{1}{2}}, \mathbf{\bar{\frac{1}{2}}})$ equivalent to the $\mathbf{4}$ of $\mathsf{SO}(3,1)$. The Yang-Mills field strength thus reads as[7]

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] =: F_{\alpha\dot{\alpha},\beta\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} + \varepsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} , \tag{IV.46}$$

and the action takes the form

$$\begin{aligned} S = \int \mathrm{d}^4x \ \mathrm{tr} \Big\{ & f^{\dot{\alpha}\dot{\beta}} f_{\dot{\alpha}\dot{\beta}} + f^{\alpha\beta} f_{\alpha\beta} + \nabla^{\alpha\dot{\alpha}}\phi^{ij}\nabla_{\alpha\dot{\alpha}}\phi_{ij} + \tfrac{1}{8}[\phi^{ij},\phi^{kl}][\phi_{ij},\phi_{kl}] + \\ & + \varepsilon^{\alpha\beta}\varepsilon^{\dot{\beta}\dot{\gamma}}(\chi^i_\alpha(\nabla_{\beta\dot{\beta}}\bar{\chi}_{i\dot{\gamma}}) - (\nabla_{\beta\dot{\beta}}\chi^i_\alpha)\bar{\chi}_{i\dot{\gamma}}) - \varepsilon^{\alpha\beta}\chi^k_\alpha[\chi^l_\beta,\phi_{kl}] - \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{i\dot{\alpha}}[\bar{\chi}_{j\dot{\beta}},\phi^{ij}] \Big\} . \end{aligned} \tag{IV.47}$$

In this notation, we can order the field content according to helicity. The fields $(f_{\alpha\beta}, \chi^i_\alpha, \phi_{ij}, \bar{\chi}_{i\dot{\alpha}}, f_{\dot{\alpha}\dot{\beta}})$ are of helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$, respectively.

§13 Equations of motion. The equations of motion of $\mathcal{N} = 4$ SYM theory are easily obtained by varying (IV.47) with respect to the different fields. For the spinors, we obtain the equations

$$\varepsilon^{\alpha\beta}\nabla_{\alpha\dot{\alpha}}\chi^i_\beta + [\phi^{ij}, \bar{\chi}_{j\dot{\alpha}}] = 0 , \tag{IV.48a}$$

$$\varepsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\bar{\chi}_{i\dot{\beta}} + [\phi_{ij}, \chi^j_\alpha] = 0 , \tag{IV.48b}$$

and the bosonic fields are governed by the equations

$$\varepsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\gamma\dot{\alpha}} f_{\dot{\beta}\dot{\gamma}} + \varepsilon^{\alpha\beta}\nabla_{\alpha\dot{\gamma}} f_{\beta\gamma} = \tfrac{1}{2}[\nabla_{\gamma\dot{\gamma}}\phi_{ij}, \phi^{ij}] + \{\chi^i_\gamma, \bar{\chi}_{i\dot{\gamma}}\} , \tag{IV.48c}$$

$$\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\nabla_{\beta\dot{\beta}}\phi_{ij} - \tfrac{1}{2}[\phi^{kl}, [\phi_{kl}, \phi_{ij}]] = \tfrac{1}{2}\varepsilon_{ijkl}\varepsilon^{\alpha\beta}\{\chi^k_\alpha, \chi^l_\beta\} + \varepsilon^{\dot{\alpha}\dot{\beta}}\{\bar{\chi}_{i\dot{\alpha}}, \bar{\chi}_{j\dot{\beta}}\} . \tag{IV.48d}$$

[7] The following equations include the decomposition of the field strength into self-dual and anti-self-dual parts, see the next section.

§14 Remark on superspace formulation. There is a formulation of $\mathcal{N}=4$ SYM theory in the $\mathcal{N}=1$ superfield formalism. The field content is reconstructed from three chiral superfields plus a vector superfield and the corresponding action reads [101, 164]

$$\begin{aligned} S = \mathrm{tr}\int \mathrm{d}^4x \Bigg(\int \mathrm{d}^4\theta\ \mathrm{e}^{-V}\Phi_I^\dagger \mathrm{e}^V\Phi^I + \tfrac{1}{4}\left(\int \mathrm{d}^2\theta\ \tfrac{1}{4}W^\alpha W_\alpha + c.c.\right) + \\ +\ \mathrm{i}\tfrac{\sqrt{2}}{3!}\left(\int \mathrm{d}^2\theta\ \varepsilon_{IJK}\Phi^I[\Phi^J,\Phi^K] + \int \mathrm{d}^2\bar\theta\ \varepsilon^{IJK}\Phi_I^\dagger[\Phi_J^\dagger,\Phi_K^\dagger]\right)\Bigg)\ , \end{aligned} \tag{IV.49}$$

where I,J,K run from 1 to 3. A few remarks are in order here: Only a $\mathsf{SU}(3)\times\mathsf{U}(1)$ subgroup of the R-symmetry group $\mathsf{SU}(4)$ is manifest. This is the R-symmetry group of $\mathcal{N}=3$ SYM theory, and this theory is essentially equivalent to $\mathcal{N}=4$ SYM theory as mentioned above. Furthermore, one should stress that this is an $\mathcal{N}=1$ formalism only, and far from a manifestly off-shell supersymmetric formulation of the theory. In fact, such a formulation would require an infinite number of auxiliary fields. For further details, see [164].

§15 Constraint equations for $\mathcal{N}=4$ SYM theory. Similarly to the ten-dimensional SYM theory, one can derive the equations of motion of $\mathcal{N}=4$ SYM theory in four dimensions from a set of constraint equations on $\mathbb{R}^{4|16}$. They read as

$$\begin{gathered} \{\nabla_{\alpha i},\nabla_{\beta j}\} \;=\; -2\varepsilon_{\alpha\beta}\phi_{ij}\ ,\quad \{\bar\nabla^i_{\dot\alpha},\bar\nabla^j_{\dot\beta}\} \;=\; -2\varepsilon_{\dot\alpha\dot\beta}\phi^{ij}\ , \\ \{\nabla_{\alpha i},\bar\nabla^j_{\dot\beta},\} \;=\; -2\delta_i^j\nabla_{\alpha\dot\beta}\ , \end{gathered} \tag{IV.50}$$

where we introduced the covariant derivatives

$$\nabla_{\alpha i} \;=\; D_{\alpha i} + \{\!\!\{\omega_{\alpha i},\cdot\}\!\!\}\ ,\quad \nabla^i_{\dot\alpha} \;=\; \bar D^i_{\dot\alpha} - \{\!\!\{\bar\omega^i_{\dot\alpha},\cdot\}\!\!\}\ ,\quad \nabla_{\alpha\dot\alpha} \;=\; \partial_{\alpha\dot\alpha} + \{\!\!\{A_{\alpha\dot\alpha},\cdot\}\!\!\}\ . \tag{IV.51}$$

We can now define the superfields whose components will be formed by the field content of $\mathcal{N}=4$ SYM theory. As such, we have the bosonic curvature

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}] \;=:\; F_{\alpha\dot\alpha\beta\dot\beta} \;=\; \varepsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} + \varepsilon_{\alpha\beta}f_{\dot\alpha\dot\beta} \tag{IV.52}$$

and the two superspinor fields

$$[\nabla_{\alpha i},\nabla_{\beta\dot\beta}] \;=:\; \varepsilon_{\alpha\beta}\bar\chi_{i\dot\beta} \quad\text{and}\quad [\bar\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}] \;=:\; \varepsilon_{\dot\alpha\dot\beta}\chi^i_\beta\ . \tag{IV.53}$$

Using the graded Bianchi identities (cf. (III.12)) for all possible combinations of covariant derivatives introduced above, we obtain the equations of motion of $\mathcal{N}=4$ SYM theory (IV.48) with all the fields being superfields.

§16 Superfield expansions. It can be shown that the equations (IV.48) with all fields being superfields are satisfied if and only if they are satisfied to zeroth order in the superfield expansion. To prove this, we need to calculate explicitly the shape of this expansion, which can always be done following a standard procedure: We first impose a transverse gauge condition

$$\theta^{\alpha i}\omega_{\alpha i} - \bar\theta^{\dot\alpha}_i\bar\omega^i_{\dot\alpha} \;=\; 0\ , \tag{IV.54}$$

which allows us to introduce the fermionic Euler operator

$$\mathcal{D} \;=\; \theta\nabla + \bar\theta\bar\nabla \;=\; \theta D + \bar\theta\bar D\ . \tag{IV.55}$$

Together with the constraint equations (IV.50), we then easily obtain

$$(1+\mathcal{D})\omega_{\alpha i} \;=\; 2\bar{\theta}^{\dot{\alpha}}_i A_{\alpha\dot{\alpha}} - 2\varepsilon_{\alpha\beta}\theta^{\beta j}\phi_{ij}\ , \tag{IV.56a}$$

$$(1+\mathcal{D})\bar{\omega}^i_{\dot{\alpha}} \;=\; 2\theta^{\alpha i}A_{\alpha\dot{\alpha}} - \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{ijkl}\bar{\theta}^{\dot{\beta}}_j\phi_{kl}\ , \tag{IV.56b}$$

and with the graded Bianchi-identities we can calculate

$$\mathcal{D}A_{\alpha\dot{\alpha}} \;=\; -\varepsilon_{\alpha\beta}\theta^{i\beta}\bar{\chi}_{i\dot{\alpha}} + \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}_i\chi^i_\alpha\ , \tag{IV.56c}$$

$$\mathcal{D}\phi_{ij} \;=\; \varepsilon_{ijkl}\theta^{k\alpha}\chi^l_\alpha - \bar{\theta}^{\dot{\alpha}}_i\bar{\chi}_{j\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}_j\bar{\chi}_{i\dot{\alpha}}\ , \tag{IV.56d}$$

$$\mathcal{D}\chi^i_\alpha \;=\; -2\theta^{i\beta}f_{\alpha\beta} + \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{iklm}\theta^{\beta j}[\phi_{lm},\phi_{jk}] - \varepsilon^{ijkl}\bar{\theta}^{\dot{\alpha}}_j\nabla_{\alpha\dot{\alpha}}\phi_{kl}\ , \tag{IV.56e}$$

$$\mathcal{D}\bar{\chi}_{i\dot{\alpha}} \;=\; 2\theta^{j\alpha}\nabla_{\alpha\dot{\alpha}}\phi_{ij} + 2\bar{\theta}^{\dot{\beta}}_i f_{\dot{\alpha}\dot{\beta}} + \tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{jklm}\bar{\theta}^{\dot{\beta}}_j[\phi_{lm},\phi_{ik}]\ . \tag{IV.56f}$$

From the above equations, one can recursively reconstruct the exact field expansion of the superfields whose zeroth order components form the $\mathcal{N}=4$ supermultiplet. In the following, we will only need a detailed expansion in θ which, up to quadratic order in the θs, is given by

$$A_{\alpha\dot{\alpha}} = \overset{\circ}{A}_{\alpha\dot{\alpha}} + \varepsilon_{\alpha\beta}\overset{\circ}{\bar{\chi}}_{i\dot{\alpha}}\theta^{i\beta} - \varepsilon_{\alpha\beta}\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{\phi}_{ij}\theta^{i\beta}\theta^{j\gamma} \;+\;\cdots\ , \tag{IV.57a}$$

$$\phi_{ij} = \overset{\circ}{\phi}_{ij} - \varepsilon_{ijkl}\overset{\circ}{\chi}{}^l_\alpha\theta^{k\alpha} - \varepsilon_{ijkl}(\delta^l_m\overset{\circ}{f}_{\beta\alpha} + \tfrac{1}{4}\varepsilon_{\beta\alpha}\varepsilon^{lnpq}[\overset{\circ}{\phi}_{pq},\overset{\circ}{\phi}_{mn}])\theta^{k\alpha}\theta^{m\beta} \;+\;\cdots\ , \tag{IV.57b}$$

$$\begin{aligned}\chi^i_\alpha = {}& \overset{\circ}{\chi}{}^i_\alpha - (2\delta^i_j\overset{\circ}{f}_{\beta\alpha} + \tfrac{1}{2}\varepsilon_{\beta\alpha}\varepsilon^{iklm}[\overset{\circ}{\phi}_{lm},\overset{\circ}{\phi}_{jk}])\theta^{\beta j} + {}\\ &\Big\{\tfrac{1}{2}\delta^i_j\varepsilon^{\dot{\alpha}\dot{\beta}}(\varepsilon_{\gamma\alpha}\overset{\circ}{\nabla}_{\beta\dot{\alpha}}\overset{\circ}{\bar{\chi}}_{k\dot{\beta}} + \varepsilon_{\gamma\beta}\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{\bar{\chi}}_{k\dot{\beta}}) - {}\\ &\tfrac{1}{4}\varepsilon_{\alpha\beta}\varepsilon^{ipmn}(\varepsilon_{jkpq}[\overset{\circ}{\phi}_{mn},\overset{\circ}{\chi}{}^q_\gamma] + \varepsilon_{mnkq}[\overset{\circ}{\phi}_{jp},\overset{\circ}{\chi}{}^q_\gamma])\Big\}\,\theta^{\beta j}\theta^{k\gamma} \;+\;\cdots\ ,\end{aligned} \tag{IV.57c}$$

$$\bar{\chi}_{i\dot{\alpha}} = \overset{\circ}{\bar{\chi}}_{i\dot{\alpha}} + 2\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{\phi}_{ij}\theta^{j\alpha} + (\varepsilon_{ijkl}\overset{\circ}{\nabla}_{\alpha\dot{\alpha}}\overset{\circ}{\chi}{}^l_\beta + \varepsilon_{\alpha\beta}[\overset{\circ}{\phi}_{ij},\overset{\circ}{\bar{\chi}}_{k\dot{\alpha}}])\theta^{j\alpha}\theta^{k\beta} \;+\;\cdots\ . \tag{IV.57d}$$

Therefore, the equations (IV.48) with all the fields being superfields are indeed equivalent to the $\mathcal{N}=4$ SYM equations.

IV.2.3 Supersymmetric self-dual Yang-Mills theories

In the following, we will always restrict ourselves to four dimensional spacetimes with Euclidean ($\varepsilon=-1$) or Kleinian ($\varepsilon=+1$) signature. Furthermore, we will label the Graßmann variables on $\mathbb{R}^{4|2\mathcal{N}}_R$ by $\eta^{\dot{\alpha}}_i$, and since the Weyl spinors χ and $\bar{\chi}$ are no longer related via complex conjugation we redenote $\bar{\chi}$ by $\tilde{\chi}$.

§17 Self-dual Yang-Mills theory. Self-dual Yang-Mills (SDYM) fields on $\mathbb{R}^{4,0}$ and $\mathbb{R}^{2,2}$ are solutions to the self-duality equations

$$F_{\mu\nu} \;=\; \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \quad\text{or}\quad F \;=\; *F \tag{IV.58}$$

which are equivalently written in spinor notation as

$$f_{\dot{\alpha}\dot{\beta}} \;:=\; -\frac{1}{2}\varepsilon^{\alpha\beta}(\partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}} + [A_{\alpha\dot{\alpha}},A_{\beta\dot{\beta}}]) \;=\; 0\ . \tag{IV.59}$$

Solutions to these equations form a subset of the solution space of Yang-Mills theory. If such a solution is of finite energy, it is called an *instanton*. Recall that an arbitrary Yang-Mills field strength decomposes into a self-dual $f_{\alpha\beta}$ and an anti-self-dual part $f_{\dot{\alpha}\dot{\beta}}$:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} \;=\; \varepsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta} + \varepsilon_{\alpha\beta}f_{\dot{\alpha}\dot{\beta}}\ , \tag{IV.60}$$

where the former is of helicity +1 and has equal electric and magnetic components and the latter is of helicity −1 and has magnetic and electric components of opposite signs. Furthermore, it is known that the only local symmetries of the self-dual Yang-Mills equations (for a semisimple gauge group G) are the conformal group[8] and the gauge symmetry [226].

§18 Supersymmetric extensions of the SDYM equations. As the self-dual Yang-Mills equations form a subsector of the full Yang-Mills theory, a possible supersymmetric extension of the self-duality equations can be obtained by taking the full set of SYM field equations and imposing certain constraints on them. These constraints have to include (IV.58) and keep the resulting set of equations invariant under supersymmetry transformations. This works for SYM theories with $\mathcal{N} \leq 3$, and the field content of the full $\mathcal{N}$-extended SYM theory splits into a self-dual supermultiplet and an anti-self-dual supermultiplet:

	$h = 1$	$h = \frac{1}{2}$	$h = 0$	$h = -\frac{1}{2}$	$h = -1$
$\mathcal{N} = 0$	$f_{\alpha\beta}$				$f_{\dot\alpha\dot\beta}$
$\mathcal{N} = 1$	$f_{\alpha\beta}$	λ_α		$\lambda_{\dot\alpha}$	$f_{\dot\alpha\dot\beta}$
$\mathcal{N} = 2$	$f_{\alpha\beta}$	λ^i_α	$\phi^{[12]}$ $\phi_{[12]}$	$\lambda_{\dot\alpha i}$	$f_{\dot\alpha\dot\beta}$
$\mathcal{N} = 3$	$f_{\alpha\beta}$	λ_α χ^i_α	$\phi^{[ij]}$ $\phi_{[ij]}$	$\chi_{\dot\alpha i}$ $\lambda_{\dot\alpha}$	$f_{\dot\alpha\dot\beta}$
$\mathcal{N} = 4$	$f_{\alpha\beta}$	χ^i_α	$\phi^{[ij]} = \frac{1}{2}\varepsilon^{ijkl}\phi_{[kl]}$	$\chi_{\dot\alpha i}$	$f_{\dot\alpha\dot\beta}$

(IV.61)

where each column consists of fields with a certain helicity and each row contains a supermultiplet for a certain value of $\mathcal{N}$. The indices $i, j, \ldots$ always run from 1 to $\mathcal{N}$. From the table (IV.61), we see that for $\mathcal{N}$=4, the situation is more complicated, as the SYM multiplet $(f_{\alpha\beta}, \chi^{\alpha i}, \phi^{ij}, \tilde\chi_{\dot\alpha i}, f_{\dot\alpha\dot\beta})$, where the fields have the helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$, is irreducible. By introducing an additional field $G_{\dot\alpha\dot\beta}$ with helicity -1, which takes in some sense the place of $f_{\dot\alpha\dot\beta}$, one can circumvent this problem (see e.g. [260, 71]). The set of physical fields for $\mathcal{N}$=4 SYM theory consists of the self-dual and the anti-self-dual field strengths of a gauge potential $\mathcal{A}_{\alpha\dot\alpha}$, four spinors χ^i_α together with four spinors $\tilde\chi_{\dot\alpha i} \sim \varepsilon_{ijkl}\tilde\chi^{jkl}_{\dot\alpha}$ of opposite chirality and six real (or three complex) scalars $\phi^{ij} = \phi^{[ij]}$. For $\mathcal{N}$=4 super SDYM theory, the multiplet is joined by an additional spin-one field $G_{\dot\alpha\dot\beta} \sim \varepsilon_{ijkl}G^{ijkl}_{\dot\alpha\dot\beta}$ with helicity −1 and the multiplet is – after neglecting the vanishing anti-self-dual field strength $f_{\dot\alpha\dot\beta}$ – identified with the one of $\mathcal{N} = 4$ SYM theory.

§19 Equations of motion. Using the above mentioned auxiliary field $G_{\dot\alpha\dot\beta}$, we arrive at the following equations of motion:

$$\begin{aligned}
f_{\dot\alpha\dot\beta} &= 0\ ,\\
\nabla_{\alpha\dot\alpha}\chi^{\alpha i} &= 0\ ,\\
\Box\phi^{ij} &= -\tfrac{\varepsilon}{2}\{\chi^{\alpha i}, \chi^j_\alpha\}\ ,\\
\nabla_{\alpha\dot\alpha}\tilde\chi^{\dot\alpha ijk} &= +2\varepsilon\,[\phi^{[ij}, \chi^{k]}_\alpha]\ ,\\
\varepsilon^{\dot\alpha\dot\gamma}\nabla_{\alpha\dot\alpha}G^{ijkl}_{\dot\gamma\dot\delta} &= +\varepsilon\{\chi^{[i}_\alpha, \tilde\chi^{jkl]}_{\dot\delta}\} - \varepsilon\,[\phi^{[ij}, \nabla_{\alpha\dot\delta}\phi^{kl]}]\ ,
\end{aligned} \tag{IV.62}$$

where we introduced the shorthand notations $\Box := \frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla^{\alpha\dot\alpha}$ and $\varepsilon = \pm 1$ distinguishes between Kleinian and Euclidean signature on the spacetime under consideration.

[8]The conformal group on $\mathbb{R}^{p,q}$ is given by $\mathsf{SO}(p + 1, q + 1)$.

§20 Action for $\mathcal{N}=4$ SDYM theory. The action reproducing the above equations of motion was first given in [260] and reads with our scaling of fields as

$$S \;=\; \int \mathrm{d}^4x\ \mathrm{tr}\,\left(G^{\dot\alpha\dot\beta}f_{\dot\alpha\dot\beta}+\tfrac{\varepsilon}{2}\varepsilon_{ijkl}\tilde\chi^{\dot\alpha ijk}\nabla_{\alpha\dot\alpha}\chi^{\alpha l}+\tfrac{\varepsilon}{2}\varepsilon_{ijkl}\phi^{ij}\Box\phi^{kl}+\varepsilon_{ijkl}\phi^{ij}\chi^{\alpha k}\chi^l_\alpha\right)\;, \tag{IV.63}$$

where $G_{\dot\alpha\dot\beta}:=\frac{1}{4!}\varepsilon_{ijkl}G^{ijkl}_{\dot\alpha\dot\beta}$. Note that although the field content appearing in this action is given by the multiplet $(f_{\alpha\beta},\chi^{\alpha i},\phi^{ij},\tilde\chi_{\dot\alpha i},f_{\dot\alpha\dot\beta},G_{\dot\alpha\dot\beta})$, $f_{\dot\alpha\dot\beta}$ vanishes due to the SDYM equations of motion and the supermultiplet of non-trivial fields is $(f_{\alpha\beta},\chi^{\alpha i},\phi^{ij},\tilde\chi_{\dot\alpha i},G_{\dot\alpha\dot\beta})$. These degrees of freedom match exactly those of the full $\mathcal{N}$=4 SYM theory and often it is stated that they are the same. Following this line, one can even consider the full $\mathcal{N}$=4 SYM theory and $\mathcal{N}$=4 SDYM theory as the same theories on linearized level, which are only distinguished by different interactions.

§21 Constraint equations. Similarly to the case of the full $\mathcal{N}=4$ SYM theory, one can obtain the equations of motion (IV.62) also from a set of constraint equations. These constraint equations live on the chiral superspace $\mathbb{R}^{4|2\mathcal{N}}_R$ with coordinates $(x^{\alpha\dot\alpha},\eta^{\dot\alpha}_i)$ and read explicitly as

$$\begin{aligned}[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]+[\nabla_{\alpha\dot\beta},\nabla_{\beta\dot\alpha}] \;=\; 0\;,\quad [\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]+[\nabla^i_{\dot\beta},\nabla_{\beta\dot\alpha}] \;=\; 0\;,\\ \{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}+\{\nabla^i_{\dot\beta},\nabla^j_{\dot\alpha}\} \;=\; 0\;,\end{aligned} \tag{IV.64}$$

where we have introduced covariant derivatives

$$\nabla_{\alpha\dot\alpha} \;:=\; \frac{\partial}{\partial x^{\alpha\dot\alpha}}+\mathcal{A}_{\alpha\dot\alpha}\quad\text{and}\quad \nabla^i_{\dot\alpha} \;:=\; \frac{\partial}{\partial\eta^{\dot\alpha}_i}+\mathcal{A}^i_{\dot\alpha}\;. \tag{IV.65}$$

Note that the gauge potentials $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ are functions on the chiral superspace $\mathbb{R}^{4|2\mathcal{N}}_R$. Equations (IV.64) suggest the introduction of the following self-dual super gauge field strengths:

$$\begin{aligned}[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}f_{\alpha\beta}(x,\eta)\;,\quad [\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}f^i_\beta(x,\eta)\;,\\ \{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}f^{ij}(x,\eta)\;,\end{aligned} \tag{IV.66}$$

and by demanding that f^{ij} is antisymmetric and $f_{\alpha\beta}$ is symmetric, these equations are equivalent to (IV.64). The lowest components of $f_{\alpha\beta}$, f^i_α and f^{ij} will be the SDYM field strength, the spinor field χ^i_α and the scalars ϕ^{ij}, respectively. By using Bianchi identities for the self-dual super gauge field strengths, one can show that these definitions yield superfield equations which agree in zeroth order with the component equations of motion (IV.62) [71].

To show the actual equivalence of the superfield equations with the equations (IV.62), one proceeds quite similarly to the full SYM case, cf. §16. We impose the transverse gauge condition $\eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha}=0$ and introduce an Euler operator

$$\mathcal{D} \;:=\; \eta^{\dot\alpha}_i\nabla^i_{\dot\alpha} \;=\; \eta^{\dot\alpha}_i\partial^i_{\dot\alpha}\;, \tag{IV.67}$$

which yields the following relations:

$$\begin{aligned}\mathcal{D}f_{\alpha\beta} &= \tfrac{1}{2}\,\eta^{\dot\alpha}_i\nabla_{(\alpha\dot\alpha}\chi^i_{\beta)}\\ \mathcal{D}\chi^j_\alpha &= 2\,\eta^{\dot\alpha}_i\nabla_{\alpha\dot\alpha}\phi^{ij}\\ \mathcal{D}\phi^{jk} &= \eta^{\dot\alpha}_i\tilde\chi^{ijk}_{\dot\alpha}\\ \mathcal{D}\tilde\chi^{jkl}_{\dot\beta} &= \eta^{\dot\alpha}_i(G^{ijkl}_{\dot\alpha\dot\beta}+\varepsilon_{\dot\alpha\dot\beta}[\phi^{i[j},\phi^{kl]}])\\ \mathcal{D}G^{jklm}_{\dot\beta\dot\gamma} &= -\eta_{i(\dot\beta}\left(\tfrac{2}{3}[\phi^{i[j},\tilde\chi^{klm]}_{\dot\gamma)}]-\tfrac{1}{3}[\phi^{[jk},\tilde\chi^{lm]i}_{\dot\gamma)}]\right)\end{aligned} \tag{IV.68}$$

as well as

$$
\begin{aligned}
(1+\mathcal{D})\mathcal{A}^i_{\dot\alpha} &= 2\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\phi^{ij} \ , \\
\mathcal{D}\mathcal{A}_{\alpha\dot\alpha} &= -\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\chi^i_\alpha \ ,
\end{aligned}
\tag{IV.69}
$$

from which the field expansion can be reconstructed explicitly. For our purposes, it will always be sufficient to know that

$$
\begin{aligned}
\mathcal{A}_{\alpha\dot\alpha} &= A_{\alpha\dot\alpha} - \varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\chi^i_\alpha + \ldots - \tfrac{1}{12}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\eta^{\dot\gamma}_j\eta^{\dot\delta}_k\eta^{\dot\varepsilon}_l\nabla_{\alpha\dot\gamma}G^{ijkl}_{\dot\delta\dot\varepsilon} \ , \\
\mathcal{A}^i_{\dot\alpha} &= \varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\phi^{ik} + \tfrac{2}{3}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\eta^{\dot\gamma}_k\tilde\chi^{ijk}_{\dot\gamma} + \tfrac{1}{4}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\eta^{\dot\gamma}_k\eta^{\dot\delta}_l\left(G^{ijkl}_{\dot\gamma\dot\delta} + \varepsilon_{\dot\gamma\dot\delta}\ldots\right) \ ,
\end{aligned}
\tag{IV.70}
$$

as this already determines the field content completely.

§22 From SYM theory to super SDYM theory. Up to $\mathcal{N}=3$, solutions to the supersymmetric SDYM equations form a subset of the corresponding full SYM equations. By demanding an additional condition, one can restrict the constraint equations of the latter to the ones of the former [256, 280]. For $\mathcal{N}=1$, the condition to impose is $[\nabla_{i\alpha}, \nabla_{\beta\dot\beta}] = 0$, while for $\mathcal{N}=2$ and $\mathcal{N}=3$, one has to demand that $\{\nabla_{i\alpha}, \nabla_{j\beta}\} = \varepsilon_{\alpha\beta}\phi_{ij} = 0$. For $\mathcal{N}=4$, one can use the same condition as for $\mathcal{N}=3$, but one has to drop the usual reality condition $\phi_{ij} = \frac{1}{2}\varepsilon_{ijkl}\bar\phi^{kl}$, which renders the fourth supersymmetry nonlinear. Alternatively, one can follow the discussion in [297], where $\mathcal{N}=4$ SYM and $\mathcal{N}=4$ SDYM theories are considered as different weak coupling limits of an underlying field theory including an auxiliary field.

IV.2.4 Instantons

§23 Meaning of instantons. The dominant contribution to the partition function

$$
\mathcal{Z} := \int \mathscr{D}\varphi\, \mathrm{e}^{-S_E[\varphi,\partial_\mu\varphi]}
\tag{IV.71}
$$

of a quantum field theory defined by a (Euclidean) action S_E stems from the minima of the action functional $S_E[\varphi,\partial_\mu\varphi]$. In non-Abelian gauge theories, one calls the local minima, which exist besides the global one, *instantons*. Instantons therefore cannot be studied perturbatively, but they are non-perturbative effects.

Although they did not give rise to an explanation of quark confinement, instantons found various other applications in QCD and supersymmetric gauge theories. In mathematics, they are related to certain topological invariants on four-manifolds.

§24 Instantons in Yang-Mills theory. Consider now such a non-Abelian gauge theory on Euclidean spacetime $\mathbb{R}^4$, which describes the dynamics of a gauge potential A, a Lie algebra valued connection one-form on a bundle E, and its field strength $F = \mathrm{d}A + A\wedge A$. The corresponding Yang-Mills energy is given by the functional $-\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{tr}\,(F\wedge *F)$.

We will restrict our considerations to those gauge configurations with finite energy, i.e. the gauge potential has to approach pure gauge at infinity. This essentially amounts to considering the theory on S^4 instead of $\mathbb{R}^4$. We can then define the topological invariant $-\frac{1}{2(2\pi)^2}\int_{\mathbb{R}^4}\mathrm{tr}\,(F\wedge F)$, which is the *instanton number* and counts instantons contained in the considered configuration. Note that this invariant corresponds to a nontrivial second Chern character of the curvature F. Recall that one can write this second Chern character in terms of first and second Chern classes, see section II.2.1, §21.

We can decompose the energy functional into

$$
0 \le \tfrac{1}{2}\int_{\mathbb{R}^4}\mathrm{tr}\,((*F + \mathrm{e}^{-\mathrm{i}\theta}F)\wedge(F + \mathrm{e}^{i\theta} * F)) = \int_{\mathbb{R}^4}\mathrm{tr}\,(*F\wedge F + 2\cos\theta F\wedge F)
\tag{IV.72}
$$

for all real θ. Therefore we have

$$\tfrac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}\,(*F\wedge F) \;\leq\; \tfrac{1}{2}\left|\int_{\mathbb{R}^4} \mathrm{tr}\,(F\wedge F)\right| \tag{IV.73}$$

The configurations satisfying this bound are called BPS, cf. §34, and they form minima of the energy functional. For such configurations, either the self-dual or the anti-self-dual Yang-Mills equation holds:

$$F = \pm * F\ . \tag{IV.74}$$

The name instantons stems from the fact that these configurations are localized at spacetime points.

In our conventions, an instanton is a self-dual gauge field configuration with positive topological charge k. Anti-instantons have negative such charge and satisfy the anti-self-duality equations.

§25 Abelian instantons. From the above definition of the instanton number, it is clear that in the Abelian case, where $F = \mathrm{d}A$, no instanton solutions can exist:

$$\begin{aligned} -\frac{1}{2(2\pi)^2}\int_{\mathbb{R}^4} \mathrm{tr}\,(F\wedge F) \;&=\; -\frac{1}{2(2\pi)^2}\int_{\mathbb{R}^4} \mathrm{tr}\,\mathrm{d}(A\wedge F)\\ &=\; -\frac{1}{2(2\pi)^2}\int_{S^3} \mathrm{tr}\,(A\wedge F) \;=\; 0\ , \end{aligned} \tag{IV.75}$$

where S^3 is the sphere at spatial infinity, on which the curvature F vanishes. Note, however, that the situation is different on noncommutative spacetime, where Abelian instantons do exist.

§26 Moduli space of instantons. On a generic four-dimensional Riemann manifold M, the moduli space of instantons is the space of self-dual gauge configurations modulo gauge transformations. It is noncompact and for k $\mathsf{U}(N)$ instantons of dimension

$$4Nk - \frac{N^2-1}{2}(\chi+\sigma)\ , \tag{IV.76}$$

where χ and σ are the Euler characteristics and the signature of M, respectively.

§27 Construction of instantons. There is a number of methods for constructing instantons, which are almost all inspired by twistor geometry. We will discuss them in detail in section VII.8. There, one finds in particular a discussion of the well-known *ADHM construction* of instantons.

§28 Supersymmetric instantons. Note furthermore that in $\mathcal{N}=1$ supersymmetric gauge theories, instanton configurations break half of the supersymmetries, as they appear on the right-hand side of the supersymmetry transformations, cf. (IV.45). From (IV.45) we also see that this holds for supersymmetric instanton configurations up to $\mathcal{N}=3$.

IV.2.5 Related field theories

In this section, we want to briefly discuss two related field theories which will become important in the later discussion: $\mathcal{N}=8$ SYM theory in three dimensions and the super Bogomolny model. These theories are obtained by reduction of $\mathcal{N}=4$ SYM theory and $\mathcal{N}$-extended SDYM theory from four to three dimensions.

§29 Dimensional reduction $\mathbb{R}^4 \to \mathbb{R}^3$. Recall that the rotation group $\mathsf{SO}(4)$ of $(\mathbb{R}^4, \delta_{\mu\nu})$ is locally isomorphic to $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R \cong \mathsf{Spin}(4)$. The rotation group $\mathsf{SO}(3)$ of $(\mathbb{R}^3, \delta_{ab})$ with $a, b = 1, 2, 3$ is locally $\mathsf{SU}(2) \cong \mathsf{Spin}(3)$, which can be interpreted as the diagonal group $\mathrm{diag}(\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R)$ upon dimensional reduction to three dimensions. Therefore, the distinction between undotted, i.e. $\mathsf{SU}(2)_L$, and dotted, i.e. $\mathsf{SU}(2)_R$, indices disappears.

Explicitly, the dimensional reduction $\mathbb{R}^4 \to \mathbb{R}^3$ is now performed by introducing the new coordinates[9]

$$\begin{aligned} y^{\dot\alpha\dot\beta} &:= -\mathrm{i}x^{(\dot\alpha\dot\beta)} \quad \text{and} \quad x^{[\dot\alpha\dot\beta]} = -\varepsilon^{\dot\alpha\dot\beta}x^2 \\ \text{with } y^{\dot1\dot1} &= -\bar y^{\dot2\dot2} = (-\mathrm{i}x^4 - x^3) =: y \quad \text{and} \quad y^{\dot1\dot2} = \bar y^{\dot1\dot2} = -x^1 \end{aligned} \tag{IV.77}$$

together with the derivatives

$$\partial_{\dot\alpha\dot\alpha} := \frac{\partial}{\partial y^{\dot\alpha\dot\alpha}} \quad \text{and} \quad \partial_{\dot1\dot2} := \frac{1}{2}\frac{\partial}{\partial y^{\dot1\dot2}} \; . \tag{IV.78}$$

More abstractly, this splitting corresponds to the decomposition $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$ of the irreducible real vector representation $\mathbf{4}$ of the group $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$ into two irreducible real representations $\mathbf{3}$ and $\mathbf{1}$ of the group $\mathsf{SU}(2)$.

The four-dimensional gauge potential $A_{\alpha\dot\alpha}$ is split into a three-dimensional gauge potential $A_{(\dot\alpha\dot\beta)}$ and a Higgs field Φ

$$B_{\dot\alpha\dot\beta} = A_{\dot\alpha\dot\beta} - \tfrac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi \; , \tag{IV.79}$$

which motivates the introduction of the following differential operator and covariant derivative:

$$\nabla_{\dot\alpha\dot\beta} := \partial_{\dot\alpha\dot\beta} + B_{\dot\alpha\dot\beta} \quad \text{and} \quad D_{\dot\alpha\dot\beta} := \nabla_{(\dot\alpha\dot\beta)} = \partial_{\dot\alpha\dot\beta} + A_{\dot\alpha\dot\beta} \; . \tag{IV.80}$$

§30 Yang-Mills-Higgs theory. Yang-Mills-Higgs theory is defined in d dimensions by the action

$$S = \int \mathrm{d}^d x \;\, \mathrm{tr}\, \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \nabla_\mu\phi\nabla^\mu\phi - \tfrac{\gamma}{4}(\phi\phi^* - 1)^2\right) \; , \tag{IV.81}$$

where F is as usually the field strength of a gauge potential and ϕ is a complex scalar. The potential term can in principle be chosen arbitrarily, but renormalizability restricts it severely. The equations of motion of this theory read

$$\nabla_\mu F^{\mu\nu} = -[\nabla^\nu\phi, \phi] \quad \text{and} \quad \nabla^\mu\nabla_\mu\phi = \gamma\phi(\phi\phi^* - 1) \; . \tag{IV.82}$$

In our considerations, we will only be interested in a three-dimensional version of this theory with vanishing potential term $\gamma = 0$, which can be obtained from four-dimensional Yang-Mills theory via a dimensional reduction.

§31 $\mathcal{N} = 8$ SYM theory in three dimensions. This theory is obtained by dimensionally reducing $\mathcal{N} = 1$ SYM theory in ten dimensions to three dimensions, or, equivalently, by dimensionally reducing four-dimensional $\mathcal{N} = 4$ SYM theory to three dimensions. As a result, the 16 real supercharges are re-arranged in the latter case from four spinors transforming as a $\mathbf{2}_\mathbb{C}$ of $\mathsf{Spin}(3,1) \cong \mathsf{SL}(2,\mathbb{C})$ into eight spinors transforming as a $\mathbf{2}$ of $\mathsf{Spin}(2,1) \cong \mathsf{SL}(2,\mathbb{R})$.

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[9]The fact that we dimensionally reduce by the coordinate x^2 is related to our sigma matrix convention.

The automorphism group of the supersymmetry algebra is $\mathsf{Spin}(8)$, and the little group of the remaining Lorentz group $\mathsf{SO}(2,1)$ is trivial. As massless particle content, we therefore expect bosons transforming in the $\mathbf{8}_v$ and fermions transforming in the $\mathbf{8}_c$ of $\mathsf{Spin}(8)$. One of the bosons will, however, appear as a dual gauge potential on $\mathbb{R}^3$ after dimensional reduction, and therefore only a $\mathsf{Spin}(7)$ R-symmetry group is manifest in the action and the equations of motion. Altogether, we have a gauge potential A_a with $a=1,\ldots,3$, seven real scalars ϕ^i with $i=1,\ldots,7$ and eight spinors $\chi^j_{\dot\alpha}$ with $j=1,\ldots,8$.

Moreover, recall that in four dimensions, $\mathcal{N}=3$ and $\mathcal{N}=4$ super Yang-Mills theories are equivalent on the level of field content and corresponding equations of motion. The only difference[10] is found in the manifest R-symmetry groups which are $\mathsf{SU}(3)\times\mathsf{U}(1)$ and $\mathsf{SU}(4)$, respectively. This equivalence obviously carries over to the three-dimensional situation: $\mathcal{N}=6$ and $\mathcal{N}=8$ super Yang-Mills theories are equivalent regarding their field content and the equations of motion.

§32 The super Bogomolny model. We start from the $\mathcal{N}$-extended supersymmetric SDYM equations on $\mathbb{R}^4$, i.e. the first $\mathcal{N}$ equations of (IV.62) in which the R-symmetry indices $i,j,\ldots$ are restricted to $1,\ldots,\mathcal{N}$. After performing the dimensional reduction as presented in §29, one arrives at the field content

$$A_{\dot\alpha\dot\beta},\chi^i_{\dot\alpha},\Phi,\phi^{ij},\tilde\chi_{i\dot\alpha},G_{\dot\alpha\dot\beta} \tag{IV.83}$$

with helicities $(1,\frac{1}{2},0,0,-\frac{1}{2},-1)$, where we used the shorthand notations

$$\tilde\chi_{i\dot\alpha} \;:=\; \tfrac{1}{3!}\varepsilon_{ijkl}\tilde\chi^{jkl}_{\dot\alpha} \quad\text{and}\quad G_{\dot\alpha\dot\beta} \;:=\; \tfrac{1}{4!}\varepsilon_{ijkl}G^{ijkl}_{\dot\alpha\dot\beta}\ . \tag{IV.84}$$

The supersymmetric extension of the Bogomolny equations now read

$$\begin{aligned} f_{\dot\alpha\dot\beta} &= -\tfrac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi\ ,\\ \varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}\chi^i_{\dot\gamma} &= -\tfrac{\mathrm{i}}{2}[\Phi,\chi^i_{\dot\alpha}]\ ,\\ \triangle\phi^{ij} &= -\tfrac{1}{4}[\Phi,[\phi^{ij},\Phi]]+\varepsilon^{\dot\alpha\dot\beta}\{\chi^i_{\dot\alpha},\chi^j_{\dot\beta}\}\ ,\\ \varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}\tilde\chi_{i\dot\gamma} &= -\tfrac{\mathrm{i}}{2}[\tilde\chi_{i\dot\alpha},\Phi]+2\mathrm{i}[\phi_{ij},\chi^j_{\dot\alpha}]\ ,\\ \varepsilon^{\dot\beta\dot\gamma}D_{\dot\alpha\dot\beta}G_{\dot\gamma\dot\delta} &= -\tfrac{\mathrm{i}}{2}[G_{\dot\alpha\dot\delta},\Phi]+\mathrm{i}\{\chi^i_{\dot\alpha},\tilde\chi_{i\dot\delta}\}-\tfrac{1}{2}[\phi_{ij},D_{\dot\alpha\dot\delta}\phi^{ij}]+\tfrac{\mathrm{i}}{4}\varepsilon_{\dot\alpha\dot\delta}[\phi_{ij},[\Phi,\phi^{ij}]]\ . \end{aligned} \tag{IV.85}$$

Here, we have used the fact that we have a decomposition of the field strength in three dimensions according to

$$F_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \;=\; [D_{\dot\alpha\dot\beta},D_{\dot\gamma\dot\delta}] \;=:\; \varepsilon_{\dot\beta\dot\delta}f_{\dot\alpha\dot\gamma}+\varepsilon_{\dot\alpha\dot\gamma}f_{\dot\beta\dot\delta} \tag{IV.86}$$

with $f_{\dot\alpha\dot\beta}=f_{\dot\beta\dot\alpha}$. We have also introduced the abbreviation $\triangle:=\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\gamma\dot\delta}D_{\dot\alpha\dot\gamma}D_{\dot\beta\dot\delta}$.

For $\mathcal{N}=8$, one can write down the following action functional leading to the equations (IV.85):

$$\begin{aligned} S_{\mathrm{sB}} \;=\; \int \mathrm{d}^3x\ \mathrm{tr}\Big\{&G^{\dot\alpha\dot\beta}\left(f_{\dot\alpha\dot\beta}+\tfrac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi\right)+\mathrm{i}\varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}\chi^i_{\dot\alpha}D_{\dot\delta\dot\beta}\tilde\chi_{i\dot\gamma}+\\ &+\tfrac{1}{2}\phi_{ij}\triangle\phi^{ij}-\tfrac{1}{2}\varepsilon^{\dot\alpha\dot\delta}\chi^i_{\dot\alpha}[\tilde\chi_{i\dot\delta},\Phi]-\varepsilon^{\dot\alpha\dot\gamma}\phi_{ij}\{\chi^i_{\dot\alpha},\chi^j_{\dot\gamma}\}+\tfrac{1}{8}[\phi_{ij},\Phi][\phi^{ij},\Phi]\Big\}\ . \end{aligned} \tag{IV.87}$$

In this expression, we have again used the shorthand notation $\phi_{ij}:=\frac{1}{2!}\varepsilon_{ijkl}\phi^{kl}$.

[10]In the complexified case, one has an additional condition which takes the shape $\phi_{ij}=\frac{1}{2}\varepsilon_{ijkl}\phi^{kl}$, cf. [290].

§33 Constraint equations. Similarly to the SYM and the (super-)SDYM equations, one can give a set of constraint equations on $\mathbb{R}^{3|2\mathcal{N}}$, which are equivalent to the super Bogomolny equations on $\mathbb{R}^3$. For this, we introduce the first-order differential operators $\nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta}$ and $D^i_{\dot\alpha} = \frac{\partial}{\partial\eta_i^{\dot\alpha}} + \mathcal{A}^i_{\dot\alpha} =: \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}$, where $\mathcal{B}_{\dot\alpha\dot\beta} := \mathcal{A}_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi$. Then the appropriate constraint equations read as

$$[\nabla_{\dot\alpha\dot\gamma}, \nabla_{\dot\beta\dot\delta}] =: \varepsilon_{\dot\gamma\dot\delta}\Sigma_{\dot\alpha\dot\beta}\ , \quad [D^i_{\dot\alpha}, \nabla_{\dot\beta\dot\gamma}] =: \mathrm{i}\varepsilon_{\dot\alpha\dot\gamma}\Sigma^i_{\dot\beta} \quad \text{and} \quad \{D^i_{\dot\alpha}, D^j_{\dot\beta}\} =: \varepsilon_{\dot\alpha\dot\beta}\Sigma^{ij}\ , \tag{IV.88}$$

where $\Sigma_{\dot\alpha\dot\beta} = \Sigma_{\dot\beta\dot\alpha}$ and $\Sigma^{ij} = -\Sigma^{ji}$. Note that the first equation in (IV.88) immediately shows that $f_{\dot\alpha\dot\beta} = -\frac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi$ and thus the contraction of the first equation of (IV.88) with $\varepsilon^{\dot\gamma\dot\delta}$ gives $\Sigma_{\dot\alpha\dot\beta} = f_{\dot\alpha\dot\beta} - \frac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi = 2f_{\dot\alpha\dot\beta}$. The graded Bianchi identities for the differential operators $\nabla_{\dot\alpha\dot\beta}$ and $D^i_{\dot\alpha}$ yield in a straightforward manner further field equations, which allow us to identify the superfields $\Sigma^i_{\dot\alpha}$ and Σ^{ij} with the spinors $\chi^i_{\dot\alpha}$ and the scalars ϕ^{ij}, respectively. Moreover, $\tilde\chi_{i\dot\alpha}$ is given by $\tilde\chi_{i\dot\alpha} := \frac{1}{3}\varepsilon_{ijkl}D^j_{\dot\alpha}\phi^{kl}$ and $G_{\dot\alpha\dot\beta}$ is defined by $G_{\dot\alpha\dot\beta} := -\frac{1}{4}D^i_{(\dot\alpha}\tilde\chi_{i\dot\beta)}$. Collecting the above information, one obtains the superfield equations for $\mathcal{A}_{\dot\alpha\dot\beta}$, $\chi^i_{\dot\alpha}$, Φ, ϕ^{ij}, $\tilde\chi_{i\dot\alpha}$ and $G_{\dot\alpha\dot\beta}$ which take the same form as (IV.85) but with all the fields now being superfields. Thus, the projection of the superfields onto the zeroth order components of their η-expansions gives (IV.85).

Similarly to all the previous constraint equations, one can turn to transverse gauge and introduce the Euler operator $\mathcal{D} := \eta_i^{\dot\alpha}D^i_{\dot\alpha}$ to recover the component fields in the superfield expansion of the superconnection $\mathcal{A}$. The explicit result is obtained straightforwardly to be

$$\begin{aligned}\mathcal{B}_{\dot\alpha\dot\beta} &= \overset{\circ}{\mathcal{B}}_{\dot\alpha\dot\beta} - \mathrm{i}\varepsilon_{\dot\beta\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\overset{\circ}{\chi}{}^{j_1}_{\dot\alpha} + \tfrac{1}{2!}\varepsilon_{\dot\beta\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\nabla_{\dot\alpha\dot\gamma_2}\overset{\circ}{\phi}{}^{j_1j_2} - \tfrac{1}{2\cdot3!}\varepsilon_{\dot\beta\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\varepsilon^{j_1j_2j_3k}\nabla_{\dot\alpha\dot\gamma_2}\overset{\circ}{\tilde\chi}_{k\dot\gamma_3} - \\ &\quad - \tfrac{1}{4!}\varepsilon_{\dot\beta\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\eta^{\dot\gamma_4}_{j_4}\varepsilon^{j_1j_2j_3j_4}\nabla_{\dot\alpha\dot\gamma_2}\overset{\circ}{G}_{\dot\gamma_3\dot\gamma_4} + \cdots\end{aligned} \tag{IV.89a}$$

$$\begin{aligned}\mathcal{A}^i_{\dot\alpha} &= \tfrac{1}{2!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\overset{\circ}{\phi}{}^{ij_1} - \tfrac{1}{3!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\varepsilon^{ij_1j_2k}\overset{\circ}{\tilde\chi}_{k\dot\gamma_2} + \\ &\quad + \tfrac{3}{2\cdot4!}\varepsilon_{\dot\alpha\dot\gamma_1}\eta^{\dot\gamma_1}_{j_1}\eta^{\dot\gamma_2}_{j_2}\eta^{\dot\gamma_3}_{j_3}\varepsilon^{ij_1j_2j_3}\overset{\circ}{G}_{\dot\gamma_2\dot\gamma_3} + \cdots\end{aligned} \tag{IV.89b}$$

The equations (IV.88) are satisfied for these expansions if the supersymmetric Bogomolny equations (IV.85) hold for the physical fields appearing in the above expansions and vice versa.

§34 BPS monopoles. The Bogomolny equations appear also as the defining equation for Bogomolny-Prasad-Sommerfield (BPS) monopole configurations [37, 233], see also [122]. We start from the Yang-Mills-Higgs Lagrangian[11] given in §30, and note that its energy functional for static configurations (A_a, ϕ) is given by

$$E = \tfrac{1}{4}\int \mathrm{d}^3x\ \operatorname{tr}(F_{ab}F_{ab} + 2D_a\phi D_a\phi)\ . \tag{IV.90}$$

To guarantee finite energy, we have to demand that

$$\lim_{|r|\to\infty}\operatorname{tr}(F_{ab}F_{ab}) = 0 \quad \text{and} \quad \lim_{|r|\to\infty}\operatorname{tr}(D_a\phi D_a\phi) = 0 \tag{IV.91}$$

sufficiently rapidly. The energy functional has a lower bound, which can be calculated to be

$$\begin{aligned}E &= -\tfrac{1}{4}\int \mathrm{d}^3x\ \operatorname{tr}(F_{ab} \mp \varepsilon_{abc}D_c\phi)(F_{ab} \mp \varepsilon_{abd}D_d\phi) \mp \int \mathrm{d}^3x\ \operatorname{tr}\left(\tfrac{1}{2}\varepsilon_{abc}F_{bc}D_c\phi\right) \\ &= -\tfrac{1}{4}\int \mathrm{d}^3x\ \operatorname{tr}(F_{ab} \mp \varepsilon_{abc}D_c\phi)(F_{ab} \mp \varepsilon_{abd}D_d\phi) \pm 4\pi Q \ \geq\ 4\pi|Q|\ .\end{aligned} \tag{IV.92}$$

[11] For convenience, we will switch again to vector indices $a, b, \ldots$ ranging from 1 to 3 in the following.

Here, we could choose the absolute value as one of the bounds always becomes trivial. We found the magnetic charge Q which can also be understood as the magnetic flux through a sphere around the origin with infinite radius:

$$Q \;=\; -\frac{1}{8\pi}\int \mathrm{d}^3x \;\;\mathrm{tr}\left(\varepsilon_{abc}F_{ab}D_c\phi\right) \;=\; -\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{d}s_a \;\;\mathrm{tr}\;\left(\tfrac{1}{2}\varepsilon_{abc}F_{bc}\phi\right)\;. \qquad \text{(IV.93)}$$

The configurations (A_a, ϕ), which satisfy the bound (IV.92) are called *BPS monopoles* and necessarily fulfill the (first order) Bogomolny equations

$$F_{ab} \;=\; \varepsilon_{abc}D_c\phi\;. \qquad \text{(IV.94)}$$

Inversely, those finite energy configurations (A_a, ϕ) which satisfy the Bogomolny equations (IV.94) are BPS monopoles.

§35 Monopole solutions. A twistor-inspired solution generating technique, the Nahm construction, is presented in section VII.8.

IV.3 Chern-Simons theory and its relatives

In this section, we briefly review basic and relevant facts on Chern-Simons theory. A broader discussion can be found in [99] and [89]. Subsequently, we present some related models, which we will encounter later on. In particular, we will present a holomorphic Chern-Simons theory [293], which will play a vital rôle in chapter VII.

IV.3.1 Basics

§1 Motivation. Chern-Simons theory is a completely new type of gauge theories, which was accidently discovered by Shiing-Shen Chern and James Harris Simons when studying Pontryagin densities of 3-manifolds [62]. It is crucial in 3-manifold topology and knot theory and its partition function defines the Witten-Reshetikhin-Turaev invariant, a topological invariant of 3-manifolds. Furthermore, perturbation theory gives rise to an infinite number of other topological invariants.

Chern-Simons theories are deeply connected to anyons, particles living in two dimension which have magnetic flux tied to their electric charge and – considering a large wavelength limit – to a description of the Landau problem of charged particles moving in a plane under the influence of a magnetic field perpendicular to the plane.

§2 Abelian Chern-Simons and Maxwell theory. The difference between Chern-Simons theory and ordinary Maxwell theory is easiest seen comparing the Lagrangians and the equations of motion:

$$\mathcal{L}_{\mathrm{M}} \;=\; -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu J^\mu\;, \qquad \partial_\mu F^{\mu\nu} \;=\; J^\nu\;, \qquad \text{(IV.95a)}$$

$$\mathcal{L}_{\mathrm{CS}} \;=\; \tfrac{\kappa}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu\;, \qquad \tfrac{\kappa}{2}\varepsilon^{\mu\nu\rho}F_{\nu\rho} \;=\; J^\mu\;, \qquad \text{(IV.95b)}$$

Gauge invariance is not obvious, as the Lagrangian is not exclusively defined in terms of the invariant field strength $F_{\mu\nu}$. Nevertheless, one easily checks that gauge transforming $\mathcal{L}_{\mathrm{CS}}$ leads to a total derivative, which vanishes for manifolds without boundary.

§3 Solutions. The solutions to the Chern-Simons field equations $F_{\mu\nu} = \frac{1}{\kappa}\varepsilon_{\mu\nu\rho}J^\rho$ are trivial for vanishing source. To get nontrivial solution, there are several possibilities: One can consider couplings to matter fields and to a Maxwell term (the latter provides a new mass generation formalism for gauge fields besides the Higgs mechanism), nontrivial topology and boundaries of the configuration space or generalize the action to non-Abelian gauge fields and incorporating gravity.

§4 Non-Abelian Chern-Simons theory. Consider a vector bundle E over a three-dimensional real manifold M with a connection one-form A. Non-Abelian Chern-Simons theory is then defined by the action

$$\mathcal{L}_{CS} = \kappa\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\,\left(A_\mu\partial_\nu A_\rho + \tfrac{2}{3}A_\mu A_\nu A_\rho\right) \tag{IV.96}$$

Under a transformation δA_μ, the Lagrangian changes according to

$$\delta\mathcal{L}_{\mathrm{CS}} = \kappa\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\,(\delta A_\mu F_{\nu\rho}) \tag{IV.97}$$

with the standard non-Abelian field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The equations of motion take the same form as in the Abelian case

$$\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} = J^\mu\ . \tag{IV.98}$$

Under the non-Abelian gauge transformation, the Lagrangian transforms into

$$\mathcal{L}'_{\mathrm{CS}} = \mathcal{L}_{\mathrm{CS}} - (\text{tot. derivative}) - w(g) \tag{IV.99}$$

where $w(g)$ describes a winding number

$$\int_M w(g) = 8\pi^2\kappa N\ ,\quad N\in\mathbb{Z}\ . \tag{IV.100}$$

This gives rise to a quantization condition for κ if we demand that the partition function $\mathrm{e}^{\mathrm{i}S_{CS}}$ is invariant under gauge transformations

§5 Topological invariance. Note that the energy-momentum tensor of Chern-Simons theory vanishes:

$$T^{\mu\nu} = \frac{2}{\sqrt{\det g}}\frac{\delta S_{CS}}{\delta g_{\mu\nu}} = 0\ , \tag{IV.101}$$

which is due to the fact that $\mathcal{L}_{\mathrm{CS}}$ is independent of the metric. Therefore, Chern-Simons theory is a *topological field theory.*

§6 Quantization. Canonical quantization of the system is straightforward as the components of the gauge fields are canonically conjugate to each other:

$$[A_i(\vec{x}), A_j(\vec{y})] = \frac{\mathrm{i}}{\kappa}\varepsilon_{ij}\delta(\vec{x}-\vec{y})\ , \tag{IV.102}$$

where $i,j = 1,2$.

IV.3.2 Holomorphic Chern-Simons theory

Holomorphic Chern-Simons theory is besides super Yang-Mills theory the most important field theory we will consider. Its omnipresence is simply due to the fact that the open topological B-model on a Calabi-Yau threefold containing n space-filling D5-branes is equivalent to holomorphic Chern-Simons theory on the same Calabi-Yau manifold with gauge group $\mathsf{GL}(n,\mathbb{C})$, as we will see in section V.3.4.

§7 Setup. We start from a complex d-dimensional manifold M over which we consider a holomorphic principal G-bundle P, where G is a semisimple Lie (matrix) group with Lie algebra $\mathfrak{g}$. Consider furthermore a connection one-form (i.e. a Lie algebra valued one-form) A on P, which is carried over to the associated holomorphic vector bundle $E\to M$ of P. We define the corresponding field strength by $F = \mathrm{d}A + A\wedge A$, and denote by $A^{0,1}$ and $F^{0,2}$ the $(0,1)$-part and the $(0,2)$-part of A and F, respectively. Note that $F^{0,2} = \bar{\partial}A^{0,1} + A^{0,1}\wedge A^{0,1}$.

§8 Equations of motion. Analogously to Chern-Simons theory without sources, the equations of motion of holomorphic Chern-Simons theory simply read

$$F^{0,2} \;=\; \bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1} \;=\; 0 \;, \tag{IV.103}$$

and thus $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ defines a holomorphic structure on E, see §6 in section II.2.1. One can state that the Dolbeault description of holomorphic vector bundles is in fact a description via holomorphic Chern-Simons theory.

§9 Action. If M is a Calabi-Yau threefold and thus comes with a holomorphic $(3,0)$-form $\Omega^{3,0}$, one can write down an action of holomorphic Chern-Simons theory which reproduces the equation (IV.103):

$$S_{\text{hCS}} \;=\; \tfrac{1}{2}\int_M \Omega^{3,0} \wedge \operatorname{tr}\left(A^{0,1} \wedge \bar{\partial}A^{0,1} + \tfrac{2}{3}A \wedge A \wedge A\right) \;. \tag{IV.104}$$

This action has been introduced in [293].

§10 Remarks. In his paper [293], Witten remarks that hCS theory is superficially non-renormalizable by power counting but that its symmetries suggest that it should be finite at quantum level. This conclusion is in agreement with holomorphic Chern-Simons theory being equivalent to a string theory.

IV.3.3 Related field theories

§11 Topological BF-theory. This theory [36, 131] is an extension of Chern-Simons theory to manifolds with arbitrary dimension. Consider a semisimple Lie matrix group G with Lie algebra $\mathfrak{g}$. Furthermore, let M be a real manifold of dimension d, P a principal G-bundle over M and A a connection one-form on P. The associated curvature is – as usual – given by $F = \mathrm{d}A + A \wedge A$. Then the action of topological BF-theory is given by

$$S_{\text{BF}} \;=\; \int_M \operatorname{tr}(B \wedge F) \;, \tag{IV.105}$$

where B is a $(d-2)$-form in the adjoint representation of the gauge group G. That is, a gauge transformation $g \in \Gamma(P)$ act on the fields A and B according to

$$A \;\mapsto\; g^{-1}Ag + g^{-1}\mathrm{d}g \quad \text{and} \quad B \;\mapsto\; g^{-1}Bg \;. \tag{IV.106}$$

The equations of motion of (IV.105) read as

$$F \;=\; 0 \quad \text{and} \quad \mathrm{d}B + A \wedge B - (-1)^d B \wedge A \;=\; 0 \;, \tag{IV.107}$$

and thus BF-theory describes flat connections and d_A-closed $(d-2)$-forms on an d-dimensional real manifold.

§12 Holomorphic BF-theory. Holomorphic BF-theory [227, 142, 143] is an extension of topological BF-theory to the complex situation. As such, it can also be considered as an extension of holomorphic Chern-Simons theory to complex manifolds of complex dimensions different from three. Let us consider the same setup as above, but now with M a complex manifold of (complex) dimension d. Then the corresponding action reads

$$S_{\text{hBF}} \;=\; \int_M \operatorname{tr}(B \wedge F^{0,2}) \;, \tag{IV.108}$$

where B is here a $(d, d-2)$-form on M in the adjoint representation of the gauge group G and $F^{0,2}$ is the $(0,2)$-part of the curvature F. If M is a Calabi-Yau manifold, there is a natural holomorphic volume form $\Omega^{d,0}$ on M, and one can alternatively introduce the action

$$S_{\mathrm{hBF}} \;=\; \int_M \Omega^{d,0}\wedge \mathrm{tr}\,(B\wedge F^{0,2})\;, \tag{IV.109}$$

where B is now a $(0,d-2)$-form on M. The equations of motion in the latter case read as

$$\bar\partial A^{0,1}+A^{0,1}\wedge A^{0,1} \;=\; 0 \quad\text{and}\quad \bar\partial B + A^{0,1}\wedge B-(-1)^d B\wedge A^{0,1} \;=\; 0\;. \tag{IV.110}$$

This theory is sometimes called holomorphic θBF-theory, where $\theta=\Omega^{d,0}$.

We will encounter an example for such a holomorphic BF-theory when discussing the topological B-model on the mini-supertwistor space in section VII.6.

IV.4 Conformal field theories

A *conformal field theory* is a (quantum) field theory, which is invariant under (local) conformal, i.e. angle-preserving, coordinate transformations. Such field theories naturally arise in string theory, quantum field theory, statistical mechanics and condensed matter physics. Usually, conformal field theories are considered in two dimensions, but e.g. also $\mathcal{N}=4$ super Yang-Mills theory in four dimensions is conformal, even at quantum level. Among the many available introductions to conformal field theory, very useful ones are e.g. [102, 247, 74]. A very concise introduction can moreover be found in the first chapter of [221].

IV.4.1 CFT basics

§1 The conformal group. Infinitesimal conformal transformations $x^\mu\rightarrow x^\mu+\varepsilon^\mu$ have to preserve the square of the line element up to a local factor $\Omega(x)$, and from

$$\mathrm{d}s^2 \;\rightarrow\; \mathrm{d}s^2+(\partial_\mu\varepsilon_\nu+\partial_\nu\varepsilon_\mu)\mathrm{d}x^\mu\mathrm{d}x^\nu\;, \tag{IV.111}$$

we therefore conclude that $(\partial_\mu\varepsilon_\nu+\partial_\nu\varepsilon_\mu)\sim\eta_{\mu\nu}$. On the two-dimensional plane with complex coordinates $z=x^1+\mathrm{i}x^2$, these equations are simply the Cauchy-Riemann equations

$$\partial_1\varepsilon_1 \;=\; \partial_2\varepsilon_2 \quad\text{and}\quad \partial_1\varepsilon_2 \;=\; -\partial_2\varepsilon_1\;. \tag{IV.112}$$

Two-dimensional, local conformal transformations are thus given by holomorphic functions and these transformations are generated by

$$\ell_n \;=\; -z^{n+1}\partial_z \quad\text{and}\quad \bar\ell_n \;=\; -\bar z^{n+1}\partial_{\bar z}\;, \tag{IV.113}$$

which are the generators of the *Witt-algebra*[12]

$$[\ell_m,\ell_n] \;=\; (m-n)\ell_{m+n}\;,\quad [\bar\ell_m,\bar\ell_n] \;=\; (m-n)\bar\ell_{m+n}\;,\quad [\ell_m,\bar\ell_n] \;=\; 0\;. \tag{IV.114}$$

On the compactification $\mathbb{C}P^1$ of the complex plane $\mathbb{C}$, the *global* conformal transformations are the so-called *Möbius transformations*, which are maps $z\mapsto\frac{az+b}{cz+d}$ with $ad-bc=1$. Note that these maps form a group $\cong\mathsf{SL}(2,\mathbb{C})/\mathbb{Z}_2\cong\mathsf{SO}(3,1)$ and map circles on the sphere onto circles. They are generated by ℓ_{-1},ℓ_0,ℓ_1 and their complex conjugates.

[12]the algebra of Killing vector fields on the Riemann sphere.

§2 Exemplary theory. To briefly discuss relevant properties of conformal field theories, we will use an exemplary theory, which is introduced in this paragraph. Consider the two-dimensional field theory given by the action

$$S = \tfrac{1}{4\pi} \int \mathrm{d}^2 z\ \partial X \bar{\partial} X\ , \qquad \text{(IV.115)}$$

where $X = X(z, \bar{z})$ is a function[13] on $\mathbb{C}$ and ∂ and $\bar{\partial}$ denote derivatives with respect to z and $\bar{z}$. Furthermore, the normalization of the measure $\mathrm{d}^2 z$ is chosen such that $\int \mathrm{d}^2 z \delta^2(z, \bar{z}) = 1$. The equation of motion following from this action simply reads $\partial\bar{\partial} X(z, \bar{z}) = 0$ and the solutions to these equations are harmonic functions $X(z, \bar{z})$.

§3 Operator equation. On the quantum level, the above mentioned equation of motion is only true up to contact terms, as one easily derives

$$\begin{aligned} 0 &= \int \mathscr{D}X \frac{\delta}{\delta X(z,\bar{z})} \mathrm{e}^{-S} X(z', \bar{z}') \\ &= \langle \delta^2(z - z', \bar{z} - \bar{z}') \rangle + \tfrac{1}{2\pi} \partial_z \bar{\partial}_{\bar{z}} \langle X(z, \bar{z}) X(z', \bar{z}') \rangle\ . \end{aligned} \qquad \text{(IV.116)}$$

Such an equation is called an *operator equation*. By introducing *normal ordering*

$$: \mathcal{O}(X) :\ =\ \exp\left(\tfrac{1}{2} \int \mathrm{d}^2 z \mathrm{d}^2 \bar{z}'\ \ln |z - z'|^2 \frac{\delta}{\delta X(z, \bar{z})} \frac{\delta}{\delta X(z', \bar{z}')} \right) \mathcal{O}(X)\ , \qquad \text{(IV.117)}$$

we can cast the operator equation (IV.116) into the classical form

$$\partial_z \partial_{\bar{z}} : X(z, \bar{z}) X(z', \bar{z}') :\ =\ 0\ , \qquad \text{(IV.118)}$$

where

$$: X(z, \bar{z}) X(z', \bar{z}') :\ =\ X(z, \bar{z}) X(z', \bar{z}') + \ln |z - z'|^2\ . \qquad \text{(IV.119)}$$

Taylor expanding the above equation, we obtain an example of an *operator product expansion*:

$$X(z, \bar{z}) X(0, 0)\ =\ -\ln |z|^2 + : X^2(0, 0) : + z : X \partial X(0, 0) : + \bar{z} : X \bar{\partial} X(0, 0) : + \ldots\ .$$

§4 Energy-momentum tensor. The *energy-momentum tensor* naturally appears as Noether current for conformal transformations. Consider an infinitesimal such transformation $z' = z + \varepsilon g(z)$, which leads to a field transformation $\delta X = -g(z) \partial X - \bar{g}(\bar{z}) \bar{\partial} X$. The Noether currents are $j(z) = \mathrm{i} g(z) T(z)$ and $\bar{\jmath}(\bar{z}) = \mathrm{i} \bar{g}(\bar{z}) \tilde{T}(\bar{z})$, where we have in our exemplary theory

$$T(z)\ =\ -\tfrac{1}{2} : \partial X \partial X : \quad \text{and} \quad \tilde{T}(\bar{z})\ =\ -\tfrac{1}{2} : \bar{\partial} X \bar{\partial} X :\ . \qquad \text{(IV.120)}$$

From the condition that in the divergence $\bar{\partial} j - \partial \bar{\jmath}$ of j, each term has to vanish separately[14], and the fact that the energy-momentum tensor is the Noether current for rigid translations, one derives that the only nontrivial components of the tensor T are $T_{zz} = T(z)$, $T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z})$. (Back in real coordinates $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$, this is equivalent to the energy-momentum tensor having vanishing trace, and one can also take this property as a definition for a conformal field theory.) One can derive furthermore that in any

[13] The notation $X(z, \bar{z})$ here merely implies that X is a priori a general, not necessarily holomorphic function. Sometimes, however, it is also helpful to consider a complexified situation, in which z and $\bar{z}$ are independent, complex variables.

[14] g and $\bar{g}$ are "linearly independent"

given conformal field theory, the operator product expansion of the *energy-momentum tensor* $T(z)$ is given by

$$T(z)T(w) \;=\; \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \dots\ , \qquad \text{(IV.121)}$$

where c is called the *central charge* of the theory.

§5 Radial quantization. Let us take a short glimpse at quantum aspects of conformal field theories. For this, we compactify the complex plane along the x-axis to an infinitely long cylinder and map it via $z \mapsto \mathrm{e}^z$ to the annular region $\mathbb{C}^\times$. Time now runs radially and equal time lines are circles having the origin as their center. The equal time commutators of operators can here be easily calculated via certain contour integrals. Take e.g. charges $Q_i[C] = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} j_i$ and three circles C_1, C_2, C_3 with constant times $t_1 > t_2 > t_3$. Then the expression

$$Q_1[C_1]Q_2[C_2] - Q_1[C_3]Q_2[C_2]\ , \qquad \text{(IV.122)}$$

which vanishes classically, will turn into the commutator

$$[Q_1, Q_2][C_2] \;=\; \oint_{C_2} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \mathrm{Res}_{z\to z_2} j_1(z) j_2(z_2) \qquad \text{(IV.123)}$$

when considered as an expectation value, i.e. when inserted into the path integral. The residue arises by deforming $C_1 - C_3$ to a contour around z_2, which is possible as there are no further poles present. The operator order yielding the commutator is due to the fact that any product of operators inserted into the path integral will be automatically time-ordered, which corresponds to a *radial ordering* in our situation.

§6 Virasoro algebra. Upon radial quantization, the mode expansion of the energy-momentum tensor $T(z) = \sum_n L_n z^{-n-2}$ and $\tilde{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}$ together with the inverse relations

$$L_m \;=\; \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^{m+1} T(z) \quad \text{and} \quad \bar{L}_m \;=\; -\oint_C \frac{\mathrm{d}\bar{z}}{2\pi\mathrm{i}} \bar{z}^{m+1} \tilde{T}(\bar{z}) \qquad \text{(IV.124)}$$

then lead immediately to the *Virasoro algebra*

$$[L_n, L_m] \;=\; (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{m+n,0}\ . \qquad \text{(IV.125)}$$

This algebra is the central extension of the Witt algebra (IV.114).

§7 Canonical quantization. To canonically quantize our exemplary model (IV.115), we can use the fact that any harmonic field[15] X can be (locally) expanded as the sum of a holomorphic and an antiholomorphic function. That is, we expand ∂X as a Laurent series in z with coefficients α_m and $\bar{\partial} X$ in $\bar{z}$ with coefficients $\tilde{\alpha}_m$. Integration then yields

$$X \;=\; x - \mathrm{i}\tfrac{\alpha'}{2} p \ln|z|^2 + \mathrm{i}\sqrt{\tfrac{2}{\alpha'}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha_m}{z^m} + \frac{\tilde{\alpha}_m}{\bar{z}^m}\right)\ , \qquad \text{(IV.126)}$$

where we singled out the zeroth order in both series and identified the log-terms arising from ∂X and $\bar{\partial} X$ with translations and thus momentum. The radial quantization procedure then yields the relations

$$[\alpha_m, \alpha_n] \;=\; [\tilde{\alpha}_m, \tilde{\alpha}_n] \;=\; m\delta_{m+n} \quad \text{and} \quad [x, p] \;=\; \mathrm{i}\ . \qquad \text{(IV.127)}$$

We will examine the spectrum of this theory for 26 fields X^μ in section V.1.2.

[15] i.e. $\partial\bar{\partial} X = 0$

§8 Primary fields. A *tensor* or *primary field* $\phi(w)$ in a conformal field theory transforms under general conformal transformations as

$$\phi'(z',\bar{z}') = (\partial_z z')^{-h_\phi}(\partial_{\bar{z}}\bar{z}')^{-\bar{h}_\phi}\phi(z,\bar{z}) \ , \tag{IV.128}$$

where h_ϕ and $\bar{h}_\phi$ are the *conformal weights* of the field $\phi(w)$. Furthermore, $h_\phi + \bar{h}_\phi$ determine its *scaling dimension*, i.e. its behavior under scaling, and $h_\phi - \bar{h}_\phi$ is the field's spin. With the energy-momentum tensor $T(z)$, such a field ϕ has the following operator product expansion:

$$T(z)\phi(w) = \frac{h_\phi}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + \dots \ . \tag{IV.129}$$

For the modes appearing in the expansion $\phi(z) = \sum_n \phi_n z^{-n-h_\phi}$, we thus have the algebra

$$[L_n, \phi_m] = (n(h_\phi - 1) - m)\phi_{m+n} \ . \tag{IV.130}$$

§9 Current algebras. Currents in a conformal field theory are (1,0)-tensor $j^a(z)$ with the operator product expansion

$$j^a(z)j^b(0) \sim \frac{k^{ab}}{z^2} + \mathrm{i}\frac{f^{ab}{}_c}{z}j^c(0) \ . \tag{IV.131}$$

The Laurent expansion $j^a(z) = \sum_{m=-\infty}^{\infty} \frac{j^a_m}{z^{m+1}}$ then leads to the *current algebra* or *Kac-Moody algebra*

$$[j^a_m, j^b_n] = mk^{ab}\delta_{m+n,0} + \mathrm{i}f^{ab}{}_c j^c_{m+n} \ . \tag{IV.132}$$

§10 Further theories. The exemplary theory (IV.115) is certainly one of the most important conformal field theories. Further examples are given by the bc- and the $\beta\gamma$-systems

$$S_{bc} = \int \mathrm{d}^2 z \ b\bar{\partial}c \quad \text{and} \quad S_{\beta\gamma} = \int \mathrm{d}^2 z \ \beta\bar{\partial}\gamma \ , \tag{IV.133}$$

which serve e.g. as Faddeev-Popov ghosts for the Polyakov string and the superstring, see also section V.2.1. In the former theory, the fields b and c are anticommuting fields and tensors of weight $(\lambda, 0)$ and $(1-\lambda, 0)$. This theory is purely holomorphic and in the operator product expansion of the energy-momentum tensor, an additional contribution to the central charge of $c = -3(2\lambda-1)^2 + 1$ and $\tilde{c} = 0$ appears for each copy of the bc-system. The $\beta\gamma$-system has analogous properties, but the fields β and γ are here commuting and the central charge contribution has an opposite sign. Recall the relation between $\beta\gamma$-systems and local Calabi-Yau manifolds of type $\mathcal{O}(a) \oplus \mathcal{O}(-2-a) \to \mathbb{C}P^1$ discussed in section II.3.2, §11. The case of a bc-system with equal weights $h_b = h_c = \frac{1}{2}$ will be important when discussing the superstring. Here, one usually relabels $b \to \psi$ and $c \to \bar{\psi}$

IV.4.2 The $\mathcal{N} = 2$ superconformal algebra

§11 Constituents. The $\mathcal{N} = 2$ superconformal algebra (SCA) is generated by the energy-momentum tensor $T(z)$, two supercurrents $G^+(z)$ and $G^-(z)$, which are primary fields of the Virasoro algebra with weight $\frac{3}{2}$ and a $\mathsf{U}(1)$ current $J(z)$, which is a primary field of weight 1. The supercurrents $G^\pm(z)$ have $\mathsf{U}(1)$ charges ± 1. This mostly fixes the operator product expansion of the involved generators to be

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \ , \\
T(z)G^{\pm}(w) &= \frac{3/2}{(z-w)^2}\, G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{z-w} + \cdots \ , \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \cdots \ , \\
G^{+}(z)G^{-}(w) &= \frac{2\,c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} + \cdots \ , \\
J(z)G^{\pm}(w) &= \pm \frac{G^{\pm}(w)}{z-w} + \cdots \ , \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \cdots \ ,
\end{aligned}
\tag{IV.134}
$$

where the dots stand for regular terms. Additionally to the mode expansion of the energy-momentum tensor given in §6, we have the mode expansions for the two supercurrents and the $\mathsf{U}(1)$ current

$$
G^{\pm}(z) = \sum_{n=-\infty}^{\infty} G^{\pm}_{n\pm\eta\pm\frac{1}{2}} z^{-(n\pm\eta\pm\frac{1}{2})-\frac{3}{2}} \quad \text{and} \quad J(z) = \sum_{n=-\infty}^{\infty} J_n z^{-n-1} \ , \tag{IV.135}
$$

where $\eta \in [-\frac{1}{2}, \frac{1}{2})$. The latter parameter is responsible for the boundary conditions of the supercurrents, and by substituting $z \to \mathrm{e}^{2\pi \mathrm{i}} z$, we obtain

$$
G^{\pm}(\mathrm{e}^{2\pi\mathrm{i}} z) = -\mathrm{e}^{\mp 2\pi \mathrm{i}(\eta + \frac{1}{2})} G^{\pm}(z) \ , \tag{IV.136}
$$

and therefore in superstring theory, $\eta = -\frac{1}{2}$ will correspond to the Neveu-Schwarz (NS) sector, while $\eta = 0$ is related with the Ramond (R) sector, cf. section V.2.2.

§12 The algebra. The operator product expansion essentially fixes the algebra in terms of the modes introduced for the generators. First, we have the Virasoro algebra

$$
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{m+n,0} \ . \tag{IV.137}
$$

Second, there is the algebra for the $\mathsf{U}(1)$ current and its commutation relation with the Virasoro generators

$$
\begin{aligned}
[J_m, J_n] &= \tfrac{c}{3} m\, \delta_{m+n,0} \ , \\
[L_n, J_m] &= -m\, J_{m+n} \ .
\end{aligned}
\tag{IV.138}
$$

Eventually, there are the relations involving the two supercurrents $G^{\pm}$:

$$
\begin{aligned}
\left[L_n, G^{\pm}_{m\pm a}\right] &= \left(\tfrac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a} \ , \\
\left[J_n, G^{\pm}_{m\pm a}\right] &= \pm\, G^{\pm}_{m+n\pm a} \ , \\
\left\{G^{+}_{n+a}, G^{-}_{m-a}\right\} &= 2\, L_{m+n} + (n-m+2a)\, J_{n+m} \tfrac{c}{3} \left((n+a)^2 - \tfrac{1}{4}\right) \delta_{m+n,0} \ ,
\end{aligned}
\tag{IV.139}
$$

where we used the shorthand notation $a = \eta + \frac{1}{2}$.

§13 The $\mathcal{N} = (2,2)$ SCA. This algebra is obtained by adding a second, right-moving $\mathcal{N} = 2$ SCA algebra with generators $\tilde{T}(\bar{z})$, $\tilde{G}^{\pm}(\bar{z})$ and $\tilde{J}(\bar{z})$.

§14 Representations of the $\mathcal{N} = (2,2)$ SCA. There are three well-established representations of the $\mathcal{N} = 2$ SCA. Most prominently, one can define a supersymmetric nonlinear sigma model in two dimensions, which possesses $\mathcal{N} = 2$ superconformal symmetry. In section V.3.1, we will discuss such models in more detail. Furthermore, there are the Landau-Ginzburg theories discussed in §8, section IV.1.2 and the so-called minimal models. For more details, see [111, 246].

Chapter V

String Theory

String theory is certainly the most promising and aesthetically satisfying candidate for a unification of the concepts of quantum field theory and general relativity. Although there is still no realistic string theory describing accurately all the measured features of the (known) elementary particles, "existence proofs" for standard-model-like theories arising from string theories have been completed, see e.g. [185]. One of the most important current problems is the selection of the correct background in which string theory should be discussed; the achievement of moduli stabilization (see [73] and references therein) show that there is progress in this area. Among the clearly less appealing approaches is the "landscape"-concept discussed in [266].

The relevant literature to this chapter is [109, 108, 218, 219, 221, 250, 267] (general and $\mathcal{N}=1$ string theory), [191, 192, 171] ($\mathcal{N}=2$ string theory), [223, 257, 144] (D-branes), [292, 111, 128, 130] (topological string theory and mirror symmetry).

V.1 String theory basics

In this section, we will briefly recall the elementary facts on the bosonic string. This theory can be regarded as a toy model to study features which will also appear in the later discussion of the superstring.

V.1.1 The classical string

§1 Historical remarks. Strings were originally introduced in the late 1960s to describe confinement in a quantum field theory of the strong interaction, but during the next years, QCD proved to be the much more appropriate theory. Soon thereafter it was realized that the spectrum of an oscillating string contains a spin-2 particle which behaves as a graviton and therefore string theory should be used for unification instead of a model of hadrons. After the first "superstring revolution" in 1984/1985, string theory had become an established branch of theoretical physics and the five consistent superstring theories had been discovered. In the second superstring revolution around 1995, dualities relating these five string theories were found, giving a first taste of non-perturbative string theory. Furthermore, one of the most important objects of study in string theory today, the concept of the so-called D-branes, had been introduced.

§2 Bosonic string actions. Consider a two-dimensional (pseudo-)Riemannian manifold Σ described locally by coordinates σ^0 and σ^1 and a metric $\gamma_{\alpha\beta}$ of Minkowski signature $(-1,+1)$. This space is called the *worldsheet* of the string and is the extended analogue of the worldline of a particle. Given a further Riemannian manifold M and a map $X:\Sigma\to M$ smoothly embedding the worldsheet Σ of the string into the *target space* M, we can write down a string action (the *Polyakov action*)

$$S \;=\; -\frac{T}{2}\int \mathrm{d}^2\sigma\;\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\;, \qquad \text{(V.1)}$$

where T is the tension of the string. This meaning becomes even clearer, when we recast this action into the form of the *Nambu-Goto action*

$$S \;=\; -T\int \mathrm{d}^2\sigma\;\sqrt{-\det\partial_\alpha X^\mu\partial_\beta X_\mu}\;, \tag{V.2}$$

which is equal to $-T$ times the area of the worldsheet of the string. Note that more frequently, one encounters the constants $l_s = \sqrt{\frac{1}{2\pi T}}$, which is the *string length* and the *Regge slope* $\alpha' = \frac{1}{2\pi T} = l_s^2$.

In general, we have σ^0 run in an arbitrary interval of "time" and σ^1 run between 0 and π if the "spatial part" of the worldsheet is noncompact and between 0 and 2π else.

§3 Equations of motion. The equations of motion obtained by varying (V.1) with respect to the worldsheet metric read as

$$\partial_\alpha X^\mu\partial_\beta X_\mu \;=\; \tfrac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X_\mu\;, \tag{V.3}$$

which implies that the induces metric $h_{\alpha\beta} := \partial_\alpha X^\mu\partial_\beta X_\mu$ is proportional to the worldsheet metric.

§4 Closed and Neumann boundary conditions. To determine the variation with respect to X, we have to impose boundary conditions on the worldsheet. The simplest case is the one of periodic boundary conditions, in which the spatial part of the worldsheet becomes compact:

$$\begin{gathered} X^\mu(x,2\pi) \;=\; X^\mu(x,0)\;,\quad \partial^\mu X^\nu(x,2\pi) \;=\; \partial^\mu X^\nu(x,0)\;,\\ \gamma_{\alpha\beta}(x,2\pi) \;=\; \gamma_{\alpha\beta}(x,0)\;. \end{gathered} \tag{V.4}$$

This describes a closed string, where all boundary terms clearly vanish. The same is true if we demand that

$$n^\alpha\partial_\alpha X^\mu \;=\; 0 \quad\text{on}\quad \partial\Sigma\;, \tag{V.5}$$

where n^α is normal to $\partial\Sigma$, as the boundary term in the variation of the action with respect to X is evidently proportional to $\partial_\alpha X^\mu$. Taking a flat, rectangular worldsheet, (V.5) reduces to $\partial_1 X^\mu = 0$. These conditions are called *Neumann boundary conditions* and describe an open string whose endpoints can move freely in the target space. Both the closed and the Neumann boundary conditions yield

$$\partial^\alpha\partial_\alpha X^\mu \;=\; 0 \tag{V.6}$$

as further equations of motion.

§5 Dirichlet boundary conditions. One can also impose so-called *Dirichlet boundary conditions*, which state that the endpoints of a string are fixed in the spatial direction:

$$\partial_0 X^\mu \;=\; 0\;. \tag{V.7}$$

However, these boundary conditions by themselves have some unpleasant features: Not only do they break Poincaré symmetry, but they also have momentum flowing off the endpoints of the open strings. The true picture is that open strings with Dirichlet-boundary conditions end on subspaces of the target space, so called *D-branes*, which we will study in section V.4. For this reason, we will restrict ourselves here to open strings with Neumann boundary conditions.

§6 Symmetries. The Polyakov action has a remarkable set of symmetries:

▷ Poincaré symmetry in the target space

▷ Diffeomorphism invariance on the worldsheet

▷ Weyl-invariance on the worldsheet

Weyl-invariance means that the action is invariant under a local rescaling of the worldsheet metric and thus the worldsheet action is *conformally invariant.*

§7 The energy-momentum tensor. The variation of the action with respect to the worldsheet metric yields the *energy-momentum tensor*

$$T^{\alpha\beta} \;=\; -4\pi\sqrt{-\det\gamma}\frac{\delta}{\delta\gamma_{\alpha\beta}}S\ . \tag{V.8}$$

Since the worldsheet metric is a dynamical field, the energy-momentum tensor vanishes classically. Furthermore, the trace of this tensor has to vanish already due to Weyl-invariance:

$$T_\alpha{}^\alpha \sim \gamma_{\alpha\beta}\frac{\delta}{\delta\gamma_{\alpha\beta}}S \;=\; 0\ . \tag{V.9}$$

V.1.2 Quantization

§8 Canonical quantization. To quantize classical string theory given by the Polyakov action (V.1), we first fix the gauge for the worldsheet metric. In *conformally flat gauge*, we have $(\gamma^{\alpha\beta}) = \mathrm{e}^{\phi(\sigma)}\mathrm{diag}(-1,+1)$ and the action (V.1) reduces to D copies of our exemplary theory (IV.115) from section IV.4.1, for which we already discussed the quantization procedure. Note, however, that the creation and annihilation operators receive an additional index for the D dimensions of spacetime and their algebra is modified to

$$[\alpha^\mu_m,\alpha^\nu_n] \;=\; [\tilde\alpha^\mu_m,\tilde\alpha^\nu_n] \;=\; m\delta_{m+n}\eta^{\mu\nu}_{(M)}\ , \tag{V.10}$$

where $\eta^{\mu\nu}_{(M)}$ is the Minkowski metric on the target space manifold M. We thus have a quantum mechanical system consisting of the tensor product of $2D$ harmonic oscillators and a free particle. We therefore derive the states in our theory from vacua $|k,0\rangle$, which are eigenstates of the momentum operators $p^\mu = \alpha^\mu_0 = \tilde\alpha^\mu_0$. These vacuum states are furthermore annihilated by the operators α^μ_m, $\tilde\alpha^\mu_m$ with $m<0$. The remaining operators with $m>0$ are the corresponding creation operators. Due to the negative norm of the oscillator states in the time direction on the worldsheet, canonical quantization by itself is insufficient and one needs to impose the further constraints

$$(L_0-a)|\psi\rangle \;=\; (\tilde L_0-a)|\psi\rangle \;=\; 0 \quad\text{and}\quad L_n|\psi\rangle \;=\; \tilde L_n|\psi\rangle \;=\; 0 \quad\text{for}\quad n\;>\;0\ . \tag{V.11}$$

This is a consequence of the above applied naïve gauge fixing procedure and to be seen analogously to the Gupta-Bleuler quantization prescription in quantum electrodynamics.

Note that in the case of closed strings, one has an additional independent copy of the above Fock-space.

§9 BRST quantization. A more modern approach to quantizing the bosonic string is the BRST approach, from which the above Virasoro constrains follow quite naturally: The physical states belong here to the cohomology of the BRST operator. We will not discuss this procedure but refer to the review material on string theory, in particular to [218].

§10 Virasoro generators and creation/annihilation operators. To expand the Virasoro generators in terms of the creation and annihilation operators used above, we insert the Laurent expansion (IV.126) for X^μ into the energy-momentum tensor (IV.120) and read of the coefficients of the total Laurent expansion. We find

$$L_0 \;=\; \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}(\alpha_{-n}^{\mu}\alpha_{\mu n}) + \varepsilon_0 \;=\; \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty}(\alpha_{-n}^{\mu}\alpha_{\mu n}) + \varepsilon_0\ , \tag{V.12}$$

where ε_0 is a normal ordering constant, and

$$L_m \;=\; \tfrac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}^{\mu}\alpha_{\mu n}:\ , \tag{V.13}$$

where $:\cdot:$ denotes creation-annihilation normal ordering. For the quantum operator L_0, the vacuum energy is formally $\varepsilon_0 = \frac{d-2}{2}\zeta(-1)$, where $\zeta(-1) = \sum_n n = -\frac{1}{12}$ after regularization.

§11 Conformal anomaly. The *conformal anomaly* or *Weyl anomaly* is the quantum anomaly of local worldsheet symmetries. One can show that the anomaly related to Weyl invariance is proportional to the central charge of the underlying conformal field theory. Since the appropriate ghost system for gauging the worldsheet symmetries is a *bc*-system with $\lambda = 2$, the central charge is proportional to $D - 26$, where D is the number of bosons (X^μ). Thus, the *critical dimension* of bosonic string theory, i.e. the dimension for which the total central charge vanishes, is $D = 26$. From this, it also follows that $a = 1$.

§12 Open string spectrum. The constraint $L_0 = a$ in (V.11) is essentially the mass-shell condition

$$m^2 \;=\; -p_0^2 \;=\; -\frac{1}{2\alpha'}\alpha_0^2 \;=\; \frac{1}{\alpha'}(N-a) \;:=\; \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n\right)\ . \tag{V.14}$$

The ground state $|k,0\rangle$, for which $N = 0$ has thus mass $-k^2 = m^2 = -\frac{a}{\alpha'} < 0$ and is in fact tachyonic. Therefore, bosonic string theory is actually not a consistent quantum theory and should be regarded as a pedagogical toy model[1].

The first excited level is given by $N = 1$ and consists of oscillator states of the form $\alpha_{-1}^{\mu}|k;0\rangle = \zeta_\mu\alpha_{-1}|k;0\rangle$, where ζ_μ is some polarization vector. As mass, we obtain $m^2 = \frac{1}{\alpha'}(1-a)$ and from

$$L_1|k;\zeta\rangle \;=\; \sqrt{2\alpha'}(k\cdot\zeta)|k;0\rangle \tag{V.15}$$

together with the physical state condition $L_1|k;\zeta\rangle = 0$, it follows that $k \cdot \zeta = 0$. As mentioned in §11, $a = 1$ and therefore the first excited level is massless. (Other values of a would have led to further tachyons and ghost states of negative norm.) Furthermore, due to the polarization condition, we have $d - 2 = 24$ independent polarization states. These states therefore naturally correspond to a massless spin 1 vector particle.

Higher excited states become significantly massive and are usually discarded with the remark that they practically decouple.

[1] *Tachyon condensation* might be a remedy to this problem, but we will not go into details here.

§13 Closed string spectrum. In the case of closed strings, the physical state conditions (V.11) need to hold for both copies L_n and $\tilde{L}_n$ of the Virasoro generators, corresponding to the right- and left-moving sectors. Adding and subtracting the two conditions for L_0 and $\tilde{L}_0$ yields equations for the action of the Hamiltonian H and the momentum P on a physical state, which amount to invariance under translations in space and time. These considerations lead to the two conditions

$$m^2 \;=\; \frac{4}{\alpha'}(N-1) \quad \text{and} \quad N \;=\; \tilde{N}\;, \tag{V.16}$$

where the first equation is the new mass-shell condition and the second is the so-called level-matching condition.

The ground state $|k;0,0\rangle$ is evidently again a spin 0 tachyon and therefore unstable.

The first excited level is of the form

$$|k;\zeta\rangle \;=\; \zeta_{\mu\nu}(\alpha^{\mu}_{-1}|k;0\rangle \otimes \tilde{\alpha}^{\nu}_{-1}|k;0\rangle \tag{V.17}$$

and describes massless states satisfying the polarization condition $k^\mu\zeta_{\mu\nu}=0$. The polarization tensor $\zeta_{\mu\nu}$ can be further decomposed into a symmetric, an antisymmetric and a trace part according to

$$\zeta_{\mu\nu} \;=\; g_{\mu\nu}+B_{\mu\nu}+\eta_{\mu\nu}\Phi\;. \tag{V.18}$$

The symmetric part here corresponds to a spin 2 *graviton field*, the antisymmetric part is called the *Neveu-Schwarz B-field* and the scalar field Φ is the spin 0 *dilaton*.

§14 Chan-Paton factors. We saw above that open strings contain excitations related to Abelian gauge bosons. To lift them to non-Abelian states, one attaches non-dynamical degrees of freedom to the endpoints of the open string, which are called *Chan-Paton factors*. Here, one end will carry the fundamental representation and the other end the antifundamental representation of the gauge group. Assigning Chan-Paton factors to both ends leads correspondingly to an adjoint representation. Note that in the discussion of scattering amplitudes, one has to appropriately take traces over the underlying matrices.

V.2 Superstring theories

There are various superstring theories which have proven to be interesting to study. Most conveniently, one can classify these theories with the number of supersymmetries (p,q) which square to translations along the left- and right-handed light cone in the 1+1 dimensions of the worldsheet. The bosonic string considered above and living in 26 dimensions has supersymmetry $\mathcal{N}=(0,0)$. The type IIA and type IIB theories, which are of special interest in this thesis, have supersymmetry $\mathcal{N}=(1,1)$. The type I theories are obtained from the type II ones by orbifolding with respect to worldsheet parity and the heterotic string theories have supersymmetry $\mathcal{N}=(0,1)$. Interestingly, it has been possible to link all of the above supersymmetric string theories to a master theory called M-theory [294], on which we do not want to comment further.

Besides the above theories with one supersymmetry, there are the $\mathcal{N}=2$ string theories with supersymmetry $\mathcal{N}=(2,2)$ or heterotic supersymmetry $\mathcal{N}=(2,1)$. We will discuss the former case at the end of this section. The latter has target space $\mathbb{R}^{2,2}$ for the right-handed sector and $\mathbb{R}^{2,2}\times T^8$ for the left-handed sector. Also this theory has been conjectured to be related to M-theory [192].

V.2.1 $\mathcal{N}=1$ superstring theories

§1 Preliminary remarks. The motivation for turning to superstring theories essentially consists of two points: First of all, the bosonic spectrum contains a tachyon as we saw above and therefore bosonic string theory is inconsistent as a quantum theory. Second, to describe reality, we will eventually need some fermions in the spectrum and therefore bosonic string theory cannot be the ultimate answer. One might add a third reason for turning to superstrings: The critical dimension of bosonic string theory, 26, is much less aesthetical than the critical dimensions of $\mathcal{N}=1$ superstring theory, 10, which includes the beautiful mathematics of Calabi-Yau manifolds into the target space compactification process.

Note that there are several approaches to describe the superstring, see also section V.4.5, §16. Here, we will follow essentially the *Ramond-Neveu-Schwarz* (RNS) formulation, which uses two-dimensional worldsheet supersymmetry and the additional *Gliozzi-Scherck-Olive* (GSO) projection to ensure also target space supersymmetry. Furthermore, the GSO projection guarantees a tachyon-free spectrum and modular invariance.

§2 Superstring action. A straightforward generalization of the bosonic string action is given by

$$S=\tfrac{1}{4\pi}\int \mathrm{d}^2z\ \left(\tfrac{2}{\alpha'}\partial X^\mu\bar\partial X_\mu+\psi^\mu\bar\partial\psi_\mu+\tilde\psi^\mu\partial\tilde\psi_\mu\right)\ , \tag{V.19}$$

where the two fermionic fields ψ^μ and $\tilde\psi^\mu$ are holomorphic and antiholomorphic fields, respectively. Recall that we already discussed the conformal field theories for the ψ^μ and the $\tilde\psi^\mu$ in section IV.4.1.

§3 Boundary conditions. From the equations of motion, we get two possible boundary conditions leading to two sectors:

$$\begin{array}{lll}\text{Ramond (R)} & \psi^\mu(0,\tau)=\tilde\psi^\mu(0,\tau) & \psi^\mu(\pi,\tau)=\tilde\psi^\mu(\pi,\tau)\ ,\\ \text{Neveu-Schwarz (NS)} & \psi^\mu(0,\tau)=-\tilde\psi^\mu(0,\tau) & \psi^\mu(\pi,\tau)=\tilde\psi^\mu(\pi,\tau)\ .\end{array}$$

It is useful to unify these boundary condition in the equations

$$\psi^\mu(z+2\pi)\ =\ \mathrm{e}^{2\pi\mathrm{i}\nu}\psi^\mu(z)\quad\text{and}\quad\tilde\psi^\mu(\bar z+2\pi)\ =\ \mathrm{e}^{-2\pi\mathrm{i}\tilde\nu}\tilde\psi^\mu(\bar z)\ , \tag{V.20}$$

over the complex plane, with $\nu,\tilde\nu\in\{0,\frac{1}{2}\}$.

§4 Superconformal symmetry. Recall that in the bosonic case, the Virasoro generators appeared as Laurent coefficients in the expansion of the energy-momentum tensor, which in turn is the Noether current for conformal transformations. For superconformal transformations, we have the additional supercurrents

$$T_F(z)\ =\ \mathrm{i}\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial X_\mu(z)\quad\text{and}\quad T_F(\bar z)\ =\ \mathrm{i}\sqrt{\frac{2}{\alpha'}}\tilde\psi^\mu(\bar z)\bar\partial X_\mu(\bar z)\ . \tag{V.21}$$

Their Laurent expansions are given by

$$T_F(z)\ =\ \sum_{r\in\mathbb{Z}+\nu}\frac{G_r}{z^{r+3/2}}\quad\text{and}\quad\tilde T_F(\bar z)\ =\ \sum_{r\in\mathbb{Z}+\tilde\nu}\frac{\tilde G_r}{\bar z^{r+3/2}}\ , \tag{V.22}$$

where ν and $\tilde\nu$ label again the applied boundary conditions. The Laurent coefficients complete the Virasoro algebra to its $\mathcal{N}=(1,0)$ and $\mathcal{N}=(0,1)$ supersymmetric extension.

The former reads

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0} \ , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0} \ , \\
[L_m, G_r] &= \frac{m-2r}{2} G_{m+r} \ .
\end{aligned}
\tag{V.23}
$$

This algebra is also called the *Ramond algebra* for r, s integer and the *Neveu-Schwarz algebra* for r, s half-integer.

§5 Critical dimension. Since in each of the holomorphic (right-moving) and antiholomorphic (left-moving) sectors, each boson contributes 1 and each fermion contributes $\frac{1}{2}$ to the central charge, the total central charge is $c = \frac{3}{2}D$. This central charge has to compensate the contribution from the superconformal ghosts, which is $-26 + 11$, and thus the critical dimension of $\mathcal{N} = 1$ superstring theory is 10.

§6 Preliminary open superstring spectrum. The mode expansions of the right- and left-moving fermionic fields read

$$
\psi^\mu(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}} \quad \text{and} \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r \in \mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}} \ , \tag{V.24}
$$

and after canonical quantization, one arrives at the algebra

$$
\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \ . \tag{V.25}
$$

The normal ordering constant a appearing in (V.11) is found to be $a = 0$ in the Ramond sector and $a = \frac{1}{2}$ in the Neveu-Schwarz sector. The physical state conditions (V.11) are extended by the demand

$$
G_r|\text{phys}\rangle = 0 \quad \text{for} \quad r > 0 \ , \tag{V.26}
$$

and the level number N is modified to

$$
N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r>0} r\psi_{-r} \cdot \psi_r \ . \tag{V.27}
$$

Equipped with these results, we see that the NS ground state is again tachyonic, and has mass $m^2 = -\frac{1}{2\alpha'}$. The first excited levels consist of the massless states $\psi^\mu_{-\frac{1}{2}}|k; 0\rangle_{\text{NS}}$, and can again be related to spacetime gauge potentials.

The R sector contains (massless) zero modes ψ^μ_0 satisfying the ten-dimensional Dirac algebra $\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}$ and all states in the R sector are spacetime fermions. Note that already in the ground state, we cannot expect spacetime supersymmetry between the R and NS sectors due to the strong difference in the number of states.

§7 Preliminary closed string spectrum. There are evidently four different pairings of the fermion boundary conditions for closed strings giving rise to the four sectors of the closed superstring. Spacetime bosons are contained in the NS-NS and the R-R sectors, while spacetime fermions are in the NS-R and the R-NS sectors.

While the NS-NS ground state contains again a tachyon, the remaining states in the first levels form all expected states of supergravity.

§8 GSO projection. To obtain (local) supersymmetry in the target space of the theory, we have to apply the so-called *Gliozzi-Scherck-Olive* (GSO) projection, which eliminates certain states from the naïve superstring spectrum.

Explicitly, the GSO projection acts on the NS sector by keeping states with an odd number of ψ excitations, while removing all other states. This clearly eliminates the tachyonic NS-vacuum, and the ground states become massless. More formally, one can apply the projection operator $P_{\mathrm{GSO}} = \frac{1}{2}(1-(-1)^F)$, where F is the fermion number operator.

In the R sector, we apply the same projection operator, but replace $(-1)^F$ by

$$(-1)^F \;\rightarrow\; \pm\Gamma(-1)^F\ , \tag{V.28}$$

where Γ is the ten-dimensional chirality operator $\Gamma = \Gamma^0 \dots \Gamma^9$. This projection reduces the zero modes in the R ground state to the appropriate number to match the new massless NS ground states. This is an indication for the fact that the GSO projection indeed restores spacetime supersymmetry.

Note that the massless ground states of the theory are characterized by their representation of the little group $\mathsf{SO}(8)$ of the Lorentz group $\mathsf{SO}(1,9)$. The NS sector is just $\mathbf{8}_v$, while there the two possibilities for the R sector, depending on the choice of GSO projection: $\mathbf{8}_s$ and $\mathbf{8}_c$.

§9 Green-Schwarz action. For completeness sake, let us give the covariant Green-Schwarz action of the type IIB superstring in Nambu-Goto form, which will be needed in the definition of the IKKT model in section VIII.1.2. The action reads as

$$\begin{aligned} S_{\mathrm{GS}} \;=\; -T\int \mathrm{d}^2\sigma\, \Big(\sqrt{-\tfrac{1}{2}\Sigma} + \mathrm{i}\varepsilon^{ab}\partial_a X^\mu(\bar\theta^1\Gamma_\mu\partial_b\theta^1 + \bar\theta^2\Gamma_\mu\partial_b\theta^2) \\ + \varepsilon^{ab}\bar\theta^1\Gamma^\mu\partial_a\theta^1\bar\theta^2\Gamma_\mu\partial_b\theta^2\Big)\ , \end{aligned} \tag{V.29}$$

where θ^1 and θ^2 are Majorana-Weyl spinors in ten-dimensions and

$$\Sigma^{\mu\nu} \;=\; \varepsilon^{ab}\Pi^\mu_a\Pi^\nu_b \quad \text{with} \quad \Pi^\mu_a \;=\; \partial_a X^\mu - \mathrm{i}\bar\theta^1\Gamma^\mu\partial_a\theta^1 + \mathrm{i}\bar\theta^2\Gamma^\mu\partial_a\theta^2\ . \tag{V.30}$$

V.2.2 Type IIA and type IIB string theories

Recall that we had two different possibilities of defining the fermion number operator in equation (V.28). For open strings, both choices are in principle equivalent but for closed strings, the relative sign between left- and right-moving sectors are important.

§10 Type IIA. In this case, we choose the opposite GSO projections for the left- and the right-moving sectors. The resulting theory is therefore non-chiral, and the field content can be classified under the little group $\mathsf{SO}(8)$ as $(\mathbf{8}_v\oplus\mathbf{8}_s)\otimes(\mathbf{8}_v\oplus\mathbf{8}_c)$

§11 Type IIB. Here, we choose the same GSO projection on both sectors, which will lead to a chiral theory with field content $(\mathbf{8}_v\oplus\mathbf{8}_s)\otimes(\mathbf{8}_v\oplus\mathbf{8}_s)$.

§12 The R-R sectors. Constructing vertex operators for the R-R sector states leads to antisymmetric tensors G of even rank n for type IIA and odd rank n for type IIB, satisfying Maxwell equations. Thus, we get the following potentials C with $G = \mathrm{d}C$:

type IIA		$C_{(1)}$	$C_{(3)}$	$C_{(5)}$	$C_{(7)}$
type IIB	$C_{(0)}$	$C_{(2)}$	$C_{(4)}$	$C_{(6)}$	$C_{(8)}$

Each potential of rank k has a Hodge dual of rank $8-k$ via

$$*\mathrm{d}C_{(k)} \;=\; \mathrm{d}\tilde{C}_{(8-k)}\ , \tag{V.31}$$

since the target space has dimension 10.

§13 Compactification. To obtain a phenomenologically relevant theory, one evidently has to reduce the number of large dimensions from ten to four. Since the geometry of spacetime is determined dynamically, one can imagine that there are certain solutions to the string theory under consideration, which correspond to a compactification of the theory on a six-dimensional manifold. Particularly nice such manifolds besides the six-dimensional torus are Calabi-Yau manifolds and compactifying a ten-dimensional string theory on such a space often yields standard model like physics, with many parameters as masses, coupling constants, numbers of quark and lepton families determined by the explicit geometry of the chosen Calabi-Yau threefold.

V.2.3 T-duality for type II superstrings

In this section, let us briefly describe the symmetry called T-duality in string theory. This symmetry has no analogue in field theory and is therefore truly stringy.

§14 T-duality for closed strings. Assume we quantize one of the nine spatial dimensions on a circle as $X^9 \sim X^9 + 2\pi R$, where R is the radius of the circle. It follows that the momentum along this direction is quantized $p_0^9 = \frac{n}{R}$ with $n \in \mathbb{Z}$. Recall the expansion of the string embedding function

$$X^\mu(\tau,\sigma) \;=\; x_0^\mu + \tilde{x}_0^\mu + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \dots\ , \tag{V.32}$$

where the dots denote oscillator terms. Moreover, the center of mass spacetime momentum reads as

$$p_0^\mu \;=\; \sqrt{\frac{1}{2\alpha'}}\,(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\ . \tag{V.33}$$

With the quantization condition, we thus obtain $\alpha_0^9 + \tilde{\alpha}_0^9 = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}}$. The compactification also constrains the coordinate in the X^9 direction according to

$$X^9(\tau,\sigma+2\pi) \;=\; X^9(\tau,\sigma) + 2\pi w R\ , \tag{V.34}$$

where the integer w is the *winding number* and describes, how often the closed string is wound around the compactified direction. Together with the expansion (V.32), we derive the relation

$$\alpha_0^9 - \tilde{\alpha}_0^9 \;=\; wR\sqrt{\frac{2}{\alpha'}}\ . \tag{V.35}$$

Putting (V.33) and (V.35) together, we obtain furthermore that

$$\alpha_0^9 \;=\; \sqrt{\alpha'}2\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) \quad\text{and}\quad \tilde{\alpha}_0^9 \;=\; \sqrt{\alpha'}2\left(\frac{n}{R} - \frac{wR}{\alpha'}\right), \tag{V.36}$$

and the mass formula for the spectrum gets modified to

$$\begin{aligned} m^2 \;&=\; -p_\mu p^\mu \;=\; \frac{2}{\alpha'}(\alpha_0^9)^2 + \frac{4}{\alpha'}(N-1) = \frac{2}{\alpha'}(\tilde{\alpha}_0^9)^2 + \frac{4}{\alpha'}(\tilde{N}-1) \\ &=\; \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\tilde{N}-2)\ . \end{aligned} \tag{V.37}$$

We see that there are essentially two towers of states in the game: the tower of Kaluza-Klein momentum states and the tower of winding states. Noncompact states are obtained for $n = w = 0$. In the limit of large radius $R \to \infty$, the winding states become very massive and thus disappear, while the momentum states form a continuum. In the opposite limit $R \to 0$, the momentum states decouple and the winding states become continuous.

Note that all the formulæ are symmetric under the interchange

$$n \leftrightarrow w \quad \text{and} \quad R \leftrightarrow \frac{\alpha'}{R} , \tag{V.38}$$

and this symmetry is called *T-duality.* In terms of zero-modes, this symmetry corresponds to

$$\alpha_0^9 \leftrightarrow \alpha_0^9 \quad \text{and} \quad \tilde{\alpha}_0^9 \leftrightarrow -\tilde{\alpha}_0^9 . \tag{V.39}$$

Note that T-duality therefore corresponds to a parity transformation of the right-movers.

§15 T-duality for open strings. As open strings cannot wrap around the compact dimensions, they are dimensionally reduced by T-duality in the limit $R \to 0$. Although the interior of the open strings still vibrate in all ten dimensions, the endpoints are restricted to a nine-dimensional subspace. This is also seen by adding the mode expansion of the open string with reversed parity of the right-movers, which causes the momentum in the T-dualized direction to vanish. The nine-dimensional subspace is naturally explained in the language of D-branes, see section V.4.

§16 T-duality for type II superstrings. We saw that T-duality corresponds to a parity change of the right-movers. By target space supersymmetry, it must therefore also change the parity of the right-moving fermion fields. This inverts the choice of sign in the GSO projection and eventually turns the GSO projection for type IIA theory into the GSO projection of type IIB theory. T-dualizing any odd number of target space dimensions thus maps the two different type II superstring theories into each other, while T-dualizing an even number of dimensions does not modify the superstring theory's type.

V.2.4 String field theory

§17 Motivation. String field theory (SFT) is an attempt to describe string theory in a background independent manner. All the excitations of the string are encoded in an infinite number of fields, which in turn are recombined in a single *string field* $\mathcal{A}$. After quantizing this field, we have – roughly speaking – an operator $\hat{\mathcal{A}}$ for every string in the target space. There are different SFTs, which describe the dynamics of the string field. In the following, we will only be interested in the Chern-Simons-like version formulated by Witten [291].

Although SFT found several successful applications, there are also conceptual drawbacks. First of all, the close strings are still missing or at least hidden in Witten's successful formulation. Second, and most importantly, it contradicts the principle derived from M-theory that branes and strings should be equally fundamental.

§18 Cubic SFT. Take a $\mathbb{Z}$-graded algebra $\mathfrak{A}$ with an associative product $\star$ and a derivative Q with $Q^2 = 0$ and $\widetilde{Q\mathcal{A}} = \tilde{\mathcal{A}}+1$ for any $\mathcal{A} \in \mathfrak{A}$. Assume furthermore a map $\int : \mathfrak{A} \to \mathbb{C}$ which gives non-vanishing results only for elements of grading 3 and respects the grading, i.e. $\int \mathcal{A} \star \mathcal{B} = (-1)^{\tilde{\mathcal{A}}\tilde{\mathcal{B}}} \int \mathcal{B} \star \mathcal{A}$. The (formal) action of cubic SFT is then

$$S = \tfrac{1}{2} \int \left(\mathcal{A} \star Q\mathcal{A} + \tfrac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A} \right) . \tag{V.40}$$

This action is invariant under the gauge transformations $\delta\mathcal{A} = Q\varepsilon - \varepsilon \star \mathcal{A} + \mathcal{A} \star \varepsilon$. It can furthermore be easily extended to allow for Chan-Paton factors by replacing $\mathfrak{A}$ by $\mathfrak{A} \otimes \mathfrak{gl}(n, \mathbb{C})$ and $\int$ by $\int \otimes \operatorname{tr}$.

§19 Physical interpretation. The physical interpretation of the above construction is the following: $\mathcal{A}$ is a *string field* encoding all possible excitations of an open string. The operator $\star$ glues the halves of two open strings together, forming a third one and the operator $\int$ folds an open string and glues its two halves together [291].

V.2.5 The $\mathcal{N} = 2$ string

§20 Introduction. Besides the bosonic string theory having a 26-dimensional target space (and some consistency problems due to a tachyon in the spectrum) and the super string theory with $\mathcal{N} = 1$ worldsheet supersymmetry having a 10-dimensional target space, the $\mathcal{N} = 2$ string living naturally in 4 dimensions received much attention as a toy model. In our consideration, this string will essentially serve as a model for some D-brane configurations arising in the context of twistor geometry. For more details see [191, 190, 171, 105] and references therein.

§21 Action. The action of the $\mathcal{N} = 2$ string is given by a two-dimensional $\mathcal{N} = 2$ supergravity model with chiral matter coupled to it. The $\mathcal{N} = 2$ supergravity multiplet here consists of a zweibein e^a_α, a complex gravitino $(\chi_\alpha, \chi^*_\alpha)$ and a $\mathsf{U}(1)$ gauge potential A_α. The chiral matter is captured by the components of a $\mathcal{N} = 2$ chiral superfield $X^i \sim x^i + \theta\psi^i$, where $i = 1, \ldots, d$ and d is the target space dimension. The corresponding action reads as

$$S = \int \mathrm{d}^2 z \;\sqrt{\eta}\big(\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha x_i \partial_\beta \bar{x}^i + \mathrm{i}\bar{\psi}^i \not{D}\psi_i + A_\alpha \bar{\psi}^i\gamma^\alpha\psi_i + \\ + (\partial_\alpha \bar{x}^i + \bar{\psi}^i\chi_\alpha)\chi_\beta\gamma^\alpha\gamma^\beta\psi_i + \mathrm{c.c.}\big)\;. \tag{V.41}$$

§22 Critical dimension. As usual, the critical dimension is calculated by adding all the contributions of the necessary ghosts systems. Here, we have again one bc-ghost system for worldsheet reparameterizations, a complex $\beta\gamma$-system for the supersymmetry and a $b'c'$-system with weights $(1, 0)$ for the $\mathsf{U}(1)$-symmetry. Together with the matter fields, we have $c = -2 + D$ in total, where D is the complex dimension. Thus, the $\mathcal{N} = 2$ string has critical dimension 4.

§23 Spectrum and symmetries. In the following we will always assume the metric on the target space $\mathbb{R}^4$ of the $\mathcal{N} = 2$ string to be either Euclidean[2] or Kleinian, i.e. $\eta_{\mu\nu} = \mathrm{diag}(+1, +1, -1, -1)$. The underlying worldsheet theory [40] is $\mathcal{N} = 2$ supergravity coupled to two $\mathcal{N} = 2$ massless chiral multiplets, the latter forming the ordinary sigma model describing a string. The corresponding action is $\mathcal{N} = 2$ supersymmetric and Weyl invariant on the worldsheet. Furthermore,there is a global $\mathsf{U}(1, 1)$ target space symmetry.

§24 Amplitudes. Upon quantization, one finds a single massless open string state $|k\rangle$ in the spectrum, which can be endowed with Chan-Paton factors. On the interaction side, the structure of amplitudes is rather simple. All n-point functions with $n > 3$ vanish identically for both open and closed strings. The lower amplitudes give rise to the effective field theory.

§25 Effective field theory. It has been shown in [212] that the $\mathcal{N} = 2$ open string is equivalent to self-dual Yang-Mills theory in 2+2 dimensions. It was also proven there that the $\mathcal{N} = 2$ closed string is equivalent to self-dual supergravity. In [260], it was argued that the appropriate field theory is rather a fully supersymmetrized version, and thus the $\mathcal{N} = 2$ critical string should correspond to $\mathcal{N} = 4$ supersymmetric self-dual Yang-Mills theory. Note that the D-branes of $\mathcal{N} = 2$ string theory will be discussed in V.4.5, §15.

[2] Considering a Euclidean target space, however, yields no propagating degrees of freedom.

V.3 Topological string theories

Topological string theories are obtained from the physical description of strings moving on Calabi-Yau manifolds after twisting the field content which turns the usual supersymmetric sigma model into a topological field theory. They describe subsectors of the physical string, which are under control and suited for extensive study. We will be mostly interested in the so-called topological B-model, as it nicely reduces to holomorphic Chern-Simons theory.

Besides the topological field theories which we will obtain in the following by twisting the field content of a nonlinear sigma model, there are further field theories giving rise to a topological string representation of the $\mathcal{N} = (2,2)$ superconformal algebra: the Landau-Ginzburg model and the so-called minimal models.

V.3.1 The nonlinear sigma model and its twists

§1 Sigma models. A theory which contains a scalar field ϕ mapping some spacetime to some (usually Riemannian) manifold X is called a *sigma model*. The sigma model is called *linear* if the target manifold X is a linear space, otherwise it is called *nonlinear*.

§2 Nonlinear sigma model. The most convenient starting point of discussing topological strings is the standard nonlinear sigma model in two dimensions which describes maps Φ from a Riemann surface Σ to a target manifold X with Riemannian metric (g_{IJ}) and Riemann tensor (R_{IJKL}). This model is defined by the action

$$\begin{aligned} S \;=\; 2t\int_\Sigma \mathrm{d}^2 z\ \big(&\tfrac{1}{2}g_{IJ}(\phi)\partial_z\phi^I\partial_{\bar z}\phi^J + \tfrac{\mathrm{i}}{2}g_{IJ}(\phi)\psi_-^I D_z\psi_-^J + \\ &\tfrac{\mathrm{i}}{2}g_{IJ}\psi_+^I D_{\bar z}\psi_+^J + \tfrac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L\big)\ , \end{aligned} \tag{V.42}$$

where D_z is the pullback of the Levi-Civita connection Γ^I_{JK} on TX and the ϕ^I are coordinates on X. If we denote the canonical and anticanonical line bundles[3] over Σ by K and $\bar K$, the fermions are sections of the following bundles:

$$\psi_+^I \in \Gamma(K^{1/2}\otimes\Phi^*(TX)) \quad \text{and} \quad \psi_-^I \in \Gamma(\bar K^{1/2}\otimes\Phi^*(TX))\ . \tag{V.43}$$

The supersymmetry transformations leaving (V.42) invariant are given by

$$\begin{aligned} \delta\phi^I &= \mathrm{i}\varepsilon_-\psi_+^I + \mathrm{i}\varepsilon_+\psi_-^I\ , \\ \delta\psi_+^I &= -\varepsilon_-\partial_z\phi^I - \mathrm{i}\varepsilon_+\psi_-^K\Gamma^I_{KM}\psi_+^M\ , \\ \delta\psi_-^I &= -\varepsilon_+\partial_{\bar z}\phi^I - \mathrm{i}\varepsilon_-\psi_+^K\Gamma^I_{KM}\psi_-^M\ . \end{aligned} \tag{V.44}$$

§3 $\mathcal{N}=2$ supersymmetry. If X is Kähler, we gain additional $\mathcal{N}=2$ supersymmetry: The indices $I,J,K,\dots$ split into holomorphic and antiholomorphic parts: $i,\bar\imath,\dots$ and we have the following field content:

$$\begin{aligned} &\phi^i\in T^{1,0}X\ ,\quad \psi_+^i\in K^{1/2}\otimes\Phi^*(T^{1,0}X)\ ,\quad \psi_-^i\in\bar K^{1/2}\otimes\Phi^*(T^{1,0}X)\ ,\\ &\phi^{\bar\imath}\in T^{0,1}X\ ,\quad \psi_+^{\bar\imath}\in K^{1/2}\otimes\Phi^*(T^{0,1}X)\ ,\quad \psi_-^{\bar\imath}\in\bar K^{1/2}\otimes\Phi^*(T^{0,1}X) \end{aligned} \tag{V.45}$$

together with the action

$$S \;=\; 2t\int_\Sigma \mathrm{d}^2 z\ \left(\tfrac{1}{2}g_{IJ}(\phi)\partial_z\phi^I\partial_{\bar z}\phi^J + \mathrm{i}\psi_-^{\bar\imath}D_z\psi_-^i g_{\bar\imath i} + \mathrm{i}\psi_+^{\bar\imath}D_z\psi_+^i g_{\bar\imath i} + R_{i\bar\imath j\bar\jmath}\psi_+^i\psi_+^{\bar\imath}\psi_-^j\psi_-^{\bar\jmath}\right)\ .$$

[3] i.e. bundles of one-forms of type $(1,0)$ and $(0,1)$, respectively

The supersymmetry transformations under which this action is invariant are given by

$$\begin{aligned} \delta\phi^i &= \mathrm{i}\alpha_-\psi_+^i + \mathrm{i}\alpha_+\psi_-^i\ , & \delta\phi^{\bar\imath} &= \mathrm{i}\tilde\alpha_-\psi_+^{\bar\imath} + \mathrm{i}\tilde\alpha_+\psi_-^{\bar\imath}\ , \\ \delta\psi_+^i &= -\tilde\alpha_-\partial_z\phi^i - \mathrm{i}\alpha_+\psi_-^j\Gamma^i_{jm}\psi_+^m\ , & \delta\psi_+^{\bar\imath} &= -\alpha_-\partial_z\phi^{\bar\imath} - \mathrm{i}\tilde\alpha_+\psi_-^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar m}\psi_+^{\bar m}\ , \\ \delta\psi_-^i &= -\tilde\alpha_+\partial_{\bar z}\phi^i - \mathrm{i}\alpha_-\psi_+^j\Gamma^i_{jm}\psi_-^m\ , & \delta\psi_-^{\bar\imath} &= -\alpha_+\partial_{\bar z}\phi^{\bar\imath} - \mathrm{i}\tilde\alpha_-\psi_+^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar m}\psi_-^{\bar m}\ , \end{aligned} \tag{V.46}$$

where the infinitesimal fermionic parameters $\alpha_+, \tilde\alpha_+$ and $\alpha_-, \tilde\alpha_-$ are holomorphic sections of $\bar K^{1/2}$ and $K^{1/2}$, respectively.

§4 Twist of the nonlinear sigma model. The nonlinear sigma model defined in the previous paragraph can now be twisted in two possible ways resulting in the topological A- and B-model. On each pair of spinors $(\psi_+^i, \psi_+^{\bar\imath})$, we can apply the following twists:

$$\begin{array}{rll} \text{untwisted} & \psi_+^i \in K^{1/2}\otimes\Phi^*(T^{1,0}X) & \psi_+^{\bar\imath} \in K^{1/2}\otimes\Phi^*(T^{0,1}X) \\ +\ \text{twist} & \psi_+^i \in \Phi^*(T^{1,0}X) & \psi_+^{\bar\imath} \in K\otimes\Phi^*(T^{0,1}X) \\ -\ \text{twist} & \psi_+^i \in K\otimes\Phi^*(T^{1,0}X) & \psi_+^{\bar\imath} \in \Phi^*(T^{0,1}X) \end{array}$$

Analogous twists can be applied on the pairs $(\psi_-^i, \psi_-^{\bar\imath})$ with K replaced by $\bar K$.

Equally well, one can consider this as a modification of the underlying energy-momentum tensor by

$$T(z) \ \to\ T_{\mathrm{top}}(z) \ =\ T(z) \pm \tfrac12\partial J(z)\ , \tag{V.47}$$
$$\tilde T(\bar z) \ \to\ \tilde T_{\mathrm{top}}(\bar z) \ =\ \tilde T(\bar z) \pm \tfrac12\bar\partial\tilde J(\bar z)\ . \tag{V.48}$$

Combining the twists on the (ψ_+, ψ_-) sectors, we arrive again at two possible total twists: the *A-twist* $(-,+)$ and the *B-twist* $(-,-)$. Here, only the relative sign of the twists in the two sectors matters, as other combinations are obtained by complex conjugation. Half-twisted models have not aroused much attention.

V.3.2 The topological A-model

We will not consider the topological A-model in detail but just give a rough outline for completeness sake only.

§5 Field content. Due to the properties of the Graßmann coordinates interpreted as sections of different bundles over Σ, we follow [292] and rename the fields according to

$$\chi^i \ =\ \psi_+^i\ ,\quad \chi^{\bar\imath} \ =\ \psi_-^{\bar\imath}\ ,\quad \psi_z^{\bar\imath} \ =\ \psi_+^{\bar\imath}\ ,\quad \psi_{\bar z}^i \ =\ \psi_-^i\ . \tag{V.49}$$

The action thus reads in the new coordinates as

$$S \ =\ 2t\int_\Sigma \mathrm{d}^2z\ \left(\tfrac12 g_{IJ}\partial_z\phi^I\partial_{\bar z}\phi^J + \mathrm{i}\psi_z^{\bar\imath}D_{\bar z}\chi^i g_{\bar\imath i} + \mathrm{i}\psi_z^i D_z\chi^{\bar\imath}g_{\bar\imath i} - R_{i\bar\imath j\bar\jmath}\psi_{\bar z}^i\psi_z^{\bar\imath}\chi^j\chi^{\bar\jmath}\right)\ . \tag{V.50}$$

§6 Supersymmetry transformations. The supersymmetry transformations of the nonlinear sigma model become topological transformation laws after performing the A-twist. They are easily derived by setting $\alpha_+ = \tilde\alpha_- = 0$ and by introducing a BRST operator Q arising from the topological transformation laws as $\delta(\cdot) = -\mathrm{i}\alpha\{Q,\cdot\}$, one can rewrite the action as

$$S = \mathrm{i}t\int_\Sigma \mathrm{d}^2z\ \{Q,V\} + t\int_\Sigma \Phi^*(J) \tag{V.51}$$

with

$$V = g_{i\bar{\jmath}}(\psi_z^{\bar{\imath}}\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{\imath}}\psi_{\bar{z}}^j) \int_\Sigma \Phi^*(J) = \int_\Sigma \mathrm{d}^2 z \left(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{\jmath}}g_{i\bar{\jmath}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{\jmath}}g_{i\bar{\jmath}}\right) . \quad \text{(V.52)}$$

The latter expression is the pull-back of the Kähler form $J = -\mathrm{i}g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}z^{\bar{\jmath}}$ and its integral depends only on the cohomology class of J and the homotopy class of Φ. In general, one considers normalizations such that this integral equals $2\pi n$, where n is an integer called the *instanton number* or the *degree*.

Note that the reformulation done in (V.51) actually shows that the topological A-model is indeed a topological field theory.

§7 Observables. Given an n-form W, one can map it to a corresponding local operator $\mathcal{O}_W$ by replacing $\mathrm{d}z^i$ and $\mathrm{d}\bar{z}^{\bar{\imath}}$ in the basis of one-forms by χ^i and $\chi^{\bar{\imath}}$, respectively. Furthermore, it is $\{Q, \mathcal{O}_W\} = -\mathcal{O}_{\mathrm{d}W}$, where d is the de-Rham differential. Thus, there is a consistent map from the BRST-cohomology of local operators to the de Rham cohomology, and (when restricting to local operators) we can represent observables by elements of the de Rham cohomology. There is an additional "physical state condition", which reduces the de Rham cohomology to its degree $(1,1)$-subset. This subset corresponds to deformations of the Kähler form and the topological A-model therefore describes deformations of the Kähler moduli of its target space.

V.3.3 The topological B-model

The topological B-model and its open string equivalent, holomorphic Chern-Simons theory will concern us mostly in the later discussion, so let us be more explicit at this point.

§8 Reformulation and supersymmetry. We follow again [292] and define the following new coordinates:

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} , \qquad \theta_i = g_{i\bar{\imath}}(\psi_+^{\bar{\imath}} - \psi_-^{\bar{\imath}}) , \qquad \rho_z^i = \psi_+^i , \qquad \rho_{\bar{z}}^i = \psi_-^i , \quad \text{(V.53)}$$

where ρ_z^i and $\rho_{\bar{z}}^i$ are now one-forms with values in $\Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$, respectively. After this redefinition, the action becomes

$$S = t\int_\Sigma \mathrm{d}^2 z \Big(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + \mathrm{i}\eta^{\bar{\imath}}(D_z\rho_{\bar{z}}^i + D_{\bar{z}}\rho_z^i)g_{i\bar{\imath}} + \\ + \mathrm{i}\theta_i(D_{\bar{z}}\rho_z^i - D_z\rho_{\bar{z}}^i) + R_{i\bar{\imath}j\bar{\jmath}}\rho_z^i\rho_{\bar{z}}^j\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}\Big) . \quad \text{(V.54)}$$

The supersymmetry transformations are reduced by $\alpha_\pm = 0$ and $\tilde{\alpha}_\pm = \alpha$ to

$$\delta\phi^i = 0 , \quad \delta\phi^{\bar{\imath}} = \mathrm{i}\alpha\eta^{\bar{\imath}} , \quad \delta\eta^{\bar{\imath}} = \delta\theta_i = 0 , \quad \delta\rho^i = -\alpha\mathrm{d}\phi^i . \quad \text{(V.55)}$$

One can define a BRST operator from $-\mathrm{i}\alpha\{Q,\cdot\} = \delta(\cdot)$ satisfying $Q^2 = 0$ modulo equations of motion. With its help, one can write

$$S = \mathrm{i}t\int\{Q,V\} + tW \quad \text{(V.56)}$$

with

$$V = g_{i\bar{\jmath}}\left(\rho_z^i\partial_{\bar{z}}\phi^{\bar{\jmath}} + \rho_{\bar{z}}^i\partial_z\phi^{\bar{\jmath}}\right) \quad \text{and} \quad W = \int_\Sigma \left(-\theta_i D\rho^i - \tfrac{\mathrm{i}}{2}R_{i\bar{\imath}j\bar{\jmath}}\rho^i\wedge\rho^j\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}\right) . \quad \text{(V.57)}$$

Since one can show that the B-model is independent of the complex structure on Σ as well as the Kähler metric on X, this model is a topological field theory. Furthermore, the theory is mostly independent of $t \in \mathbb{R}^+$, as the first term in the action (V.56) changes by a term $\{Q, \cdot\}$ and the second term can be readjusted by a redefinition of $\theta \to \theta/t$. Thus the only dependence of correlation functions on t stems from θ-dependence of the observables. As this dependence can be clearly factored out, one can perform all calculations in the large t-limit, and this renders the B-model much simpler than the A-model: In this weak coupling limit, one can simply expand around the bosonic minima of the action, which are constant maps $\Phi : \Sigma \to X$, and thus the path integral becomes an ordinary integral over X.

§9 Anomalies. One can show that if X is not a Calabi-Yau manifold, the topological B-model is anomalous. This condition is stricter than for the A-model, and interestingly reduces our target spaces to the mathematically most appealing ones.

§10 Ghost number. The B-model has a $\mathbb{Z}$-grading from a quantum number called the *ghost number*. Putting $\tilde{Q} = 1$ and $\tilde{\phi} = 0$, we obtain from the BRST algebra (V.55) that $\tilde{\eta} = 1$ and $\tilde{\theta} = -1$. One can show that for a Calabi-Yau manifold X of complex dimension d, a correlation function vanishes for genus g, unless its total ghost number equals $2d(1-g)$.

§11 Observables. In the A-model, we could take the de Rham cohomology as a model for our local operators. In the case of the topological B-model, the situation is slightly more difficult. We have to consider forms in the Dolbeault cohomology which take values in the exterior algebra of the tangent bundle of X. Consider an element V of $\Lambda^q T^{1,0}X \otimes \Omega^{0,p}$ given by

$$V \;=\; V_{\bar{\imath}_1 \dots \bar{\imath}_p}{}^{j_1 \dots j_q} \mathrm{d}\bar{z}^{\bar{\imath}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{\imath}_p} \frac{\partial}{\partial z^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial z^{j_q}} \; . \tag{V.58}$$

We can again map V to a local operator $\mathcal{O}_V$ by replacing the one-forms $\mathrm{d}z^{\bar{\imath}}$ by $\eta^{\bar{\imath}}$ and the vector fields $\frac{\partial}{\partial z_j}$ by ψ_j. One then finds that

$$\{Q, \mathcal{O}_V\} \;=\; -\mathcal{O}_{\bar{\partial} V} \tag{V.59}$$

and thus, we can consider the sheaf cohomology $\oplus_{p,q} H^{0,p}(X, \Lambda^q T^{1,0}X)$ as the space of local operators in the topological B-model. The BRST operator Q is naturally mapped to the Dolbeault operator $\bar{\partial}$. As in the topological A-model, the Dolbeault cohomology is reduced to a subset by a physical state condition: the group of Beltrami differentials, introduced in II.4.1, §4. Thus, it describes deformations of the complex structure moduli of X.

§12 Correlation functions. Given a set of points x_α on Σ, the correlation function

$$\langle \prod_\alpha \mathcal{O}_{\mathcal{V}_\alpha}(x_\alpha) \rangle \; , \tag{V.60}$$

vanishes, unless the wedge product of all $\mathcal{V}_\alpha$ is an element of $H^{0,d}(X, \Lambda^d T^{1,0}X)$, where d is the dimension of the Calabi-Yau manifold: Any such element can be transformed into a top form by multiplying with the square of the holomorphic volume form $\Omega^{d,0}$. This top-form is then integrated over, since, as stated above, the path integral reduces to an integral over the Calabi-Yau manifold X in the case of the topological B-model.

§13 Comparison of the topological models. The topological A-model suffers some drawbacks compared to the topological B-model: The moduli space of consistent maps from the worldsheet to the target space does not reduce as nicely as in the case of the topological B-model and thus the calculation of the path integral is considerably more difficult. This fact is also related to the additional instanton corrections the partition function of the topological A-model receives.

However, the A-model is not as strictly restricted to having a Calabi-Yau manifold as its target space as the topological B-model. Furthermore, physical quantities are more easily interpreted in the framework of the A-twist. Therefore, one often starts from the A-model and uses mirror symmetry, the T-duality on the level of topological sigma models, to switch to the B-model and perform the calculations there. A mirror transformation of the results leads then back to the A-model.

V.3.4 Equivalence to holomorphic Chern-Simons theory

In this section, we will briefly describe the arguments for the equivalence of the open topological B-model with a Calabi-Yau threefold as target space and holomorphic Chern-Simons theory on the Calabi-Yau threefold as presented in [293].

§14 Argumentation via coupling. We start from a worldsheet Σ which has a disjoint union of circles C_i as its boundary $\partial\Sigma$. For our strings to end within the Calabi-Yau manifold M, we assume the target space to be filled with a stack of n D5-branes, that is, we have a rank n vector bundle E over M with a gauge potential A, see section V.4.1, §2. Let us examine the consistency condition for coupling the open topological B-model to the gauge potential. This coupling is accomplished by adding the following term to the Feynman path integral:

$$\int \mathscr{D}\Phi_i \exp(-S[\Phi_i]) \cdot \Pi_i \operatorname{tr} P \exp \oint_{C_i} \phi^*(\tilde{A}) \ , \tag{V.61}$$

where $\phi : \Sigma \to M$ and $\tilde{A} = \phi^*(A) - \mathrm{i}\eta^{\bar{\imath}} F_{\bar{\imath}j}\rho^j$ is the adjusted gauge potential. Preservation of the BRST symmetry then demands that $F^{0,2} := \partial A^{0,1} + A^{0,1} \wedge A^{0,1} = 0$, where $A^{0,1}$ is the (0,1)-part of the gauge potential A, see also the discussion in the section V.4. Thus, we can only couple the topological B-model consistently to a gauge potential if its (0,2)-part of the curvature vanishes. Via topological arguments, one can furthermore show that the only degrees of freedom contained in the open topological B-model is the gauge potential, and we can reduce this model to the action of holomorphic Chern-Simons theory with the equations of motion $F^{0,2} = 0$.

§15 Argumentation via SFT. Considering the open string field theory presented in section V.2.4, one can also show that the open topological B-model reduces to the holomorphic Chern-Simons action [293].

§16 Summary. Altogether, we can state that the open topological B-model describes the dynamics of holomorphic structures $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ on its target space. Note that it is possible to extend the equivalence between the open topological B-model and holomorphic Chern-Simons theory to the case of target spaces which are Calabi-Yau supermanifolds [297].

V.3.5 Mirror symmetry

§17 Mirror symmetry and T-duality. One of the most important symmetries in string theory is T-duality, which inverts the radius of a compactified dimension and thus exchanges winding and momentum modes in the corresponding direction, see section V.2.3. This symmetry links e.g. type IIA and type IIB superstring theories. On the level of the embedded topological string theories, this symmetry might translate into the conjectured mirror symmetry. The target space Calabi-Yau manifolds would then come in mirror pairs, and mirror symmetry would exchange Kähler and complex structure deformations. The complete statement of the mirror conjecture is that A-type topological string theory with a Calabi-Yau manifold M as a target space is fully equivalent to B-type topological string theory with a Calabi-Yau manifold W as a target space, where M and W are mirror pairs, i.e. they have Hodge numbers with $h^{1,1}_M = h^{2,1}_W$ and $h^{2,1}_M = h^{1,1}_W$ in the three-dimensional case. Such mirror pairs of Calabi-Yau manifolds are usually constructed via orbifolding varieties in complex and weighted projective spaces or using toric geometry, see e.g. [111] for examples.

§18 Mirror CFTs. We mentioned above that the topological A- and B-models are independent on the complex structure and Kähler moduli, respectively, and that this independence is due to the Q-exactness of the moduli in the respective theories. In this sense, the two models are complementary, and it is indeed possible to consider not only the mirror symmetry of Calabi-Yau manifolds, but mirror symmetry of the whole field theories. A number of examples for such mirror pairs of conformal field theory has indeed been found. Mirror symmetry has furthermore been extended from the set of CFTs defined via a nonlinear sigma model action having a Calabi-Yau manifold as a target space to more general models. Among those are nonlinear sigma models with non-compact or local Calabi-Yau manifolds as target space, Landau-Ginzburg models and minimal models. This extension has in fact been necessary since a direct calculation of a mirror theory of a nonlinear sigma model can, e.g., yield a Landau-Ginzburg theory.

§19 Consequences of mirror symmetry. Mirror symmetry might be of vast importance in string theory. First of all, one expects it to give rise to a number of new string dualities, similarly to the new dualities found with the help of T-duality. Second, it is already a major calculatory tool within topological string theory. As we saw above, the B-model often allows for a mathematically more tractable description, while the A-model is often more closely related to physically interesting quantities. One could thus imagine to work essentially in the A-model and switch via mirror symmetry to the B-model, whenever a calculation is to be performed. Eventually, the results can then be retranslated to the A-model.

Mirror symmetry even found applications in mathematics, when it was used for finding all the numbers n_d of rational degree d curves lying in the quintic embedded in $\mathbb{C}P^4$ [55]. This result obtained by physicists was preceded by more than a century of efforts by mathematicians. Mirror symmetry related here the complicated problem of enumerative geometry to a much simpler problem in complex geometry.

V.4 D-Branes

Certainly one of the turning points in the development of string theory was the discovery that besides the fundamental string, there are further objects, the so-called D-branes, which unavoidably arise in string theories, and that these D-branes are sources in the Ramond-Ramond sector with a nearly arbitrary worldvolume dimension [222]. Roughly speaking, a D-brane is a hypersurface on which open strings with Dirichlet boundary condition can end, and which absorb the momentum flowing off the endpoints of the string. Note that in our conventions, a Dp-brane will denote a D-brane with a worldvolume of dimensions $(1, p)$ and (a, b) with $a + b = p$ in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ critical superstring theories, respectively.

V.4.1 Branes in type II superstring theory

§1 The NS-five brane. As we saw before, the NS-NS-sector contains an antisymmetric tensor of rank two which has a Hodge dual $B_{(6)}$ by $*\mathrm{d}B_{(2)} = \mathrm{d}B_{(6)}$. This potential couples naturally to the world volume of a five-dimensional object, the NS-five brane:

$$S \;=\; Q_{\mathrm{NS5}} \int_{\mathcal{M}_6} B_{(6)}. \tag{V.62}$$

The NS5-brane exists in both type IIA and type IIB superstring theories.

§2 D-branes in the R-R sector. Generally speaking, there are two different points of views for these D-branes. First, one can understand a Dp-brane as a p-dimensional hyperplane on which open strings end. Second, a Dp-brane is a brane-like soliton of type IIA or type IIB supergravity in ten dimensions.

Recall from section V.2 that there are higher-form potentials in the R-R sector of type II string theory. It is only natural to introduce sources to which these potentials can couple electrically. This gives rise to hypersurfaces, the Dp-branes, with a $(p+1)$-dimensional worldvolume $\mathcal{M}_{\mathrm{D}p}$ which couple to the potentials $C_{(i)}$ via [222]

$$\mu_p \int_{\mathcal{M}_{\mathrm{D}p}} C_{(p+1)} \;, \tag{V.63}$$

where μ_p is the corresponding charge. Since the higher-form potentials are of even and odd rank in type IIA and type IIB string theory, respectively, this construction yields D$0, 2, 4$ and 6-branes in type IIA and D$(-1), 1, 3, 5$ and 7-branes in type IIB string theory.

A stack of n such Dp-branes naturally comes with a rank n vector bundle E over their $p+1$-dimensional worldvolume together with a connection one-form A. This field arises from the Chan-Paton factors attached as usually to the ends of an open string.

In the following, we will mostly discuss D-branes within type IIB superstring theory.

§3 D-brane dynamics. The action for a D9-brane is the *Born-Infeld action*

$$S_{\mathrm{BI}} \;=\; \frac{1}{(4\pi^2\alpha')^5 g_s} \int \mathrm{d}^{10}x \, \sqrt{-\det(\eta_{\mu\nu} + T^{-1}F_{\mu\nu})} \;, \tag{V.64}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension and $F_{\mu\nu}$ the field strength of the gauge potential A living on the D-brane's worldvolume. The actions for lower-dimensional D-branes are obtained by dimensional reduction, which converts the gauge potential components A^μ in the reduced dimensions to Higgs-fields X^i. Expanding the determinant and taking the field theory limit $\alpha' \to 0$ in which all massive string modes decouple, yields the ten-dimensional Yang-Mills equations (or a dimensional reduction thereof).

The theory describing the dynamics in the worldvolume is therefore $\mathcal{N} = 1$ super Yang-Mills theory reduced from ten dimensions to the worldvolume of the D-brane. The resulting Higgs fields describe the motion of the D-brane in the directions of the target space normal to the worldvolume of the D-brane.

Note that on curved spaces, one often has to consider twisted supersymmetry as linear realizations may not be compatible with the geometry [33]. One therefore uses (a supersymmetric extension of) the Hermitian Yang-Mills equations[4]

$$F^{0,2} \;=\; F^{2,0} \;=\; 0 \quad \text{and} \quad k^{d-1}\wedge F \;=\; \gamma k^d \;, \tag{V.65}$$

[4]Depending on their explicit shape, these equations are also called generalized Hitchin equations, Donaldson-Uhlenbeck-Yau equations and Hermite-Einstein equations.

which are also reduced appropriately from ten to $p+1$ dimensions, see e.g. [138]. Here, k is the Kähler form of the target space and γ is the slope of E, i.e. a constant encoding information about the first Chern class of the vector bundle E. These equations imply the (dimensionally reduced, supersymmetric) Yang-Mills equations.

V.4.2 Branes within branes

§4 Instanton configurations. From comparing the amplitude of a closed string being exchanged between two parallel D-branes to the equivalent one-loop open string vacuum amplitude, one derives for the coupling μ_p in (V.63) that $\mu_p = (2\pi)^{-p}\alpha'^{-\frac{1}{2}(p+1)}$. Furthermore, the anomalous coupling of gauge brane fields with bulk fields have to satisfy certain conditions which restrict them to be given by

$$\mu_p \int_{\mathcal{M}_{\mathrm{D}p}} \sum_i i^* C_{(i)} \wedge \operatorname{tr} \mathrm{e}^{2\pi\alpha' F + B} \wedge \sqrt{\hat{A}(4\pi^2\alpha' R)}\ , \tag{V.66}$$

where i is the embedding of $\mathcal{M}_{\mathrm{D}p}$ into spacetime and $\hat{A}$ is the A-roof genus[5] (Dirac genus), which is equivalent to the Todd class if $\mathcal{M}$ is a Calabi-Yau manifold. By expanding the exponent and the Dirac genus in (V.66), one picks up a term

$$\mu_p \frac{(2\pi\alpha')^2}{2} \int C_{p-3} \wedge \operatorname{tr} F \wedge F\ , \tag{V.67}$$

and thus one learns that instanton configurations on E give rise to D$(p-4)$-branes, where each instanton carries exactly one unit of D$(p-4)$-brane charge. Similarly, the first term from the expansion of the Dirac genus gives rise to D$(p-4)$-branes when $\mathcal{M}_{\mathrm{D}p}$ is wrapped on a surface with non-vanishing first Pontryagin class, e.g. on a K3 surface.

A bound state of a stack of Dp-branes with a D(p-4)-brane can therefore be described in two possible ways. On the one hand, we can look at this state from the perspective of the higher-dimensional Dp-brane. Here, we find that the D(p-4) brane is described by a gauge field strength F on the bundle E over the worldvolume of the Dp-brane with a nontrivial second Chern character $ch_2(E)$. The instanton number (the number of D(p-4) branes) is given by the corresponding second Chern class. In particular, the bound state of a stack of BPS D3-branes with a D(-1)-brane is given by a self-dual field strength $F = *F$ on E with $-\frac{1}{8\pi^2}\int F \wedge F = 1$. On the other hand, one can adapt the point of view of the D(p-4)-brane inside the Dp-brane and consider the dimensional reduction of the $\mathcal{N} = 1$ super Yang-Mills equations from ten dimensions to the worldvolume of the D(p-4)-branes. To complete the picture, one has to add strings with one end on the Dp-brane and the other one on the D(p-4)-branes. Furthermore, one has to take into account that the presence of the Dp-brane will halve the number of supersymmetries once more, usually to a chiral subsector. In the case of the above example of D3- and D(-1)-branes, this will give rise to the ADHM equations discussed in section VII.8.

§5 Monopole configurations. Similarly, one obtains monopole configurations [51], but here the D-brane configuration, consisting of a bound state of Dp-D(p-2)-branes, is slightly more involved. One can again discuss this configuration from both the perspectives of the D3- and the D1-branes. From the perspective of the D1-branes, the bound state is described by the Nahm equations presented in section VII.8.4.

[5]If the normal bundle $\mathcal{N}$ of $\mathcal{M}_{\mathrm{D}p}$ in spacetime has non-vanishing curvature $R_{\mathcal{N}}$, we additionally have to divide by $\sqrt{\hat{A}(4\pi^2\alpha' R_{\mathcal{N}})}$.

V.4.3 Physical B-branes

§6 Boundary conditions. As stated above, a D-brane in type II string theory is a Ramond-Ramond charged BPS state. When compactifying this theory on Calabi-Yau manifolds, one has to consider boundary conditions corresponding to BPS states in the appropriate $\mathcal{N} = 2$ superconformal field theory (SCFT), and there are precisely two possibilities: the so-called A-type boundary condition and the B-type boundary condition [211]. Therefore, D-branes on Calabi-Yau manifolds come in two kinds: A-branes and B-branes. We will only be concerned with the latter ones.

Recall that the $\mathcal{N} = (2,2)$ superconformal algebra is generated by a holomorphic set of currents $T(z), G^{\pm}(z), J(z)$ and an antiholomorphic one $\tilde{T}(\bar{z}), \tilde{G}^{\pm}(\bar{z}), \tilde{J}^{\pm}(\bar{z})$. The B-type boundary condition is then given by

$$G^{\pm}(z) \;=\; \tilde{G}^{\pm}(\bar{z}) \quad \text{and} \quad J(z) \;=\; \tilde{J}(\bar{z}) \;, \tag{V.68}$$

see e.g. [76].

§7 Dynamics of a stack of D-branes. Consider now a stack of n Dp-branes[6] which are B-branes in a d-dimensional Calabi-Yau manifold M with Kähler form k. As the open strings living on a brane come with Chan-Paton degrees of freedom, our Dp-branes come with a vector bundle E of rank n and a gauge theory determining the connection A on E. Let us denote the field strength corresponding to A by F. The dynamics of A is then governed by the generalized Hitchin equations [123] (cf. equations (V.65))

$$F^{0,2} \;=\; F^{2,0} \;=\; 0 \;, \tag{V.69a}$$

$$k^{d-1} \wedge F \;=\; \gamma k^d \;, \tag{V.69b}$$

$$\bar{\partial}_A X_i \;=\; 0 \quad \text{and} \quad [X_i, X_j] \;=\; 0 \;, \tag{V.69c}$$

where γ is again a constant determined by the magnetic flux of the gauge bundle. The fields X_i represent the normal motions of the B-brane in M.

§8 The six-dimensional case. Consider now the case $p + 1 = d = 6$. Then we are left with equations (V.69a) and (V.69b), which can also be obtained from the instanton equations of a twisted maximally supersymmetric Yang-Mills theory, reduced from ten to six dimensions [33, 138, 204]. It is not clear whether there is any difference to the holomorphic Chern-Simons theory obtained by Witten in [293] as argued in [204]: equations (V.69a) are obviously the correct equations of motion and (V.69b) combines with $\mathsf{U}(N)$ gauge symmetry to a $\mathsf{GL}(N, \mathbb{C})$ gauge symmetry.

Lower-dimensional branes, as e.g. D2- and D0-branes correspond to gauge configurations with nontrivial second and third Chern classes, respectively, and thus they are instantons of this maximally supersymmetric Yang-Mills theory [204].

§9 Remark concerning topological A-branes. For quite some time, only special Lagrangian submanifolds were thought to give rise to a topological A-brane. For a Calabi-Yau threefold compactification, this would imply that those branes always have a world-volume of real dimension three. However, Kapustin and Orlov have shown [149] that it is necessary to extend this set to coisotropic Lagrangian submanifolds, which allow for further odd-dimensional topological A-branes.

[6] Here, a Dp-brane has $p + 1$ real dimensions in the Calabi-Yau manifold.

V.4.4 Topological B-branes

§10 Holomorphic submanifolds. Recall that the complex structure I of the Calabi-Yau manifold does not interchange normal and tangent directions of a boundary consistently defined in the topological B-model. Therefore, a D-brane should wrap a holomorphically embedded submanifold $\mathcal{C}$ of the ambient Calabi-Yau manifold M and this restriction will preserve the topological symmetry of our model [22, 211]. Thus, there are topological B-branes with worldvolumes of dimension $0, 2, 4$ and 6.

§11 Chan-Paton degrees. Furthermore, the open topological strings ending on a stack of B-branes will also carry Chan-Paton degrees of freedom, which in turn will lead to a complex vector bundle E over $\mathcal{C}$. However, one is forced to impose a boundary condition [293, 129]: the vanishing of the variation of the action from the boundary term. This directly implies that the curvature F of E is a 2-form of type (1,1) and in particular $F^{0,2}$ vanishes. Therefore, the underlying gauge potential $A^{0,1}$ defines a holomorphic structure and E becomes a holomorphic vector bundle. The gauge theory describing the D-brane dynamics is holomorphic Chern-Simons theory, as shown in [293]. Note that the equations of motion

$$F^{0,2} \;=\; 0\;. \tag{V.70}$$

differ from the one of their BPS-cousins only by the second equation in (V.65). This equation is a (BPS) stability condition on the vector bundle E.

§12 Lower-dimensional B-branes. B-branes, which do not fill the complete Calabi-Yau manifold M are described by dimensional reductions of hCS theory [275, 203] and we have again additional (Higgs) fields X_i, which are holomorphic sections of the normal bundle of the worldvolume $\mathcal{C}$ in M with values in $\mathsf{GL}(n,\mathbb{C})$ satisfying $[X_i, X_j] = 0$. They describe fluctuations of the B-branes in the normal directions. Explicitly, the equations governing the fields present due to the B-branes read as

$$F^{0,2} \;=\; F^{2,0} \;=\; 0\;,\quad \bar{\partial}_A X_i \;=\; 0\;,\quad [X_i, X_j] \;=\; 0\;. \tag{V.71}$$

These equations are a subset of the generalized Hitchin equations (V.69a)-(V.69c). The missing equation (V.69b) completes (V.69a) to the Hermitian Yang-Mills equations. According to a theorem by Donaldson, Uhlenbeck and Yau (see e.g. [11] and references therein), the existence of a Hermitian Yang-Mills connection is equivalent to E being μ-stable, which in turn is equivalent to the BPS condition at large radius.

For the latter remark, recall that the actually appropriate description of B-branes is the derived category of coherent sheaves, see e.g. [11] and references therein. A topological B-brane is simultaneously a physical B-brane if it satisfies some stability condition which is equivalent to the B-type BPS condition. Thus, we saw above that the physical B-branes are a subsector of the topological B-branes.

§13 Topological and physical D-branes. As one can nicely embed the topological open string into the physical open string (and therefore physical D-branes into topological ones), we expect that lower-dimensional topological branes, which are bound states in a topological D5-brane should appear as gauge configurations in six-dimensional twisted super Yang-Mills theory with nontrivial Chern classes. In particular, a D2-brane should correspond to an instanton and thus to a nontrivial second Chern class [203]. The term in the partition function capturing this kind of instantons is $\exp(-\int_M k \wedge ch_2)$, where k is the Kähler form of the ambient Calabi-Yau manifold[7] M.

[7]Note at this point that while the A-model and the B-model depend on the Kähler structure moduli and the complex structure moduli, respectively, the rôle is interchanged for D-branes: Lagrangian submanifolds

Note that we will completely ignore closed strings interacting with the B-branes. Their vertex operators would give rise to deformations of the complex structure described by a Beltrami differential, cf. section II.4.1, §4. The theory governing these deformations is the Kodaira-Spencer theory of gravity [32].

V.4.5 Further aspects of D-branes

§14 Non-BPS branes. There are essentially two reasons for extending the analysis of D-branes to non-BPS [258] ones: First, stable non-BPS branes are part of the spectrum and lead to non-trivial but calculable results in different limits of the string coupling. Second, they give rise to worldvolume gauge theories with broken supersymmetry and might therefore play an important rôle in string compactifications yielding phenomenologically relevant models.

As an example [258], let us consider a D$2p$-$\bar{\text{D}}2p$-pair of BPS branes in type IIA superstring theory. This configuration is invariant under orbifolding with respect to $(-1)^{F_L}$, where F_L is the spacetime fermion number of the left-movers. The bulk, however, will be described by type IIB superstring theory after this orbifolding. This operation projects out modes in the open string spectrum which would correspond to separating the two D-branes. Thus, we arrive at a single object, a non-BPS D$2p$-brane in type IIB superstring theory. However, the tachyonic mode, which is present from the very beginning for a D-brane-anti-D-brane pair, is not projected out.

Although the non-BPS D-brane considered above was unstable due to the existence of a tachyonic mode, there are certain orbifold/orientifold compactifications in which the tachyonic modes are projected out and therefore the non-BPS D-brane becomes stable.

§15 D-branes in $\mathcal{N} = 2$ string theory. Considering D-branes in critical $\mathcal{N} = 2$ string theory is not as natural as in ten-dimensional superstring theories since the NS sector is connected to the R sector via the $\mathcal{N} = 2$ spectral flow, and it is therefore sufficient to consider the purely NS part of the $\mathcal{N} = 2$ string. Nevertheless, one can confine the endpoints of the open strings in this theory to certain subspaces and impose Dirichlet boundary conditions to obtain objects which we will call D-branes in $\mathcal{N} = 2$ string theory. Although the meaning of these objects has not yet been completely established, there seem to be a number of safe statements we can recollect. First of all, the effective field theory of these D-branes is four-dimensional (supersymmetric) SDYM theory reduced to the appropriate worldvolume [192, 105]. The four-dimensional SDYM equations are nothing but the Hermitian Yang-Mills equations:

$$F^{2,0} \;=\; F^{0,2} \;=\; 0 \quad \text{and} \quad k \wedge F \;=\; 0\;, \tag{V.72}$$

where k is again the Kähler form of the background. The Higgs-fields arising in the reduction process describe again fluctuations of the D-branes in their normal directions.

As is familiar from the topological models yielding hCS theory, we can introduce A- and B-type boundary conditions for the D-branes in $\mathcal{N} = 2$ critical string theory. For the target space $\mathbb{R}^{2,2}$, the A-type boundary conditions are compatible with D-branes of worldvolume dimension (0,0), (0,2), (2,0) and (2,2) only [146, 105].

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couple naturally to the holomorphic 3-form of a Calabi-Yau, while holomorphic submanifolds naturally couple to the Kähler form [204], as seen in this example.

§16 Super D-branes. There are three approaches of embedding worldvolumes into target spaces when Graßmann directions are involved. First, one has the Ramond-Neveu-Schwarz (RNS) formulation [207, 236], which maps a super worldvolume to a bosonic target space. This approach only works for a spinning particle and a spinning string; no spinning branes have been constructed so far. However, this formulation allows for a covariant quantization. Second, there is the Green-Schwarz (GS) formulation [110], in which a bosonic worldvolume is mapped to a target space which is a supermanifold. In this approach, the well-known κ-symmetry appears as a local worldvolume fermionic symmetry. Third, there is the doubly-supersymmetric formulation (see [265] and references therein), which unifies in some sense both the RNS and GS approaches. In this formulation, an additional superembedding condition is imposed, which reduces the worldvolume supersymmetry to the κ-symmetry of the GS approach.

In the following, we will often work implicitly with the doubly supersymmetric approach.

§17 Geometric engineering. It is easily possible to engineer certain D-brane configurations, which, when put in certain Calabi-Yau compactifications, give rise to a vast variety of field theories in four dimensions [152, 197]. Most prominently, one realizes $\mathcal{N}=2$ supersymmetric gauge theories from compactifications of type II string theories. In particular, objects arising in field theory, as e.g. the Seiberg-Witten torus, are easily interpreted within such a compactification scheme.

Let us consider a popular example, which was developed in [46] and studied e.g. in [147] and [78]. We start from the algebraic variety

$$xy \;=\; z^2 - t^{2n} \tag{V.73}$$

in $\mathbb{C}^4$. For $n=1$, this is just the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{C}P^1$ with a rigid $\mathbb{C}P^1$ at its tip, see section II.3.3. For $n>1$, the geometry also contains a $\mathbb{C}P^1$ with normal bundle $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ but the deformation of the sphere inside this bundle is obstructed at n-th order, which can be described by a superpotential $W(\phi)$, which is a polynomial of $n+1$th order, where $t^2\sim\phi$. We have therefore the coordinates

$$\lambda_+ \;=\; \frac{1}{\lambda_-}~,\quad z^1_+ \;=\; z^1_-~,\quad \lambda_+ z^2_+ \;=\; \lambda_- z^2_- + W'(z^1_+) \tag{V.74}$$

on the two patches $\mathcal{U}_\pm$ covering the deformation of $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ together with the identification

$$z^1_+ \;=\; z^1_- \;=\; t~,\quad z^2_+ \;=\; \tfrac{1}{2}x~,\quad z^2_- \;=\; \tfrac{1}{2}y \quad\text{and}\quad z \;=\; (2\lambda_+ z^2_+ - W'(z^1_+))~. \tag{V.75}$$

This geometry has rigid $\mathbb{C}P^1$s at the critical points of the superpotential $W(z^1_+)$.

Without the deformation by $W(z^1_+)$, wrapping n D5-branes around the $\mathbb{C}P^1$ of $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ yields an $\mathcal{N}=2$ $\mathsf{U}(n)$ gauge theory on the remaining four dimensions of the D5-branes, which are taken to extend in ordinary spacetime. The deformation by $W(z^1_+)$ breaks $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$ with vacua at the critical points of the superpotential. One can now distribute the D5-branes among the i critical points of $W(z^1_+)$, each corresponding to a rigid $\mathbb{C}P^1$, and thus break the gauge group according to $\mathsf{U}(n)\to\mathsf{U}(n_1)\times\ldots\times\mathsf{U}(n_i)$, with $n_1+\ldots+n_i=n$.

V.4.6 Twistor string theory

A good source for further information and deeper review material about the developments in twistor string theory is [181] and [234]. The paper in which twistor string theory was considered for the first time is [297].

§18 Motivation. Even after half a century of intense research, we still do not completely understand quantum chromodynamics. The most prominent point is probably the phenomenon called *confinement*, i.e. the fact that quarks are permanently confined inside a bound state as the coupling constant becomes large at low energies. To answer this and more questions, string-gauge theory dualities are important. The most prominent example is certainly the AdS/CFT correspondence [187].

Witten's motivation for the construction of twistor string theory was originally to find an alternative description of the string theory side in the AdS/CFT correspondence, which is suited for describing the small gauge coupling limit. The existence of a radically different such description is in fact plausible, as many theories change drastically their shape when considered in a certain regime or after dualities have been applied. One aspect should, however, remain conserved: the symmetry group $P\mathsf{SU}(2,2|4)$ or $P\mathsf{SU}(4|4)$ of the target space $AdS_5 \times S^5$. The most natural space with this symmetry group is probably the supertwistor space[8] $\mathbb{C}P^{3|4}$. Since this space is in fact a Calabi-Yau supermanifold, one can study the topological B-model having this space as a target space.

§19 Twistor string theory. Consider the supertwistor space $\mathbb{C}P^{3|4}$ with a stack of n almost space-filling D5-branes. Here, "almost space-filling" means that the fermionic coordinates extend in the holomorphic directions only, while all the antiholomorphic directions are completely ignored. Twistor string theory is now simply the topological B-model with $\mathbb{C}P^{3|4}$ as its target space and the above given D-brane configuration. This model can be shown to be equivalent to holomorphic Chern-Simons theory on $\mathbb{C}P^{3|4}$ describing holomorphic structures on a rank n complex vector bundle. The power of twistor string theory in describing gauge theories arises from the twistor correspondence and the Penrose-Ward transform, see chapter VII.

§20 Further twistor string theories. Further topological string theories with a supertwistor space as target space have been considered. First, following the proposal in [297], the superambitwistor space has been considered in [203] and [2] as a target space for the topological B-model. In particular, a mirror conjecture was established between the superambitwistor space and the supertwistor space previously discussed by Witten. In [63], the discussion was extended to the mini-supertwistor space, which will probably be the mirror of the mini-superambitwistor space introduced in [243]. All of these spaces and their rôle in twistor geometry will be extensively discussed in chapter VII.

[8] We will consider twistor spaces in more detail in chapter VII.

Chapter VI

Non-(anti)commutative Field Theories

In this chapter, we will be concerned with noncommutative deformations of spacetime and non-anticommutative deformations of superspace. Both noncommutativity[1] and non-anticommutativity naturally arise in type II string theories put in a constant NS-NS B-field background [67] and a constant R-R graviphoton background [69], respectively. Therefore these deformations seem to be unavoidable when studying string theory in nontrivial backgrounds. Moreover, they can provide us with interesting toy models which are well-suited for studying features of string theories (as e.g. non-locality) which do not appear within ordinary field theories.

VI.1 Comments on noncommutative field theories

Over the last decade, there has been an immense effort by string theorists to improve our understanding of string dynamics in nontrivial backgrounds. Most prominently, Seiberg and Witten [255] discovered that superstring theory in a constant Kalb-Ramond 2-form background can be formulated in terms of field theories on noncommutative spacetimes upon taking the so-called Seiberg-Witten zero slope limit. Subsequently, these noncommutative variants of ordinary field theories were intensely studied, revealing many interesting new aspects, such as UV/IR mixing [198], the vastness of nontrivial classical solutions to the field equations[2] and the nonsingular nature of the noncommutative instanton moduli spaces, see e.g. [206]. It turned out that as low energy effective field theories, noncommutative field theories exhibit many manifestations of stringy features descending from the underlying string theory. Therefore, these theories have proven to be an ideal toy model for studying string theoretic questions which otherwise remain intractable, as e.g. tachyon condensation [257, 68, 1, 165, 148] and further dynamical aspects of strings [116] (for recent work, see e.g. [289, 230]). Noncommutativity has also been used as a means to turn a field theory into a matrix model [176]. The results of this publication are presented in section VIII.3.

VI.1.1 Noncommutative deformations

§1 Deformation of the coordinate algebra. In ordinary quantum mechanics, the coordinate algebra on the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ is deformed to the Heisenberg algebra

$$[\hat{x}^i, \hat{p}_j] \;=\; \mathrm{i}\hbar\delta^i_j \;, \quad [\hat{x}^i, \hat{x}^j] \;=\; [\hat{p}_i, \hat{p}_j] \;=\; 0 \;. \tag{VI.1}$$

[1]Note that it has become common usage to call a space noncommutative, while gauge groups with the analogous property are called non-Abelian.

[2]See also the discussion in sections VII.8 and VIII.3.2.

As these relations have not been verified to very low distances (i.e. very high energies), a natural (relativistic) generalization would look like

$$[\hat{x}^\mu, \hat{x}^\nu] \;=\; \mathrm{i}\theta^{\mu\nu} \;, \tag{VI.2}$$

where $\theta^{\mu\nu}$ is a constant of dimension $[L]^2$. Clearly, by such a deformation, the Poincaré group is broken down to the stabilizer subgroup of the deformation tensor θ. The deformation of the space $\mathbb{R}^4$ with coordinates satisfying the algebra (VI.2) will be denoted by $\mathbb{R}^4_\theta$ and called *noncommutative spacetime.*

The first discussion of noncommutative spaces in a solid mathematical framework has been presented by Alain Connes [66]. Since then, noncommutative geometry has been used in various areas of theoretical physics as e.g. in the description of the quantum Hall effect in condensed matter physics and in particular in string theory.

For a review containing a rather formal introduction to noncommutative geometry, see [279]. Further useful review papers are [87] and [268].

§2 Noncommutativity from string theory. In 1997, noncommutative geometry was shown to arise in certain limits of M-theory and string theory on tori [67, 86]; several further appearances have been discovered thereafter. Let us here briefly recall the analysis of [255].

Consider open strings in flat space and in the background of a constant Neveu-Schwarz B-field on a D-brane with the action

$$S = \tfrac{1}{4\pi\alpha'}\int_\Sigma g_{MN}\partial_\alpha X^M \partial^\alpha X^N - \tfrac{\mathrm{i}}{2}\int_{\partial\Sigma} B_{MN} X^M \partial_T X^N \;, \tag{VI.3}$$

where Σ denotes the string worldsheet and ∂_T is a derivative tangential to the boundary $\partial\Sigma$. We assume the latter subspace to be mapped to the worldvolume of the D-brane. For our purposes, it is enough to restrict ourselves to the case where Σ is the disc and map it conformally to the upper half plane $H = \{z \in \mathbb{C}\,|\,\mathrm{Im}(z) > 0\}$. The resulting equations of motion read

$$g_{MN}(\partial_z - \bar{\partial}_{\bar{z}})X^N + 2\pi\alpha' B_{MN}(\partial_z + \bar{\partial}_{\bar{z}})X^N\big|_{z=\bar{z}} \;=\; 0 \;, \tag{VI.4}$$

from which we can calculate a propagator $\langle X^M(z)X^N(z')\rangle$. At the boundary $\partial\Sigma = \mathbb{R} \subset \mathbb{C}$ of Σ, where the open string vertex operators live we are interested in, this propagator reads

$$\begin{aligned}\langle X^M(\tau)X^N(\tau')\rangle \;&=\; -\alpha' G^{MN}\log(\tau-\tau')^2 + \tfrac{\mathrm{i}}{2}\theta^{MN}\mathrm{sign}(\tau-\tau')\\ \text{with}\quad \theta^{MN} \;&=\; 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{[MN]} \;,\end{aligned} \tag{VI.5}$$

where $\tau \in \mathbb{R}$ parameterizes the boundary. Recall now that one can calculate commutators of operators from looking at the short distance limit of operator products. From (VI.5), we get

$$[X^M(\tau), X^N(\tau)] \;=\; T\left(X^M(\tau)X^N(\tau^-) - X^M(\tau)X^N(\tau^+)\right) \;=\; \mathrm{i}\theta^{MN} \;. \tag{VI.6}$$

Thus, the target space in our string configuration indeed proves to carry a noncommutative coordinate algebra. Note, however, that to be accurate, one has to carefully consider a zero slope limit $\alpha' \to 0$ to decouple more complicated string effects.

§3 Two-oscillator Fock space. In the following, let us restrict ourselves to four dimensions and consider a self-dual ($\kappa = 1$) or an anti-self-dual ($\kappa = -1$) deformation tensor $\theta^{\mu\nu}$, which has components

$$\theta^{12} \;=\; -\theta^{21} \;=\; \kappa\theta^{34} \;=\; -\kappa\theta^{43} \;=\; \theta \;>\; 0\ . \tag{VI.7}$$

After introducing the annihilation operators[3]

$$a_1 \;=\; x^1 - \mathrm{i}\varepsilon x^2 \quad \text{and} \quad a_2 \;=\; -\kappa\varepsilon x^3 + \mathrm{i}\varepsilon x^4\ , \tag{VI.8}$$

we find the appropriate representation space of the algebra (VI.2) to be the two-oscillator Fock space $\mathcal{H} = \mathrm{span}\{|n_1, n_2\rangle | n_1, n_2 \in \mathbb{N}\}$ with

$$|n_1, n_2\rangle \;=\; \frac{1}{\sqrt{n_1!n_2!}} \left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2} |0,0\rangle\ . \tag{VI.9}$$

One can therefore picture functions on $\mathbb{R}^4_\theta$ as (the tensor product of two) infinite-dimensional matrices representing operators on $\mathcal{H}$.

§4 Derivatives and integrals. The derivatives on noncommutative spacetimes are given by inner derivations of the Heisenberg algebra (VI.2). We can define

$$\partial_\mu \;\rightarrow\; \hat{\partial}_\mu \;:=\; -\mathrm{i}\theta_{\mu\nu}[\hat{x}^\nu, \cdot]\ , \tag{VI.10}$$

where $\theta_{\mu\nu}$ is the inverse of $\theta^{\mu\nu}$. This definition yields $\hat{\partial}_\mu \hat{x}^\nu = \delta^\nu_\mu$, analogously to the undeformed case. Furthermore, due to the commutator in the action, the Leibniz rule holds as usual.

Integration is correspondingly defined by taking the trace over the Fock space $\mathcal{H}$ representing the noncommutative space

$$\int \mathrm{d}^4 x \;\rightarrow\; (2\pi\theta)^2\, \mathrm{tr}_{\mathcal{H}}\ . \tag{VI.11}$$

The analogue to the fact that the integral over a total derivative vanishes is here the vanishing of the trace of a commutator. Equally well as the former does not hold for arbitrary functions, the latter does not hold for arbitrary operators [87].

§5 Moyal-Weyl correspondence. The Moyal-Weyl correspondence maps the operator formalism of noncommutative geometry to the star-product formalism, i.e.

$$(\hat{f}(\hat{x}), \cdot) \longleftrightarrow (f(x), \star)\ . \tag{VI.12}$$

This map can be performed by a double Fourier transform using the formulæ

$$\hat{f}(\hat{x}) \;=\; \int \mathrm{d}\alpha\ \mathrm{e}^{\mathrm{i}\alpha\hat{x}}\phi(\alpha) \quad \text{and} \quad \phi(\alpha) \;=\; \int \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}\alpha x} f(x)\ . \tag{VI.13}$$

Consistency then requires the star product to be defined according to

$$(f \star g)(x) \;:=\; f(x) \exp\left(\tfrac{\mathrm{i}}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right) g(x)\ , \tag{VI.14}$$

and the noncommutative deformation of spacetime is then written as $[x^\mu \overset{\star}{,} x^\nu] = \mathrm{i}\theta^{\mu\nu}$. Note furthermore that the star product is associative: $(f \star g) \star h = f \star (g \star h)$ and behaves as one would expect under complex conjugation: $(f \star g)^* = g^* \star f^*$. Under the integral, we have the identities

$$\int \mathrm{d}^4 x\ (f \star g)(x) \;=\; \int \mathrm{d}^4 x\ (g \star f)(x) \;=\; \int \mathrm{d}^4 x\ (f \cdot g)(x)\ . \tag{VI.15}$$

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[3]The constant $\varepsilon = \pm 1$ here distinguishes between a metric of Kleinian signature $(2,2)$ for $\varepsilon = +1$ and a metric of Euclidean signature $(4,0)$ for $\varepsilon = -1$ on $\mathbb{R}^4$.

VI.1.2 Features of noncommutative field theories

§6 Noncommutative gauge theories. As found above, a derivative is mapped to a commutator on noncommutative spaces. We can extend the arising commutator by a gauge potential $\hat{A}_\mu$, which is a Lie algebra valued function on the noncommutative space. We thus arrive at

$$\nabla_\mu \;\to\; [\hat{X}_\mu, \cdot] \quad \text{with} \quad \hat{X}_\mu \;=\; -\mathrm{i}\theta_{\mu\nu}\hat{x}^\nu \otimes \mathbb{1}_{\mathcal{G}} + \hat{A}_\mu \;, \tag{VI.16}$$

where $\mathbb{1}_{\mathcal{G}}$ is the unit of the gauge group $\mathcal{G}$ corresponding to the Lie algebra under consideration. The field strength is then given by

$$\hat{F}_{\mu\nu} \;=\; [\hat{X}_\mu, \hat{X}_\nu] + \mathrm{i}\theta_{\mu\nu} \otimes \mathbb{1}_{\mathcal{G}} \;, \tag{VI.17}$$

where the last term compensates the noncommutativity of the bare derivatives. The Yang-Mills action becomes

$$S = \mathrm{tr}_{\mathcal{H}} \otimes \mathrm{tr}_{\mathcal{G}} \left([\hat{X}_\mu, \hat{X}_\nu] + \mathrm{i}\theta_{\mu\nu} \otimes \mathbb{1}_{\mathcal{G}}\right)^2 \;, \tag{VI.18}$$

which is the action of a matrix model with infinite-dimensional matrices.

§7 Gauge transformations. The action of gauge transformations $\hat{g}$ is found by trivially translating their action from the commutative case, i.e.

$$\hat{A}_\mu \;\to\; \hat{g}^{-1}\hat{A}_\mu\hat{g} + \hat{g}^{-1}\hat{\partial}_\mu\hat{g} \;. \tag{VI.19}$$

Let us now switch to the star product formalism and consider the noncommutative analogue to infinitesimal Abelian gauge transformations $\delta A_\mu = \partial_\mu\lambda$, which reads

$$\delta A_\mu \;=\; \partial_\mu\lambda + \lambda \star A_\mu - A_\mu \star \lambda \;. \tag{VI.20}$$

We thus see that even in the case of an Abelian gauge group, the group of gauge transformations is a non-Abelian one.

It is important to stress that in noncommutative Yang-Mills theory, not all gauge groups are admissible. This is due to the fact that the corresponding Lie algebras may no longer close under star multiplication. As an example, consider the gauge group $\mathsf{SU}(2)$: The commutator $[x^\mu \mathrm{i}\sigma^3 \overset{\star}{,} x^\nu \mathrm{i}\sigma^3] = -\mathrm{i}\theta^{\mu\nu}\mathbb{1}_2$ is not an element of $\mathfrak{su}(2)$.

§8 Seiberg-Witten map. The last observations seem intuitively to forbid the following statement: There is a map, called *Seiberg-Witten map* [255], which links gauge equivalent configurations in a commutative gauge theory to gauge equivalent configurations in its noncommutative deformation, thus rendering both theories equivalent via field redefinitions. The idea is to regularize the low-energy effective theory of open strings in a B-field background in two different ways, once using Pauli-Villars and once with the point-splitting procedure. In the former case, we obtain the ordinary Born-Infeld action yielding commutative Yang-Mills theory as the effective theory. In the latter case, however, we obtain a noncommutative variant of the Born-Infeld action, which gives rise to a noncommutative gauge theory. Since the effective action should be independent of the regularization process, both theories should be equivalent and connected via a Seiberg-Witten map.

Consistency conditions imposed by the existence of a Seiberg-Witten map like

$$\hat{A}(A + \delta_\lambda A) \;=\; \hat{A}(A) + \delta_{\hat{\Lambda}}\hat{A}(A) \;, \tag{VI.21}$$

where λ and $\hat{\Lambda}$ describe infinitesimal commutative and noncommutative gauge transformations, respectively, prove to be a helpful calculatory tool. We will make use of a similar condition in a non-anticommutative deformed situation in section VI.2.2, §7.

§9 UV/IR mixing. One of the hopes for noncommutative field theories was that the divergencies which are ubiquitous in ordinary quantum field theory would be tamed by the noncommutativity of spacetime, since the latter implies non-locality, which does the job in string theory. The situation, however, is even worse: besides some infinities inherited from the commutative theory, certain ultraviolet singularities get mapped to peculiar infrared singularities, even in massive scalar theories. This phenomenon is known under the name or *UV/IR mixing* and was first studied in [198] and [278].

§10 Noncommutative instantons. Instantons in noncommutative gauge theories have some peculiar properties. First of all, it is possible to have non-trivial instantons even for gauge group $\mathsf{U}(1)$ as discussed in [205]. This is due to the above presented fact that even for an Abelian gauge group, the group of gauge transformations is non-Abelian. In [205], it has moreover been shown that a suitable deformation can resolve the singularities in the instanton moduli space.

VI.2 Non-anticommutative field theories

Expanding essentially on the analysis of [213], Seiberg [254] showed[4] that there is a deformation of Euclidean $\mathcal{N} = 1$ superspace in four dimensions which leads to a consistent supersymmetric field theory with half of the supersymmetries broken. The idea was to deform the algebra of the anticommuting coordinates θ to the Clifford algebra

$$\{\hat{\theta}^A, \hat{\theta}^B\} \;=\; C^{A,B} \;, \tag{VI.22}$$

which arises from considering string theory in a background with a constant graviphoton field strength. This discovery triggered many publications, in particular, non-anticommutativity for extended supersymmetry was considered, as well [141, 97, 245, 70].

An alternative approach, which was followed in [96], manifestly preserves supersymmetry but breaks chirality. This has many disadvantages, as without chiral superfields, it is e.g. impossible to define super Yang-Mills theory in the standard superspace formalism.

In section VI.3, we will present an approach in which supersymmetry *and* chirality are manifestly and simultaneously preserved, albeit in a twisted form.

VI.2.1 Non-anticommutative deformations of superspaces

§1 Associativity of the star product. In the Minkowski case, one can show that the deformations preserving the associativity of the star product all satisfy

$$\{\hat{\theta}^A, \hat{\theta}^B\} \;=\; \{\hat{\bar{\theta}}^A, \hat{\bar{\theta}}^B\} \;=\; \{\hat{\theta}^A, \hat{\bar{\theta}}^B\} \;=\; 0 \;. \tag{VI.23}$$

These deformations are clearly too trivial, but one can circumvent this problem by turning to Euclidean spacetime. Here, the most general deformation compatible with associativity of the star product reads

$$\{\hat{\theta}^A, \hat{\theta}^B\} \;\neq\; 0 \quad \text{and} \quad \{\hat{\bar{\theta}}^A, \hat{\bar{\theta}}^B\} \;=\; \{\hat{\theta}^A, \hat{\bar{\theta}}^B\} \;=\; 0 \;, \tag{VI.24}$$

which is possible, as θ and $\bar{\theta}$ are no longer related by complex conjugation. For this reason, we will always consider superspaces which have an Euclidean metric on their bodies in the following. To justify our use of the (Minkowski) superfield formalism, we can assume to temporarily work with complexified spacetime and field content and impose appropriate reality conditions after all calculations have been performed.

[4] For earlier work in this area, see [249, 96, 156, 69].

§2 The deformed superspace $\mathbb{R}^{4|4\mathcal{N}}_\hbar$. The canonical deformation of $\mathbb{R}^{4|4\mathcal{N}}$ to $\mathbb{R}^{4|4\mathcal{N}}_\hbar$ amounts to putting

$$\{\hat{\theta}^{\alpha i},\hat{\theta}^{\beta j}\} \;=\; \hbar C^{\alpha i,\beta j}\;, \tag{VI.25}$$

where the hats indicate again the deformed Graßmann coordinates in the operator representation.

As in the case of the noncommutative deformation, one can equivalently deform the algebra of superfunctions $\mathcal{S}$ on $\mathbb{R}^{4|4\mathcal{N}}$ to an algebra $\mathcal{S}_\star$, in which the product is given by the Moyal-type star product

$$f\star g \;=\; f\exp\left(-\tfrac{\hbar}{2}\overleftarrow{Q}_{\alpha i}C^{\alpha i,\beta j}\overrightarrow{Q}_{\beta j}\right)g\;, \tag{VI.26}$$

where $\overleftarrow{Q}_{\alpha i}$ and $\overrightarrow{Q}_{\beta j}$ are supercharges acting from the right and the left, respectively. Recall that in our convention for superderivatives, we have

$$\theta^{\alpha i}\overleftarrow{Q}_{\beta j} \;=\; -\delta^i_j\delta^\alpha_\beta\;. \tag{VI.27}$$

Contrary to the case of noncommutative deformations, an $\hbar$ was inserted into the definition of the deformation (VI.25) to indicate the different orders. Since the star product (VI.26) is a finite sum due to the nilpotency of the Graßmann variables, power expansions in the deformation parameter are even more important than in the noncommutative case.

All commutators involving this star multiplication will be denoted by a $\star$, e.g. the graded commutator will read as

$$[\![f \overset{\star}{,} g]\!] \;:=\; f\star g-(-1)^{\tilde{f}\tilde{g}}g\star f\;. \tag{VI.28}$$

The new coordinate algebra obtained from this deformation reads as

$$\begin{gathered}[x^{\alpha\dot\alpha}\overset{\star}{,}x^{\beta\dot\beta}] \;=\; -\hbar C^{\alpha i,j\beta}\bar\theta^{\dot\alpha}_i\bar\theta^{\dot\beta}_j\;,\quad [x^{\alpha\dot\alpha}\overset{\star}{,}\theta^{\beta j}] \;=\; -\hbar C^{\alpha i,j\beta}\bar\theta^{\dot\alpha}_i\;,\\ \{\theta^{\alpha i}\overset{\star}{,}\theta^{\beta j}\} \;=\; \hbar C^{\alpha i,j\beta}\end{gathered} \tag{VI.29}$$

and all other supercommutators vanish, but after changing to the chiral coordinates

$$(y^{\alpha\dot\alpha}:=x^{\alpha\dot\alpha}+\theta^{\alpha i}\bar\theta^{\dot\alpha}_i,\theta^{\alpha i},\bar\theta^{\dot\alpha}_i)\;, \tag{VI.30}$$

the coordinate algebra simplifies to

$$[y^{\alpha\dot\alpha}\overset{\star}{,}y^{\beta\dot\beta}] \;=\; 0\;,\quad [y^{\alpha\dot\alpha}\overset{\star}{,}\theta^{\beta j}] \;=\; 0\;,\quad [\theta^{\alpha i}\overset{\star}{,}\theta^{\beta j}] \;=\; \hbar C^{\alpha i,\beta j}\;. \tag{VI.31}$$

This deformation has been shown to arise in string theory from open superstrings of type IIB in the background of a constant graviphoton field strength [213, 254, 69].

§3 Deformed supersymmetry algebra. The corresponding deformed algebra of superderivatives and supercharges reads as[5]

$$\begin{aligned}\{D_{\alpha i}\overset{\star}{,}D_{\beta j}\} &\;=\; 0\;,\quad \{\bar D^i_{\dot\alpha}\overset{\star}{,}\bar D^j_{\dot\beta}\} \;=\; 0\;,\\ \{D_{\alpha i}\overset{\star}{,}\bar D^j_{\dot\beta}\} &\;=\; -2\delta^i_j\partial_{\alpha\dot\beta} \;=\; -2\mathrm{i}\delta^j_i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\;,\\ \{Q_{\alpha i}\overset{\star}{,}Q_{\beta j}\} &\;=\; 0\;,\\ \{\bar Q^i_{\dot\alpha}\overset{\star}{,}\bar Q^j_{\dot\beta}\} &\;=\; 4\hbar C^{\alpha i,\beta j}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta} \;=\; -4\hbar C^{\alpha i,\beta j}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\partial_\mu\partial_\nu\;,\\ \{Q_{\alpha i}\overset{\star}{,}\bar Q^j_{\dot\beta}\} &\;=\; 2\delta^j_i\partial_{\alpha\dot\beta} \;=\; 2\mathrm{i}\delta^j_i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\;.\end{aligned} \tag{VI.32}$$

[5]For further reference, we present the algebra both in spinor and vector notation.

By inspection of this deformed algebra, it becomes clear that the number of supersymmetries is reduced to $\mathcal{N}/2$, since those generated by $\bar{Q}^i_{\dot{\alpha}}$ are broken. On the other hand, it still allows for the definition of chiral and anti-chiral superfields as the algebra of the superderivatives $D_{\alpha i}$ and $\bar{D}^i_{\dot{\alpha}}$ is undeformed. Because of this, graded Bianchi identities are also retained, e.g.

$$\{\!\![\nabla_a \overset{\star}{,} \{\!\![\nabla_b \overset{\star}{,} \nabla_c]\!\!\}]\!\!\} + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}\{\!\![\nabla_b \overset{\star}{,} \{\!\![\nabla_c \overset{\star}{,} \nabla_a]\!\!\}]\!\!\} + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}\{\!\![\nabla_c \overset{\star}{,} \{\!\![\nabla_a \overset{\star}{,} \nabla_b]\!\!\}]\!\!\} \;=\; 0\;.$$

§4 Consequence for field theories. Field theories on non-anticommutative superspaces are usually defined by replacing all ordinary products in the action written in the $\mathcal{N} = 1$ superfield formalism by star products. First of all, such theories will evidently have non-Hermitian Lagrangians since – roughly speaking – chiral parts of the action will get deformed, while anti-chiral parts remain unchanged. This, however, allows for renormalizable theories which have terms in their Lagrangian with mass dimension larger than 4 [42]. Of particular interest to our work is the question of renormalizability of non-anticommutative field theories and here specifically of the $\mathcal{N} = \frac{1}{2}$ Wess-Zumino model, as discussed in [273, 43, 44, 114, 42, 241, 184, 26]. For more recent work on the renormalizability of non-anticommutative super Yang-Mills theory, see e.g. [115].

VI.2.2 Non-anticommutative $\mathcal{N} = 4$ SYM theory

§5 Idea. In the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$, one has appropriate superspace formalisms at hand, which allow for a direct deformation of supersymmetric field theories by deforming their actions in these formalisms. In the cases[6] $\mathcal{N} = 3$ and $\mathcal{N} = 4$, however, there is no such formalism. Instead, one can use the constraint equations (IV.50) on $\mathbb{R}^{4|16}$, which are equivalent to the $\mathcal{N} = 4$ SYM equations as discussed in section IV.2.2. By considering these constraint equations on the deformed space $\mathbb{R}^{4|16}_\hbar$, one finds the equations of motion of the corresponding deformed theory.

§6 Deformed constraint equations. We start from the constraint equations of $\mathcal{N} = 4$ SYM theory introduced in IV.2.2 on $\mathbb{R}^{4|16}$ and follow the discussion of the undeformed case. On the deformed space $\mathbb{R}^{4|16}_\hbar$, they read as

$$\begin{gathered}\{\tilde{\nabla}_{\alpha i} \overset{\star}{,} \tilde{\nabla}_{\beta j}\} \;=\; -2\varepsilon_{\alpha\beta}\tilde{\phi}_{ij}\;, \quad \{\tilde{\bar{\nabla}}^i_{\dot{\alpha}} \overset{\star}{,} \tilde{\bar{\nabla}}^j_{\dot{\beta}}\} \;=\; -2\varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\bar{\phi}}^{ij}\;,\\ \{\tilde{\nabla}_{\alpha i} \overset{\star}{,} \tilde{\bar{\nabla}}^j_{\dot{\beta}},\} \;=\; -2\delta^j_i\tilde{\nabla}_{\alpha\dot{\beta}}\;,\end{gathered} \tag{VI.33}$$

where we will use a tilde[7] to label fields living on the deformed superspace $\mathbb{R}^{4|16}_\hbar$. The covariant derivatives are obtained from super gauge potentials

$$\tilde{\nabla}_{\alpha i} \;=\; D_{\alpha i} + \{\!\![\tilde{\omega}_{\alpha i} \overset{\star}{,} \cdot]\!\!\}\;, \quad \tilde{\bar{\nabla}}^i_{\dot{\alpha}} \;=\; \bar{D}^i_{\dot{\alpha}} - \{\!\![\tilde{\bar{\omega}}^i_{\dot{\alpha}} \overset{\star}{,} \cdot]\!\!\}\;, \quad \tilde{\nabla}_{\alpha\dot{\alpha}} \;=\; \partial_{\alpha\dot{\alpha}} + \{\!\![\tilde{A}_{\alpha\dot{\alpha}} \overset{\star}{,} \cdot]\!\!\}\;, \tag{VI.34}$$

and we define additionally the superspinor fields

$$[\tilde{\nabla}_{\alpha i} \overset{\star}{,} \tilde{\nabla}_{\beta\dot{\beta}}] \;=:\; \varepsilon_{\alpha\beta}\tilde{\bar{\chi}}_{i\dot{\beta}} \quad\text{and}\quad [\tilde{\bar{\nabla}}^i_{\dot{\alpha}} \overset{\star}{,} \tilde{\nabla}_{\beta\dot{\beta}}] \;=:\; \varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\chi}^i_\beta\;. \tag{VI.35}$$

[6] These cases are essentially equivalent, see section IV.2.2.

[7] Appearing over an exponent, the tilde still denotes the corresponding parity.

Proceeding further along the lines of the undeformed case, we finally arrive at the $\mathcal{N}=4$ SYM equations with all commutators replaced by star-commutators:

$$\begin{aligned} \tilde{\nabla}_{\alpha\dot{\alpha}}\tilde{\chi}^{i\beta} + \tfrac{1}{2}\varepsilon^{ijkl}[\tilde{\phi}_{kl} \overset{\star}{,} \bar{\chi}_{j\dot{\alpha}}] &= 0 , \\ \tilde{\nabla}_{\alpha\dot{\alpha}}\tilde{\bar{\chi}}_i^{\dot{\beta}} + [\tilde{\phi}_{ij} \overset{\star}{,} \tilde{\chi}^j_\alpha] &= 0 , \\ \varepsilon^{\dot{\alpha}\dot{\beta}}\tilde{\nabla}_{\gamma\dot{\alpha}}\tilde{f}_{\dot{\beta}\dot{\gamma}} + \varepsilon^{\alpha\beta}\tilde{\nabla}_{\alpha\dot{\gamma}}\tilde{f}_{\beta\gamma} &= \tfrac{1}{4}\varepsilon^{ijkl}[\tilde{\nabla}_{\gamma\dot{\gamma}}\tilde{\phi}_{ij} \overset{\star}{,} \tilde{\phi}_{kl}] + \{\tilde{\chi}^i_\gamma \overset{\star}{,} \tilde{\bar{\chi}}_{i\dot{\gamma}}\} , \\ \tilde{\nabla}_{\alpha\dot{\alpha}}\tilde{\nabla}^{\alpha\dot{\alpha}}\tilde{\phi}_{ij} - \tfrac{1}{4}\varepsilon^{klmn}[\tilde{\phi}_{mn} \overset{\star}{,} [\tilde{\phi}_{kl} \overset{\star}{,} \tilde{\phi}_{ij}]] &= \tfrac{1}{2}\varepsilon_{ijkl}\{\tilde{\chi}^k_\alpha \overset{\star}{,} \tilde{\chi}^{l\beta}\} + \{\tilde{\bar{\chi}}_{i\dot{\alpha}} \overset{\star}{,} \tilde{\bar{\chi}}^{\dot{\alpha}}_j\} . \end{aligned} \qquad \text{(VI.36)}$$

Recall that all the fields appearing in the above equations are in fact superfields on the deformed space $\mathbb{R}^{4|16}_\hbar$, and we still have to extract the zeroth order components and their deformed equations of motion.

§7 The Seiberg-Witten map. While the derivation of the superfield expansion in the undeformed case was quite simple by imposing transverse gauge and using the recursion operator $\mathcal{D}$, we face some difficulties in the deformed case. Using again this Euler operator would lead to a highly nonlinear system of algebraic equations and the complete knowledge of the superfield expansion, i.e. about 2^{16} terms for every field, is needed to calculate corrections even to first order in $\hbar$.

Therefore we suggest an alternative approach based on a generalization of the Seiberg-Witten map, cf. section VI.1.2, §8, which will yield the expansion of the superfields order by order in $\hbar$. For this, let us choose $\tilde{\omega}_{\alpha i}$ as the fundamental field of our theory, i.e. all the other fields $\tilde{\bar{\omega}}^i_{\dot{\alpha}}$, $\tilde{A}_{\alpha\dot{\alpha}}$, $\tilde{\phi}_{ij}$, $\tilde{\chi}^i_\alpha$ and $\tilde{\bar{\chi}}_{i\dot{\alpha}}$ are fixed for a certain $\tilde{\omega}_{\alpha i}$ by the constraint equations (VI.33) and the definitions (VI.35).

First recall that infinitesimal gauge transformations of the undeformed and the deformed gauge potential are given by

$$\delta_\lambda\omega_{\alpha i} = D_{\alpha i}\lambda + [\omega_{\alpha i}, \lambda] \quad \text{and} \quad \delta_{\tilde{\Lambda}}\tilde{\omega}_{\alpha i} = D_{\alpha i}\tilde{\Lambda} + [\tilde{\omega}_{\alpha i} \overset{\star}{,} \tilde{\Lambda}] , \qquad \text{(VI.37)}$$

respectively, where λ and $\tilde{\Lambda}$ are even superfields parameterizing the transformation. Analogously to the noncommutative formula (VI.21), the starting point is then the equation

$$\tilde{\omega}_{\alpha i}(\omega + \delta_\lambda\omega, \bar{\omega} + \delta_\lambda\bar{\omega}) = \tilde{\omega}_{\alpha i}(\omega, \bar{\omega}) + \delta_{\tilde{\Lambda}}\tilde{\omega}_{\alpha i}(\omega, \bar{\omega}) . \qquad \text{(VI.38)}$$

§8 Explicit solution. To obtain the explicit form of the Seiberg-Witten map, we can use the consistency condition that two successive gauge transformations should, e.g. for a superfield $\tilde{\psi}$ in the fundamental representation, satisfy

$$[\delta_{\tilde{\Lambda}}, \delta_{\tilde{\Sigma}}]\tilde{\psi} = -[\tilde{\Lambda} \overset{\star}{,} \tilde{\Sigma}] \star \tilde{\psi} = \delta_{[\tilde{\Lambda} \overset{\star}{,} \tilde{\Sigma}]}\tilde{\psi} . \qquad \text{(VI.39)}$$

By the Seiberg-Witten map, gauge equivalent solutions get mapped to deformed gauge equivalent solutions, and thus we can restrict ourselves to gauge transformations of the type $\delta_\lambda\tilde{\psi} = -\tilde{\Lambda}_\lambda(\omega, \bar{\omega}) \star \tilde{\psi}$. Then one can simplify the above consistency condition to

$$\delta_\lambda\tilde{\Lambda}_\sigma - \delta_\sigma\tilde{\Lambda}_\lambda + [\tilde{\Lambda}_\lambda \overset{\star}{,} \tilde{\Lambda}_\sigma] = \tilde{\Lambda}_{[\lambda,\sigma]} . \qquad \text{(VI.40)}$$

As for all the fields in our deformed theory, we assume that also $\tilde{\Lambda}$ is a polynomial[8] in $\hbar$ and considering the first order of $\hbar$ in (VI.40), we arrive at

$$\delta_\lambda\tilde{\Lambda}^1_\sigma - \delta_\sigma\tilde{\Lambda}^1_\lambda + [\lambda, \tilde{\Lambda}^1_\sigma] + [\tilde{\Lambda}^1_\lambda, \sigma] - \tfrac{1}{2}C^{\alpha i,\beta j}[\partial_{\alpha i}\lambda, \partial_{j\beta}\sigma] = \tilde{\Lambda}^1_{[\lambda,\sigma]} . \qquad \text{(VI.41)}$$

[8]In principle, it could also be any power series.

Although it is not straightforward, it is possible to guess the solution to this equation, which is given by

$$\begin{aligned} \tilde{\Lambda}_\lambda &= \lambda - \tfrac{\hbar}{4} C^{\alpha i,\beta j}[\partial_{\alpha i}\lambda, \Omega_{\beta j}] + \mathcal{O}(\hbar^2) \,, \\ \Omega_{\alpha i} &:= \omega_{\alpha i} + \bar{\theta}^{\dot\beta}_j \left(\bar{D}^j_{\dot\beta}\omega_{\alpha i} + D_{\alpha i}\bar{\omega}^j_{\dot\beta} + \{\bar{\omega}^j_{\dot\beta}, \omega_{\alpha i}\} \right) \,. \end{aligned} \tag{VI.42}$$

To verify this solution, note that infinitesimal gauge transformations act on $\Omega_{\alpha i}$ as $\delta_\lambda \Omega_{\alpha i} = \partial_{\alpha i}\lambda + [\Omega_{\alpha i}, \lambda]$, and therefore we have $\tilde{\Lambda}_\lambda = \lambda - \frac{\hbar}{4} C^{\alpha i,\beta j}[\partial_{\alpha i}\lambda, \Omega_{\beta j}] + \mathcal{O}(\hbar^2)$.

§9 Field expansion. Let us now consider the first order in $\hbar$ of the second equation in (VI.38), which reads

$$\delta_\lambda \tilde{\omega}^1_{\alpha i} = D_{\alpha i}\tilde{\Lambda}^1 + [\tilde{\omega}^1_{\alpha i}, \lambda] + [\omega_{\alpha i}, \tilde{\Lambda}^1] + \tfrac{1}{2} C^{\beta j, k\gamma}\{\partial_{\beta j}\omega_{\alpha i}, \partial_{k\gamma}\lambda\} \,. \tag{VI.43}$$

With our above result for $\tilde{\Lambda}^1_\lambda$, one finds after some algebraic manipulations that

$$\tilde{\omega}^1_{\alpha i} = \tfrac{1}{4} C^{\beta j, k\gamma}\{\Omega_{\beta j}, \partial_{k\gamma}\omega_{\alpha i} + R_{k\gamma,\alpha i}\} \tag{VI.44}$$

with

$$R_{\alpha i,\beta j} := \partial_{\alpha i}\omega_{\beta j} + D_{\beta j}\Omega_{\alpha i} + \{\omega_{\beta j}, \Omega_{\alpha i}\} \,. \tag{VI.45}$$

Now that we have the definition of our fundamental field, we can work through the constraint equations (VI.33) and the definitions (VI.35) to obtain the first order in $\hbar$ of the other fields. From the first constraint equation, we immediately obtain

$$\tilde{\phi}^1_{ij} = \tfrac{1}{2}\varepsilon^{\alpha\beta}\nabla_{(\alpha i}\tilde{\omega}^1_{\beta j)} + \tfrac{1}{8}\varepsilon^{\alpha\beta} C^{m\delta, n\varepsilon}\{\partial_{m\delta}\omega_{\alpha i}, \partial_{n\varepsilon}\omega_{\beta j}\} \,, \tag{VI.46}$$

where the parentheses denote, as usual, symmetrization with appropriate weight. From this solution, we can use the second constraint equation to solve for the first order term in $\tilde{\bar{\omega}}^{j\,1}_{\dot\beta}$. Together with the assumption that $\bar{\nabla}^i_{\dot\alpha}\tilde{\bar{\omega}}^{j\,1}_{\dot\beta} = \bar{\nabla}^j_{\dot\beta}\tilde{\bar{\omega}}^{i\,1}_{\dot\alpha}$, we find the equation

$$\begin{aligned} \bar{\nabla}^i_{\dot\alpha}\tilde{\bar{\omega}}^{j\,1}_{\dot\beta} &= \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{ijkl}\tilde{\phi}^1_{kl} + \tfrac{1}{4}C^{m\delta,n\varepsilon}\{\partial_{m\delta}\bar{\omega}^i_{\dot\alpha}, \partial_{n\varepsilon}\bar{\omega}^j_{\dot\beta}\} \\ &= \bar{D}^i_{\dot\alpha}\tilde{\bar{\omega}}^{j\,1}_{\dot\beta} - \{\bar{\omega}^i_{\dot\alpha}, \tilde{\bar{\omega}}^{j\,1}_{\dot\beta}\} \,. \end{aligned} \tag{VI.47}$$

Recall that in the undeformed case, we used transverse gauge to break super gauge invariance to ordinary gauge symmetry. Here, we can impose a similar condition to simplify the situation:

$$\theta\tilde{\omega} - \bar{\theta}\tilde{\bar{\omega}} = \theta^{\alpha i}\tilde{\omega}_{\alpha i} + \bar{\theta}^{\dot\alpha}_i\tilde{\bar{\omega}}^i_{\dot\alpha} = 0 \,, \tag{VI.48}$$

which is separately valid to all orders in $\hbar$. From this, we obtain the further relation $\tilde{\bar{\omega}}^{i\,1}_{\dot\alpha} = \bar{D}^i_{\dot\alpha}(\theta\tilde{\omega}^1) - \bar{\theta}^{\dot\beta}_j \bar{D}^i_{\dot\alpha}\tilde{\bar{\omega}}^{j\,1}_{\dot\beta}$, which turns equation (VI.47) into

$$\bar{D}^i_{\dot\alpha}\tilde{\bar{\omega}}^{j\,1}_{\dot\beta} - \bar{\theta}^{\dot\gamma}_l[\bar{\omega}^i_{\dot\alpha}, \bar{D}^j_{\dot\beta}\tilde{\bar{\omega}}^{l\,1}_{\dot\gamma}] = K^{ij}_{\dot\alpha\dot\beta} \,, \tag{VI.49}$$

where we have abbreviated

$$K^{ij}_{\dot\alpha\dot\beta} := \tfrac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{ijkl}\tilde{\phi}^1_{kl} + \tfrac{1}{4}C^{m\delta,n\varepsilon}\{\partial_{m\delta}\bar{\omega}^i_{\dot\alpha}, \partial_{n\varepsilon}\bar{\omega}^j_{\dot\beta}\} + \{\bar{\omega}^i_{\dot\alpha}, \bar{D}^j_{\dot\beta}(\theta\tilde{\omega}^1)\} \,. \tag{VI.50}$$

The expression for $\tilde{\bar{\omega}}^1$ is found by iterating the equation (VI.49), which becomes a little technical. We obtain

$$\bar{D}_{\bar{A}}\tilde{\bar{\omega}}^1_{\bar{B}} = \sum_{|\bar{I}|\,\leq\,8} (-1)^{\lfloor\frac{|\bar{I}|}{2}\rfloor}\,\bar{\theta}^{\bar{I}}\,\{\!\!\{\bar{\omega}, K\}\!\!\}_{\bar{I},\bar{A}\bar{B}} \,, \tag{VI.51}$$

where "$\lfloor\ \rfloor$" denotes the Gauß bracket, $|I|$ the length of the multiindex I and

$$\{\!\!\{\bar\omega, K\}\!\!\}_{\bar I,\bar A\bar B} \;:=\; \{\!\!\{\bar\omega_{\bar A}, \{\!\!\{\bar\omega_{\bar B}, \{\!\!\{\bar\omega_{\bar A_1}, \cdots \{\!\!\{\bar\omega_{\bar A_{|\bar I|-2}}, K_{\bar A_{|\bar I|-1}\bar A_{|\bar I|}}\}\!\!\}\cdots\}\!\!\}\}\!\!\}\}\!\!\} \;. \qquad \text{(VI.52)}$$

The sum in (VI.51) is finite as the order of $\bar\theta$ increases during the iteration. The first order contribution is thus given by

$$\tilde{\bar\omega}^1_{\bar A} \;=\; \bar D_{\bar A}(\theta\tilde\omega^1) - \bar\theta^{\bar B}\sum_{|\bar I|\,\leq\,8}(-1)^{\lfloor\frac{|\bar I|}{2}\rfloor}\,\bar\theta^{\bar I}\,\{\!\!\{\bar\omega, K\}\!\!\}_{\bar I,\bar A\bar B} \;. \qquad \text{(VI.53)}$$

From here on, it is easy to write down the first order deformation of the remaining fields via the third constraint equation and the definitions (VI.35):

$$\begin{aligned}
\tilde A^1_{\alpha\dot\beta} &= \tfrac{1}{8}(\nabla_{\alpha i}\tilde{\bar\omega}^{i\,1}_{\dot\beta} - \bar\nabla^i_{\dot\beta}\tilde\omega^1_{\alpha i} + \tfrac{1}{2}C^{m\delta,n\varepsilon}\{\partial_{m\delta}\omega_{\alpha i}, \partial_{n\varepsilon}\bar\omega^i_{\dot\beta}\})\\
\tilde{\bar\chi}^1_{i\dot\beta} &= -\tfrac{1}{2}\varepsilon^{\alpha\beta}(\nabla_{\alpha i}\tilde A^1_{\beta\dot\beta} - \nabla_{\beta\dot\beta}\tilde\omega^1_{\alpha i} + \tfrac{1}{2}C^{m\delta,n\varepsilon}\{\partial_{m\delta}\omega_{\alpha i}, \partial_{n\varepsilon}A_{\beta\dot\beta}\})\\
\tilde\chi^{i\,1}_{\beta} &= -\tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}(\bar\nabla^i_{\dot\alpha}\tilde A^1_{\beta\dot\beta} + \nabla_{\beta\dot\beta}\tilde{\bar\omega}^{i\,1}_{\dot\alpha} - \tfrac{1}{2}C^{m\delta,n\varepsilon}\{\partial_{m\delta}\bar\omega^i_{\dot\alpha}, \partial_{n\varepsilon}A_{\beta\dot\beta}\})
\end{aligned} \qquad \text{(VI.54)}$$

§10 Deformed field equations. So far, we computed the first order deformations in $\hbar$ of the superfields and by restricting to their zeroth order components, we obtained the deformations of the $\mathcal{N}=4$ SYM multiplet. It remains, however, to calculate the zeroth order components of the superfield equations (VI.36). For this, we need to know the explicit zeroth order form of products $\theta^I\star\theta^J$ with I, J being multiindices. By induction, one can easily prove that

$$\begin{aligned}
\theta^{A_1}\star\ldots\star\theta^{A_n} &= \theta^{A_1}\ldots\theta^{A_n} + \sum \text{contractions}\\
&= \theta^{A_1}\ldots\theta^{A_n} + \sum_{i<j}\theta^{A_1}\ldots\overbrace{\theta^{A_i}\cdots\theta}^{}{}^{A_j}\ldots\theta^{A_n} + \ldots\;,
\end{aligned} \qquad \text{(VI.55)}$$

which resembles a fermionic Wick theorem and where a contraction is defined as

$$\overbrace{\theta^{A_i}\theta}^{}{}^{A_j} \;:=\; \tfrac{\hbar}{2}C^{A_i,A_j}\;. \qquad \text{(VI.56)}$$

Note that signs appearing from the grading have to be taken into account. For $n=2$, (VI.55) is obvious, and for $n>2$ one can show that

$$(\theta^{A_1}\cdots\theta^{A_n})\star\theta^{A_{n+1}} \;=\; \theta^{A_1}\cdots\theta^{A_n}\theta^{A_{n+1}} \;+\; \sum_{i=1}^{n}\,\theta^{A_1}\cdots\overbrace{\theta^{A_i}\cdots\;\theta}^{}{}^{A_{n+1}}\;, \qquad \text{(VI.57)}$$

which proves (VI.55) by induction. Since we are interested only in the zeroth order terms in (VI.55), let us define the projection operator $\pi_\circ$, which extracts these terms. Then we have

$$\begin{aligned}
\pi_\circ(\theta^I\star\theta^J) &= \pi_\circ((\theta^{A_1}\cdots\theta^{A_n})\star(\theta^{B_1}\cdots\theta^{B_m}))\\
&= \delta_{nm}\frac{(-1)^{\frac{n}{2}(n-1)}\hbar^n}{2^n\,n!}\sum_{\{i,j\}}\varepsilon_{i_1\cdots i_n}\varepsilon_{j_1\cdots j_n}C^{A_{i_1},B_{j_1}}\cdots C^{A_{i_n},B_{j_n}}\;,
\end{aligned} \qquad \text{(VI.58)}$$

which is rather obvious, and we have $\pi_\circ(\theta^I\star\theta^J) = \pi_\circ(\theta^J\star\theta^I)$ as a corollary.

To compute all the commutators appearing in the equations of motion (VI.36), let us expand every superfield as

$$\tilde f \;=\; \overset{\circ}{\tilde f} + \sum_I \tilde f_I\theta^I + \text{terms containing }\bar\theta\;. \qquad \text{(VI.59)}$$

Given two superfunctions $\tilde{f}$ and $\tilde{g}$, we obtain for the three possible cases of gradings $(0,0)$, $(1,1), (0,1)$ for the pair $(\tilde{f}, \tilde{g})$ the following results:

$$
\begin{aligned}
\pi_\circ([\tilde{f} \overset{\star}{,} \tilde{g}]) &= [\overset{\circ}{\tilde{f}}, \overset{\circ}{\tilde{g}}] + \sum_{|I|=|J|} (-1)^{\tilde{I}} [\tilde{f}_I, \tilde{g}_J]\ \pi_\circ(\theta^I \star \theta^J)\ , \\
\pi_\circ(\{\tilde{f} \overset{\star}{,} \tilde{g}\}) &= \{\overset{\circ}{\tilde{f}}, \overset{\circ}{\tilde{g}}\} + \sum_{|I|=|J|} \{\tilde{f}_I, \tilde{g}_J\}\ \pi_\circ(\theta^I \star \theta^J)\ , \\
\pi_\circ([\tilde{f} \overset{\star}{,} \tilde{g}]) &= [\overset{\circ}{\tilde{f}}, \overset{\circ}{\tilde{g}}] + \sum_{|I|=|J|} (\tilde{f}_I \tilde{g}_J - (-1)^{\tilde{I}} \tilde{g}_J \tilde{f}_I)\ \pi_\circ(\theta^I \star \theta^J)\ ,
\end{aligned}
\tag{VI.60}
$$

Now we have all the necessary ingredients to derive the field equations of $\mathcal{N}=4$ SYM theory on $\mathbb{R}_\hbar^{4|16}$. The equations of motion for the eight Weyl spinors read

$$
\begin{aligned}
\varepsilon^{\alpha\beta} \overset{\circ}{\tilde{\nabla}}_{\alpha\dot{\alpha}} \overset{\circ}{\tilde{\chi}}{}^i_\beta + \tfrac{1}{2}\varepsilon^{ijkl}[\overset{\circ}{\tilde{\phi}}_{kl}, \overset{\circ}{\tilde{\tilde{\chi}}}_{j\dot{\alpha}}] &= -\,\varepsilon^{\alpha\beta} \sum_{|I|=|J|} (\tilde{A}_{\alpha\dot{\alpha}|I} \tilde{\chi}^i_{\beta|J} - (-1)^{\tilde{I}} \tilde{\chi}^i_{\beta|J} \tilde{A}_{\alpha\dot{\alpha}|I})\, T^{IJ} \\
&\quad - \tfrac{1}{2}\varepsilon^{ijkl} \sum_{|I|=|J|} (\tilde{W}_{kl|I} \tilde{\tilde{\chi}}_{j\dot{\alpha}|J} - (-1)^{\tilde{I}} \tilde{\tilde{\chi}}_{j\dot{\alpha}|J} \tilde{\phi}_{kl|I})\, T^{IJ}\ , \\
\varepsilon^{\dot{\alpha}\dot{\beta}} \overset{\circ}{\tilde{\nabla}}_{\alpha\dot{\alpha}} \overset{\circ}{\tilde{\tilde{\chi}}}_{i\dot{\beta}} + [\overset{\circ}{\tilde{\phi}}_{ij}, \overset{\circ}{\tilde{\chi}}{}^j_\alpha] &= -\,\varepsilon^{\dot{\alpha}\dot{\beta}} \sum_{|I|=|J|} (\tilde{A}_{\alpha\dot{\alpha}|I} \tilde{\tilde{\chi}}_{i\dot{\beta}|J} - (-1)^{\tilde{I}} \tilde{\tilde{\chi}}_{i\dot{\beta}|J} \tilde{A}_{\alpha\dot{\alpha}|I})\, T^{IJ} \\
&\quad - \sum_{|I|=|J|} (\tilde{\phi}_{ij|I} \tilde{\chi}^j_{\alpha|J} - (-1)^{\tilde{I}} \tilde{\chi}^j_{\alpha|J} \tilde{\phi}_{ij|I})\, T^{IJ}\ ,
\end{aligned}
$$

where we introduced $T^{IJ} := \pi_\circ(\theta^I \star \theta^J)$ for brevity. For the bosonic fields, the equations of motion read as

$$
\begin{aligned}
&\varepsilon^{\dot{\alpha}\dot{\beta}} \overset{\circ}{\tilde{\nabla}}_{\gamma\dot{\alpha}} \overset{\circ}{\tilde{f}}_{\dot{\beta}\dot{\gamma}} + \varepsilon^{\alpha\beta} \overset{\circ}{\tilde{\nabla}}_{\alpha\dot{\gamma}} \overset{\circ}{\tilde{f}}_{\beta\gamma} - \tfrac{1}{4}\varepsilon^{ijkl}[\overset{\circ}{\tilde{\nabla}}_{\gamma\dot{\gamma}} \overset{\circ}{\tilde{\phi}}_{ij}, \overset{\circ}{\tilde{\phi}}_{kl}] - \{\overset{\circ}{\tilde{\chi}}{}^i_\gamma, \overset{\circ}{\tilde{\tilde{\chi}}}_{i\dot{\gamma}}\} = \\
&\qquad - \sum_{|I|=|J|} (-1)^{\tilde{I}} \left\{ \varepsilon^{\dot{\alpha}\dot{\beta}} [\tilde{A}_{\gamma\dot{\alpha}|I}, \tilde{f}_{\dot{\beta}\dot{\gamma}|J}] + \varepsilon^{\alpha\beta}[\tilde{A}_{\alpha\dot{\gamma}|I}, \tilde{f}_{\beta\gamma|J}] \right\} T^{IJ} \\
&\qquad\qquad + \sum_{|I|=|J|} \left\{ (-1)^{\tilde{I}} \tfrac{1}{4}\varepsilon^{ijkl}[(\tilde{\nabla}_{\gamma\dot{\gamma}} \tilde{\phi}_{ij})_I, \tilde{\phi}_{kl|J}] + \{\tilde{\chi}^i_{\gamma|I}, \tilde{\tilde{\chi}}_{i\dot{\gamma}|J}\} \right\} T^{IJ}\ , \\
&\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} \overset{\circ}{\tilde{\nabla}}_{\alpha\dot{\alpha}} \overset{\circ}{\tilde{\nabla}}_{\beta\dot{\beta}} \overset{\circ}{\tilde{\phi}}_{ij} - \tfrac{1}{4}\varepsilon^{klmn}[\overset{\circ}{\tilde{\phi}}_{mn}, [\overset{\circ}{\tilde{\phi}}_{kl}\ , \overset{\circ}{\tilde{\phi}}_{ij}]] = \tfrac{1}{2}\varepsilon_{ijkl}\varepsilon^{\alpha\beta}\{\overset{\circ}{\tilde{\chi}}{}^k_\alpha, \overset{\circ}{\tilde{\chi}}{}^l_\beta\} + \varepsilon^{\dot{\alpha}\dot{\beta}}\{\overset{\circ}{\tilde{\tilde{\chi}}}_{i\dot{\alpha}}, \overset{\circ}{\tilde{\tilde{\chi}}}_{j\dot{\beta}}\} \\
&\quad - \sum_{|I|=|J|} (-1)^{\tilde{I}} \left\{ \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} [\tilde{A}_{\alpha\dot{\alpha}|I}, (\tilde{\nabla}_{\beta\dot{\beta}} \tilde{\phi}_{ij})_J] - \tfrac{1}{4}\varepsilon^{klmn}[\tilde{\phi}_{mn|I}\ , [\tilde{\phi}_{kl}, \tilde{\phi}_{ij}]_J] \right\} T^{IJ} \\
&\qquad\qquad + \sum_{|I|=|J|} \left\{ \tfrac{1}{2}\varepsilon_{ijkl}\varepsilon^{\alpha\beta}\{\tilde{\chi}^k_{\alpha|I}, \tilde{\chi}{}^{\ l}_{\beta|J}\} + \varepsilon^{\dot{\alpha}\dot{\beta}}\{\tilde{\tilde{\chi}}_{i\dot{\alpha}|I}, \tilde{\tilde{\chi}}_{j\dot{\beta}|J}\} \right\} T^{IJ}\ .
\end{aligned}
$$

§11 Deformed $\mathcal{N}=4$ multiplet. It now remains to calculate the bodies of the deformed superfields. This is a rather lengthy but straightforward calculation which yields the following results:

$$
\begin{aligned}
\tilde{\overset{\circ}{\phi}}_{ij} &= \overset{\circ}{\phi}_{ij} + \tfrac{\hbar}{2}\, C^{m\delta,n\varepsilon}\, \varepsilon_{\delta\varepsilon}\{\overset{\circ}{\phi}_{mi}, \overset{\circ}{\phi}_{jn}\} + \mathcal{O}(\hbar^2)\ , \\
\tilde{\overset{\circ}{A}}_{\alpha\dot\beta} &= \overset{\circ}{A}_{\alpha\dot\beta} + \tfrac{\hbar}{4}\, C^{m\delta,n\varepsilon}\, \varepsilon_{\alpha\delta}\{\overset{\circ}{\phi}_{mn}, \overset{\circ}{A}_{\varepsilon\dot\beta}\} + \mathcal{O}(\hbar^2)\ , \\
\tilde{\overset{\circ}{\bar\chi}}_{i\dot\beta} &= \overset{\circ}{\bar\chi}_{i\dot\beta} + \tfrac{\hbar}{96}\, C^{m\delta,n\varepsilon}\, [11\varepsilon_{\delta\varepsilon}(\{\overset{\circ}{\phi}_{mn}, \overset{\circ}{\bar\chi}_{i\dot\beta}\} - 2\{\overset{\circ}{\phi}_{in}, \overset{\circ}{\bar\chi}_{m\dot\beta}\}) \\
&\qquad - 5(\varepsilon_{mnij}\{\overset{\circ}{A}_{\delta\dot\beta}, \overset{\circ}{\chi}{}^j_\varepsilon\})] + \mathcal{O}(\hbar^2)\ , \\
\tilde{\overset{\circ}{\chi}}{}^i_\beta &= \overset{\circ}{\chi}{}^i_\beta + \tfrac{\hbar}{16}\, C^{m\delta,n\varepsilon}\, [\{\overset{\circ}{\phi}_{mn}, \tfrac{4}{3}\varepsilon_{\varepsilon\delta}\overset{\circ}{\chi}{}^i_\beta - \tfrac{11}{3}\varepsilon_{\delta\beta}\overset{\circ}{\chi}{}^i_\varepsilon\} \\
&\qquad - \delta^i_m\{\overset{\circ}{\phi}_{ln}, \tfrac{4}{3}\varepsilon_{\varepsilon\delta}\overset{\circ}{\chi}{}^l_\beta + \tfrac{7}{3}\varepsilon_{\varepsilon\beta}\overset{\circ}{\chi}{}^l_\delta - \tfrac{2}{3}\varepsilon_{\delta\beta}\overset{\circ}{\chi}{}^l_\varepsilon\} \\
&\qquad - \varepsilon_{\beta\varepsilon}\varepsilon^{\dot\alpha\dot\beta}\{\overset{\circ}{A}_{\delta\dot\alpha}, 12\delta^i_m\overset{\circ}{\bar\chi}_{n\dot\beta} - \tfrac{1}{2}\delta^i_n\overset{\circ}{\bar\chi}_{m\dot\beta}\}] + \mathcal{O}(\hbar^2)\ .
\end{aligned}
\tag{VI.61}
$$

To obtain the final equations of motion for the $\mathcal{N}=4$ SYM multiplet, one has to substitute these expressions into the deformed field equations. All the remaining superfield components can be replaced with the corresponding undeformed components, as we are only interested in terms of first order in $\hbar$ and T^{IJ} is at least of this order. This will eventually give rise to equations of the type

$$
\varepsilon^{\alpha\beta}\overset{\circ}{\nabla}_{\alpha\dot\alpha}\overset{\circ}{\chi}{}^i_\beta + \tfrac{1}{2}\varepsilon^{ijkl}[\overset{\circ}{W}_{kl}, \overset{\circ}{\bar\chi}_{j\dot\alpha}] \;=\; \mathcal{O}(\hbar)\ , \tag{VI.62}
$$

but actually performing this task leads to both unenlightening and complicated looking expressions, so we refrain from writing them down. To proceed in a realistic manner, one can constrain the deformation parameters to obtain manageable equations of motion.

For instance, in order to compare the deformed equations of motion with Seiberg's deformed $\mathcal{N}=1$ equations[9] [254], one would have to restrict the deformation matrix $C^{\alpha i,\beta j}$ properly and to put some of the fields, e.g., $\tilde\phi_{ij}$, to zero.

§12 Remarks on the Seiberg-Witten map. Generalizing the string theory side of the derivation of Seiberg-Witten maps seems to be nontrivial. The graviphoton used to deform the fermionic coordinates belongs to the R-R sector, while the gauge field strength causing the deformation in the bosonic case sits in the NS-NS sector. This implies that the field strengths appear on different footing in the vertex operators of the appropriate string theory (type II with $\mathcal{N}=2$, $d=4$). A first step might be to consider a "pure gauge" configuration in which the gluino and gluon field strengths vanish. The corresponding vertex operator in Berkovits' hybrid formalism on the boundary of the worldsheet of an open string contains the terms

$$
V \;=\; \frac{1}{2\alpha'}\int \mathrm{d}\tau\ (\dot\theta^\alpha\omega_\alpha + \dot X^\mu A_\mu - \mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\dot\theta^\alpha\bar\theta^{\dot\alpha}A_\mu)\ , \tag{VI.63}
$$

with the formal (classical) gauge transformations $\delta_\lambda\omega_\alpha = D_\alpha\lambda$ and $\delta_\lambda A_\mu = \partial_\mu\lambda$. From here, one may proceed exactly as in [251] using the deformation of [254]: regularization of the action by Pauli-Villars[10] and point-splitting procedures lead to an undeformed and a deformed gauge invariance, respectively. Although on flat Euclidean space, pure gauge is trivial, the two different gauge transformations obtained are not.

More general, a Seiberg-Witten map is a translation rule between two physically equivalent field theories. The fact that our choice of deformation generically breaks half

[9] or similarly in the case of the deformed $\mathcal{N}=2$ equations in $\mathcal{N}=1$ superspace language [6]

[10] Pauli-Villars was applied to supergravity, e.g., in [100].

of the supersymmetry is not in contradiction with the existence of a Seiberg-Witten map, but may be seen analogously to the purely bosonic case: in both the commutative and noncommutative theories, particle Lorentz invariance is broken which is due to the background field (B-field).

VI.3 Drinfeld-twisted supersymmetry

Another development which attracted much attention recently began with the realization that noncommutative field theories, although manifestly breaking Poincaré symmetry, can be recast into a form which is invariant under a twist-deformed action of the Poincaré algebra [210, 57, 59]. In this framework, the commutation relation $[x^\mu, x^\nu] = \mathrm{i}\Theta^{\mu\nu}$ is understood as a result of the non-*co*commutativity of the coproduct of a twisted Hopf Poincaré algebra acting on the coordinates. This result can be used to show that the representation content of Moyal-Weyl-deformed theories is identical to that of their undeformed Lorentz invariant counterparts. Furthermore, theorems in quantum field theory which require Lorentz invariance for their proof can now be carried over to the Moyal-Weyl-deformed case using twisted Lorentz invariance. For related works, see also [18, 158, 81, 196, 106, 10, 53, 183, 9, 58].

The following section is based on the paper [136] and presents an extension of the analysis of [57, 59] to supersymmetric field theories on non-anticommutative superspaces. We will use Drinfeld-twisted supersymmetry to translate properties of these field theories into the non-anticommutative situation, where half of the supersymmetries are broken.

Note that Drinfeld-twisted supersymmetry was already considered in the earlier publication [157] and there is some overlap with our discussion in the case $\mathcal{N} = 1$. The analysis of extended supersymmetries presented in this reference differs from the one we will propose here. Furthermore, our discussion will include several new applications of the re-gained twisted supersymmetry. In the paper [303], which appeared simultaneously with [136], Drinfeld-twisted $\mathcal{N} = (1,1)$ supersymmetry has been discussed. More recent work in this area is, e.g., [300, 19, 301, 304, 235].

VI.3.1 Preliminary remarks

§1 Hopf algebra. A *Hopf algebra* is an algebra H over a field $\mathbb{K}$ together with a *product* m, a *unit* $\mathbb{1}$, a *coproduct* $\Delta : H \to H \otimes H$ satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, a *counit* $\varepsilon : H \to \mathbb{K}$ satisfying $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ and $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$ and an *antipode* $S : H \to H$ satisfying $m(S \otimes \mathrm{id})\Delta = \varepsilon\mathbb{1}$ and $m(\mathrm{id} \otimes S)\Delta = \varepsilon\mathbb{1}$. The maps Δ, ε and S are unital maps, that is $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$, $\varepsilon(\mathbb{1}) = 1$ and $S(\mathbb{1}) = \mathbb{1}$.

§2 Hopf superalgebra. Recall from section III.2.1 that a *superalgebra* is a supervector space endowed with *i)* an associative multiplication respecting the grading and *ii)* the graded commutator $[\![a, b]\!] = ab - (-1)^{\tilde{a}\tilde{b}}ba$. We fix the following rule for the interplay between the multiplication and the tensor product $\otimes$ in a superalgebra:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) \;=\; (-1)^{\tilde{a}_2\tilde{b}_1}(a_1 b_1 \otimes a_2 b_2) \; . \tag{VI.64}$$

A superalgebra is called a *Hopf superalgebra* if it is endowed with a graded coproduct[11] Δ and a counit ε, both of which are graded algebra morphisms, i.e.

$$\Delta(ab) \;=\; \sum (-1)^{\tilde{a}_{(2)}\tilde{b}_{(1)}} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \quad \text{and} \quad \varepsilon(ab) \;=\; \varepsilon(a)\varepsilon(b) \; , \tag{VI.65}$$

[11]In Sweedler's notation with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, this amounts to $\tilde{a} \equiv \tilde{a}_{(1)} + \tilde{a}_{(2)} \mod 2$.

and an antipode S which is a graded algebra anti-morphism, i.e.

$$S(ab) \;=\; (-1)^{\tilde{a}\tilde{b}} S(b)S(a) \;. \tag{VI.66}$$

As usual, one furthermore demands that Δ, ε and S are unital maps, that Δ is coassociative and that ε and S are counital. For more details, see [45] and references therein.

§3 An extended graded Baker-Campbell-Hausdorff formula First, note that $\mathrm{e}^{A\otimes B}\mathrm{e}^{-A\otimes B}$ is indeed equal to $\mathbb{1}\otimes\mathbb{1}$ for any two elements A, B of a superalgebra. This is clear for $\tilde{A}=0$ or $\tilde{B}=0$. For $\tilde{A}=\tilde{B}=1$ it is most instructively gleaned from

$$\left(\mathbb{1}\otimes\mathbb{1} + A\otimes B - \tfrac{1}{2}A^2\otimes B^2 + \ldots\right)\left(\mathbb{1}\otimes\mathbb{1} - A\otimes B - \tfrac{1}{2}A^2\otimes B^2 - \ldots\right) \;=\; \mathbb{1}\otimes\mathbb{1}\;.$$

Now, for elements A_I, B_J, D of a graded algebra, where the parities of the elements A_I and B_J are all equal $\tilde{A}=\tilde{A}_I=\tilde{B}_J$ and $\{\!\![A_I, A_J]\!\!\} = \{\!\![B_I, B_J]\!\!\} = 0$, we have the relation

$$\begin{aligned}&\mathrm{e}^{C^{IJ}A_I\otimes B_J}\,(D\otimes\mathbb{1})\,\mathrm{e}^{-C^{KL}A_K\otimes B_L}\\ &\quad=\;\sum_{n=0}^{\infty}\frac{(-1)^{n\tilde{A}\tilde{D}+\frac{n(n-1)}{2}\tilde{A}}}{n!}C^{I_1J_1}\ldots C^{I_nJ_n}\{\!\![A_{I_1},\{\!\![\ldots\{\!\![A_{I_n},D]\!\!\}]\!\!\}]\!\!\}\otimes B_{J_1}\ldots B_{J_n}\;.\end{aligned} \tag{VI.67}$$

Proof: To verify this relation, one can simply adapt the well-known iterative proof via a differential equation. First note that

$$\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(C^{KL}A_K\otimes B_L) \;=\; (C^{KL}A_K\otimes B_L)\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}\;. \tag{VI.68}$$

Then define the function

$$F(\lambda) \;:=\; \mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}\;, \tag{VI.69}$$

which has the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) \;=\;& (C^{MN}A_M\otimes B_N)\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}\\ &-\mathrm{e}^{\lambda C^{IJ}A_I\otimes B_J}(D\otimes 1)\mathrm{e}^{-\lambda C^{KL}A_K\otimes B_L}(C^{MN}A_M\otimes B_N)\;.\end{aligned} \tag{VI.70}$$

Thus, we have the identity $\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) = [(C^{MN}A_M\otimes B_N), F(\lambda)]$, which, when applied recursively together with the Taylor formula, leads to

$$F(1) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\left[C^{I_1J_1}A_{I_1}\otimes B_{J_1}\left[\ldots\left[C^{I_nJ_n}A_{I_n}\otimes B_{J_n}, D\otimes\mathbb{1}\right]\ldots\right]\right]\;. \tag{VI.71}$$

Also recursively, one easily checks that

$$\begin{aligned}&\left[C^{I_1J_1}A_{I_1}\otimes B_{J_1}\left[\ldots\left[C^{I_nJ_n}A_{I_n}\otimes B_{J_n}, D\otimes\mathbb{1}\right]\ldots\right]\right]\\ &\quad=\;(-1)^{\tilde{A}\tilde{D}}(-1)^{\kappa}C^{I_1J_1}\ldots C^{I_nJ_n}\{\!\![A_{I_1},\{\!\![\ldots\{\!\![A_{I_n},D]\!\!\}]\!\!\}]\!\!\}\otimes B_{J_1}\ldots B_{J_n}\;,\end{aligned} \tag{VI.72}$$

where κ is given by $\kappa = (n-1)\tilde{A} + (n-2)\tilde{A} + \ldots + \tilde{A}$. Furthermore, we have

$$(-1)^{\kappa} \;=\; (-1)^{n^2-\sum_{i=1}^n i} \;=\; (-1)^{n^2+\sum_{i=1}^n i} \;=\; (-1)^{\frac{n(n-1)}{2}}\;, \tag{VI.73}$$

which, together with the results above, proves formula (VI.67). This extended graded Baker-Campbell-Hausdorff formula also generalizes straightforwardly to the case when $D\otimes\mathbb{1}$ is replaced by $\mathbb{1}\otimes D$.

VI.3.2 Drinfeld twist of the Euclidean super Poincaré algebra

§4 Euclidean super Poincaré algebra. The starting point of our discussion is the ordinary Euclidean super Poincaré algebra[12] $\mathfrak{g}$ on $\mathbb{R}^{4|4\mathcal{N}}$ without central extensions, which generates the isometries on the space $\mathbb{R}^{4|4\mathcal{N}}$. More explicitly, we have the generators of translations P_μ, the generators of four-dimensional rotations $M_{\mu\nu}$ and the $4\mathcal{N}$ supersymmetry generators $Q_{\alpha i}$ and $\bar{Q}^i_{\dot{\alpha}}$. Recall from section III.1.1 that they satisfy the algebra

$$\begin{aligned}
[P_\rho, M_{\mu\nu}] &= \mathrm{i}(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu) \ , \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}(\delta_{\mu\rho}M_{\nu\sigma} - \delta_{\mu\sigma}M_{\nu\rho} - \delta_{\nu\rho}M_{\mu\sigma} + \delta_{\nu\sigma}M_{\mu\rho}) \ , \\
[P_\mu, Q_{\alpha i}] &= 0 \ , \qquad [P_\mu, \bar{Q}^i_{\dot{\alpha}}] = 0 \ , \\
[M_{\mu\nu}, Q_{i\alpha}] &= \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta} \ , \quad [M_{\mu\nu}, \bar{Q}^{i\dot{\alpha}}] = \mathrm{i}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{i\dot{\beta}} \ , \\
\{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} &= 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu \ , \qquad \{Q_{\alpha i}, Q_{\beta j}\} = \{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\} = 0 \ .
\end{aligned} \tag{VI.74}$$

Recall furthermore that the Casimir operators of the Poincaré algebra used for labelling representations are P^2 and W^2, where the latter is the square of the Pauli-Ljubanski operator

$$W_\mu = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma \ . \tag{VI.75}$$

This operator is, however, not a Casimir of the super Poincaré algebra; instead, there is a supersymmetric variant: the (superspin) operator $\widetilde{C}^2$ defined as the square of

$$\widetilde{C}_{\mu\nu} = \widetilde{W}_\mu P_\nu - \widetilde{W}_\nu P_\mu, \tag{VI.76}$$

where $\widetilde{W}_\mu := W_\mu - \frac{1}{4}\bar{Q}^i_{\dot{\alpha}}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu Q_{i\alpha}$.

§5 Universal enveloping algebra. A *universal enveloping algebra* $\mathcal{U}(\mathfrak{a})$ of a Lie algebra $\mathfrak{a}$ is an associative unital algebra together with a Lie algebra homomorphism $h : \mathfrak{a} \to \mathcal{U}(\mathfrak{a})$, satisfying the following universality property: For any further associative algebra A with homomorphism $\phi : \mathfrak{a} \to A$, there exists a unique homomorphism $\psi : \mathcal{U}(\mathfrak{a}) \to A$ of associative algebras, such that $\phi = \psi \circ h$. Every Lie algebra has an universal enveloping algebra, which is unique up to algebra isomorphisms.

§6 The universal enveloping algebra of $\mathfrak{g}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Euclidean super Poincaré algebra $\mathfrak{g}$ is a cosupercommutative Hopf superalgebra with counit and coproduct defined by $\varepsilon(\mathbb{1}) = 1$ and $\varepsilon(x) = 0$ otherwise, $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ and $\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1}$ otherwise.

§7 Drinfeld twist. Given a Hopf algebra H with coproduct Δ, a counital 2-cocycle $\mathcal{F}$ is a counital element of $H \otimes H$, which has an inverse and satisfies

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F} \ , \tag{VI.77}$$

where we used the common shorthand notation $\mathcal{F}_{12} = \mathcal{F} \otimes \mathbb{1}$, $\mathcal{F}_{23} = \mathbb{1} \otimes \mathcal{F}$ etc. As done in [57], such a counital 2-cocycle $\mathcal{F} \in H \otimes H$ can be used to define a twisted Hopf algebra[13] $H^{\mathcal{F}}$ with a new coproduct given by

$$\Delta^{\mathcal{F}}(Y) := \mathcal{F}\Delta(Y)\mathcal{F}^{-1} \ . \tag{VI.78}$$

The element $\mathcal{F}$ is called a *Drinfeld twist*; such a construction was first considered in [88].

---

[12] or inhomogeneous super Euclidean algebra

[13] This twisting amounts to constructing a quasitriangular Hopf algebra, as discussed, e.g., in [61].

§8 The Drinfeld twist of $\mathcal{U}(\mathfrak{g})$. For our purposes, i.e. to recover the canonical algebra of non-anticommutative coordinates (VI.25), we choose the Abelian twist $\mathcal{F} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ defined by

$$\mathcal{F} \;=\; \exp\left(-\frac{\hbar}{2} C^{\alpha i,\beta j} Q_{\alpha i} \otimes Q_{\beta j}\right) \;. \tag{VI.79}$$

As one easily checks, $\mathcal{F}$ is indeed a counital 2-cocycle: First, it is invertible and its inverse is given by $\mathcal{F}^{-1} = \exp\left(\frac{\hbar}{2} C^{\alpha i,\beta j} Q_{\alpha i} \otimes Q_{\beta j}\right)$. (Because the $Q_{\alpha i}$ are nilpotent, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are not formal series but rather finite sums.) Second, $\mathcal{F}$ is counital since it satisfies the conditions

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} \;=\; \mathbb{1} \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon)\mathcal{F} \;=\; \mathbb{1} \;, \tag{VI.80}$$

as can be verified without difficulty. Also, the remaining cocycle condition (VI.77) turns out to be fulfilled since

$$\begin{aligned} \mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} \;&=\; \mathcal{F}_{12} \exp\left(-\frac{\hbar}{2} C^{\alpha i,\beta j} (Q_{\alpha i} \otimes \mathbb{1} + \mathbb{1} \otimes Q_{\alpha i}) \otimes Q_{\beta j}\right) \;, \\ \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F} \;&=\; \mathcal{F}_{23} \exp\left(-\frac{\hbar}{2} C^{\alpha i,\beta j} Q_{\alpha i} \otimes (Q_{\beta j} \otimes \mathbb{1} + \mathbb{1} \otimes Q_{\beta j})\right) \end{aligned} \tag{VI.81}$$

yields, due to the (anti)commutativity of the $Q_{\alpha i}$,

$$\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} \;=\; \mathcal{F}_{23}\mathcal{F}_{12}\mathcal{F}_{13} \;, \tag{VI.82}$$

which is obviously true.

§9 Twisted multiplication and coproduct. Note that after introducing this Drinfeld twist, the multiplication in $\mathcal{U}(\mathfrak{g})$ and the action of $\mathfrak{g}$ on the coordinates remain the same. In particular, the representations of the twisted and the untwisted algebras are identical. It is only the action of $\mathcal{U}(\mathfrak{g})$ on the tensor product of the representation space, given by the coproduct, which changes.

Let us be more explicit on this point: the coproduct of the generator P_μ does not get deformed, as P_μ commutes with $Q_{\beta j}$:

$$\Delta^{\mathcal{F}}(P_\mu) \;=\; \Delta(P_\mu) \;. \tag{VI.83}$$

For the other generators of the Euclidean super Poincaré algebra, the situation is slightly more complicated. Due to the rule $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\tilde{a}_2 \tilde{b}_1}(a_1 b_1 \otimes a_2 b_2)$, where $\tilde{a}$ denotes the Graßmann parity of a, we have the relations[14] (cf. equation (VI.67))

$$\begin{aligned} &\mathcal{F}\,(D \otimes \mathbb{1})\,\mathcal{F}^{-1} \;= \\ &\quad \sum_{n=0}^{\infty} \frac{(-1)^{n\tilde{D} + \frac{n(n-1)}{2}}}{n!} \left(-\frac{\hbar}{2}\right)^n C^{I_1 J_1} \ldots C^{I_n J_n} \{\!\![Q_{I_1}, \{\!\![\ldots \{\!\![Q_{I_n}, D]\!\!\}]\!\!\}]\!\!\} \otimes Q_{J_1} \ldots Q_{J_n} \;, \\ &\mathcal{F}\,(\mathbb{1} \otimes D)\,\mathcal{F}^{-1} \;= \\ &\quad \sum_{n=0}^{\infty} \frac{(-1)^{n\tilde{D} + \frac{n(n-1)}{2}}}{n!} \left(-\frac{\hbar}{2}\right)^n C^{I_1 J_1} \ldots C^{I_n J_n} Q_{I_1} \ldots Q_{I_n} \otimes \{\!\![Q_{J_1}, \{\!\![\ldots \{\!\![Q_{J_n}, D]\!\!\}]\!\!\}]\!\!\} \;, \end{aligned} \tag{VI.84}$$

where $\{\!\![\cdot, \cdot]\!\!\}$ denotes the graded commutator. From this, we immediately obtain

$$\Delta^{\mathcal{F}}(Q_{\alpha i}) \;=\; \Delta(Q_{\alpha i}) \;. \tag{VI.85}$$

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[14]Here, I_k and J_k are multi-indices, e.g. $I_k = i_k \alpha_k$.

Furthermore, we can also derive the expressions for $\Delta^{\mathcal{F}}(M_{\mu\nu})$ and $\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot\gamma})$, which read

$$\Delta^{\mathcal{F}}(M_{\mu\nu}) = \Delta(M_{\mu\nu}) + \frac{\mathrm{i}\hbar}{2}C^{\alpha i,\beta j}\left[(\sigma_{\mu\nu})_\alpha{}^\gamma Q_{i\gamma}\otimes Q_{\beta j} + Q_{\alpha i}\otimes(\sigma_{\mu\nu})_\beta{}^\gamma Q_{j\gamma}\right] , \quad \text{(VI.86)}$$

$$\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot\gamma}) = \Delta(\bar{Q}^k_{\dot\gamma}) + \hbar C^{\alpha i,\beta j}\left[\delta^k_i\sigma^\mu_{\alpha\dot\gamma}P_\mu\otimes Q_{\beta j} + Q_{\alpha i}\otimes\delta^k_j\sigma^\mu_{\beta\dot\gamma}P_\mu\right] . \quad \text{(VI.87)}$$

The twisted coproduct of the Pauli-Ljubanski operator W_μ becomes

$$\Delta^{\mathcal{F}}(W_\mu) = \Delta(W_\mu) - \frac{\mathrm{i}\hbar}{4}C^{\alpha i,j\beta}\varepsilon_{\mu\nu\rho\sigma}\left(Q_{i\alpha}\otimes(\sigma^{\nu\rho})_\beta{}^\gamma Q_{j\gamma}P^\sigma + (\sigma^{\nu\rho})_\alpha{}^\gamma Q_{i\gamma}P^\sigma\otimes Q_{j\beta}\right) , \quad \text{(VI.88)}$$

while for its supersymmetric variant $\widetilde{C}_{\mu\nu}$, we have

$$\begin{aligned}
\Delta^{\mathcal{F}}(\widetilde{C}_{\mu\nu}) &= \Delta(\widetilde{C}_{\mu\nu}) - \frac{\hbar}{2}C^{\alpha i,j\beta}\left[Q_{\alpha i}\otimes Q_{j\beta},\Delta(\widetilde{C}_{\mu\nu})\right] \\
&= \Delta(\widetilde{C}_{\mu\nu}) - \frac{\hbar}{2}C^{\alpha i,j\beta}\left(\left[Q_{\alpha i},\widetilde{C}_{\mu\nu}\right]\otimes Q_{j\beta} + Q_{i\alpha}\otimes\left[Q_{j\beta},\widetilde{C}_{\mu\nu}\right]\right) \\
&= \Delta(\widetilde{C}_{\mu\nu}) ,
\end{aligned} \quad \text{(VI.89)}$$

since $[Q_{i\alpha},\widetilde{C}_{\mu\nu}] = 0$ by construction.

§10 Representation on the algebra of superfunctions. Given a representation of the Hopf algebra $\mathcal{U}(\mathfrak{g})$ in an associative algebra consistent with the coproduct Δ, one needs to adjust the multiplication law after introducing a Drinfeld twist. If $\mathcal{F}^{-1}$ is the inverse of the element $\mathcal{F}\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ generating the twist, the new product compatible with $\Delta^{\mathcal{F}}$ reads

$$a\star b := m^{\mathcal{F}}(a\otimes b) := m\circ\mathcal{F}^{-1}(a\otimes b) , \quad \text{(VI.90)}$$

where m denotes the ordinary product $m(a\otimes b) = ab$.

Let us now turn to the representation of the Hopf superalgebra $\mathcal{U}(\mathfrak{g})$ on the algebra $\mathcal{S} := C^\infty(\mathbb{R}^4)\otimes\Lambda_{4\mathcal{N}}$ of superfunctions on $\mathbb{R}^{4|4\mathcal{N}}$. On $\mathcal{S}$, we have the standard representation of the super Poincaré algebra in chiral coordinates $(y^\mu,\theta^{\alpha i},\bar\theta^{\dot\alpha}_i)$:

$$\begin{aligned}
P_\mu f &= \mathrm{i}\partial_\mu f , \qquad & M_{\mu\nu}f &= \mathrm{i}(y_\mu\partial_\nu - y_\nu\partial_\mu)f , \\
Q_{\alpha i}f &= \frac{\partial}{\partial\theta^{\alpha i}}f , \qquad & \bar{Q}^i_{\dot\alpha}f &= \left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}_i}f + 2\mathrm{i}\theta^{\alpha i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\right)f ,
\end{aligned} \quad \text{(VI.91)}$$

where f is an element of $\mathcal{S}$. After the twist, the multiplication m becomes the twist-adapted multiplication $m^{\mathcal{F}}$ (VI.90), which reproduces the coordinate algebra of $\mathbb{R}^{4|4\mathcal{N}}_\hbar$, e.g. we have

$$\begin{aligned}
\{\theta^{\alpha i}\overset{\star}{,}\theta^{\beta j}\} &:= m^{\mathcal{F}}(\theta^{\alpha i}\otimes\theta^{\beta j}) + m^{\mathcal{F}}(\theta^{\beta j}\otimes\theta^{\alpha i}) \\
&= \theta^{\alpha i}\theta^{\beta j} + \frac{\hbar}{2}C^{\alpha i,\beta j} + \theta^{\beta j}\theta^{\alpha i} + \frac{\hbar}{2}C^{\beta j,\alpha i} \\
&= \hbar C^{\alpha i,\beta j} .
\end{aligned} \quad \text{(VI.92)}$$

Thus, we have constructed a representation of the Euclidean super Poincaré algebra on $\mathbb{R}^{4|4\mathcal{N}}_\hbar$ by employing $\mathcal{S}_\star$, thereby making twisted supersymmetry manifest.

VI.3.3 Applications

We saw in the above construction of the twisted Euclidean super Poincaré algebra that our description is equivalent to the standard treatment of Moyal-Weyl-deformed superspace. We can therefore use it to define field theories via their Lagrangians, substituting all products by star products, which then will be invariant under twisted super Poincaré transformations. This can be directly carried over to quantum field theories, replacing the products between operators by star products. Therefore, twisted super Poincaré invariance, in particular twisted supersymmetry, will always be manifest.

As a consistency check, we want to show that the tensor $C^{\alpha i,\beta j} := \{\theta^{\alpha i}, \theta^{\beta j}\}_\star$ is invariant under twisted super Poincaré transformations before tackling more advanced issues. Furthermore, we want to relate the representation content of the deformed theory with that of the undeformed one by scrutinizing the Casimir operators of this superalgebra. Eventually, we will turn to supersymmetric Ward-Takahashi identities and their consequences for renormalizability.

§11 Invariance of $C^{\alpha i,\beta j}$. The action of the twisted supersymmetry charge on $C^{\alpha i,\beta j}$ is given by

$$\begin{aligned}
\hbar Q^{\mathcal{F}}_{k\gamma} C^{\alpha i,\beta j} &= Q^{\mathcal{F}}_{k\gamma}\left(\{\theta^{\alpha i} \overset{\star}{,} \theta^{\beta j}\}\right) \\
&:= m^{\mathcal{F}} \circ \left(\Delta^{\mathcal{F}}(Q_{k\gamma})(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{j\beta} \otimes \theta^{i\alpha})\right) \\
&= m^{\mathcal{F}} \circ \left(\Delta(Q_{k\gamma})(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{j\beta} \otimes \theta^{i\alpha})\right) \\
&= m \circ \mathcal{F}^{-1}(\delta^i_k \delta^\alpha_\gamma \otimes \theta^{j\beta} + \delta^j_k \delta^\beta_\gamma \otimes \theta^{i\alpha} - \theta^{i\alpha} \otimes \delta^j_k \delta^\beta_\gamma - \theta^{j\beta} \otimes \delta^i_k \delta^\alpha_\gamma) \\
&= m(\delta^i_k \delta^\alpha_\gamma \otimes \theta^{j\beta} + \delta^j_k \delta^\beta_\gamma \otimes \theta^{i\alpha} - \theta^{i\alpha} \otimes \delta^j_k \delta^\beta_\gamma - \theta^{j\beta} \otimes \delta^i_k \delta^\alpha_\gamma) \\
&= 0\ .
\end{aligned} \tag{VI.93}$$

Similarly, we have

$$\begin{aligned}
\hbar(\bar{Q}^k_{\dot\gamma})^{\mathcal{F}} C^{\alpha i,\beta j} &= m^{\mathcal{F}} \circ \left(\Delta^{\mathcal{F}}(\bar{Q}^k_{\dot\gamma})(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i})\right) \\
&= m^{\mathcal{F}} \circ \left(\Delta(\bar{Q}^k_{\dot\gamma})(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i})\right) \\
&= 0\ ,
\end{aligned} \tag{VI.94}$$

and

$$\hbar P^{\mathcal{F}}_{\mu\nu} C^{\alpha i,\beta j} = m^{\mathcal{F}} \circ \left(\Delta(P_\mu)(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i})\right) = 0\ . \tag{VI.95}$$

For the action of the twisted rotations and boosts, we get

$$\begin{aligned}
\hbar M^{\mathcal{F}}_{\mu\nu} C^{\alpha i,\beta j} &= m^{\mathcal{F}} \circ \left(\Delta^{\mathcal{F}}(M_{\mu\nu})(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i})\right) \\
&= m \circ \mathcal{F}^{-1}\mathcal{F}\Delta(M_{\mu\nu})\mathcal{F}^{-1}(\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i}) \\
&= m(\mathbb{1} \otimes M_{\mu\nu} + M_{\mu\nu} \otimes \mathbb{1})\left((\theta^{\alpha i} \otimes \theta^{\beta j} + \theta^{\beta j} \otimes \theta^{\alpha i}) - \hbar C^{\alpha i,\beta j}\mathbb{1} \otimes \mathbb{1}\right) \\
&= 0\ ,
\end{aligned} \tag{VI.96}$$

where we made use of $M_{\mu\nu} = \mathrm{i}(y_\mu \partial_\nu - y_\nu \partial_\mu)$. Thus, $C^{\alpha i,\beta j}$ is invariant under the twisted Euclidean super Poincaré transformations, which is a crucial check of the validity of our construction.

§12 Representation content. An important feature of noncommutative field theories was demonstrated recently [57, 59]: they share the same representation content as their commutative counterparts. Of course, one would expect this to also hold for non-anticommutative deformations, in particular since the superfields defined, e.g., in [254] on a deformed superspace have the same set of components as the undeformed ones.

To decide whether the representation content in our case is the same as in the commutative theory necessitates checking whether the twisted action of the Casimir operators

$P^2 = P_\mu \star P^\mu$ and $\widetilde{C}^2 = \widetilde{C}_{\mu\nu} \star \widetilde{C}^{\mu\nu}$ on elements of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is altered with respect to the untwisted case. But since we have already shown in (VI.83) and (VI.89) that the actions of the operators P_μ and $\widetilde{C}_{\mu\nu}$ remain unaffected by the twist, it follows immediately that the operators P^2 and $\widetilde{C}^2$ are still Casimir operators in the twisted case. Together with the fact that the representation space considered as a module is not changed, this proves that the representation content is indeed the same.

§13 Chiral rings and correlation functions. As discussed in section IV.1.3, the chiral rings of operators in supersymmetric quantum field theories are cohomology rings of the supercharges $Q_{i\alpha}$ and $\bar{Q}^i_{\dot\alpha}$ and correlation functions which are built out of elements of a single such chiral ring have peculiar properties.

In [254], the anti-chiral ring was defined and discussed for non-anticommutative field theories. The chiral ring, however, lost its meaning: the supersymmetries generated by $\bar{Q}^i_{\dot\alpha}$ are broken, cf. (VI.32), and therefore the vacuum is expected to be no longer invariant under this generator. Thus, the $\bar{Q}$-cohomology is not relevant for correlation functions of chiral operators.

In our approach to non-anticommutative field theory, *twisted* supersymmetry is manifest and therefore the chiral ring can be treated similarly to the untwisted case as we want to discuss in the following.

Let us assume that the Hilbert space $\mathcal{H}$ of our quantum field theory carries a representation of the Euclidean super Poincaré algebra $\mathfrak{g}$, and that there is a unique, $\mathfrak{g}$ invariant vacuum state $|0\rangle$. Although the operators $Q_{\alpha i}$ and $\bar{Q}^i_{\dot\alpha}$ are not related via Hermitian conjugation when considering supersymmetry on Euclidean spacetime, it is still natural to assume that the vacuum is annihilated by both supercharges. The reasoning for this is basically the same as the one employed in [254] to justify the use of Minkowski superfields on Euclidean spacetime: one can obtain a complexified supersymmetry algebra on Euclidean space from a complexified supersymmetry algebra on Minkowski space.[15] Furthermore, it has been shown that in the non-anticommutative situation, just as in the ordinary undeformed case, the vacuum energy of the Wess-Zumino model is not renormalized [43].

We can now define the ring of chiral and anti-chiral operators by the relations

$$\{\!\![\bar{Q} \overset{\star}{,} \mathcal{O}]\!\!\} \;=\; 0 \quad \text{and} \quad \{\!\![Q \overset{\star}{,} \bar{\mathcal{O}}]\!\!\} \;=\; 0\ , \tag{VI.97}$$

respectively. In a correlation function built from chiral operators, $\bar{Q}$-exact terms, i.e. terms of the form $\{\!\![\bar{Q} \overset{\star}{,} A]\!\!\}$, do not contribute as is easily seen from

$$\begin{aligned}
\langle \{\!\![\bar{Q} \overset{\star}{,} A]\!\!\} \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n \rangle \; &= \langle \{\!\![\bar{Q} \overset{\star}{,} A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n]\!\!\} \rangle \pm \langle A \star \{\!\![\bar{Q} \overset{\star}{,} \mathcal{O}_1]\!\!\} \star \ldots \star \mathcal{O}_n \rangle \\
&\quad \pm \ldots \pm \langle A \star \mathcal{O}_1 \star \ldots \star \{\!\![\bar{Q} \overset{\star}{,} \mathcal{O}_n]\!\!\} \rangle \\
&= \langle \bar{Q} A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n \rangle \pm \langle A \star \mathcal{O}_1 \star \ldots \star \mathcal{O}_n \star \bar{Q} \rangle \\
&= 0\ ,
\end{aligned} \tag{VI.98}$$

where we used that $\bar{Q}$ annihilates both $\langle 0|$ and $|0\rangle$, completely analogously to the case of untwisted supersymmetry. Therefore, the relevant operators in the chiral ring consist of the $\bar{Q}$-closed modulo the $\bar{Q}$-exact operators. The same argument holds for the anti-chiral

[15] One can then perform all superspace calculations and impose suitable reality conditions on the component fields in the end.

ring after replacing $\bar{Q}$ with Q, namely

$$\begin{aligned}\langle \{\![Q \overset{\star}{,} A]\!\} \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n \rangle &= \langle \{\![Q \overset{\star}{,} A \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n]\!\} \rangle \pm \langle A \star \{\![Q \overset{\star}{,} \bar{\mathcal{O}}_1]\!\} \star \ldots \star \bar{\mathcal{O}}_n \rangle \\ &\quad \pm \ldots \pm \langle A \star \bar{\mathcal{O}}_1 \star \ldots \star \{\![Q \overset{\star}{,} \bar{\mathcal{O}}_n]\!\} \rangle \\ &= \langle QA \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n \rangle \pm \langle A \star \bar{\mathcal{O}}_1 \star \ldots \star \bar{\mathcal{O}}_n \star Q \rangle \\ &= 0 \ . \end{aligned} \qquad \text{(VI.99)}$$

§14 Twisted supersymmetric Ward-Takahashi identities. The above considered properties of correlation functions are particularly useful since they imply a twisted supersymmetric Ward-Takahashi identity: any derivative with respect to the bosonic coordinates of an anti-chiral operator annihilates a purely chiral or anti-chiral correlation function, cf. section IV.1.3, §12. Recall that this is due to the fact that $\partial \sim \{Q, \bar{Q}\}$ and therefore any derivative gives rise to a Q-exact term, which causes an anti-chiral correlation function to vanish. Analogously, the bosonic derivatives of chiral correlation functions vanish. Thus, the correlation functions are independent of the bosonic coordinates, and we can move the operators to a far distance of each other, also in the twisted supersymmetric case:

$$\langle \bar{\mathcal{O}}_1(x_1) \star \ldots \star \bar{\mathcal{O}}_n(x_n) \rangle \;=\; \langle \bar{\mathcal{O}}_1(x_1^\infty) \rangle \star \ldots \star \langle \bar{\mathcal{O}}_n(x_n^\infty) \rangle \ . \qquad \text{(VI.100)}$$

and we discover again that these correlation functions *clusterize.*

Another direct consequence of (VI.98) is the holomorphic dependence of the chiral correlation functions on the coupling constants, i.e.

$$\frac{\partial}{\partial \bar{\lambda}} \langle \mathcal{O}_1 \star \ldots \star \mathcal{O}_n \rangle \;=\; 0 \ . \qquad \text{(VI.101)}$$

This follows in a completely analogous way to the ordinary supersymmetric case, and for an example, see again IV.1.3, §12.

§15 Comments on non-renormalization theorems. A standard perturbative non-renormalization theorem for $\mathcal{N} = 1$ supersymmetric field theory states that every term in the effective action can be written as an integral over $\mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$. It has been shown in [43] that this theorem also holds in the non-anticommutative case. The same is then obviously true in our case of twisted and therefore unbroken supersymmetry, and the proof carries through exactly as in the ordinary case.

Furthermore, in a supersymmetric nonlinear sigma model, the superpotential is not renormalized. A nice argument for this fact was given in [252]. Instead of utilizing Feynman diagrams and supergraph techniques, one makes certain naturalness assumptions about the effective superpotential. These assumptions turn out to be strong enough to enforce a non-perturbative non-renormalization theorem.

In the following, let us demonstrate this argument in a simple case, following closely [8]. Take a nonlinear sigma model with superpotential

$$\mathcal{W} \;=\; \tfrac{1}{2} m \Phi^2 + \tfrac{1}{3} \lambda \Phi^3 \ , \qquad \text{(VI.102)}$$

where $\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F$ is an ordinary chiral superfield. The assumptions we impose on the effective action are the following:

▷ Supersymmetry is also a symmetry of the effective superpotential.

▷ The effective superpotential is holomorphic in the coupling constants.

- ▷ Physics is smooth and regular under the possible weak-coupling limits.
- ▷ The effective superpotential preserves the $\mathsf{U}(1) \times \mathsf{U}(1)_R$ symmetry of the original superpotential with charge assignments $\Phi : (1,1)$, $m : (-2,0)$, $\lambda : (-3,-1)$ and $\mathrm{d}^2\theta : (0,-2)$.

It follows that the effective superpotential must be of the form

$$\mathcal{W}_{\text{eff}} \;=\; m\Phi W\left(\frac{\lambda\Phi}{m}\right) \;=\; \sum_i a_i\lambda^i m^{1-i}\Phi^{i+2} \;, \tag{VI.103}$$

where W is an arbitrary holomorphic function of its argument. Regularity of physics in the two weak-coupling limits $\lambda \to 0$ and $m \to 0$ then implies that $\mathcal{W}_{\text{eff}} = \mathcal{W}$.

To obtain an analogous non-renormalization theorem in the non-anticommutative setting, we make similar assumptions about the effective superpotential as above. We start from

$$\mathcal{W}_\star \;=\; \tfrac{1}{2}m\Phi\star\Phi + \tfrac{1}{3}\lambda\Phi\star\Phi\star\Phi \;, \tag{VI.104}$$

and assume the following:

- ▷ *Twisted* supersymmetry is a symmetry of the effective superpotential. Note that this assumption is new compared to the discussion in [43]. Furthermore, arguments substantiating that the effective action can always be written in terms of star products have been given in [44].
- ▷ The effective superpotential is holomorphic in the coupling constants. (This assumption is equally natural as in the supersymmetric case, since it essentially relies on the existence of chiral and anti-chiral rings, which we proved above for our setting.)
- ▷ Physics is smooth and regular under the possible weak-coupling limits.
- ▷ The effective superpotential preserves the $\mathsf{U}(1) \times \mathsf{U}(1)_R$ symmetry of the original superpotential with charge assignments $\Phi : (1,1)$, $m : (-2,0)$, $\lambda : (-3,-1)$, $\mathrm{d}^2\theta : (0,-2)$ and, additionally, $C^{\alpha i,\beta j} : (0,2)$, $|C| \sim C^{\alpha i,\beta j}C_{\alpha i,\beta j} : (0,4)$.

At first glance, it seems that one can now construct more $\mathsf{U}(1) \times \mathsf{U}(1)_R$-symmetric terms in the effective superpotential due to the new coupling constant C; however, this is not true. Taking the $C \to 0$ limit, one immediately realizes that C can never appear in the denominator of any term. Furthermore, it is not possible to construct a term containing C in the nominator, which does not violate the regularity condition in at least one of the other weak-coupling limits. Altogether, we arrive at an expression similar to (VI.103)

$$\mathcal{W}_{\text{eff},\star} \;=\; \sum_i a_i\lambda^i m^{1-i}\Phi^{\star i+2} \;, \tag{VI.105}$$

and find that $\mathcal{W}_{\text{eff},\star} = \mathcal{W}_\star$.

To compare this result with the literature, first note that, in a number of papers, it has been shown that quantum field theories in four dimensions with $\mathcal{N} = \frac{1}{2}$ supersymmetry are renormalizable to all orders in perturbation theory [273]-[26]. This even remains true for generic $\mathcal{N} = \frac{1}{2}$ gauge theories with arbitrary coefficients, which do not arise as a $\star$-deformation of $\mathcal{N} = 1$ theories. However, the authors of [43, 114], considering the non-anticommutative Wess-Zumino model we discussed above, add certain terms to the action by hand, which seem to be necessary for the model to be renormalizable. This would clearly contradict our result $\mathcal{W}_{\text{eff},\star} = \mathcal{W}_\star$. We conjecture that this contradiction is merely a seeming one and that it is resolved by a resummation of all the terms in the perturbative expansion. A similar situation was encountered in [44], where it was found that one could not write certain terms of the effective superpotential using star products, as long as they were considered separately. This obstruction, however, vanished after a resummation of the complete perturbative expansion and the star product was found to be sufficient to write down the complete effective superpotential.

Clearly, the above result is stricter than the result obtained in [43], where less constraint terms in the effective superpotential were assumed. However, we should stress that it is still unclear to what extend the above assumptions on $\mathcal{W}_{\text{eff},\star}$ are really natural. This question certainly deserves further and deeper study, which we prefer to leave to future work.

Chapter VII

Twistor Geometry

The main reason we will be interested in twistor geometry is its use in describing the solutions to certain Yang-Mills equations by holomorphic vector bundles on a corresponding twistor space, which allows us to make contact with holomorphic Chern-Simons theory. We will completely ignore the gravity aspect of twistor theory.

In this chapter, we will first deal with the twistor correspondence and its underlying geometrical structure. Then we will discuss in detail the Penrose-Ward transform, which maps solutions to certain gauge field equations to certain holomorphic vector bundles over appropriate twistor spaces.

The relevant literature to this chapter consists of [217, 284, 195, 299, 118, 98, 228].

VII.1 Twistor basics

The twistor formalism was initially introduced by Penrose to give an appropriate framework for describing both general relativity and quantum theory. For this, one introduces so-called twistors, which – like the wave-function – are intrinsically complex objects but allow for enough algebraic structure to encode spacetime geometry.

VII.1.1 Motivation

§1 Idea. As mentioned above, the basic motivation of twistor theory was to find a common framework for describing both general relativity and quantum mechanics. However, twistors found a broad area of application beyond this, e.g. in differential geometry.

In capturing both relativity and quantum mechanics, twistor theory demands some modifications of both. For example, it allows for the introduction of nonlinear elements into quantum mechanics, which are in agreement with some current interpretations of the measurement process: The collapse of the wave-function contradicts the principle of unitary time evolution, and it has been proposed that this failure of unitarity is due to some overtaking nonlinear gravitational effects.

The main two ingredients of twistor theory are non-locality in spacetime and analyticity (holomorphy) in an auxiliary complex space, the *twistor space*. This auxiliary space can be thought of as the space of light rays at each point in spacetime. Given an observer in a four-dimensional spacetime at a point p, his *celestial sphere*, i.e. the image of planets, suns and galaxies he sees around him, is the backward light cone at p given by the 2-sphere

$$t = -1 \quad \text{and} \quad x^2+y^2+z^2 = 1 \ . \tag{VII.1}$$

From this, we learn that the twistor space of $\mathbb{R}^4$ is $\mathbb{R}^4 \times S^2$. On the other hand, this space will be interpreted as the complex vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over the Riemann sphere $\mathbb{C}P^1$. The prescription for switching between the twistor space and spacetime is called the *twistor* or *Klein correspondence*.

Non-locality of the fields in a physical theory is achieved by encoding the field information at a point in spacetime into holomorphic functions on twistor space. By choosing an appropriate description, one can cause the field equations to vanish on twistor space, i.e. holomorphy of a function on twistor space automatically guarantees that the corresponding field satisfies its field equations.

§2 Two-spinors. Recall our convention for switching between vector and spinor indices:

$$x^{\alpha\dot{\alpha}} = -\mathrm{i}\sigma_\mu^{\alpha\dot{\alpha}}x^\mu = -\tfrac{\mathrm{i}}{2}\begin{pmatrix} -\mathrm{i}x^0-\mathrm{i}x^3 & -\mathrm{i}x^1-x^2 \\ -\mathrm{i}x^1+x^2 & -\mathrm{i}x^0+\mathrm{i}x^3 \end{pmatrix}, \tag{VII.2}$$

where we used here the σ-matrices appropriate for signature $(-+++)$. The inverse transformation is given by $x^\mu = \frac{\mathrm{i}}{2}\,\mathrm{tr}\,(\sigma^\mu_{\alpha\dot{\alpha}}(x^{\alpha\dot{\alpha}}))$.

The norm of such a vector is easily obtained from $||x||^2 = \eta_{\mu\nu}x^\mu x^\nu = \det(x^{\alpha\dot{\alpha}}) = \frac{1}{2}x_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}}$. From this formula, we learn that

$$\eta_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\ , \tag{VII.3}$$

where $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor in two dimensions. We choose again the convention $\varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = -1$ which implies that $\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}$, see also section III.2.3, §22.

Recall now that any vector $x^\mu \sim x^{\alpha\dot{\alpha}}$ can be decomposed into four (commuting) two-spinors according to

$$x^{\alpha\dot{\alpha}} = \tilde{\lambda}^\alpha\lambda^{\dot{\alpha}} + \kappa^\alpha\tilde{\kappa}^{\dot{\alpha}}\ . \tag{VII.4}$$

If the vector x^μ is real then $\tilde{\lambda}$ and $\tilde{\kappa}$ are related to λ and κ by complex conjugation. If the real vector x^μ is a null and future-pointing vector, its norm vanishes, and one can hence drop one of the summands in (VII.4)

$$x^{\alpha\dot{\alpha}} = \kappa^\alpha\bar{\kappa}^{\dot{\alpha}} \quad \text{with} \quad \bar{\kappa}^{\dot{\alpha}} = \overline{\kappa^\alpha}\ , \tag{VII.5}$$

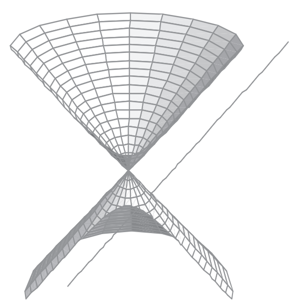
where κ^α is an $\mathsf{SL}(2,\mathbb{C})$-spinor, while $\bar{\kappa}^{\dot{\alpha}}$ is an $\overline{\mathsf{SL}(2,\mathbb{C})}$-spinor. Spinor indices are raised and lowered with the ε-tensor, i.e.

$$\kappa_\alpha = \varepsilon_{\alpha\beta}\kappa^\beta \quad \text{and} \quad \bar{\kappa}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\kappa}_{\dot{\beta}}\ . \tag{VII.6}$$

As the spinors κ^α and $\bar{\kappa}^{\dot{\alpha}}$ are commuting, i.e. they are *not* Graßmann-valued, we have

$$\kappa_\alpha\kappa^\alpha = \bar{\kappa}^{\dot{\alpha}}\bar{\kappa}_{\dot{\alpha}} = 0\ . \tag{VII.7}$$

§3 Light rays. A light ray in Minkowski space is parameterized by the equations $x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + tp^{\alpha\dot{\alpha}}$. For a general light ray, one can reparameterize this description such that $x_0^{\alpha\dot{\alpha}}$ is a null vector. For light rays which intersect the light cone of the origin more than once, i.e. light rays which lie in a null hyperplane through the origin, one chooses $x_0^{\alpha\dot{\alpha}}$ to be orthogonal to $p^{\alpha\dot{\alpha}}$ with respect to the Minkowski metric. After decomposing the vectors into spinors, we have



$$x^{\alpha\dot{\alpha}} = c\omega^\alpha\bar{\omega}^{\dot{\alpha}} + t\lambda^{\dot{\alpha}}\bar{\lambda}^\alpha \quad \text{and} \quad x^{\alpha\dot{\alpha}} = \zeta^\alpha\lambda^{\dot{\alpha}} + \bar{\zeta}^{\dot{\alpha}}\bar{\lambda}^\alpha + t\lambda^{\dot{\alpha}}\bar{\lambda}^\alpha \tag{VII.8}$$

in the general and special cases, respectively. We can reduce both cases to the single equation

$$\omega^\alpha = \mathrm{i}x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}} \tag{VII.9}$$

by assuming $c = -\mathrm{i}(\bar{\omega}^{\dot{\alpha}}\lambda_{\dot{\alpha}})^{-1}$ in the general case and $\omega^\alpha = \mathrm{i}\bar{\lambda}^\alpha(\bar{\zeta}^{\dot{\beta}}\lambda_{\dot{\beta}})$ in the special case. For all points $x^{\alpha\dot{\alpha}}$ on a light ray, equation (VII.9), the *incidence relations* hold.

§4 Twistors. A *twistor* Z^i is now defined as a pair of two-spinors $(\omega^\alpha, \lambda_{\dot\alpha})$ which transform under a translation of the origin $0 \to r^{\alpha\dot\alpha}$ as

$$(\omega^\alpha, \lambda_{\dot\alpha}) \;\to\; (\omega^\alpha - \mathrm{i}r^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \lambda_{\dot\alpha}) \ . \tag{VII.10}$$

The spinors ω^α and $\lambda_{\dot\alpha}$ are usually called the *primary* and *secondary spinor parts* of the twistor Z^i.

The twistor space $\mathbb{T}$ is thus a four-dimensional complex vector space, and we can introduce an anti-linear involution $\tau : \mathbb{T} \to \mathbb{T}^\vee$ by defining

$$Z^i \;=\; (\omega^\alpha, \lambda_{\dot\alpha}) \;\mapsto\; (\bar\lambda_\alpha, \bar\omega^{\dot\alpha}) \;=\; \bar Z_i \ . \tag{VII.11}$$

Furthermore, we can define a Hermitian inner product $h(Z,U)$ for two twistors $Z^i = (\omega^\alpha, \lambda_{\dot\alpha})$ and $U^i = (\sigma^{\dot\alpha}, \mu_\alpha)$ via

$$h(Z,U) \;=\; Z^i\bar U_i \;=\; \omega^\alpha\bar\mu_\alpha + \lambda_{\dot\alpha}\bar\sigma^{\dot\alpha} \ , \tag{VII.12}$$

which is not positive definite but of signature $(++--)$. This leads to the definition of *null-twistors*, for which $Z^i\bar Z_i = 0$. Since this constraint is a real equation, the null-twistors form a real seven-dimensional subspace $\mathbb{T}_N$ in $\mathbb{T}$. Furthermore, $\mathbb{T}_N$ splits $\mathbb{T}$ into two halves: $\mathbb{T}^\pm$ with twistors of positive and negative norm.

As a relative phase between ω^α and $\bar\lambda^\alpha$ does not affect the underlying light ray, we can assume that $\omega^\alpha\bar\lambda_\alpha$ is purely imaginary. Then the twistor underlying the light ray becomes a null twistor, since $Z^i\bar Z_i = 2\mathrm{Re}(\omega^\alpha\bar\lambda_\alpha)$.

There is a nice way of depicting twistors based on the so-called Robinson congruence. An example is printed on the back of this thesis. For more details, see appendix E.

§5 Light rays and twistors. We call a twistor *incident* to a spacetime point $x^{\alpha\dot\alpha}$ if its spinor parts satisfy the incidence relations (VII.9).

After we restrict $x^{\alpha\dot\alpha}$ to be real, there is clearly a null twistor Z^i incident to all points $x^{\alpha\dot\alpha}$ on a particular light ray, which is unique up to complex rescaling. On the other hand, every null twistor with non-vanishing secondary spinor (λ-) part corresponds to a light ray. The remaining twistors correspond to light rays through infinity and can be interpreted by switching to the conformal compactification of Minkowski space. This is easily seen by considering the incidence relations (VII.9), where – roughly speaking – a vanishing secondary spinor part implies infinite values for $x^{\alpha\dot\alpha}$ if the primary spinor part is finite.

Since the overall scale of the twistor is redundant, we rather switch to the *projective twistor space* $\mathbb{PT} = \mathbb{C}P^3$. Furthermore, non-vanishing of the secondary spinor part implies that we take out a sphere and arrive at the space $\mathcal{P}^3 = \mathbb{C}P^3\backslash\mathbb{C}P^1$. The restriction of $\mathcal{P}^3$ to null-twistors will be denoted by $\mathcal{P}^3_N$. We can now state that there is a one-to-one correspondence between light rays in (complexified) Minkowski space M (M^c) and elements of $\mathcal{P}^3_N$ ($\mathcal{P}^3$).

§6 "Evaporation" of equations of motion. Let us make a simple observation which will prove helpful when discussing solutions to field equations in later sections. Consider a massless particle with vanishing helicity. Its motion is completely described by a four momentum $p^{\alpha\dot\alpha}$ and an initial starting point $x^{\alpha\dot\alpha}$. To each such motion, there is a unique twistor Z^i. Thus, while one needs to solve equations to determine the motion of a particle in its phase space, this is not so in twistor space: Here, the equations of motion have[1] "evaporated" into the structure of the twistor space. We will encounter a similar phenomenon later, when discussing the Penrose and the Penrose-Ward transforms, which encode solutions to certain field equations in holomorphic functions and holomorphic vector bundles on the twistor space.

§7 Quantization. The canonical commutation relations on Minkowski spacetime (the *Heisenberg algebra*) read

$$[\hat{p}_\mu, \hat{p}_\nu] \;=\; [\hat{x}^\mu, \hat{x}^\nu] \;=\; 0 \quad \text{and} \quad [\hat{x}^\nu, \hat{p}_\mu] \;=\; \mathrm{i}\hbar\delta^\nu_\mu \;, \tag{VII.13}$$

and induce canonical commutation relations for twistors:

$$[\hat{Z}^i, \hat{Z}^j] \;=\; [\hat{\bar{Z}}_i, \hat{\bar{Z}}_j] \;=\; 0 \quad \text{and} \quad [\hat{Z}^i, \hat{\bar{Z}}_j] \;=\; \hbar\delta^i_j \;. \tag{VII.14}$$

Alternatively, one can also follow the ordinary canonical quantization prescription

$$[\hat{f}, \hat{g}] \;=\; \mathrm{i}\hbar\widehat{\{f,g\}} + \mathcal{O}(\hbar^2) \tag{VII.15}$$

for the twistor variables and neglecting terms beyond linear order in $\hbar$, which yields the same result.

A representation of this algebra is easily found on the algebra of functions on twistor space by identifying $\hat{Z}^i = Z^i$ and $\hat{\bar{Z}}_i = -\hbar\frac{\partial}{\partial Z^i}$. This description is not quite equivalent to the Bargmann representation, as there one introduces complex coordinates on the phase space, while here, the underlying space is genuinely complex.

The helicity $2s = Z^i\bar{Z}_i$ is augmented to an operator, which reads in symmetrized form

$$\hat{s} \;=\; \tfrac{1}{4}(\hat{Z}^i\hat{\bar{Z}}_i + \hat{\bar{Z}}_i\hat{Z}^i) \;=\; -\frac{\hbar}{2}\left(Z^i\frac{\partial}{\partial Z^i} + 2\right) \;, \tag{VII.16}$$

and it becomes clear that an eigenstate of the helicity operator with eigenvalue $s\hbar$ must be a homogeneous twistor function $f(Z^i)$ of degree $-2s-2$. One might wonder, why this description is asymmetric in the helicity, i.e. why e.g. eigenstates of helicity ± 2 are described by homogeneous twistor functions of degree -6 and $+2$, respectively. This is due to the inherent chirality of the twistor space. By switching to the dual twistor space $\mathbb{P}\mathbb{T}^\vee$, one arrives at a description in terms of homogeneous twistor functions of degree $2s+2$.

VII.1.2 Klein (twistor-) correspondence

Interestingly, there is some work by Felix Klein [155, 154] dating back to as early as 1870, in which he discusses correspondences between points and subspaces of both $\mathbb{P}\mathbb{T}$ and the compactification of M^c. In the following section, this *Klein correspondence* or *twistor correspondence* will be developed.

§8 The correspondence $r \in M^c \leftrightarrow \mathbb{C}P^1 \subset \mathbb{P}\mathbb{T}$. The (projective) twistors satisfying the incidence relations (VII.9) for a given fixed point $r^{\alpha\dot{\alpha}} \in M^c$ are of the form $Z^i = (\mathrm{i}r^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}, \lambda_{\dot{\alpha}})$ up to a scale. Thus, the freedom we have is a projective two-spinor $\lambda_{\dot{\alpha}}$ (i.e. in particular, at most one component of $\lambda_{\dot{\alpha}}$ can vanish) and the components of this spinor are the homogeneous coordinates of the space $\mathbb{C}P^1$. This is consistent with our previous observations, as twistors incident to a certain point $r \in M^c$ describe all light rays through r, and this space is the celestial sphere $S^2 \cong \mathbb{C}P^1$ at this point. Thus, the point r in M corresponds to a projective line $\mathbb{C}P^1$ in $\mathbb{P}\mathbb{T}$.

[1] This terminology is due to [98].

§9 The correspondence $p \in \mathbb{PT} \leftrightarrow \alpha$-plane in M^c. Given a twistor Z^i in $\mathbb{PT}$, the points x which are incident with Z^i form a two-dimensional subspace of M^c, a so-called *α-plane*. These planes are completely null, which means that two arbitrary points on such a plane are always separated by a null distance. More explicitly, an α-plane corresponding to a twistor $Z^i = (\omega^\alpha, \lambda_{\dot\alpha})$ is given by

$$x^{\alpha\dot\alpha} \;=\; x_0^{\alpha\dot\alpha} + \mu^\alpha \lambda^{\dot\alpha} \;, \tag{VII.17}$$

where $x_0^{\alpha\dot\alpha}$ is an arbitrary solution to the incidence relations (VII.9) for the fixed twistor Z^i. The two-spinor μ^α then parameterizes the α-plane.

§10 Dual twistors. The incidence relations for a *dual twistor* $W_i = (\sigma_{\dot\alpha}, \mu^\alpha)$ read as

$$\sigma^{\dot\alpha} \;=\; -\mathrm{i} x^{\alpha\dot\alpha}\mu_\alpha \;. \tag{VII.18}$$

Such a twistor again corresponds to a two-dimensional totally null subspace of M^c, the so called *β-planes*. The parameterization is exactly the same as the one given in (VII.17). Note, however, that the rôle of $\lambda_{\dot\alpha}$ and μ^α have been exchanged, i.e. in this case, $\lambda^{\dot\alpha}$ parameterizes the β-plane.

§11 Totally null hyperplanes. Two α- or two β-planes either coincide or intersect in a single point. An α- and a β-plane are either disjoint or intersect in a line, which is null. The latter observation will be used in the definition of ambitwistor spaces in section VII.3.3. Furthermore, the correspondence between points in $\mathbb{PT}$ and planes in M^c breaks down in the real case.

VII.1.3 Penrose transform

The Penrose transform gives contour integral formulæ for mapping certain functions on the twistor space to solutions of the massless field equations for particles with arbitrary helicity. For our considerations, we can restrict ourselves to the subspace of $\mathbb{C}P^3$, for which $\lambda_{\dot 1} \neq 0$ and switch for simplicity to the inhomogeneous coordinates $\lambda_{\dot\alpha} := (1, \lambda)^T$.

§12 Elementary states. A useful class of functions on twistor space are the so-called *elementary states*. Let us again denote a twistor by $(Z^i) = (\omega^\alpha, \lambda_{\dot\alpha})$. Then an elementary state is given by

$$f(Z) \;=\; \frac{(C_i Z^i)^c (D_i Z^i)^d}{(A_i Z^i)^{a+1}(B_i Z^i)^{b+1}} \;, \tag{VII.19}$$

where A_i, B_i, C_i, D_i are linearly independent and $a, b, c, d \in \mathbb{N}$. The Penrose transform will relate such an elementary state to a field with helicity

$$h = \tfrac{1}{2}(a + b - c - d) \;, \tag{VII.20}$$

satisfying its massless equations of motion.

§13 Negative helicity. Consider the contour integral

$$\phi_{\dot\alpha_1 \dots \dot\alpha_{2h}}(x) \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \mathrm{d}\lambda^{\dot\alpha}\; \lambda_{\dot\alpha}\lambda_{\dot\alpha_1} \dots \lambda_{\dot\alpha_{2h}} f(Z) \;, \tag{VII.21}$$

where $\mathcal{C} \cong S^1$ is a contour in $\mathbb{C}P^1$, which is the equator $|\lambda_{\dot 1}/\lambda_{\dot 2}| = 1$ or a suitable deformation thereof, if f should become singulary on $\mathcal{C}$. The contour integral can only yield a finite result if the integrand is of homogeneity zero and thus f has to be a section of $\mathcal{O}(-2h-2)$. For these f, ϕ is a possibly non-trivial solution to the massless field equations

$$\partial^{\alpha\dot\alpha_1}\phi_{\dot\alpha_1\dots\dot\alpha_{2h}} \;=\; 0 \tag{VII.22}$$

for a field of helicity $-h$. This can be easily seen by substituting the incidence relations (VII.9) into the primary spinor part of the twistor $Z = (\mathrm{i}x^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \lambda_{\dot\alpha})$ and taking the derivative into the integral.

§14 Zero helicity. For the case of zero helicity, we can generalize the above formalism. We employ the same contour integral as in (VII.21), restricted to $h = 0$:

$$\phi(x) \;=\; \tfrac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}} \mathrm{d}\lambda^{\dot\alpha}\;\lambda_{\dot\alpha} f(Z)\;, \tag{VII.23}$$

where ϕ is now a solution to the scalar field equation

$$\Box\phi \;=\; \tfrac{1}{2}\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi \;=\; 0\;, \tag{VII.24}$$

the massless Klein-Gordon equation. Again, this fact is readily seen by pulling the derivatives into the integral and for non-vanishing ϕ, f is a section of $\mathcal{O}(-2)$.

§15 Positive helicity. For positive helicities, we have to adapt our contour integral in the following way:

$$\phi_{\alpha_1\dots\alpha_{2h}}(x) \;=\; \tfrac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}} \mathrm{d}\lambda^{\dot\alpha}\;\lambda_{\dot\alpha}\frac{\partial}{\partial\omega^{\alpha_1}}\dots\frac{\partial}{\partial\omega^{\alpha_{2h}}} f(Z)\;, \tag{VII.25}$$

which will give rise to the massless equations of motion for helicity h

$$\varepsilon^{\alpha\alpha_1}\partial_{\alpha\dot\alpha}\phi_{\alpha_1\dots\alpha_{2h}} \;=\; 0 \tag{VII.26}$$

and for nontrivial ϕ, f must be a section of $\mathcal{O}(2h-2)$. This result requires slightly more effort to be verified, but the calculation is nevertheless straightforward.

§16 Further remarks. Altogether, we saw how to construct solutions to massless field equations for fields with helicity h using functions on twistor space, which transform as sections of $\mathcal{O}(2h-2)$. One can prove that the (Abelian) Čech cohomology group $H^1(\mathcal{O}(1)\oplus\mathcal{O}(1),\mathcal{O}(2h-2))$ is isomorphic to the sheaf of solutions to the massless equations of motion for particles with helicity h. Note that our convention for h differs to another very common one used e.g. in [284] by a sign.

Furthermore, this construction is reminiscent of the "evaporation of equations of motion" for particles in the twistor approach and we will come across a generalization of this construction to Lie algebra valued fields in section VII.8.

VII.2 Integrability

The final goal of this chapter is to construct the Penrose-Ward transform for various field theories, which relates classical solutions to some equations of motion to geometric data on a twistor space. This transform is on the one hand founded on the equivalence of the Čech and Dolbeault descriptions of (topologically trivial) holomorphic vector bundles, on the other hand, its explicit construction needs the notion of linear systems and the corresponding compatibility conditions. Therefore, let us briefly comment on the property called (classical) *integrability*, which is the framework for these entities.

VII.2.1 The notion of integrability

§1 Basic Idea. Although there seems to be no general definition of integrability, quite generally speaking one can state that an integrable set of equations is an exactly solvable set of equations. Further strong hints that a set of equations may be integrable are the existence of many conserved quantities and a description in terms of algebraic geometry. Until today, there is no comprehensive algorithm to test integrability.

§2 Example. To illustrate the above remarks, let us briefly discuss an example given e.g. in [127]: The center-of-mass motion of a rigid body. Let ω be the angular velocity and I_i the principal moments of inertia. The equations of motion then take the form[2]

$$I_i\dot{\omega}_i \;=\; \varepsilon_{ijk}(I_j - I_k)\omega_j\omega_k \;. \tag{VII.27}$$

One can rescale these equations to the simpler form

$$\dot{u}_1 \;=\; u_2u_3\;,\quad \dot{u}_2 \;=\; u_3u_1 \quad\text{and}\quad \dot{u}_3 \;=\; u_1u_2\;. \tag{VII.28}$$

Let us now trace the above mentioned properties of integrable systems of equations. First, we have the conserved quantities

$$A \;=\; u_1^2 - u_2^2 \quad\text{and}\quad B \;=\; u_1^2 - u_3^2\;, \tag{VII.29}$$

since $\dot{A} = \dot{B} = 0$ by virtue of the equations of motion. Second, we find an elliptic curve by putting $y = \dot{u}_1$ and $x = u_1$ and substituting the conserved quantities in the first equation of motion $\dot{u}_1 = u_2u_3$:

$$y^2 \;=\; (x^2 - A)(x^2 - B)\;. \tag{VII.30}$$

Thus, algebraic geometry is indeed present in our example. Eventually, one can give explicit solutions, since the above algebraic equation can be recast into the standard form

$$y^2 \;=\; 4x^3 - g_2x - g_3\;, \tag{VII.31}$$

which is solved by the Weierstrass $\wp$-function with $x = \wp(u)$ and $y = \wp'(u)$. The solution is then given by $\mathrm{d}t = \mathrm{d}\wp/\wp'$.

§3 Ward conjecture. This conjecture by Richard Ward [283] states that all the integrable equations in 1+1 and 2+1 dimensions can be obtained from (anti-)self-dual Yang-Mills theory in four dimensions by dimensional reduction. On commutative spaces, this conjecture can be regarded as confirmed [195].

VII.2.2 Integrability of linear systems

§4 A simple example. In our subsequent discussion, we will have to deal with a linear system of equations, which states that some $\mathsf{GL}(n,\mathbb{C})$-valued function ψ is covariantly constant in several directions, e.g.

$$\nabla_1\psi \;=\; 0 \quad\text{and}\quad \nabla_2\psi \;=\; 0\;. \tag{VII.32}$$

Since this system is overdetermined, there can only exist a solution if a certain condition obtained by cross-differentiating is fulfilled:

$$\nabla_1\nabla_2\psi - \nabla_2\nabla_1\psi \;=:\; F_{12}\psi \;=\; 0\;. \tag{VII.33}$$

[2]These equations are the equations for a spinning top and they are related to the Nahm equations cf. section VII.8.4.

Now a sufficient and necessary condition for this to hold is $F_{12} = 0$ as ψ is invertible. This condition is called the compatibility condition of the linear system (VII.32). However, this system is too trivial to be interesting. The solutions of $F_{12} = 0$ are pure gauge and thus essentially trivial.

§5 Self-Dual Yang-Mills theory. The obvious step to make the compatibility conditions non-trivial is to introduce a so-called *spectral parameter* $\lambda \in \mathbb{C}$ and consider the linear system

$$(\nabla_1 - \lambda\nabla_3)\psi \;=\; 0 \quad \text{and} \quad (\nabla_2 + \lambda\nabla_4)\psi \;=\; 0\ . \tag{VII.34}$$

This linear system has a non-trivial solution if and only if

$$[\nabla_1 - \lambda\nabla_3, \nabla_2 + \lambda\nabla_4] \;=\; 0\ , \tag{VII.35}$$

or, written in terms of components of a Taylor expansion in the spectral parameter if

$$F_{12} \;=\; F_{14} - F_{32} \;=\; F_{34} \;=\; 0\ . \tag{VII.36}$$

The latter equations are, in a suitable basis, the self-dual Yang-Mills equations, cf. section IV.2.3, and (VII.34) is exactly the linear system we will encounter later on in the twistor formulation. Note that the system (VII.36) is underdetermined (three equations for four components) due to gauge invariance.

§6 Further examples. All the constraint equations we encountered so far, i.e. the $\mathcal{N} = 1,\ldots,4$ supersymmetrically extended self-dual Yang-Mills equations (IV.64) and also the full $\mathcal{N} = 3, 4$ super Yang-Mills theory (IV.50) can be obtained from linear systems. After introducing the simplifying spinorial notation $\lambda^{\dot\alpha} = (\lambda, -1)^T$ and $\mu^\alpha = (\mu, -1)^T$, they read as

$$\begin{aligned} \lambda^{\dot\alpha}(\partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha})\psi \;&=\; 0\ , \\ \lambda^{\dot\alpha}\left(\frac{\partial}{\partial\eta_i^{\dot\alpha}} + \mathcal{A}^i_{\dot\alpha}\right)\psi \;&=\; 0 \end{aligned} \tag{VII.37}$$

for the supersymmetric self-dual Yang-Mills equations and

$$\begin{aligned} \mu^\alpha\lambda^{\dot\alpha}(\partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha})\psi \;&=\; 0\ , \\ \lambda^{\dot\alpha}(D^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha})\psi \;&=\; 0\ , \\ \mu^\alpha(D_{\alpha i} + \mathcal{A}_{\alpha i})\psi \;&=\; 0\ . \end{aligned} \tag{VII.38}$$

in the case of the $\mathcal{N} = 3, 4$ super Yang-Mills equations. Here, i runs from 1 to $\mathcal{N}$. Note that in the latter case, really all the conditions obtained from cross-differentiating are fulfilled if the constraint equations (IV.50) hold.

VII.3 Twistor spaces and the Penrose-Ward transform

After this brief review of the basic ideas in twistor geometry and integrable systems let us now be more explicit in the spaces and and the conventions we will use. We will introduce several twistor correspondences between certain superspaces and modified and extended twistor spaces upon which we will construct Penrose-Ward transforms, relating solutions to geometric data encoded in holomorphic bundles.

VII.3.1 The twistor space

§1 Definition. We start from the complex projective space $\mathbb{C}P^3$ (the *twistor space*) with homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$ subject to the equivalence relation $(\omega^\alpha, \lambda_{\dot\alpha}) \sim (t\omega^\alpha, t\lambda_{\dot\alpha})$ for all $t \in \mathbb{C}^\times$, $\alpha = 1, 2$ and $\dot\alpha = \dot 1, \dot 2$. As we saw above, this is the space of light rays in complexified, compactified Minkowski space. To neglect the light cone at infinity, we demand that $\lambda_{\dot\alpha}$ parameterizes a $\mathbb{C}P^1$, i.e. $\lambda_{\dot\alpha} \neq (0,0)^T$. Recall that the set which we exclude by this condition is the Riemann sphere $\mathbb{C}P^1$ and the resulting space $\mathbb{C}P^3 \backslash \mathbb{C}P^1$ will be denoted by $\mathcal{P}^3$. This space can be covered by two patches $\mathcal{U}_+$ ($\lambda_{\dot 1} \neq 0$) and $\mathcal{U}_-$ ($\lambda_{\dot 2} \neq 0$) with coordinates[3]

$$\begin{aligned} z_+^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 1}} , \quad z_+^3 = \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}} =: \lambda_+ \quad \text{on } \mathcal{U}_+ , \\ z_-^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 2}} , \quad z_-^3 = \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}} =: \lambda_- \quad \text{on } \mathcal{U}_- , \end{aligned} \tag{VII.39}$$

related by

$$z_+^\alpha = z_+^3 z_-^\alpha \quad \text{and} \quad z_+^3 = \frac{1}{z_-^3} \tag{VII.40}$$

on the overlap $\mathcal{U}_+ \cap \mathcal{U}_-$. Due to (VII.40), the space $\mathcal{P}^3$ coincides with the total space of the rank two holomorphic vector bundle

$$\mathcal{P}^3 = \mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{C}P^1 , \tag{VII.41}$$

where the base manifold $\mathbb{C}P^1$ is covered by the two patches $U_\pm := \mathcal{U}_\pm \cap \mathbb{C}P^1$.

§2 Moduli space of sections. Holomorphic sections of the complex vector bundle (VII.41) are rational curves $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ defined by the equations

$$z_+^\alpha = x^{\alpha\dot 1} + \lambda_+ x^{\alpha\dot 2} \quad \text{for } \lambda_+ \in U_+ \quad \text{and} \quad z_-^\alpha = \lambda_- x^{\alpha\dot 1} + x^{\alpha\dot 2} \quad \text{for } \lambda_- \in U_- \tag{VII.42}$$

and parameterized by moduli $x = (x^{\alpha\dot\alpha}) \in \mathbb{C}^4$. After introducing the spinorial notation

$$(\lambda^+_{\dot\alpha}) := \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda^-_{\dot\alpha}) := \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} , \tag{VII.43}$$

we can rewrite (VII.42) as the *incidence relations*

$$z_\pm^\alpha = x^{\alpha\dot\alpha} \lambda^\pm_{\dot\alpha} . \tag{VII.44}$$

Note the familiarity of these equations with the incidence relations (VII.9). The meaning of (VII.44) becomes most evident, when writing down a double fibration:

$$\begin{array}{ccc} & \mathcal{F}^5 & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{P}^3 & & \mathbb{C}^4 \end{array} \tag{VII.45}$$

where $\mathcal{F}^5 := \mathbb{C}^4 \times \mathbb{C}P^1$, π_1 is the trivial projection $\pi_1(x^{\alpha\dot\alpha}, \lambda_{\dot\alpha}) = x^{\alpha\dot\alpha}$ and π_2 is given by (VII.44). We thus obtain a twistor correspondence between points and subspaces of either spaces $\mathbb{C}^4$ and $\mathcal{P}^3$:

$$\begin{aligned} \big\{ \text{projective lines } \mathbb{C}P^1_x \text{ in } \mathcal{P}^3 \big\} &\longleftrightarrow \big\{ \text{points } x \text{ in } \mathbb{C}^4 \big\} , \\ \big\{ \text{points } p \text{ in } \mathcal{P}^3 \big\} &\longleftrightarrow \big\{ \text{null } (\alpha\text{-})\text{planes } \mathbb{C}^2_p \text{ in } \mathbb{C}^4 \big\} . \end{aligned} \tag{VII.46}$$

[3] A more extensive discussion of the relation between these inhomogeneous coordinates and the homogeneous coordinates is found in appendix C.

While the first correspondence is rather evident, the second one deserves a brief remark. Suppose $\hat{x}^{\alpha\dot{\alpha}}$ is a solution to the incidence relations (VII.44) for a fixed point $p \in \mathcal{P}^3$. Then the set of all solutions is given by

$$\{x^{\alpha\dot{\alpha}}\} \quad \text{with} \quad x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}} + \mu^\alpha \lambda_\pm^{\dot{\alpha}} , \tag{VII.47}$$

where μ^α is an arbitrary commuting two-spinor and we use our standard convention of $\lambda_\pm^{\dot{\alpha}} := \varepsilon^{\dot{\alpha}\dot{\beta}}\lambda^\pm_{\dot{\beta}}$ with $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}} = 1$. One can choose to work on any patch containing p. The sets defined in (VII.47) are then called *null* or *α-planes.*

The correspondence space $\mathcal{F}^5$ is a complex five-dimensional manifold, which is covered by the two patches $\tilde{\mathcal{U}}_\pm = \pi_2^{-1}(\mathcal{U}_\pm)$.

§3 Vector fields. On the complex manifold $\mathcal{P}^3$, there is the natural basis $(\partial/\partial\bar{z}^\alpha_\pm, \partial/\partial\bar{z}^3_\pm)$ of antiholomorphic vector fields, which are related via

$$\frac{\partial}{\partial\bar{z}^\alpha_+} = \bar{z}^3_-\frac{\partial}{\partial\bar{z}^\alpha_-} \quad \text{and} \quad \frac{\partial}{\partial\bar{z}^3_+} = -(\bar{z}^3_-)^2\frac{\partial}{\partial\bar{z}^3_-} - \bar{z}^3_-\bar{z}^\alpha_-\frac{\partial}{\partial\bar{z}^\alpha_-} . \tag{VII.48}$$

on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$. The leaves to the fibration π_2 in (VII.45) are spanned by the vector fields

$$\bar{V}^\pm_\alpha = \lambda_\pm^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} , \tag{VII.49}$$

which obviously annihilate the coordinates (VII.44) on $\mathcal{P}^3$. The tangent spaces to the leaves of the fibration π_2 are evidently of dimension 2.

§4 Real structure. Recall that a real structure on a complex manifold M is defined as an antiholomorphic involution $\tau : M \to M$. On the twistor space $\mathcal{P}^3$, we can introduce three anti-linear transformations of commuting two-spinors:

$$(\omega^\alpha) \mapsto \tau_\varepsilon(\omega^\alpha) = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \varepsilon\bar{\omega}^2 \\ \bar{\omega}^1 \end{pmatrix} =: (\hat{\omega}^\alpha) , \tag{VII.50a}$$

$$\tau_0(\omega^\alpha) = (\bar{\omega}^\alpha) , \tag{VII.50b}$$

where $\varepsilon = \pm 1$. In particular, this definition implies $(\hat{\omega}_\alpha) := \tau(\omega_\alpha)$ and $(\hat{\lambda}^{\dot{\alpha}}) := \tau(\lambda^{\dot{\alpha}})$, i.e. indices are raised and lowered before τ is applied. For later reference, let us give explicitly all the possible variants of the two-spinor $\lambda^\pm_{\dot{\alpha}}$ given in (VII.43):

$$\begin{aligned} (\lambda_+^{\dot{\alpha}}) &:= \begin{pmatrix} \lambda_+ \\ -1 \end{pmatrix} , \quad (\hat{\lambda}^+_{\dot{\alpha}}) := \begin{pmatrix} \varepsilon\bar{\lambda}_+ \\ 1 \end{pmatrix} , \quad (\hat{\lambda}_+^{\dot{\alpha}}) := \begin{pmatrix} -\varepsilon \\ \bar{\lambda}_+ \end{pmatrix} , \\ (\lambda_-^{\dot{\alpha}}) &:= \begin{pmatrix} 1 \\ -\lambda_- \end{pmatrix} , \quad (\hat{\lambda}^-_{\dot{\alpha}}) := \begin{pmatrix} \varepsilon \\ \bar{\lambda}_- \end{pmatrix} , \quad (\hat{\lambda}_-^{\dot{\alpha}}) := \begin{pmatrix} -\varepsilon\bar{\lambda}_- \\ 1 \end{pmatrix} . \end{aligned} \tag{VII.51}$$

Furthermore, the transformations (VII.50a)-(VII.50b) define *three* real structures on $\mathcal{P}^3$ which in the coordinates (VII.39) are given by the formulæ

$$\tau_\varepsilon(z^1_+, z^2_+, z^3_+) = \left(\frac{\bar{z}^2_+}{\bar{z}^3_+}, \frac{\varepsilon\bar{z}^1_+}{\bar{z}^3_+}, \frac{\varepsilon}{\bar{z}^3_+}\right) , \quad \tau_\varepsilon(z^1_-, z^2_-, z^3_-) = \left(\frac{\varepsilon\bar{z}^2_-}{\bar{z}^3_-}, \frac{\bar{z}^1_-}{\bar{z}^3_-}, \frac{\varepsilon}{\bar{z}^3_-}\right) , \tag{VII.52a}$$

$$\tau_0(z^1_\pm, z^2_\pm, z^3_\pm) = (\bar{z}^1_\pm, \bar{z}^2_\pm, \bar{z}^3_\pm) . \tag{VII.52b}$$

§5 The dual twistor space. For the dual twistor space, one starts again from the complex projective space $\mathbb{C}P^3$, this time parameterized by the two-spinors $(\sigma^{\dot\alpha},\mu_\alpha)$. By demanding that $\mu_\alpha \neq (0,0)^T$, we again get a rank two holomorphic vector bundle over the Riemann sphere:

$$\mathcal{P}^3_* \;=\; \mathcal{O}(1)\oplus\mathcal{O}(1) \;\to\; \mathbb{C}P^1_* \;. \tag{VII.53}$$

One should stress that the word "dual" refers to the transformation property of the spinors and *not* to the dual line bundles $\mathcal{O}(-1)$ of the holomorphic line bundles $\mathcal{O}(1)$ contained in the twistor space. This is why we denote these spaces with a $*$ instead of a $\vee$. The dual twistor space $\mathcal{P}^3_*$ is covered again by two patches $\mathcal{U}^*_\pm$ on which we have the inhomogeneous coordinates

$$(u^{\dot\alpha}_\pm,\mu_\pm) \quad\text{with}\quad u^{\dot\alpha}_+ \;=\; \mu_+u^{\dot\alpha}_- \quad\text{and}\quad \mu_+ \;=\; \frac{1}{\mu_-} \;. \tag{VII.54}$$

Sections of the bundle (VII.53) are therefore parameterized according to

$$u^{\dot\alpha}_\pm \;=\; x^{\alpha\dot\alpha}\mu^\pm_\alpha \quad\text{with}\quad \left(\mu^+_\alpha\right) := \begin{pmatrix}1\\ \mu_+\end{pmatrix} \quad\text{and}\quad \left(\mu^-_\alpha\right) \;:=\; \begin{pmatrix}\mu_-\\ 1\end{pmatrix} \;, \tag{VII.55}$$

and one has again a double fibration analogously to (VII.45). The null planes in $\mathbb{C}^4$ corresponding to points in the dual twistor space via the above incidence relations (VII.55) are now called *β-planes.*

Note that in most of the literature on twistor spaces, our dual twistor space is called twistor space and vice versa. This is related to focusing on anti-self-dual Yang-Mills theory instead of the self-dual one, as we do.

§6 Real twistor space. It is obvious that the involution τ_{-1} has no fixed points but does leave invariant projective lines joining p and $\tau_{-1}(p)$ for any $p\in\mathcal{P}^3$. On the other hand, the involutions τ_1 and τ_0 have fixed points which form a three-dimensional real manifold

$$\mathcal{T}^3 \;=\; \mathbb{R}P^3\backslash\mathbb{R}P^1 \tag{VII.56}$$

fibred over $S^1\cong\mathbb{R}P^1\subset\mathbb{C}P^1$. The space $\mathcal{T}^3\subset\mathcal{P}^3$ is called *real twistor space.* For the real structure τ_1, this space is described by the coordinates $(z^1_\pm,\, \mathrm{e}^{\mathrm{i}\chi}\,\bar z^1_\pm,\, \mathrm{e}^{\mathrm{i}\chi})$ with $0\le\chi<2\pi$, and for the real structure τ_0, the coordinates $(z^1_\pm,\, z^2_\pm,\, \lambda_\pm)$ are real. These two descriptions are equivalent. In the following we shall concentrate mostly on the real structures $\tau_{\pm1}$ since they give rise to unified formulæ.

§7 Metric on the moduli space of real curves. The real structures introduced above naturally induce real structures on the moduli space of curves, $\mathbb{C}^4$:

$$\begin{aligned}\tau_\varepsilon(x^{1\dot1}) \;=\; \bar x^{2\dot2}\;,\quad \tau_\varepsilon(x^{1\dot2}) \;=\; \varepsilon\bar x^{2\dot1} \quad\text{with}\quad \varepsilon \;=\; \pm1\;,\\ \tau_0(x^{\alpha\dot\alpha}) \;=\; \bar x^{\alpha\dot\alpha}\;.\end{aligned} \tag{VII.57}$$

Demanding that $\tau(x^{\alpha\dot\alpha}) = x^{\alpha\dot\alpha}$ restricts the moduli space $\mathbb{C}^4$ to $\mathbb{C}^2\cong\mathbb{R}^4$, and we can extract four real coordinates via

$$\bar x^{2\dot2} \;=\; x^{1\dot1} \;=:\; -(\varepsilon x^4+\mathrm{i}x^3) \quad\text{and}\quad x^{2\dot1} \;=\; \varepsilon\bar x^{1\dot2} \;=:\; -\varepsilon(x^2-\mathrm{i}x^1)\;. \tag{VII.58}$$

Furthermore, the real moduli space comes naturally with the metric given by

$$\mathrm{d}s^2 \;=\; \det(\mathrm{d}x^{\alpha\dot\alpha}) \;=\; g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{VII.59}$$

with $g = (g_{\mu\nu}) = \mathrm{diag}(+1,+1,+1,+1)$ for the involution τ_{-1} on $\mathcal{P}^3$ and $g = \mathrm{diag}(-1,-1,+1,+1)$ for τ_1 (and τ_0). Thus, the moduli space of real rational curves of degree one in $\mathcal{P}^3$ is the Euclidean space[4] $\mathbb{R}^{4,0}$ or the Kleinian space $\mathbb{R}^{2,2}$. It is not possible to introduce a Minkowski metric on the moduli space of real sections of the twistor space $\mathcal{P}^3$. However, this will change when we consider the ambitwistor space in section VII.3.3.

§8 Diffeomorphisms in the real case. It is important to note that the diagram

$$\begin{array}{ccc} & \mathbb{R}^4\times\mathbb{C}P^1 & \\ {\scriptstyle\pi_2}\swarrow & & \searrow{\scriptstyle\pi_1} \\ \mathcal{P}^3_\varepsilon & & \mathbb{R}^4 \end{array} \qquad \text{(VII.60)}$$

describes quite different situations in the Euclidean ($\varepsilon=-1$) and the Kleinian ($\varepsilon=+1$) case. For $\varepsilon=-1$, the map π_2 is a diffeomorphism,

$$\mathcal{P}^3_{-1} \;\cong\; \mathbb{R}^{4,0}\times\mathbb{C}P^1\ , \qquad \text{(VII.61)}$$

and the double fibration (VII.60) is simplified to the non-holomorphic fibration

$$\mathcal{P}^3_{-1} \;\to\; \mathbb{R}^{4,0} \qquad \text{(VII.62)}$$

where 3 stands for complex and 4 for real dimensions. More explicitly, this diffeomorphism reads

$$\begin{aligned} x^{1\dot 1} &= \frac{z^1_+ + z^3_+\bar z^2_+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_- z^1_- + \bar z^2_-}{1+z^3_-\bar z^3_-}\ ,\quad x^{1\dot 2} = -\frac{\bar z^2_+ - \bar z^3_+ z^1_+}{1+z^3_+\bar z^3_+} = -\frac{z^3_-\bar z^2_- - z^1_-}{1+z^3_-\bar z^3_-}\ ,\\ \lambda_\pm &= z^3_\pm\ , \end{aligned} \qquad \text{(VII.63)}$$

and the patches $\mathcal{U}_\pm$ are diffeomorphic to the patches $\tilde{\mathcal{U}}_\pm$. Correspondingly, we can choose either coordinates $(z^\alpha_\pm, z^3_\pm := \lambda_\pm)$ or $(x^{\alpha\dot\alpha},\lambda_\pm)$ on $\mathcal{P}^3_{-1}$ and consider this space as a complex 3-dimensional or real 6-dimensional manifold. Note, however, that the spaces $\mathcal{P}^3_{-1}$ and $\mathbb{R}^{4,0}\times\mathbb{C}P^1$ are not biholomorphic.

In the case of Kleinian signature $(++--)$, we have a local isomorphisms

$$\mathsf{SO}(2,2)\;\simeq\;\mathsf{Spin}(2,2)\;\simeq\;\mathsf{SL}(2,\mathbb{R})\times\mathsf{SL}(2,\mathbb{R})\;\simeq\;\mathsf{SU}(1,1)\times\mathsf{SU}(1,1)\ , \qquad \text{(VII.64)}$$

and under the action of the group $\mathsf{SU}(1,1)$, the Riemann sphere $\mathbb{C}P^1$ of projective spinors decomposes into the disjoint union $\mathbb{C}P^1 = H^2_+\cup S^1\cup H^2_- = H^2\cup S^1$ of three orbits. Here, $H^2 = H^2_+\cup H^2_-$ is the two-sheeted hyperboloid and $H^2_\pm = \{\lambda_\pm\in U_\pm \,|\, |\lambda_\pm|<1\}\cong \mathsf{SU}(1,1)/\mathsf{U}(1)$ are open discs. This induces a decomposition of the correspondence space into

$$\mathbb{R}^4\times\mathbb{C}P^1 \;=\; \mathbb{R}^4\times H^2_+\cup\mathbb{R}^4\times S^1\cup\mathbb{R}^4\times H^2_- \;=\; \mathbb{R}^4\times H^2\cup\mathbb{R}^4\times S^1 \qquad \text{(VII.65)}$$

as well as a decomposition of the twistor space

$$\mathcal{P}^3 \;=\; \mathcal{P}^3_+\cup\mathcal{P}_0\cup\mathcal{P}^3_- \;=:\; \tilde{\mathcal{P}}^3\cup\mathcal{P}_0\ , \qquad \text{(VII.66)}$$

where $\mathcal{P}^3_\pm := \mathcal{P}^3|_{H^2_\pm}$ are restrictions of the holomorphic vector bundle (VII.41) to bundles over $H^2_\pm$. The space $\mathcal{P}_0 := \mathcal{P}^3|_{S^1}$ is the real 5-dimensional common boundary of the

[4]In our notation, $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g)$ is the space $\mathbb{R}^{p+q}$ with the metric $g = \mathrm{diag}(\underbrace{-1,\dots,-1}_{q},\underbrace{+1,\dots,+1}_{p})$.

spaces $\mathcal{P}^3_\pm$. There is a real-analytic bijection between $\mathbb{R}^4 \times H^2_\pm \cong \mathbb{C}^2 \times H^2_\pm$ and $\tilde{\mathcal{P}}^3_\pm$, which reads explicitly as

$$x^{1\dot{1}} = \frac{z^1_+ - z^3_+ \bar{z}^2_+}{1 - z^3_+ \bar{z}^3_+} = \frac{-\bar{z}^3_- z^1_- + \bar{z}^2_-}{1 - z^3_- \bar{z}^3_-} , \quad x^{1\dot{2}} = \frac{\bar{z}^2_+ - \bar{z}^3_+ z^1_+}{1 - z^3_+ \bar{z}^3_+} = -\frac{z^3_- \bar{z}^2_- - z^1_-}{1 - z^3_- \bar{z}^3_-} , \qquad \lambda_\pm = z^3_\pm . \tag{VII.67}$$

To indicate which spaces we are working with, we will sometimes use the notation $\mathcal{P}^3_\varepsilon$ and imply $\mathcal{P}^3_{-1} := \mathcal{P}^3$ and $\mathcal{P}^3_{+1} := \tilde{\mathcal{P}}^3 \subset \mathcal{P}^3$. The situation arising for the real structure τ_0 can – in principle – be dealt with analogously.

§9 Vector fields on $\mathcal{P}^3_\varepsilon$. On $\mathcal{P}^3_\varepsilon$, there is the following relationship between vector fields of type (0,1) in the coordinates $(z^\alpha_\pm, z^3_\pm)$ and vector fields (VII.49) in the coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\pm)$:

$$\begin{aligned} \frac{\partial}{\partial \bar{z}^1_\pm} &= -\gamma_\pm \lambda^{\dot{\alpha}}_\pm \frac{\partial}{\partial x^{2\dot{\alpha}}} =: -\gamma_\pm \bar{V}^\pm_2 , & \frac{\partial}{\partial \bar{z}^2_\pm} &= \gamma_\pm \lambda^{\dot{\alpha}}_\pm \frac{\partial}{\partial x^{1\dot{\alpha}}} =: -\varepsilon\gamma_\pm \bar{V}^\pm_1 , \\ \frac{\partial}{\partial \bar{z}^3_+} &= \frac{\partial}{\partial \bar{\lambda}_+} + \varepsilon\gamma_+ x^{\alpha\dot{1}} \bar{V}^+_\alpha , & \frac{\partial}{\partial \bar{z}^3_-} &= \frac{\partial}{\partial \bar{\lambda}_-} + \gamma_- x^{\alpha\dot{2}} \bar{V}^-_\alpha , \end{aligned} \tag{VII.68}$$

where we introduced the factors

$$\gamma_+ = \frac{1}{1 - \varepsilon\lambda_+\bar{\lambda}_+} = \frac{1}{\hat{\lambda}^{\dot{\alpha}}_+ \lambda^+_{\dot{\alpha}}} \quad \text{and} \quad \gamma_- = -\varepsilon \frac{1}{1 - \varepsilon\lambda_-\bar{\lambda}_-} = \frac{1}{\hat{\lambda}^{\dot{\alpha}}_- \lambda^-_{\dot{\alpha}}} . \tag{VII.69}$$

§10 The real twistor space $\mathcal{T}^3$. The set of fixed points under the involution τ_1[5] of the spaces contained in the double fibration (VII.45) form real subsets $\mathcal{T}^3 \subset \mathcal{P}^3$, $\mathbb{R}^{2,2} \subset \mathbb{C}^4$ and $\mathbb{R}^{2,2} \times S^1 \subset \mathcal{F}^5$. Recall that the space $\mathcal{T}^3$ is diffeomorphic to the space $\mathbb{R}P^3 \backslash \mathbb{R}P^1$ (cf. (VII.56)) fibred over $S^1 \cong \mathbb{R}P^1 \subset \mathbb{C}P^1$. Thus, we obtain the real double fibration

$$\begin{array}{ccc} & \mathbb{R}^{2,2} \times S^1 & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{T}^3 & & \mathbb{R}^{2,2} \end{array} \tag{VII.70}$$

Here, π_1 is again the trivial projection and π_2 is given by equations (VII.44) with $|\lambda_\pm| = 1$.

The tangent spaces to the two-dimensional leaves of the fibration π_2 in (VII.70) are spanned by the vector fields

$$\bar{v}^+_\alpha := \lambda^{\dot{\alpha}}_+ \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \quad \text{with} \quad \bar{v}^+_2 = -\lambda_+ \bar{v}^+_1 , \tag{VII.71}$$

where $|\lambda_+| = 1$. Equivalently, one could also use the vector fields

$$\bar{v}^-_\alpha := \lambda^{\dot{\alpha}}_- \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} = \lambda_- \bar{v}^+_\alpha \quad \text{with} \quad \lambda_- = \frac{1}{\lambda_+} = \bar{\lambda}_+ . \tag{VII.72}$$

The vector fields (VII.71) and (VII.72) are the restrictions of the vector fields $\bar{V}^\pm_\alpha$ from (VII.49) to $|\lambda_\pm| = 1$.

§11 Forms. The forms $\bar{E}^a_\pm$ with $a = 1, 2, 3$ dual to the above vector fields are given by the formulæ

$$\bar{E}^\alpha_\pm = -\gamma_\pm \hat{\lambda}^\pm_{\dot{\alpha}} \mathrm{d}x^{\alpha\dot{\alpha}} , \quad \bar{E}^3_\pm = \mathrm{d}\bar{\lambda}_\pm . \tag{VII.73}$$

[5]Although τ_1 was defined on $\mathcal{P}^3$, it induces an involution on $\mathcal{F}^5$ which we will denote by the same symbol in the following.

§12 Flag manifolds. There is a nice interpretation of the double fibrations (VII.45) and its dual version in terms of flag manifolds (see §5 of section II.1.1). For this, however, we have to focus back on the full complexified compactified twistor space $\mathbb{C}P^3$. Upon fixing the full space the flags will live in to be $\mathbb{C}^4$, we can establish the following double fibration:

$$\begin{array}{ccc} & F_{12,4} & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ F_{1,4} & & F_{2,4} \end{array} \tag{VII.74}$$

Let (L_1, L_2) be an element of $F_{12,4}$, i.e. $\dim_{\mathbb{C}} L_1 = 1$, $\dim_{\mathbb{C}} L_2 = 2$ and $L_1 \subset L_2$. Thus $F_{12,4}$ fibres over $F_{2,4}$ with $\mathbb{C}P^1$ as a typical fibre, which parameterizes the freedom to choose a complex one-dimensional subspace in a complex two-dimensional vector space. The projections are defined as $\pi_2(L_1, L_2) = L_1$ and $\pi_1(L_1, L_2) = L_2$. The full connection to (VII.45) becomes obvious, when we note that $F_{1,4} = \mathbb{C}P^3 = \mathcal{P}^3 \cup \mathbb{C}P^1$ and that $F_{2,4} = G_{2,4}(\mathbb{C})$ is the complexified and compactified version of $\mathbb{R}^4$. The advantage of the formulation in terms of flag manifolds is related to the fact that the projections are immediately clear: one has to shorten the flags to suit the structure of the flags of the base space.

The compactified version of the "dual" fibration is

$$\begin{array}{ccc} & F_{23,4} & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ F_{3,4} & & F_{2,4} \end{array} \tag{VII.75}$$

where $F_{3,4}$ is the space of hyperplanes in $\mathbb{C}^4$. This space is naturally dual to the space of lines, as every hyperplane is fixed by a vector orthogonal to the elements of the hyperplane. Therefore, we have $F_{3,4} = F^*_{1,4} = \mathbb{C}P^3_* \supset \mathcal{P}^3_*$.

VII.3.2 The Penrose-Ward transform

The Penrose-Ward transform gives a relation between solutions to the self-dual Yang-Mills equations on $\mathbb{C}^4$ and a topologically trivial, holomorphic vector bundle E over the twistor space $\mathcal{P}^3$, which becomes holomorphically trivial upon restriction to holomorphic submanifolds $\mathbb{C}P^1 \subset \mathcal{P}^3$. We will first discuss the complex case, and end this section with a remark on the simplifications in the real setting.

§13 The holomorphic bundle over $\mathcal{P}^3$. We start our considerations from a rank n holomorphic vector bundle E over the twistor space $\mathcal{P}^3$. We assume that E is topologically trivial, i.e. one can split its transition function f_{+-} according to $f_{+-} = \psi_+^{-1}\psi_-$, where $\psi_\pm$ are smooth functions on the patches $\mathcal{U}_\pm$. For the Penrose-Ward transform to work, we have to demand additionally that E becomes holomorphically trivial, when we restrict it to an arbitrary section of the vector bundle $\mathcal{P}^3 \to \mathbb{C}P^1$. We will comment in more detail on this condition in §17.

§14 Pull-back to the correspondence space. The pull-back bundle $\pi_2^* E$ over the correspondence space $\mathbb{C}^4 \times \mathbb{C}P^1$ has a transition function $\pi_2^* f_{+-}$, which due to its origin as a pull-back satisfies the equations

$$\bar{V}^{\pm}_{\alpha}(\pi_2^* f_{+-}) \;=\; 0\ . \tag{VII.76}$$

Furthermore, due to the holomorphic triviality of E on any $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$, we can split the transition function according to

$$\pi_2^* f_{+-}(x^{\alpha\dot{\alpha}}\lambda^{\pm}_{\dot{\alpha}}, \lambda^{\pm}_{\dot{\alpha}}) \;=\; \psi_+^{-1}(x^{\alpha\dot{\alpha}}, \lambda_+)\psi_-(x^{\alpha\dot{\alpha}}, \lambda_-)\ , \tag{VII.77}$$

where $\psi_\pm$ are holomorphic, matrix-group-valued functions in the coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\pm)$ of the correspondence space. This is easily seen by pulling back the restrictions of E to the $\mathbb{C}P^1_x$ for each $x^{\alpha\dot{\alpha}}$ separately. This guarantees a splitting holomorphic in $\lambda_\pm$ parameterizing the $\mathbb{C}P^1_x$. Since the embedding of the $\mathbb{C}P^1_x$ in the twistor space $\mathcal{P}^3$ is holomorphically described by the moduli $x^{\alpha\dot{\alpha}}$, the splitting is furthermore holomorphic in the latter coordinates.

§15 Construction of a gauge potential. On the correspondence space, we obtain from (VII.76) together with (VII.77) the equation

$$\psi_+ \bar{V}^+_\alpha \psi_+^{-1} \;=\; \psi_- \bar{V}^+_\alpha \psi_-^{-1} \tag{VII.78}$$

over $\tilde{\mathcal{U}}_+ \cap \tilde{\mathcal{U}}_-$. One can expand ψ_+, ψ_+^{-1} and ψ_-, ψ_-^{-1} as power series in λ_+ and $\lambda_- = \lambda_+^{-1}$, respectively. Upon substituting the expansions into equations (VII.78), one sees that both sides in (VII.78) must be linear in λ_+; this is a generalized Liouville theorem. One can introduce Lie algebra valued fields A_α whose dependence on $\lambda_\pm$ is made explicit in the formulæ

$$A^+_\alpha \;:=\; \lambda^{\dot{\alpha}}_+ A_{\alpha\dot{\alpha}} \;=\; \lambda^{\dot{\alpha}}_+ \psi_+ \partial_{\alpha\dot{\alpha}} \psi_+^{-1} \;=\; \lambda^{\dot{\alpha}}_+ \psi_- \partial_{\alpha\dot{\alpha}} \psi_-^{-1}\ . \tag{VII.79}$$

The matrix-valued functions $A_{\alpha\dot{\alpha}}(x)$ can be identified with the components of a gauge potential $A_{\alpha\dot{\alpha}}\mathrm{d}x^{\alpha\dot{\alpha}} + A_{\mathbb{C}P^1}$ on the correspondence space $\mathbb{C}^4 \times \mathbb{C}P^1$ with $A_{\mathbb{C}P^1} = 0$: the component $A_{\bar{\lambda}_+}$, vanishes as

$$A_{\bar{\lambda}_+} \;=\; \psi_+ \partial_{\bar{\lambda}_+} \psi_+^{-1} \;=\; 0\ . \tag{VII.80}$$

§16 Linear system and the SDYM equations. The equations (VII.78) can be recast into a linear system

$$(\bar{V}^+_\alpha + A^+_\alpha)\psi_+ \;=\; 0\ , \tag{VII.81}$$

with similar equations for ψ_-. We encountered this linear system already in section VII.2.2, §5, and we briefly recall that the compatibility conditions of this linear system are

$$[\bar{V}^+_\alpha + A^+_\alpha\, , \bar{V}^+_\beta + A^+_\beta] \;=\; \lambda^{\dot{\alpha}}_+ \lambda^{\dot{\beta}}_+ [\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}}\, , \partial_{\beta\dot{\beta}} + A_{\beta\dot{\beta}}] \;=:\; \lambda^{\dot{\alpha}}_+ \lambda^{\dot{\beta}}_+ F_{\alpha\dot{\alpha},\beta\dot{\beta}} \;=\; 0\ . \tag{VII.82}$$

To be satisfied for all $(\lambda^{\dot{\alpha}}_+)$, this equation has to vanish to all orders in λ_+ separately, from which we obtain the self-dual Yang-Mills (SDYM) equations

$$F_{1\dot{1},2\dot{1}} \;=\; 0\ , \quad F_{1\dot{2},2\dot{2}} \;=\; 0\ , \quad F_{1\dot{1},2\dot{2}} + F_{1\dot{2},2\dot{1}} \;=\; 0 \tag{VII.83}$$

for a gauge potential $(A_{\alpha\dot{\alpha}})$. Recall that in the spinorial notation $F_{\alpha\dot{\alpha},\beta\dot{\beta}} = \varepsilon_{\alpha\beta} f_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}$ the SDYM equations $F = *F$ are rewritten as

$$f_{\dot{\alpha}\dot{\beta}} \;=\; 0\ , \tag{VII.84}$$

i.e. the part of $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ symmetric in the indices $\dot{\alpha}\dot{\beta}$ vanishes.

§17 Holomorphic triviality. Let us comment on the condition of holomorphic triviality of E upon reduction on subspaces $\mathbb{C}P^1_x$ in slightly more detail. For this, recall that every rank n holomorphic vector bundle over $\mathbb{C}P^1$ is (holomorphically) equivalent to a direct sum of line bundles

$$\mathcal{O}(i_1) \oplus \ldots \oplus \mathcal{O}(i_n)\ , \tag{VII.85}$$

i.e., it is uniquely determined by a set of integers $(i_1, \ldots, i_n)$. Furthermore, the sum of the i_k is a topological invariant and each of the i_k is a holomorphic invariant up to permutation.

Now consider a rank n holomorphic vector bundle E over the twistor space $\mathcal{P}^3$. The set of equivalence classes of such vector bundles E which become holomorphically equivalent to the bundle (VII.85) when restricted to any projective line $\mathbb{C}P^1_x$ in $\mathcal{P}^3$ will be denoted by $\mathcal{M}(i_1, \ldots, i_n)$. The moduli space of holomorphic vector bundles on $\mathcal{P}^3$ contains then all of the above moduli spaces:

$$\mathcal{M} \supset \bigcup_{i_1,\ldots,i_n} \mathcal{M}(i_1, \ldots, i_n) \ . \qquad \text{(VII.86)}$$

Furthermore, $\mathcal{M}$ contains also those holomorphic vector bundles whose restrictions to different $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ are not holomorphically equivalent.

The focus of interest in the literature, including this thesis, is in general the moduli subspace

$$\mathcal{M}(0, \ldots, 0) \qquad \text{(VII.87)}$$

which is clearly a true subset of (VII.86) and bijective to the moduli space of solutions to the SDYM equations in four dimensions. Although a generalization of Ward's construction to the cases $\mathcal{M}(i_1, \ldots, i_n)$ for arbitrary $(i_1, \ldots, i_n)$ were studied in the literature e.g. by Leiterer in [177], a thorough geometric interpretation was only given for some special cases of $(i_1, \ldots, i_n)$. It seems that the situation has not been yet completely clarified. From (VII.86) together with the Leiterer examples, it is, however, quite evident that the moduli space $\mathcal{M}(0, \ldots, 0)$ is not a dense subset of the moduli space $\mathcal{M}$. Furthermore, the statement is irrelevant in the most important recent application of the Penrose-Ward-transform in twistor string theory, cf. section V.4.6: a perturbative expansion in the vicinity of the vacuum solution, which corresponds to a trivial transition function $f_{+-} = \mathbb{1}_n$. There, the necessary property of holomorphic triviality after restriction to any $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ follows immediately from Kodaira's theorem.

§18 Holomorphic Chern-Simons equations. One can also obtain a linear system directly on the twistor space by using the splitting of the transition function f_{+-} of E into smooth functions via

$$f_{+-} = \hat{\psi}_+^{-1} \hat{\psi}_- \ . \qquad \text{(VII.88)}$$

Such a splitting exists, as E was assumed to be topologically trivial. Note furthermore that f_{+-} is the transition function of a *holomorphic* vector bundle and therefore satisfies

$$\frac{\partial}{\partial \bar{z}^a_\pm} f_{+-} = 0 \quad \text{with} \quad a = 1, \ldots, 3 \ . \qquad \text{(VII.89)}$$

Similarly to the case of the components $A_{\alpha\dot{\alpha}}$ introduced before, we find here a gauge potential

$$\hat{A}^{0,1} = \hat{\psi}_+ \bar{\partial} \hat{\psi}_+^{-1} = \hat{\psi}_- \bar{\partial} \hat{\psi}_-^{-1} \ , \qquad \text{(VII.90)}$$

on $\mathcal{U}_+ \cap \mathcal{U}_-$, which can be extended to a gauge potential on the full twistor space $\mathcal{P}^3 = \mathcal{U}_+ \cup \mathcal{U}_-$. Note that here, we choose not to work with components, but directly with the resulting Lie algebra valued $(0,1)$-form $A^{0,1}$. Again, one can cast equations (VII.89) into a linear system

$$(\bar{\partial} + \hat{A}^{0,1}) \hat{\psi}_\pm = 0 \ , \qquad \text{(VII.91)}$$

which has the holomorphic Chern-Simons equations

$$\bar{\partial}\hat{A}^{0,1} + \hat{A}^{0,1} \wedge \hat{A}^{0,1} \ = \ 0 \tag{VII.92}$$

as its compatibility conditions. This aspect of the Penrose-Ward transform will become particularly important when dealing with the supertwistor space $\mathcal{P}^{3|4}$ in section VII.4.1. There, we will be able to give an action for holomorphic Chern-Simons theory due to the existence of a holomorphic volume form on $\mathcal{P}^{3|4}$.

One has to stress at this point that a solution to the hCS equations (VII.92) corresponds to an arbitrary holomorphic vector bundle over the twistor space $\mathcal{P}^3$, which does not necessarily satisfy the additional condition of holomorphic triviality on all the $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$. In the following, we will always imply the restriction to the appropriate subset of solutions to (VII.92) when speaking about general solutions to the hCS equations (VII.92) on $\mathcal{P}^3$. As mentioned in the previous paragraph, this restriction is irrelevant for perturbative studies. Furthermore, it corresponds to those gauge potentials, for which the component $\hat{\mathcal{A}}_{\bar{\lambda}_\pm}$ can be gauged away, as we discuss in the following paragraph.

§19 Gauge equivalent linear system. Recall that the trivializations defined by formula (VII.77) correspond to holomorphic triviality of the bundle $E|_{\mathbb{C}P^1_x}$ for any $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$. Similarly, we may consider restrictions of E to fibres $\mathbb{C}^2_\lambda$ of the fibration $\mathcal{P}^3 \to \mathbb{C}P^1$. All these restrictions are holomorphically trivial due to the contractibility of $\mathbb{C}^2_\lambda$ for any $\lambda \in \mathbb{C}P^1$. Therefore there exist regular matrix-valued functions $\breve{\psi}_\pm(z^\alpha_\pm, \lambda_\pm, \bar{\lambda}_\pm)$ depending holomorphically on $z^\alpha_\pm$ (and non-holomorphically on $\lambda_\pm$) such that

$$f_{+-} \ = \ \hat{\psi}_+^{-1}\hat{\psi}_- \ = \ (\breve{\psi}_+)^{-1}\breve{\psi}_- \ , \tag{VII.93}$$

and $\varphi := \psi_+(\breve{\psi}_+)^{-1} = \psi_-(\breve{\psi}_-)^{-1}$ defines a gauge transformation

$$\left(\frac{\partial}{\partial \bar{z}^\alpha_\pm} \lrcorner \hat{A}^{0,1}\, ,\, \hat{A}_{\bar{\lambda}_+} \ = \ 0\right) \ \overset{\varphi}{\longmapsto} \ \left(\frac{\partial}{\partial \bar{z}^\alpha_\pm} \lrcorner \breve{A}^{0,1} = 0\, ,\, \breve{A}_{\bar{\lambda}_+}\right) \tag{VII.94}$$

to a special trivialization in which only $\breve{A}_{\bar{\lambda}_+} \neq 0$ and $\frac{\partial}{\partial \bar{z}^\alpha_\pm} \lrcorner \breve{A}^{0,1} = 0$.

§20 The Euclidean case. Let us now consider the Penrose-Ward transform in the Euclidean case. The important point here is that the spaces $\mathcal{P}^3_{-1}$ and $\mathbb{R}^4 \times \mathbb{C}P^1$ become diffeomorphic, and thus the double fibration (VII.45) reduces to a single fibration

$$\mathcal{P}^3 \ \to \ \mathbb{R}^4 \ . \tag{VII.95}$$

Therefore, we have the identification (VII.68) between the vector fields $\bar{V}^\pm_\alpha$ and $\frac{\partial}{\partial \bar{z}^\alpha}$. This implies furthermore that the linear system (VII.81) and (VII.91) become (gauge) equivalent. We have for the two splittings

$$f_{+-} \ = \ \psi_+^{-1}\psi_- \ = \ \hat{\psi}_+^{-1}\hat{\psi}_- \quad \text{with} \quad \hat{\psi}_+ \ = \ \varphi^{-1}\psi_\pm \ , \tag{VII.96}$$

where φ is a globally defined, regular matrix-valued function on $\mathcal{P}^3$. Decomposing the gauge potential $\hat{A}^{0,1}$ into the components $\hat{A}^\pm_\alpha := \bar{V}^\pm_\alpha \lrcorner \hat{A}^{0,1}$, we have

$$\begin{aligned} \hat{A}^+_\alpha \ &:= \ \hat{\psi}_+\bar{V}^+_\alpha\hat{\psi}_+^{-1} \ = \ \hat{\psi}_-\bar{V}^+_\alpha\hat{\psi}_-^{-1} \quad &&= \ \varphi^{-1}(\psi_\pm\bar{V}^+_\alpha\psi_\pm^{-1})\varphi + \varphi^{-1}\bar{V}^+_\alpha\varphi \\ & &&= \ \varphi^{-1}A^+_\alpha\varphi + \varphi^{-1}\bar{V}^+_\alpha\varphi \ , \\ \hat{A}_{\bar{\lambda}_+} \ &:= \ \hat{\psi}_+\partial_{\bar{\lambda}_+}\hat{\psi}_+^{-1} \ = \ \hat{\psi}_-\partial_{\bar{\lambda}_+}\hat{\psi}_-^{-1} \quad &&= \ \varphi^{-1}\partial_{\bar{\lambda}_+}\varphi \ , \end{aligned} \tag{VII.97}$$

from which we indeed realize that φ plays the rôle of a gauge transformation.

§21 Vector bundles in the Kleinian case. Consider a real-analytic function $f^{\tau}_{+-} : \mathcal{T}^3 \to \mathsf{GL}(n,\mathbb{C})$ on the twistor space $\mathcal{T}^3$ which can be understood as an isomorphism $f^{\tau}_{+-} : E^{\tau}_{-} \to E^{\tau}_{+}$ between two trivial complex vector bundles $E^{\tau}_{\pm} \to \mathcal{T}^3$. We assume that f^{τ}_{+-} satisfies the reality condition

$$\left(f^{\tau}_{+-}\left(z^{\alpha}_{+}, \lambda_{+}\right)\right)^{\dagger} \;=\; f^{\tau}_{+-}(z^{\alpha}_{+}, \lambda_{+}) \;. \tag{VII.98}$$

Such a function f^{τ}_{+-} can be extend holomorphically into a neighborhood $\mathcal{U}$ of $\mathcal{T}^3$ in $\mathcal{P}^3$, such that the extension f_{+-} of f^{τ}_{+-} satisfies the reality condition

$$\left(f_{+-}\left(\tau_1(z^{\alpha}_{+}, \lambda_{+})\right)\right)^{\dagger} \;=\; f_{+-}(z^{\alpha}_{+}, \lambda_{+}) \;, \tag{VII.99}$$

generalizing equation (VII.98). The function f_{+-} is holomorphic on $\mathcal{U} = \mathcal{U}_+ \cap \mathcal{U}_-$ and can be identified with a transition function of a holomorphic vector bundle E over $\mathcal{P}^3 = \mathcal{U}_+ \cup \mathcal{U}_-$ which glues together two trivial bundles $E_+ = \mathcal{U}_+ \times \mathbb{C}^n$ and $E_- = \mathcal{U}_- \times \mathbb{C}^n$. Obviously, the two trivial vector bundles $E^{\tau}_{\pm} \to \mathcal{T}^3$ are restrictions of the trivial bundles $E_{\pm} \to \mathcal{U}_{\pm}$ to $\mathcal{T}^3$.

The assumption that E becomes holomorphically trivial upon reduction to a subset $\mathbb{C}P^1_x \subset \mathcal{P}^3$ implied a splitting of the transition function f_{+-},

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \;, \tag{VII.100}$$

into regular matrix-valued functions ψ_+ and ψ_- defined on $\mathcal{U}_+ = \mathcal{P}^3_+ \cup \mathcal{U}$ and $\mathcal{U}_- = \mathcal{P}^3_- \cup \mathcal{U}$ and holomorphic in $\lambda_+ \in H^2_+$ and $\lambda_- \in H^2_-$, respectively. Note that the condition (VII.99) is satisfied if

$$\psi_+^{-1}(\tau_1(x^{\alpha\dot\alpha}, \lambda_+)) \;=\; \psi_-^{\dagger}(x^{\alpha\dot\alpha}, \lambda_-) \;. \tag{VII.101}$$

Restricting (VII.100) to $S^1_x \hookrightarrow \mathbb{C}P^1_x$, we obtain

$$f^{\tau}_{+-} \;=\; (\psi^{\tau}_+)^{-1}\psi^{\tau}_- \quad \text{with} \quad (\psi^{\tau}_+)^{-1} \;=\; (\psi^{\tau}_-)^{\dagger} \;, \tag{VII.102}$$

where the $\psi^{\tau}_{\pm}$ are restrictions to $\mathbb{R}^4 \times S^1$ of the matrix-valued functions $\psi_{\pm}$ given by (VII.100) and (VII.101). Thus the initial twistor data consist of a real-analytic function[6] f^{τ}_{+-} on $\mathcal{T}^3$ satisfying (VII.98) together with a splitting (VII.102), from which we construct a holomorphic vector bundle E over $\mathcal{P}^3$ with a transition function f_{+-} which is a holomorphic extension of f^{τ}_{+-} to $\mathcal{U} \supset \mathcal{T}^3$. In other words, the space of real twistor data is the moduli space of holomorphic vector bundles $E \to \mathcal{P}^3$ with transition functions satisfying the reality conditions (VII.99).

§22 The linear system on $\mathcal{T}^3$. In the purely real setting, one considers a real-analytic $\mathsf{GL}(n,\mathbb{C})$-valued function f^{τ}_{+-} on $\mathcal{T}^3$ satisfying the Hermiticity condition (VII.98) in the context of the real double fibration (VII.70). Since the pull-back of f^{τ}_{+-} to $\mathbb{R}^4 \times S^1$ has to be constant along the fibres of π_2, we obtain the constraint equations $\bar v^+_{\alpha} f^{\tau}_{+-} = 0$ or equivalently $\bar v^-_{\alpha} f^{\tau}_{+-} = 0$ with the vector fields $\bar v^{\pm}_{\alpha}$ defined in (VII.71). Using the splitting (VII.102) of f^{τ}_{+-} on fibres S^1_x of the projection π_1 in (VII.70) and substituting $f^{\tau}_{+-} = (\psi^{\tau}_+)^{-1}\psi^{\tau}_-$ into the above constraint equations, we obtain the linear systems

$$(\bar v^+_{\alpha} + \mathcal{A}^+_{\alpha})\psi^{\tau}_+ \;=\; 0 \;, \quad \text{or} \quad (\bar v^-_{\alpha} + \mathcal{A}^-_{\alpha})\psi^{\tau}_- \;=\; 0 \;. \tag{VII.103}$$

[6] One could also consider the extension f_{+-} and the splitting (VII.102) even if f^{τ}_{+-} is not analytic, but in this case the solutions to the super SDYM equations can be singular. Such solutions are not related to holomorphic bundles.

Here, $\mathcal{A}_\pm = (\mathcal{A}^\pm_\alpha)$ are relative connections on the bundles $E^\tau_\pm$. From (VII.103), one can find $\psi^\tau_\pm$ for any given $\mathcal{A}^\pm_\alpha$ and vice versa, i.e. find $\mathcal{A}^\pm_\alpha$ for given $\psi^\tau_\pm$ by the formulæ

$$\begin{aligned} \mathcal{A}^+_\alpha &= \psi^\tau_+ \bar{v}^+_\alpha (\psi^\tau_+)^{-1} = \psi^\tau_- \bar{v}^+_\alpha (\psi^\tau_-)^{-1} , \\ \mathcal{A}^-_\alpha &= \psi^\tau_+ \bar{v}^-_\alpha (\psi^\tau_+)^{-1} = \psi^\tau_- \bar{v}^-_\alpha (\psi^\tau_-)^{-1} . \end{aligned} \tag{VII.104}$$

The compatibility conditions of the linear systems (VII.103) read

$$\bar{v}^\pm_\alpha \mathcal{A}^\pm_\beta - \bar{v}^\pm_\beta \mathcal{A}^\pm_\alpha + [\mathcal{A}^\pm_\alpha, \mathcal{A}^\pm_\beta] = 0 . \tag{VII.105}$$

Geometrically, these equations imply flatness of the curvature of the relative connections $\mathcal{A}_\pm$ on the bundles $E^\tau_\pm$ defined along the real two-dimensional fibres of the projection π_2 in (VII.70).

Recall that ψ^τ_+ and ψ^τ_- extend holomorphically in λ_+ and λ_- to H^2_+ and H^2_-, respectively, and therefore we obtain from (VII.104) that $\mathcal{A}^\pm_\alpha = \lambda^{\dot\alpha}_\pm \mathcal{A}_{\alpha\dot\alpha}$, where $\mathcal{A}_{\alpha\dot\alpha}$ does not depend on $\lambda_\pm$. Then the compatibility conditions (VII.105) of the linear systems (VII.103) reduce to the equations (VII.82). It was demonstrated above that for $\varepsilon = +1$, these equations are equivalent to the field equations of SDYM theory on $\mathbb{R}^{2,2}$. Thus, there are bijections between the moduli spaces of solutions to equations (VII.105), the field equations of SDYM theory on $\mathbb{R}^{2,2}$ and the moduli space of τ_1-real holomorphic vector bundles E over $\mathcal{P}^3$.

§23 Extension to $\tilde{\mathcal{P}}^3$. Consider now the extension of the linear systems (VII.103) to open domains $\mathcal{U}_\pm = \mathcal{P}^3_\pm \cup \mathcal{U} \supset \mathcal{T}^3$,

$$(\bar{V}^\pm_\alpha + \mathcal{A}^\pm_\alpha)\psi_\pm = 0 \quad \text{and} \quad \partial_{\bar\lambda_\pm}\psi_\pm = 0 , \tag{VII.106}$$

where here, the $\bar{V}^\pm_\alpha$ are vector fields of type $(0,1)$ on $\mathcal{U}^s_\pm := \mathcal{U}_\pm \backslash (\mathbb{R}^4 \times S^1)$ as given in (VII.49) and (VII.68). These vector fields annihilate f_{+-} and from this fact and the splitting (VII.100), one can also derive equations (VII.106). Recall that due to the existence of a diffeomorphism between the spaces $\mathbb{R}^4 \times H^2$ and $\tilde{\mathcal{P}}^3$ which is described in §8, the double fibration (VII.60) simplifies to the nonholomorphic fibration

$$\mathcal{P}^3_{+1} \to \mathbb{R}^4 . \tag{VII.107}$$

Moreover, since the restrictions of the bundle $E \to \mathcal{P}^3_{+1}$ to the two-dimensional leaves of the fibration (VII.107) are trivial, there exist regular matrix-valued functions $\hat\psi_\pm$ on $\mathcal{U}^s_\pm$ such that

$$f_{+-} = \hat\psi_+^{-1}\hat\psi_- \tag{VII.108}$$

on $\mathcal{U}^s = \mathcal{U}\backslash(\mathbb{R}^4 \times S^1)$. Additionally, we can impose the reality condition

$$\hat\psi_+^{-1}\left(x^{\alpha\dot\alpha}, \frac{1}{\bar\lambda_+}\right) = \hat\psi_-^\dagger(x^{\alpha\dot\alpha}, \lambda_-) \tag{VII.109}$$

on $\hat\psi_\pm$. Although $\mathcal{U}^s$ consists of two disconnected pieces, the functions $\hat\psi_\pm$ are not independent on each piece because of the condition (VII.109), which also guarantees (VII.99) on $\mathcal{U}^s$. The functions $\hat\psi_\pm$ and their inverses are ill-defined on $\mathbb{R}^4 \times S^1 \cong \mathcal{P}^3_0$ since the restriction of π_2 to $\mathbb{R}^4 \times S^1$ is a noninvertible projection onto $\mathcal{T}^3$, see §8. Equating (VII.99) and (VII.108), one sees that the singularities of $\hat\psi_\pm$ on $\mathbb{R}^4 \times S^1$ split off, i.e.

$$\hat\psi_\pm = \varphi^{-1}\psi_\pm , \tag{VII.110}$$

in a matrix-valued function φ^{-1} which disappears from

$$f_{+-} \;=\; \hat{\psi}_+^{-1}\hat{\psi}_- \;=\; (\psi_+^{-1}\varphi)(\varphi^{-1}\psi_-) \;=\; \psi_+^{-1}\psi_- \;. \tag{VII.111}$$

Therefore f_{+-} is a nonsingular holomorphic matrix-valued function on all of $\mathcal{U}$.

From (VII.108)-(VII.111) it follows that on $\tilde{\mathcal{P}}^3$, we have a well-defined gauge transformation generated by φ and one can introduce gauge potentials $\hat{\mathcal{A}}_+^{0,1}$ and $\hat{\mathcal{A}}_-^{0,1}$ which are defined on $\mathcal{U}_+^s$ and $\mathcal{U}_-^s$, respectively, but not on $\mathbb{R}^4\times S^1$. By construction, $\hat{\mathcal{A}}^{0,1}=(\hat{\mathcal{A}}_+^{0,1},\hat{\mathcal{A}}_-^{0,1})$ satisfies the hCS equations (VII.92) on $\tilde{\mathcal{P}}^3=\mathcal{P}_+^3\cup\mathcal{P}_-^3$, which are equivalent to the SDYM equations on $\mathbb{R}^{2,2}$. Conversely, having a solution $\hat{\mathcal{A}}^{0,1}$ of the hCS field equations on the space $\tilde{\mathcal{P}}^3$, one can find regular matrix-valued functions $\hat{\psi}_+$ on $\mathcal{U}_+^s$ and $\hat{\psi}_-$ on $\mathcal{U}_-^s$ which satisfy the reality condition (VII.109). These functions define a further function $f_{+-}^s=\hat{\psi}_+^{-1}\hat{\psi}_-:\mathcal{U}^s\to\mathsf{GL}(n,\mathbb{C})$ which can be completed to a holomorphic function $f_{+-}:\mathcal{U}\to\mathsf{GL}(n,\mathbb{C})$ due to (VII.111). The latter one can be identified with a transition function of a holomorphic vector bundle E over the full twistor space $\mathcal{P}^3$. The restriction of f_{+-} to $\mathcal{T}^3$ is a real-analytic function f_{+-}^τ which is *not constrained* by any differential equation. Thus, in the case $\varepsilon=+1$ (and also for the real structure τ_0), one can either consider two trivial complex vector bundles $E_\pm^\tau$ defined over the space $\mathcal{T}^3$ together with an isomorphism $f_{+-}^\tau:E_-^\tau\to E_+^\tau$ or a single complex vector bundle E over the space $\mathcal{P}^3$. However, the appropriate hCS theory which has the same moduli space as the moduli space of these bundles is defined on $\tilde{\mathcal{P}}^3$. Moreover, real Chern-Simons theory on $\mathcal{T}^3$ has no moduli, since its solutions correspond to flat bundles over $\mathcal{T}^3$ with constant transition functions[7] defined on the intersections of appropriate patches covering $\mathcal{T}^3$.

To sum up, there is a bijection between the moduli spaces of solutions to equations (VII.105) and to the hCS field equations on the space $\tilde{\mathcal{P}}^3$ since both moduli spaces are bijective to the moduli space of holomorphic vector bundles over $\mathcal{P}^3$. In fact, whether one uses the real supertwistor space $\mathcal{T}^3$, or works with its complexification $\mathcal{P}^3$, is partly a matter of taste. However, the complex approach is more geometrical and more natural from the point of view of an action principle and the topological B-model. For example, equations (VII.105) cannot be transformed by a gauge transformation to a set of differential equations on $\mathcal{T}^3$ as it was possible on $\tilde{\mathcal{P}}^3$ in the complex case. This is due to the fact that the transition function f_{+-}, which was used as a link between the two sets of equations in the complex case does not satisfy any differential equation after restriction to $\mathcal{T}^3$. From this we see that we cannot expect any action principle on $\mathcal{T}^3$ to yield equations equivalent to (VII.105) as we had in the complex case. For these reasons, we will mostly choose to use the complex approach in the following.

§24 Reality of the gauge potential. After imposing a reality condition on the spaces $\mathcal{P}^3$ and $\mathbb{C}^4$, we have to do so for the vector bundle and the objects it comes with, as well. Note that $A_{\alpha\dot\alpha}\mathrm{d}x^{\alpha\dot\alpha}$ will take values in the algebra of anti-Hermitian $n\times n$ matrices if $\psi_\pm$ satisfies the following condition[8]:

$$\psi_+^{-1}(x,\lambda_+) \;=\; \left(\psi_-\left(x,\frac{\varepsilon}{\bar\lambda_-}\right)\right)^\dagger \;. \tag{VII.112}$$

The anti-Hermitian gauge potential components can be calculated from (VII.79) to be

$$A_{1\dot2} \;=\; \psi_+\partial_{1\dot2}\psi_+^{-1}|_{\lambda_+=0} \;=\; -\varepsilon A_{2\dot1}^\dagger\;,\quad A_{2\dot2} \;=\; \psi_+\partial_{2\dot2}\psi_+^{-1}|_{\lambda_+=0} \;=\; -A_{1\dot1}^\dagger\;. \tag{VII.113}$$

[7] Note that these transition functions are in no way related to the transition functions f_{+-} of the bundles E over $\mathcal{P}^3$ or to the functions f_{+-}^τ defined on the whole of $\mathcal{T}^3$.

[8] Here, † means Hermitian conjugation.

§25 Explicit Penrose-Ward transform. One can make the Penrose-Ward transform more explicit. From the formula (VII.79), one obtains directly

$$A_{\alpha \dot{1}} \;=\; -\oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi \mathrm{i}} \frac{A^+_\alpha}{\lambda_+^2} \quad \text{and} \quad A_{\alpha\dot{2}} \;=\; \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi \mathrm{i}} \frac{A^+_\alpha}{\lambda_+}\;, \tag{VII.114}$$

where the contour $S^1 = \{\lambda_+ \in \mathbb{C}P^1 : |\lambda_+| = r < 1\}$ encircles $\lambda_+ = 0$. Using (VII.79), one can easily show the equivalence of (VII.114) to (VII.113). The formulæ (VII.114) define the Penrose-Ward transform

$$\mathcal{PW} : (A^+_\alpha, A_{\bar{\lambda}_+}{=}\,0) \;\mapsto\; (A_{\alpha\dot{\alpha}})\;, \tag{VII.115}$$

which together with a preceding gauge transformation

$$(\hat{A}^+_\alpha, \hat{A}_{\bar{\lambda}_+}) \overset{\varphi}{\longmapsto} (A^+_\alpha, A_{\bar{\lambda}_+}{=}\,0) \tag{VII.116}$$

maps solutions $(\hat{A}^+_\alpha, \hat{A}_{\bar{\lambda}_+})$ of the field equations of hCS theory on $\mathcal{P}^3$ to solutions $(A_{\alpha\dot{\alpha}})$ of the SDYM equations on $\mathbb{R}^4$. Conversely, any solution $(A_{\alpha\dot{\alpha}})$ of the SDYM equations corresponds to a solution $(\hat{A}^+_\alpha, \hat{A}_{\bar{\lambda}_+})$ of the field equations of hCS theory on $\mathcal{P}^3$ which directly defines the inverse Penrose-Ward transform $\mathcal{PW}^{-1}$. Note that gauge transformations[9] of $(\hat{A}^+_\alpha, \hat{A}_{\bar{\lambda}_+})$ on $\mathcal{P}^3$ and $(A_{\alpha\dot{\alpha}})$ on $\mathbb{R}^4$ do not change the transition function f_{+-} of the holomorphic bundle $E \to \mathcal{P}^3$. Therefore, we have altogether a one-to-one correspondence between equivalence classes of topologically trivial holomorphic vector bundles over $\mathcal{P}^3$ which become holomorphically trivial upon reduction to any $\mathbb{C}P^1_x \subset \mathcal{P}^3$ and gauge equivalence classes of solutions to the field equations of hCS theory on $\mathcal{P}^3$ and the SDYM equations on $\mathbb{R}^4$.

§26 Anti-self-dual gauge fields. The discussion of anti-self-dual gauge fields follows precisely the lines of the discussion of the self-dual case. The first difference noteworthy is that now the tangent spaces to the leaves of the fibration π_2 in the dual case are spanned by the vector fields $\bar{V}^\pm_{\dot{\alpha}} := \mu^\alpha_\pm \partial_{\alpha\dot{\alpha}}$ The definition of the gauge potential $(A_{\alpha\dot{\alpha}})$ is then (cf. (VII.79))

$$A^+_{\dot{\alpha}} \;:=\; \mu^\alpha_+ A_{\alpha\dot{\alpha}} \;=\; \mu^\alpha_+ \psi_+ \partial_{\alpha\dot{\alpha}} \psi_+^{-1} \;=\; \mu^\alpha_+ \psi_- \partial_{\alpha\dot{\alpha}} \psi_-^{-1}\;, \tag{VII.117}$$

which gives rise to the linear system

$$(\bar{V}^+_{\dot{\alpha}} + A^+_{\dot{\alpha}})\psi_+ \;=\; 0\;. \tag{VII.118}$$

The corresponding compatibility conditions are easily found to be

$$[\bar{V}^+_{\dot{\alpha}} + A^+_{\dot{\alpha}}\,, \bar{V}^+_{\dot{\beta}} + A^+_{\dot{\beta}}] \;=\; \mu^\alpha_+ \mu^\beta_+ [\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}}\,, \partial_{\beta\dot{\beta}} + A_{\beta\dot{\beta}}] \;=:\; \mu^\alpha_+ \mu^\beta_+ F_{\alpha\dot{\alpha},\beta\dot{\beta}} \;=\; 0\;, \tag{VII.119}$$

and these equations are equivalent to the anti-self-dual Yang-Mills equations $f_{\alpha\beta} = 0$.

§27 Example. To close this section, let us consider an explicit example for a Penrose-Ward transform, which will yield an $\mathsf{SU}(2)$ instanton. We start from a rank two holomorphic vector bundle over the real twistor space $\mathcal{P}^3_{-1}$ given by the transition function

$$f_{+-} \;=\; \begin{pmatrix} \rho & \lambda_+^{-1} \\ -\lambda_+ & 0 \end{pmatrix} \tag{VII.120}$$

[9] Let us stress that there are two gauge transformations for gauge potentials on two different spaces present in the discussion.

(a special case of the Atiyah-Ward ansatz [14]) together with the splitting

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \;=\; \begin{pmatrix} \phi+\rho_+ & -\lambda_+^{-1}\rho_+ \\ -\lambda_+ & 1 \end{pmatrix} \frac{1}{\sqrt{\phi}}\frac{1}{\sqrt{\phi}} \begin{pmatrix} \phi+\rho_- & \lambda_+^{-1} \\ \lambda\rho_- & 1 \end{pmatrix} . \qquad \text{(VII.121)}$$

Here, we decomposed the function ρ in its Laurent series

$$\rho \;=\; \sum_{n=-\infty}^{\infty} \rho_n \lambda^n \;=\; \rho_- + \phi + \rho_+ \qquad \text{(VII.122)}$$

and $\rho_\pm$ and ϕ denote the components holomorphic on $U_\pm$ and $\mathbb{C}P^1$, respectively. The gauge potential $A^\pm_\alpha = \psi_+ \bar{V}^\pm_\alpha \psi_+^{-1}$ is then easily calculated and the four-dimensional components $A_{\alpha\dot\alpha}$ are reconstructed via the formulæ (VII.114). Eventually, this calculation yields the result

$$A_\mu \;=\; \tfrac{1}{2\mathrm{i}}\bar\eta^a_{\mu\nu}\sigma^a(\phi^{\frac12}\partial_\nu\phi^{-\frac12} - \phi^{-\frac12}\partial_\nu\phi^{\frac12}) + \tfrac12 \mathbb{1}_2(\phi^{\frac12}\partial_\mu\phi^{-\frac12} + \phi^{-\frac12}\partial_\mu\phi^{\frac12}) \,. \qquad \text{(VII.123)}$$

VII.3.3 The ambitwistor space

§28 Motivation. The idea leading naturally to a twistor space of Yang-Mills theory is to "glue together" both the self-dual and the anti-self-dual subsectors to the full theory. To achieve this, we will need two copies of the twistor space, one understood as dual to the other one, and glue them together to the *ambitwistor space*. Roughly speaking, this gluing amounts to restricting to the diagonal in the two moduli spaces. From this, we can already anticipate a strange property of this space: The intersection of the α- and β-planes corresponding to points in the two twistor spaces will be null lines, but integrability along null lines is trivial. Therefore, we will have to consider infinitesimal neighborhoods of our new twistor space inside the product of the two original twistor spaces, and this is the origin of the name *ambitwistor space*. Eventually, this feature will find a natural interpretation in terms of Graßmann variables, when we will turn to the superambitwistor space in section VII.7.1. This aspect verifies incidentally the interpretation of Graßmann directions of a supermanifold as an infinitesimal "cloud of space" around its body.

§29 The quadric $\mathcal{L}^5$. Consider the product of a twistor space $\mathcal{P}^3$ with homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$ and inhomogeneous coordinates $(z^\alpha_\pm, z^3_\pm = \lambda_\pm)$ on the two patches $\mathcal{U}_\pm$ as introduced in section VII.3.1 and an analogous dual copy $\mathcal{P}^3_*$ with homogeneous coordinates $(\sigma^{\dot\alpha}, \mu_\alpha)$ and inhomogeneous coordinates $(u^{\dot\alpha}_\pm, u^{\dot 3}_\pm = \mu_\pm)$ on the two patches $\mathcal{U}^*_\pm$. The space $\mathcal{P}^3 \times \mathcal{P}^3_*$ is now naturally described by the homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha}; \sigma^{\dot\alpha}, \mu_\alpha)$. Furthermore, it is covered by the four patches

$$\mathcal{U}_{(1)} \;:=\; \mathcal{U}_+ \times \mathcal{U}^*_+ \,,\quad \mathcal{U}_{(2)} \;:=\; \mathcal{U}_- \times \mathcal{U}^*_+ \,,\quad \mathcal{U}_{(3)} \;:=\; \mathcal{U}_+ \times \mathcal{U}^*_- \,,\quad \mathcal{U}_{(4)} \;:=\; \mathcal{U}_- \times \mathcal{U}^*_- \,, \qquad \text{(VII.124)}$$

on which we have the evident inhomogeneous coordinates $(z^\alpha_{(a)}, z^3_{(a)}; u^{\dot\alpha}_{(a)}, u^{\dot 3}_{(a)})$. We can consider $\mathcal{P}^3 \times \mathcal{P}^3_*$ as a rank 4 vector bundle over the space $\mathbb{C}P^1 \times \mathbb{C}P^1_*$. The global sections of this bundle are parameterized by elements of $\mathbb{C}^4 \times \mathbb{C}^4_*$ in the following way:

$$z^\alpha_{(a)} \;=\; x^{\alpha\dot\alpha}\lambda^{(a)}_{\dot\alpha} \;;\quad u^{\dot\alpha}_{(a)} \;=\; x^{\alpha\dot\alpha}_* \mu^{(a)}_\alpha \,. \qquad \text{(VII.125)}$$

The *quadric* $\mathcal{L}^5$ is now the algebraic variety in $\mathcal{P}^3 \times \mathcal{P}^3_*$ defined by the equation

$$\omega^\alpha \mu_\alpha - \sigma^{\dot\alpha}\lambda_{\dot\alpha} \;=\; 0 \,. \qquad \text{(VII.126)}$$

Instead of (VII.126), we could have also demanded that

$$\kappa_{(a)} \;:=\; z^{\alpha}_{(a)}\mu^{(a)}_{\alpha} - u^{\dot\alpha}_{(a)}\lambda^{(a)}_{\dot\alpha} \;\stackrel{!}{=}\; 0\;, \tag{VII.127}$$

on every $\mathcal{U}_{(a)}$. These conditions – or equally well (VII.126) – are indeed the appropriate "gluing conditions" for obtaining a twistor space useful in the description of Yang-Mills theory, as we will see. In the following, we will denote the restrictions of the patches $\mathcal{U}_{(a)}$ to $\mathcal{L}^5$ by $\bar{\mathcal{U}}_{(a)} := \mathcal{U}_{(a)} \cap \mathcal{L}^5$.

§30 Double fibration. Because of the quadric condition (VII.127), the moduli $x^{\alpha\dot\alpha}$ and $x^{\alpha\dot\alpha}_{*}$ are not independent on $\mathcal{L}^5$, but one rather has the relation

$$x^{\alpha\dot\alpha} \;=\; x^{\alpha\dot\alpha}_{*}\;, \tag{VII.128}$$

which indeed amounts to taking the diagonal in the moduli space $\mathbb{C}^4 \times \mathbb{C}^4_*$. This will also become explicit in the discussion in §37. With this identification, we can establish the following double fibration using equations (VII.125):

$$\begin{array}{ccc} & \mathcal{F}^6 & \\ {}^{\pi_2}\swarrow & & \searrow{}^{\pi_1} \\ \mathcal{L}^5 & & \mathbb{C}^4 \end{array} \tag{VII.129}$$

where $\mathcal{F}^6 \cong \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ and π_1 is the trivial projection. Here, we have the correspondences

$$\begin{aligned} \big\{\text{subspaces } (\mathbb{C}P^1 \times \mathbb{C}P^1_*)_x \text{ in } \mathcal{L}^5\big\} \;&\longleftrightarrow\; \big\{\text{points } x \text{ in } \mathbb{C}^4\big\}\;,\\ \big\{\text{points } p \text{ in } \mathcal{L}^5\big\} \;&\longleftrightarrow\; \big\{\text{null lines in } \mathbb{C}^4\big\}\;. \end{aligned} \tag{VII.130}$$

The above-mentioned null lines are intersections of α-planes and the dual β-planes, as is evident from recalling the situation for both the twistor and the dual twistor space. Given a solution $(\hat{x}^{\alpha\dot\alpha})$ to the incidence relations (VII.125) for a fixed point p in $\mathcal{L}^5$, the set of points on such a null line takes the form

$$\{(x^{\alpha\dot\alpha})\} \;\;\text{with}\;\; x^{\alpha\dot\alpha} \;=\; \hat{x}^{\alpha\dot\alpha} + t\mu^{\alpha}_{(a)}\lambda^{\dot\alpha}_{(a)}\;,$$

where t is a complex parameter on the null line. The coordinates $\lambda^{\dot\alpha}_{(a)}$ and $\mu^{\alpha}_{(a)}$ can be chosen from arbitrary patches on which they are both well-defined.

§31 Vector fields The space $\mathcal{F}^6$ is covered by four patches $\tilde{\mathcal{U}}_{(a)} := \pi_2^{-1}(\bar{\mathcal{U}}_{(a)})$ and the tangent spaces to the one-dimensional leaves of the fibration $\pi_2 :\; \mathcal{F}^6 \to \mathcal{L}^5$ in (VII.129) are spanned by the holomorphic vector field

$$W^{(a)} \;:=\; \mu^{\alpha}_{(a)}\lambda^{\dot\alpha}_{(a)}\partial_{\alpha\dot\alpha}\;. \tag{VII.131}$$

§32 Flag manifolds. As for the previously discussed twistor spaces, there is a description of the double fibration (VII.129) in the compactified case in terms of flag manifolds. The ambient space of the flags is again $\mathbb{C}^4$, and the double fibration reads

$$\begin{array}{ccc} & F_{123,4} & \\ {}^{\pi_2}\swarrow & & \searrow{}^{\pi_1} \\ F_{13,4} & & F_{2,4} \end{array} \tag{VII.132}$$

where $F_{2,4} = G_{2,4}(\mathbb{C})$ is again the complexified and compactified version of $\mathbb{R}^{3,1}$. The flag manifold $F_{13,4}$ is topologically the zero locus of a quadric in $\mathbb{C}P^3 \times \mathbb{C}P^3_*$. For further details and the super generalization, see e.g. [284, 134].

§33 Real structure τ_1. The Kleinian signature $(2,2)$ is related to anti-linear transformations[10] τ_1 of spinors defined before. Recall that

$$\tau_1 \begin{pmatrix} \omega^1 & \lambda_{\dot{1}} & \sigma^{\dot{1}} & \mu_1 \\ \omega^2 & \lambda_{\dot{2}} & \sigma^{\dot{2}} & \mu_2 \end{pmatrix} = \begin{pmatrix} \bar{\omega}^2 & \bar{\lambda}_{\dot{2}} & \bar{\sigma}^{\dot{2}} & \bar{\mu}_2 \\ \bar{\omega}^1 & \bar{\lambda}_{\dot{1}} & \bar{\sigma}^{\dot{1}} & \bar{\mu}_1 \end{pmatrix}, \tag{VII.133}$$

and obviously $\tau_1^2 = 1$. Correspondingly for $(\lambda_\pm, \mu_\pm) \in \mathbb{C}P^1 \times \mathbb{C}P^1_*$, we have

$$\tau_1(\lambda_+) \;=\; \frac{1}{\bar{\lambda}_+} \;=\; \bar{\lambda}_- \,, \quad \tau_1(\mu_+) \;=\; \frac{1}{\bar{\mu}_+} \;=\; \bar{\mu}_- \tag{VII.134}$$

with stable points

$$\{\lambda, \mu \in \mathbb{C}P^1 \times \mathbb{C}P^1_* : \lambda\bar{\lambda} = 1, \mu\bar{\mu} \;=\; 1\} \;=\; S^1 \times S^1_* \subset \mathbb{C}P^1 \times \mathbb{C}P^1_* \tag{VII.135}$$

and parameterizing a torus $S^1 \times S^1_*$. For the coordinates $(x^{\alpha\dot{\alpha}})$, we have again

$$\tau_1 \begin{pmatrix} x^{1\dot{1}} & x^{1\dot{2}} \\ x^{2\dot{1}} & x^{2\dot{2}} \end{pmatrix} \;=\; \begin{pmatrix} \bar{x}^{2\dot{2}} & \bar{x}^{2\dot{1}} \\ \bar{x}^{1\dot{2}} & \bar{x}^{1\dot{1}} \end{pmatrix} \tag{VII.136}$$

and the real subspace $\mathbb{R}^4$ of $\mathbb{C}^4$ invariant under the involution τ_1 is defined by the equations

$$\bar{x}^{2\dot{2}} \;=\; x^{1\dot{1}} \;=:\; -(x^4 + \mathrm{i}x^3) \quad \text{and} \quad x^{2\dot{1}} \;=\; \bar{x}^{1\dot{2}} \;=:\; -(x^2 - \mathrm{i}x^1) \tag{VII.137}$$

with a metric $\mathrm{d}s^2 = \det(\mathrm{d}x^{\alpha\dot{\alpha}})$ of signature $(2,2)$.

§34 A τ_1-real twistor diagram. Imposing conditions (VII.137), we obtain the real space $\mathbb{R}^{2,2}$ as a fixed point set of the involution $\tau_1 : \mathbb{C}^4 \to \mathbb{C}^4$. Analogously, for the twistor space $\mathbb{C}P^3$ and its open subset $\mathcal{P}^3$, we obtain real subspaces $\mathbb{R}P^3$ and $\mathcal{T}^3$ (cf. §6). Accordingly, a real form of the space $\mathcal{F}^6$ is $\mathcal{F}^6 := \mathbb{R}^{2,2} \times S^1 \times S^1_*$, and we have a real quadric $\mathcal{L}^5 \subset \mathcal{T}^3 \times \mathcal{T}^3_*$ as the subset of fixed points of the involution[11] $\tau_1 : \mathcal{L}^5 \to \mathcal{L}^5$. This quadric is defined by equations (VII.125)-(VII.127) with the $x^{\alpha\dot{\alpha}}$ satisfying (VII.137) and $\lambda_+ = \mathrm{e}^{\mathrm{i}\chi_1} = \lambda_-^{-1}$, $\mu_+ = \mathrm{e}^{\mathrm{i}\chi_2} = \mu_-^{-1}$, $0 \le \chi_1, \chi_2 < 2\pi$. Altogether, we obtain a real form

$$\begin{array}{ccc} & \mathcal{F}^6 & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{L}^5 & & \mathbb{R}^{2,2} \end{array} \tag{VII.138}$$

of the double fibration (VII.129), where all the dimensions labelling the spaces are now *real* dimensions.

§35 The Minkowskian involution τ_M. Let us consider the manifold $\mathcal{P}^3 \times \mathcal{P}^3_*$ with homogeneous coordinates $(\omega^\alpha, \lambda_{\dot{\alpha}}; \sigma^{\dot{\alpha}}, \mu_\alpha)$. The antiholomorphic involution

$$\tau_M : \mathcal{P}^3 \times \mathcal{P}^3_* \;\to\; \mathcal{P}^3 \times \mathcal{P}^3_* \tag{VII.139}$$

gives rise to Minkowski signature on the moduli space of sections. It is defined as the map (see e.g. [189])

$$\tau_M(\omega^\alpha, \lambda_{\dot{\alpha}}; \sigma^{\dot{\alpha}}, \mu_\alpha) \;=\; (-\overline{\sigma^{\dot{\alpha}}}, \overline{\mu_\alpha}; -\overline{\omega^\alpha}, \overline{\lambda_{\dot{\alpha}}}) \tag{VII.140}$$

[10] We will not consider the map τ_0 here.

[11] Again, we use the same symbol τ_1 for maps defined on different spaces.

interchanging α-planes and β-planes. One sees from (VII.140) that the real slice in the space $\mathcal{P}^3 \times \mathcal{P}^3_*$ is defined by the equation[12]

$$\sigma^{\dot\alpha} \;=\; -\overline{\omega^{\alpha}}\ , \quad \mu_\alpha \;=\; \overline{\lambda_{\dot\alpha}}\ . \tag{VII.141}$$

Finally, for coordinates $(x^{\alpha\dot\alpha}) \in \mathbb{C}^4$, we have

$$\tau_M(x^{\alpha\dot\beta}) \;=\; -\bar{x}^{\beta\dot\alpha}\ , \tag{VII.142}$$

and the Minkowskian real slice $\mathbb{R}^{3,1} \subset \mathbb{C}^4$ is parameterized by coordinates

$$\begin{pmatrix} x^{1\dot1} & x^{1\dot2} \\ x^{2\dot1} & x^{2\dot2} \end{pmatrix}^{\dagger} \;=\; -\begin{pmatrix} x^{1\dot1} & x^{1\dot2} \\ x^{2\dot1} & x^{2\dot2} \end{pmatrix} \tag{VII.143}$$

$$\begin{aligned} x^{1\dot1} &= -\mathrm{i}x^0 - \mathrm{i}x^3\ , \qquad & x^{1\dot2} &= -\mathrm{i}x^1 - x^2\ , \\ x^{2\dot1} &= -\mathrm{i}x^1 + x^2\ , \qquad & x^{2\dot2} &= -\mathrm{i}x^0 + \mathrm{i}x^3\ , \end{aligned} \tag{VII.144}$$

with $(x^0, x^1, x^2, x^3) \in \mathbb{R}^{3,1}$ and as in (VII.59), we define

$$\mathrm{d}s^2 \;=\; \det(\mathrm{d}x^{\alpha\dot\alpha}) \;\Rightarrow\; g \;=\; \mathrm{diag}(-1, +1, +1, +1)\ . \tag{VII.145}$$

One can also introduce coordinates

$$\tilde{x}^{\alpha\dot\alpha} \;=\; \mathrm{i}x^{\alpha\dot\alpha}\ , \tag{VII.146}$$

yielding a metric with signature $(1, 3)$. Recall that the involution τ_M interchanges α-planes and β-planes and therefore exchanges opposite helicity states. It might be identified with a $\mathbb{Z}_2$-symmetry discussed recently in the context of mirror symmetry [2] and parity invariance [296].

§36 A τ_M-real twistor diagram. Recall that $(\lambda_{\dot\alpha})$ and (μ_α) are homogeneous coordinates on two Riemann spheres and the involution τ_M maps these spheres one onto another. Moreover, fixed points of the map $\tau_M : \mathbb{C}P^1 \times \mathbb{C}P^1_* \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$ form the Riemann sphere

$$\mathbb{C}P^1 \;=\; \mathrm{diag}(\mathbb{C}P^1 \times \overline{\mathbb{C}P}^1)\ , \tag{VII.147}$$

where $\overline{\mathbb{C}P}^1 (= \mathbb{C}P^1_*)$ denotes the Riemann sphere $\mathbb{C}P^1$ with the opposite complex structure. Therefore, a real slice in the space $\mathcal{F}^6 = \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ introduced in (VII.129) and characterized as the fixed point set of the involution τ_M is the space

$$\mathcal{F}^6_{\tau_M} \;:=\; \mathbb{R}^{3,1} \times \mathbb{C}P^1 \tag{VII.148}$$

of real dimension 6.

The fixed point set of the involution (VII.139) is the diagonal in the space $\mathcal{P}^3 \times \bar{\mathcal{P}}^3$, which can be identified with the complex twistor space $\mathcal{P}^3$ of real dimension 6. This involution also picks out a real quadric $\mathcal{L}^5$ defined by equations (VII.127) and the reality conditions (VII.141)-(VII.144). Thus, we obtain a real version of the double fibration (VII.129),

$$\begin{array}{ccc} & \mathcal{F}^6 & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{L}^5 & & \mathbb{R}^{3,1} \end{array} \tag{VII.149}$$

The dimensions of all spaces in this diagram are again real.

[12]Here, α and $\dot\alpha$ denote *the same* number.

§37 Yang-Mills equations from self-duality equations. Consider a vector bundle E over the space $\mathbb{C}^4 \times \mathbb{C}^4$ with coordinates $r^{\alpha\dot{\alpha}}$ and $s^{\alpha\dot{\alpha}}$. On E, we assume a gauge potential $A = A^r_{\alpha\dot{\alpha}} \mathrm{d} r^{\alpha\dot{\alpha}} + A^s_{\beta\dot{\beta}} \mathrm{d} s^{\beta\dot{\beta}}$. Furthermore, we introduce the coordinates

$$x^{\alpha\dot{\alpha}} \;=\; \tfrac{1}{2}(r^{\alpha\dot{\alpha}} + s^{\alpha\dot{\alpha}}) \quad \text{and} \quad k^{\alpha\dot{\alpha}} \;=\; \tfrac{1}{2}(r^{\alpha\dot{\alpha}} - s^{\alpha\dot{\alpha}}) \tag{VII.150}$$

on the base of E. We claim that the Yang-Mills equations $\nabla^{\alpha\dot{\alpha}} F_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0$ are then equivalent to

$$\begin{aligned}
[\nabla^r_{\alpha\dot{\alpha}}, \nabla^r_{\beta\dot{\beta}}] &= *[\nabla^r_{\alpha\dot{\alpha}}, \nabla^r_{\beta\dot{\beta}}] + \mathcal{O}(k^2) \;, \\
[\nabla^s_{\alpha\dot{\alpha}}, \nabla^s_{\beta\dot{\beta}}] &= -*[\nabla^s_{\alpha\dot{\alpha}}, \nabla^s_{\beta\dot{\beta}}] + \mathcal{O}(k^2) \;, \\
[\nabla^r_{\alpha\dot{\alpha}}, \nabla^s_{\beta\dot{\beta}}] &= \mathcal{O}(k^2) \;,
\end{aligned} \tag{VII.151}$$

where we define[13] $*F^{r,s}_{\alpha\dot{\alpha}\beta\dot{\beta}} := \frac{1}{2}\varepsilon^{r,s}_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} F^{\gamma\dot{\gamma}\delta\dot{\delta}}_{r,s}$ separately on each $\mathbb{C}^4$.

To understand this statement, note that equations (VII.151) are equivalent to

$$\begin{aligned}
[\nabla^x_{\alpha\dot{\alpha}}, \nabla^x_{\beta\dot{\beta}}] &= [\nabla^k_{\alpha\dot{\alpha}}, \nabla^k_{\beta\dot{\beta}}] + \mathcal{O}(k^2) \;, \\
[\nabla^k_{\alpha\dot{\alpha}}, \nabla^x_{\beta\dot{\beta}}] &= *[\nabla^k_{\alpha\dot{\alpha}}, \nabla^k_{\beta\dot{\beta}}] + \mathcal{O}(k^2) \;,
\end{aligned} \tag{VII.152}$$

which is easily seen by performing the coordinate change from (r, s) to (x, k). These equations are solved by the expansion [290, 139]

$$\begin{aligned}
A^k_{\alpha\dot{\alpha}} &= -\tfrac{1}{2} F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}} k^{\beta\dot{\beta}} - \tfrac{1}{3} k^{\gamma\dot{\gamma}} \nabla^{x,0}_{\gamma\dot{\gamma}} (*F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}) k^{\beta\dot{\beta}} \;, \\
A^x_{\alpha\dot{\alpha}} &= A^{x,0}_{\alpha\dot{\alpha}} - *F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}} k^{\beta\dot{\beta}} - \tfrac{1}{2} k^{\gamma\dot{\gamma}} \nabla^{x,0}_{\gamma\dot{\gamma}} (F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}}) k^{\beta\dot{\beta}} \;,
\end{aligned} \tag{VII.153}$$

if and only if $\nabla^{\alpha\dot{\alpha}}_{x,0} F^{x,0}_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0$ is satisfied. Here, a superscript 0 always denotes an object evaluated at $k^{\alpha\dot{\alpha}} = 0$. Thus we saw that a solution to the Yang-Mills equations corresponds to a solution to equations (VII.151) on $\mathbb{C}^4 \times \mathbb{C}^4$.

§38 Third order neighborhoods. As discussed before, the self-dual and anti-self-dual field strengths solving the first and second equation of (VII.151) can be mapped to certain holomorphic vector bundles over $\mathcal{P}^3$ and $\mathcal{P}^3_*$, respectively. On the other hand, the potentials given in (VII.153) are now defined on a second order infinitesimal neighborhood[14] of the diagonal in $\mathbb{C}^4 \times \mathbb{C}^4$ for which $\mathcal{O}(k^3) = 0$. In the twistor description, this potential corresponds to a transition function $f_{+-} \sim \psi_+^{-1}\psi_-$, where the Čech 0-cochain $\{\psi_\pm\}$ is a solution to the equations

$$\begin{aligned}
\lambda^{\dot{\alpha}}_\pm \left(\frac{\partial}{\partial r^{\alpha\dot{\alpha}}} + A^r_{\alpha\dot{\alpha}} \right) \psi_\pm &= \mathcal{O}(k^4) \;, \\
\mu^{\alpha}_\pm \left(\frac{\partial}{\partial s^{\alpha\dot{\alpha}}} + A^s_{\alpha\dot{\alpha}} \right) \psi_\pm &= \mathcal{O}(k^4) \;.
\end{aligned} \tag{VII.154}$$

Roughly speaking, since the gauge potentials are defined to order k^2 and since $\frac{\partial}{\partial r^{\alpha\dot{\alpha}}}$ and $\frac{\partial}{\partial s^{\alpha\dot{\alpha}}}$ contain derivatives with respect to k, the above equations can indeed be rendered exact to order k^3. The exact definition of the transition function is given by

$$f_{+-,i} \;:=\; \sum_{j=0}^{i} \psi^{-1}_{+,j} \psi_{-,i-j} \;, \tag{VII.155}$$

[13] One could also insert an i into this definition but on $\mathbb{C}^4$, this is not natural.

[14] not a thickening

where the additional indices label the order in k. On the twistor space side, a third order neighborhood in k corresponds to a third order thickening in

$$\kappa_{(a)} := z^{\alpha}_{(a)}\mu^{(a)}_{\alpha} - u^{\dot\alpha}_{(a)}\lambda^{(a)}_{\dot\alpha} . \qquad \text{(VII.156)}$$

Altogether, we see that a solution to the Yang-Mills equations corresponds to a topologically trivial holomorphic vector bundle over a third order thickening of $\mathcal{L}^5$ in $\mathcal{P}^3 \times \mathcal{P}^3_*$, which becomes holomorphically trivial, when restricted to any $\mathbb{C}P^1 \times \mathbb{C}P^1_* \hookrightarrow \mathcal{L}^5$.

VII.4 Supertwistor spaces

So far we encountered two twistor spaces: the twistor space $\mathcal{P}^3$, which is an open subset of $\mathbb{C}P^3$ and the ambitwistor space, which is a third order thickening of the quadric $\mathcal{L}^5$ in $\mathcal{P}^3 \times \mathcal{P}^3_*$. In this section, we discuss the extension of the former by Graßmann-odd directions [95]. Further extensions and the extension of the ambitwistor space will be discussed in subsequent sections.

VII.4.1 The superextension of the twistor space

§1 Complex projective superspaces. A super extension of the twistor space $\mathbb{C}P^3$ is the supermanifold $\mathbb{C}P^{3|\mathcal{N}}$ with homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha}, \eta_i)$ subject to the identification $(\omega^\alpha, \lambda_{\dot\alpha}, \eta_i) \sim (t\,\omega^\alpha, t\,\lambda_{\dot\alpha}, t\,\eta_i)$ for any nonzero complex scalar t. Here, $(\omega^\alpha, \lambda_{\dot\alpha})$ are again homogeneous coordinates on $\mathbb{C}P^3$ and η_i with $i = 1, \ldots, \mathcal{N}$ are Graßmann variables. Interestingly, this supertwistor space is a Calabi-Yau supermanifold in the case $\mathcal{N} = 4$ and one may consider the topological B-model introduced in section V.3.3 with this space as target space [297].

§2 Supertwistor spaces. Let us now neglect the super light cone at infinity similarly to the discussion in section VII.3.1. That is, we consider analogously to the space $\mathcal{P}^3 = \mathbb{C}P^3\backslash\mathbb{C}P^1 = \mathcal{O}(1) \oplus \mathcal{O}(1)$ its super extension $\mathcal{P}^{3|\mathcal{N}}$ covered by two patches, $\mathcal{P}^{3|\mathcal{N}} = \mathbb{C}P^{3|\mathcal{N}}\backslash\mathbb{C}P^{1|\mathcal{N}} = \hat{\mathcal{U}}_+ \cup \hat{\mathcal{U}}_-$, with even coordinates (VII.39) and odd coordinates

$$\eta^+_i = \frac{\eta_i}{\lambda_{\dot 1}} \text{ on } \hat{\mathcal{U}}_+ \quad \text{and} \quad \eta^-_i = \frac{\eta_i}{\lambda_{\dot 2}} \text{ on } \hat{\mathcal{U}}_- \qquad \text{(VII.157)}$$

related by

$$\eta^+_i = z^3_+\eta^-_i \qquad \text{(VII.158)}$$

on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. We see from (VII.157) and (VII.158) that the fermionic coordinates are sections of $\Pi\mathcal{O}(1)$. The supermanifold $\mathcal{P}^{3|\mathcal{N}}$ is fibred over $\mathbb{C}P^{1|0}$,

$$\mathcal{P}^{3|\mathcal{N}} \to \mathbb{C}P^{1|0} , \qquad \text{(VII.159)}$$

with superspaces $\mathbb{C}^{2|\mathcal{N}}_\lambda$ as fibres over $\lambda \in \mathbb{C}P^{1|0}$. We also have a second fibration

$$\mathcal{P}^{3|\mathcal{N}} \to \mathbb{C}P^{1|\mathcal{N}} \qquad \text{(VII.160)}$$

with $\mathbb{C}^{2|0}_\lambda$ as fibres.

§3 Global sections and their moduli. The global holomorphic sections of the bundle (VII.159) are rational curves $\mathbb{C}P^1_{x_R,\eta}\hookrightarrow\mathcal{P}^{3|\mathcal{N}}$ parameterized by moduli $(x_R,\eta) = (x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha})\in\mathbb{C}_R^{4|2\mathcal{N}}$ according to

$$\begin{aligned} z_+^\alpha &= x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^+\ , \quad \eta_i^+ = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^+ \quad \text{for}\quad (\lambda_{\dot\alpha}^+) = (1,\lambda_+)^T\ ,\quad \lambda_+\in U_+\ ,\\ z_-^\alpha &= x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^-\ , \quad \eta_i^- = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^- \quad \text{for}\quad (\lambda_{\dot\alpha}^-) = (\lambda_-,1)^T\ ,\quad \lambda_-\in U_-\ . \end{aligned} \tag{VII.161}$$

Here, the space $\mathbb{C}_R^{4|2\mathcal{N}}$ is indeed the anti-chiral superspace. Equations (VII.161) define again a supertwistor correspondence via the double fibration

$$\begin{array}{ccc} & \mathcal{F}_R^{5|2\mathcal{N}} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}_R^{4|2\mathcal{N}} \end{array} \tag{VII.162}$$

where $\mathcal{F}^{5|2\mathcal{N}}\cong\mathbb{C}_R^{4|2\mathcal{N}}\times\mathbb{C}P^1$ and the projections are defined as

$$\pi_1(x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha},\lambda_{\dot\alpha}^\pm) := (x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha}) \quad\text{and}\quad \pi_2(x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha},\lambda_{\dot\alpha}^\pm) := (x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm,\lambda_\pm,\eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm)\ . \tag{VII.163}$$

The supertwistor correspondence now reads explicitly

$$\begin{aligned} \left\{\text{projective lines } \mathbb{C}P^1_{x,\eta} \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{points } (x,\eta) \text{ in } \mathbb{C}^{4|2\mathcal{N}}\right\}\ ,\\ \left\{\text{points } p \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{null } (\alpha_R\text{-})\text{superplanes } \mathbb{C}_p^{2|2\mathcal{N}} \text{ in } \mathbb{C}^{4|2\mathcal{N}}\right\}\ . \end{aligned}$$

Given a solution $(\hat x^{\alpha\dot\alpha},\hat\eta_i^{\dot\alpha})$ to the incidence relations (VII.161) for a fixed point $p\in\mathcal{P}^{3|\mathcal{N}}$, the set of all solutions is given by

$$\{(x^{\alpha\dot\alpha},\eta_i^{\dot\alpha})\} \quad\text{with}\quad x^{\alpha\dot\alpha} = \hat x^{\alpha\dot\alpha}+\mu^\alpha\lambda_\pm^{\dot\alpha} \quad\text{and}\quad \eta_i^{\dot\alpha} = \hat\eta_i^{\dot\alpha}+\varepsilon_i\lambda_\pm^{\dot\alpha}\ , \tag{VII.164}$$

where μ^α is an arbitrary commuting two-spinor and ε_i is an arbitrary vector with Graßmann-odd entries. The sets defined in (VII.164) are then called *null* or α_R*-superplanes*, and they are of superdimension $2|\mathcal{N}$.

§4 Global sections of a different kind. In the previous paragraph, we discussed sections of the bundle (VII.159), which is naturally related to the discussion of the bosonic twistor space before. One can, however, also discuss sections of the bundle (VII.160), which will give rise to a relation to the dual supertwistor space $\mathcal{P}_*^{3|\mathcal{N}}$ and its moduli space.

The global holomorphic sections of the bundle (VII.160) are spaces $\mathbb{C}P^{1|\mathcal{N}}_{x_L,\theta}\hookrightarrow\mathcal{P}^{3|\mathcal{N}}$ defined by the equations

$$z_\pm^\alpha = x_L^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm - 2\theta^{\alpha i}\eta_i^\pm \quad\text{with}\quad (\lambda_{\dot\alpha}^\pm,\eta_i^\pm)\in\hat{\mathcal{U}}_\pm\cap\mathbb{C}P^{1|\mathcal{N}} \tag{VII.165}$$

and parameterized by the moduli $(x_L,\theta) = (x_L^{\alpha\dot\alpha},\theta^{\alpha i})\in\mathbb{C}_L^{4|2\mathcal{N}}$. Note that the moduli space is the chiral superspace $\mathbb{C}_L^{4|2\mathcal{N}}$, contrary to the anti-chiral superspace $\mathbb{C}_R^{4|2\mathcal{N}}$, which arose as the moduli space of global sections of the bundle (VII.159). Equations (VII.165) define another supertwistor correspondence,

$$\begin{array}{ccc} & \mathcal{F}_L^{5|3\mathcal{N}} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}_L^{4|2\mathcal{N}} \end{array} \tag{VII.166}$$

where $\mathcal{F}_L^{5|3\mathcal{N}} := \mathbb{C}_L^{4|2\mathcal{N}}\times\mathbb{C}P^{1|\mathcal{N}}$. The twistor correspondence here reads

$$\begin{aligned} \left\{\text{superspheres } \mathbb{C}P^{1|\mathcal{N}}_{x_L,\eta} \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{points } (x_L,\theta) \text{ in } \mathbb{C}_L^{4|2\mathcal{N}}\right\}\ ,\\ \left\{\text{points } p \text{ in } \mathcal{P}^{3|\mathcal{N}}\right\} &\longleftrightarrow \left\{\text{null } (\alpha_L\text{-})\text{superplanes } \mathbb{C}_p^{2|2\mathcal{N}} \text{ in } \mathbb{C}_L^{4|2\mathcal{N}}\right\}\ . \end{aligned}$$

§5 Relation between the moduli spaces. From (VII.161) and (VII.165) we can deduce that

$$x_R^{\alpha\dot\alpha} \;=\; x^{\alpha\dot\alpha} - \theta^{\alpha i}\eta_i^{\dot\alpha} \quad \text{and} \quad x_L^{\alpha\dot\alpha} \;=\; x^{\alpha\dot\alpha} + \theta^{\alpha i}\eta_i^{\dot\alpha}\ , \tag{VII.167}$$

where $(x^{\alpha\dot\alpha}) \in \mathbb{C}^{4|0}$ are "symmetric" (non-chiral) bosonic coordinates. Substituting the first equation of (VII.167) into (VII.161), we obtain the equations

$$z_\pm^\alpha \;=\; x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm - \theta^{\alpha i}\eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm\ , \quad \eta_i^\pm \;=\; \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm \tag{VII.168}$$

defining degree one curves $\mathbb{C}P^1_{x,\theta,\eta}\hookrightarrow\mathcal{P}^{3|\mathcal{N}}$ which are evidently parameterized by moduli $(x^{\alpha\dot\alpha}, \theta^{\alpha i}, \eta_i^{\dot\alpha}) \in \mathbb{C}^{4|4\mathcal{N}}$. Therefore we obtain a third double fibration

$$\begin{array}{ccc} & \mathcal{F}^{5|4\mathcal{N}} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|4\mathcal{N}} \end{array} \tag{VII.169}$$

with coordinates

$$(x^{\alpha\dot\alpha}\,,\,\lambda_{\dot\alpha}^\pm\,,\,\theta^{\alpha i}\,,\,\eta_i^{\dot\alpha}) \quad \text{on} \quad \mathcal{F}^{5|4\mathcal{N}} \;:=\; \mathbb{C}^{4|4\mathcal{N}}\times\mathbb{C}P^1\ , \tag{VII.170a}$$

$$(x^{\alpha\dot\alpha}\,,\,\theta^{\alpha i}\,,\,\eta_i^{\dot\alpha}) \quad \text{on} \quad \mathbb{C}^{4|4\mathcal{N}}\ , \tag{VII.170b}$$

$$z_\pm^\alpha\ ,\quad \lambda_{\dot\alpha}^\pm\ ,\quad \eta_i^\pm \quad \text{on} \quad \mathcal{P}^{3|\mathcal{N}}\ . \tag{VII.170c}$$

The definition of the projection π_1 is obvious and π_2 is defined by (VII.161) and (VII.167).

The double fibration (VII.169) generalizes both (VII.162) and (VII.166) and defines the following twistor correspondence:

$$\begin{aligned} \big\{\text{projective line } \mathbb{C}P^1_{x,\theta,\eta} \text{ in } \mathcal{P}^{3|\mathcal{N}}\big\} &\;\longleftrightarrow\; \big\{\text{points } (x,\theta,\eta) \text{ in } \mathbb{C}^{4|4\mathcal{N}}\big\}\ , \\ \big\{\text{points } p \text{ in } \mathcal{P}^{3|\mathcal{N}}\big\} &\;\longleftrightarrow\; \Big\{\text{null } (\alpha\text{-})\text{superplanes } \mathbb{C}_p^{2|3\mathcal{N}} \text{ in } \mathbb{C}^{4|4\mathcal{N}}\Big\}\ . \end{aligned}$$

§6 Vector fields. Note that one can project from $\mathcal{F}^{5|4\mathcal{N}}$ onto $\mathcal{P}^{3|\mathcal{N}}$ in two steps: first from $\mathcal{F}^{5|4\mathcal{N}}$ onto $\mathcal{F}_R^{5|2\mathcal{N}}$, which is given in coordinates by

$$(x^{\alpha\dot\alpha}\,,\,\lambda_{\dot\alpha}^\pm\,,\,\theta^{\alpha i}\,,\,\eta_i^{\dot\alpha}) \;\to\; (x_R^{\alpha\dot\alpha}\,,\,\lambda_{\dot\alpha}^\pm\,,\,\eta_i^{\dot\alpha}) \tag{VII.171}$$

with the $x_R^{\alpha\dot\alpha}$ from (VII.167), and then from $\mathcal{F}_R^{5|2\mathcal{N}}$ onto $\mathcal{P}^{3|\mathcal{N}}$, which is given in coordinates by

$$(x_R^{\alpha\dot\alpha}\,,\,\lambda_{\dot\alpha}^\pm\,,\,\eta_i^{\dot\alpha}) \;\to\; (x_R^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm\,,\,\lambda_{\dot\alpha}^\pm\,,\,\eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm)\ . \tag{VII.172}$$

The tangent spaces to the $(0|2\mathcal{N})$-dimensional leaves of the fibration (VII.171) are spanned by the vector fields

$$D_{\alpha i} \;=\; \frac{\partial}{\partial\theta^{\alpha i}} + \eta_i^{\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}} \;=:\; \partial_{\alpha i} + \eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha} \tag{VII.173}$$

on $\mathbb{C}^{4|4\mathcal{N}} \subset \mathcal{F}^{5|4\mathcal{N}}$. The coordinates $x_R^{\alpha\dot\alpha}$, $\lambda_{\dot\alpha}^\pm$ and $\eta_i^{\dot\alpha}$ belong to the kernel of these vector fields, which are also tangent to the fibres of the projection $\mathbb{C}^{4|4\mathcal{N}} \to \mathbb{C}_R^{4|2\mathcal{N}}$ onto the antichiral superspace. The tangent spaces to the $(2|\mathcal{N})$-dimensional leaves of the projection (VII.172) are spanned by the vector fields[15]

$$\bar V_\alpha^\pm \;=\; \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha}^R\ , \tag{VII.174a}$$

$$\bar V_\pm^i \;=\; \lambda_\pm^{\dot\alpha}\,\partial_{\dot\alpha}^i \quad \text{with} \quad \partial_{\dot\alpha}^i \;:=\; \frac{\partial}{\partial\eta_i^{\dot\alpha}}\ , \tag{VII.174b}$$

where $\partial_{\alpha\dot\alpha}^R = \frac{\partial}{\partial x_R^{\alpha\dot\alpha}} = \frac{\partial}{\partial x^{\alpha\dot\alpha}}$.

[15]For the definition of $\lambda_\pm^{\dot\alpha}$, see section VII.51.

§7 Dual supertwistor space. The dual supertwistor space $\mathcal{P}_*^{3|\mathcal{N}}$ is obtained from the complex projective space $\mathbb{C}P_*^{3|\mathcal{N}}$ with homogeneous coordinates $(\sigma^{\dot\alpha},\mu_\alpha,\theta^i)$ by demanding that $\mu_\alpha\neq(0,0)^T$. Thus, the space $\mathcal{P}_*^{3|\mathcal{N}}=\mathbb{C}P_*^{3|\mathcal{N}}\backslash\mathbb{C}P_*^{1|\mathcal{N}}$ is covered by the two patches $\hat{\mathcal{V}}_\pm$ with the inhomogeneous coordinates

$$u_+^{\dot\alpha}=\frac{\sigma^{\dot\alpha}}{\mu_1}\ ,\quad u_+^{\dot3}=\mu_+=\frac{\mu_2}{\mu_1}\quad\text{and}\quad\theta_+^i=\frac{\theta^i}{\mu_1}\quad\text{on}\quad\hat{\mathcal{V}}_+\ , \tag{VII.175}$$

$$w_-^{\dot\alpha}=\frac{\sigma^{\dot\alpha}}{\mu_2}\ ,\quad u_-^{\dot3}=\mu_-=\frac{\mu_1}{\mu_2}\quad\text{and}\quad\theta_-^i=\frac{\theta^i}{\mu_2}\quad\text{on}\quad\hat{\mathcal{V}}_-\ , \tag{VII.176}$$

$$u_+^{\dot\alpha}=\mu_+u_-^{\dot\alpha}\ ,\qquad\mu_+=\mu_-^{-1}\ ,\qquad\theta_+^i=\mu_+\theta_-^i\quad\text{on}\quad\hat{\mathcal{V}}_+\cap\hat{\mathcal{V}}_-\ . \tag{VII.177}$$

Sections of the bundle $\mathcal{P}_*^{3|\mathcal{N}}\to\mathbb{C}P_*^{1|0}$ (degree one holomorphic curves $\mathbb{C}P^1_{x_L,\theta}\hookrightarrow\mathcal{P}_*^{3|\mathcal{N}}$) are defined by the equations

$$w_\pm^{\dot\alpha}\ =\ x_L^{\alpha\dot\alpha}\mu_\alpha^\pm\ ,\quad\theta_\pm^i\ =\ \theta^{\alpha i}\mu_\alpha^\pm\quad\text{with}\quad(\mu_\alpha^+)\ =\ \begin{pmatrix}1\\ \mu_+\end{pmatrix}\ ,\quad(\mu_\alpha^-)\ =\ \begin{pmatrix}\mu_-\\ 1\end{pmatrix}\ , \tag{VII.178}$$

and parameterized by moduli $(x_L^{\alpha\dot\alpha},\theta^{\alpha i})\in\mathbb{C}_L^{4|2\mathcal{N}}$. Note that similarly to the supertwistor case, one can consider furthermore sections of the bundle $\mathcal{P}_*^{3|\mathcal{N}}\to\mathbb{C}P_*^{1|\mathcal{N}}$.

Equations (VII.178) give again rise to a double fibration

$$\begin{array}{ccc} & \mathcal{F}_*^{5|2\mathcal{N}} & \\ {\scriptstyle\pi_2}\swarrow & & \searrow{\scriptstyle\pi_1} \\ \mathcal{P}_*^{3|\mathcal{N}} & & \mathbb{C}_L^{4|2\mathcal{N}} \end{array} \tag{VII.179}$$

and the tangent spaces of the $(2|\mathcal{N})$-dimensional leaves of the projection $\pi_2:\mathcal{F}_*^{5|2\mathcal{N}}\to\mathcal{P}_*^{3|\mathcal{N}}$ from (VII.179) are spanned by the vector fields $\bar V_\alpha^\pm=\lambda_\pm^{\dot\alpha}\partial^L_{\alpha\dot\alpha}$ and $\bar V_i^\pm=\mu^\alpha\frac{\partial}{\partial\theta^{i\alpha}}$.

§8 Real structure. Three real structures $\tau_{\pm1},\tau_0$ can be imposed similarly to the bosonic case. We will focus on the two real structures τ_ε and define additionally to (VII.52a)

$$\tau_1(\eta_i^\pm)\ =\ \left(\frac{\bar\eta_i^\pm}{\bar z_\pm^3}\right)\ ,\qquad\tau_{-1}(\eta_1^\pm,\eta_2^\pm,\eta_3^\pm,\eta_4^\pm)\ =\ \left(\frac{\mp\bar\eta_2^\pm}{\bar z_\pm^3},\frac{\pm\bar\eta_1^\pm}{\bar z_\pm^3},\frac{\mp\bar\eta_4^\pm}{\bar z_\pm^3},\frac{\pm\bar\eta_3^\pm}{\bar z_\pm^3}\right)\ . \tag{VII.180}$$

For $\varepsilon=+1$, one can truncate the involution τ_ε to the cases $\mathcal{N}<4$, which is in the Euclidean case only possible for $\mathcal{N}=2$, see also the discussion in III.4.2, §13. The corresponding reality conditions for the fermionic coordinates on the correspondence and moduli spaces are also found in section III.4.2. As before, we will denote the real supertwistor spaces by $\mathcal{P}_\varepsilon^{3|4}$.

§9 Identification of vector fields. On $\mathcal{P}_\varepsilon^{3|4}$, there is the following relationship between vector fields of type (0,1) in the coordinates $(z_\pm^\alpha,z_\pm^3,\eta_i^\pm)$ and vector fields in the coordinates $(x_R^{\alpha\dot\alpha},\lambda_\pm,\eta_i^{\dot\alpha})$:

$$\begin{aligned}\frac{\partial}{\partial\bar z_\pm^1}\ &=\ -\gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial x_R^{2\dot\alpha}}\ =:\ -\gamma_\pm\bar V_2^\pm\ , &\qquad \frac{\partial}{\partial\bar z_\pm^2}\ &=\ \gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial x_R^{1\dot\alpha}}\ =:\ -\varepsilon\gamma_\pm\bar V_1^\pm\ ,\\ \frac{\partial}{\partial\bar z_+^3}\ &=\ \frac{\partial}{\partial\bar\lambda_+}+\varepsilon\gamma_+x_R^{\alpha\dot1}\bar V_\alpha^++\varepsilon\gamma_+\eta_i^{\dot1}\bar V_+^i\ , &\qquad \frac{\partial}{\partial\bar z_-^3}\ &=\ \frac{\partial}{\partial\bar\lambda_-}+\gamma_-x_R^{\alpha\dot2}\bar V_\alpha^-+\gamma_-\eta_i^{\dot2}\bar V_-^i\ .\end{aligned} \tag{VII.181}$$

In the Kleinian case, one obtains additionally for the fermionic vector fields

$$\frac{\partial}{\partial\bar\eta_i^\pm}\ =\ -\gamma_\pm\bar V_\pm^i\ , \tag{VII.182}$$

while in the Euclidean case, we have

$$\begin{aligned} \frac{\partial}{\partial\bar\eta_1^\pm} &= \gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_2^{\dot\alpha}} =: \gamma_\pm\bar V_\pm^2\ , & \frac{\partial}{\partial\bar\eta_2^\pm} &= -\gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_1^{\dot\alpha}} =: -\gamma_\pm\bar V_\pm^1\ ,\\ \frac{\partial}{\partial\bar\eta_3^\pm} &= \gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_4^{\dot\alpha}} =: \gamma_\pm\bar V_\pm^4\ , & \frac{\partial}{\partial\bar\eta_4^\pm} &= -\gamma_\pm\lambda_\pm^{\dot\alpha}\frac{\partial}{\partial\eta_3^{\dot\alpha}} =: -\gamma_\pm\bar V_\pm^3\ . \end{aligned} \tag{VII.183}$$

§10 Forms. It will also be useful to introduce $(0,1)$-forms $\bar E_\pm^a$ and $\bar E_i^\pm$ which are dual to $\bar V_a^\pm$ and $\bar V_\pm^i$, respectively, i.e.

$$\bar V_a^\pm \lrcorner \bar E_\pm^b = \delta_a^b \quad\text{and}\quad \bar V_\pm^i \lrcorner \bar E_j^\pm = \delta_j^i\ . \tag{VII.184}$$

Here, $\lrcorner$ denotes the interior product of vector fields with differential forms. Explicitly, the dual $(0,1)$-forms are given by the formulæ

$$\bar E_\pm^\alpha = -\gamma_\pm\hat\lambda_{\dot\alpha}^\pm \mathrm{d}x^{\alpha\dot\alpha}\ ,\quad \bar E_\pm^3 = \mathrm{d}\bar\lambda_\pm \quad\text{and}\quad \bar E_i^\pm = -\gamma_\pm\hat\lambda_{\dot\alpha}^\pm \mathrm{d}\eta_i^{\dot\alpha}\ . \tag{VII.185}$$

In the case $\mathcal{N}=4$, one can furthermore introduce the (nowhere vanishing) holomorphic volume form Ω, which is locally given as

$$\Omega_\pm := \Omega|_{\mathcal{U}_\pm} := \pm\mathrm{d}\lambda_\pm\wedge\mathrm{d}z_\pm^1\wedge\mathrm{d}z_\pm^2\mathrm{d}\eta_1^\pm\ldots\mathrm{d}\eta_4^\pm =: \pm\mathrm{d}\lambda_\pm\wedge\mathrm{d}z_\pm^1\wedge\mathrm{d}z_\pm^2\ \Omega_\pm^\eta \tag{VII.186}$$

on $\mathcal{P}^{3|4}$, independently of the real structure. The existence of this volume element implies that the Berezinian line bundle is trivial and consequently $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold [297], see also section III.2.5, §33. Note, however, that Ω is not a differential form because its fermionic part transforms as a product of Graßmann-odd vector fields, i.e. with the inverse of the Jacobian. Such forms are called integral forms.

§11 Comment on the notation. Instead of the shorthand notation $\mathcal{P}^{3|\mathcal{N}}$, we will sometimes write $(\mathcal{P}^3,\mathcal{O}_{[\mathcal{N}]})$ in the following, which makes the extension of the structure sheaf of $\mathcal{P}^3$ explicit. The sheaf $\mathcal{O}_{[\mathcal{N}]}$ is locally the tensor product of the structure sheaf of $\mathcal{P}^3$ and a Graßmann algebra of $\mathcal{N}$ generators.

VII.4.2 The Penrose-Ward transform for $\mathcal{P}^{3|\mathcal{N}}$

Similarly to the bosonic case, one can built a Penrose-Ward transform between certain holomorphic vector bundles over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ and solutions to the supersymmetric self-dual Yang-Mills equations[16] on $\mathbb{C}^4$ [256, 280, 281, 282, 269].

§12 Holomorphic bundles over $\mathcal{P}^{3|\mathcal{N}}$. In analogy to the purely bosonic discussion, let us consider a topologically trivial holomorphic vector bundle $\mathcal{E}$ over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$, which becomes holomorphically trivial, when restricted to any subset $\mathbb{C}P^1_{x,\eta}\subset\mathcal{P}^{3|\mathcal{N}}$. Note that the vector bundle $\mathcal{E}$ has ordinary, bosonic fibres and thus is *not* a supervector bundle. Since the underlying base manifold is a supermanifold, the sections of $\mathcal{E}$ are, however, vector-valued superfunctions. As usual, the bundle $\mathcal{E}\to\mathcal{P}^{3|\mathcal{N}}$ is defined by a holomorphic transition function f_{+-} which can be split according to

$$f_{+-} = \hat\psi_+^{-1}\hat\psi_-\ , \tag{VII.187}$$

where $\hat\psi_\pm$ are smooth $\mathsf{GL}(n,\mathbb{C})$-valued functions on the patches $\hat{\mathcal{U}}_\pm$ covering $\mathcal{P}^{3|\mathcal{N}}$.

[16] For a deformation of the supertwistor geometry yielding chiral mass terms, see [64].

§13 Holomorphic Chern-Simons equations. The splitting (VII.187) together with the holomorphy of the transition function

$$\frac{\partial}{\partial \bar{z}^1_\pm} f_{+-} \;=\; \frac{\partial}{\partial \bar{z}^2_\pm} f_{+-} \;=\; \frac{\partial}{\partial \bar{z}^3_\pm} f_{+-} \;=\; 0 \tag{VII.188}$$

leads to the equations

$$\hat{\psi}_+ \bar{\partial} \hat{\psi}_+^{-1} \;=\; \hat{\psi}_- \bar{\partial} \hat{\psi}_-^{-1} \;. \tag{VII.189}$$

Completely analogously to the purely bosonic case, we introduce a gauge potential $\mathcal{A}$ by

$$\hat{\mathcal{A}}^{0,1}_\pm \;=\; \hat{\psi}_\pm \bar{\partial} \hat{\psi}_\pm^{-1} \;, \tag{VII.190}$$

which fits into the linear system

$$(\bar{\partial} + \hat{\mathcal{A}}^{0,1}) \hat{\psi}_\pm \;=\; 0 \;. \tag{VII.191}$$

The compatibility conditions of this linear system are again the holomorphic Chern-Simons equations of motion

$$\bar{\partial} \hat{\mathcal{A}} + \hat{\mathcal{A}} \wedge \hat{\mathcal{A}} \;=\; 0 \;, \tag{VII.192}$$

and thus $\hat{\mathcal{A}}^{0,1}$ gives rise to a holomorphic structure on $\mathcal{P}^{3|\mathcal{N}}$.

In the following, we will always assume that we are working in a gauge for which

$$\frac{\partial}{\partial \bar{\eta}^i_\pm} \hat{\psi}_\pm \;=\; 0 \;\Leftrightarrow\; \frac{\partial}{\partial \bar{\eta}_i^\pm} \lrcorner \hat{\mathcal{A}}^{0,1} \;=\; 0 \;, \tag{VII.193}$$

for $a = 1, 2, 3$.

§14 Action for hCS theory. In the case $\mathcal{N} = 4$, the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ is a Calabi-Yau supermanifold, and thus comes with the holomorphic volume form Ω defined in (VII.186). One can therefore introduce an action functional

$$S_{\rm hCS} \;:=\; \int_{\mathscr{P}^{3|\mathcal{N}}_\varepsilon} \Omega \wedge \mathrm{tr}_{\,\mathcal{G}} \left(\hat{\mathcal{A}}^{0,1} \wedge \bar{\partial} \hat{\mathcal{A}}^{0,1} + \tfrac{2}{3} \hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1} \right) \;, \tag{VII.194}$$

where $\mathscr{P}^{3|\mathcal{N}}_\varepsilon$ is the subspace of $\mathcal{P}^{3|\mathcal{N}}_\varepsilon$ for which[17] $\bar{\eta}_i^\pm = 0$ [297]. Note that the condition (VII.193), which we introduced in the previous paragraph, is necessary for (VII.194) to be meaningful.

§15 Pull-back of $\mathcal{E}$ to the correspondence space. The pull-back of $\mathcal{E}$ along π_2 is the bundle $\pi_2^* \mathcal{E}$ with transition function π_2^* satisfying the equations

$$\bar{V}^\pm_\alpha (\pi_2^* f_{+-}) \;=\; \bar{V}^i_\pm (\pi_2^* f_{+-}) \;=\; 0 \;. \tag{VII.195}$$

These equations, together with the splitting

$$\pi_2^* f_{+-} \;=\; \psi_+^{-1} \psi_- \tag{VII.196}$$

of the transition function into group-valued holomorphic functions $\psi_\pm$ on $\pi_2^{-1}(\mathcal{U}_\pm)$, allow for the introduction of matrix-valued components of a new gauge potential,

$$\mathcal{A}^+_\alpha \;:=\; \bar{V}^+_\alpha \lrcorner \mathcal{A} \;=\; \psi_+ \bar{V}^+_\alpha \psi_+^{-1} \;=\; \psi_- \bar{V}^+_\alpha \psi_-^{-1} \;=\; \lambda^{\dot{\alpha}}_+ \mathcal{A}_{\alpha\dot{\alpha}}(x_R, \eta) \;, \tag{VII.197a}$$

$$\mathcal{A}_{\bar{\lambda}_+} \;:=\; \partial_{\bar{\lambda}_+} \lrcorner \mathcal{A} \;=\; \psi_+ \partial_{\bar{\lambda}_+} \psi_+^{-1} \;=\; \psi_- \partial_{\bar{\lambda}_+} \psi_-^{-1} \;=\; 0 \;, \tag{VII.197b}$$

$$\mathcal{A}^i_+ \;:=\; \bar{V}^i_+ \lrcorner \mathcal{A} \;=\; \psi_+ \bar{V}^i_+ \psi_+^{-1} \;=\; \psi_- \bar{\partial}^i_+ \psi_-^{-1} \;=\; \lambda^{\dot{\alpha}}_+ \mathcal{A}^i_{\dot{\alpha}}(x_R, \eta) \;. \tag{VII.197c}$$

[17] This restriction to a chiral subspace was proposed in [297] and is related to self-duality. It is not a contradiction to $\eta_i^\pm \neq 0$, but merely a restriction of all functions on $\mathcal{P}^{3|\mathcal{N}}_\varepsilon$ to be holomorphic in the $\eta_i^\pm$.

§16 Linear system and super SDYM equations. The gauge potential defined above fits into the linear system

$$(\bar{V}^+_\alpha + \mathcal{A}^+_\alpha)\psi_+ = 0 \, , \qquad \text{(VII.198a)}$$

$$\partial_{\bar{\lambda}_+}\psi_+ = 0 \, , \qquad \text{(VII.198b)}$$

$$(\bar{V}^i_+ + \mathcal{A}^i_+)\psi_+ = 0 \qquad \text{(VII.198c)}$$

of differential equations, whose compatibility conditions read

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] + [\nabla_{\alpha\dot{\beta}}, \nabla_{\beta\dot{\alpha}}] = 0 \, , \quad [\nabla^i_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] + [\nabla^i_{\dot{\beta}}, \nabla_{\beta\dot{\alpha}}] = 0 \, , \quad \{\nabla^i_{\dot{\alpha}}, \nabla^j_{\dot{\beta}}\} + \{\nabla^i_{\dot{\beta}}, \nabla^j_{\dot{\alpha}}\} = 0 \, . \qquad \text{(VII.199)}$$

Here, we have introduced covariant derivatives

$$\nabla_{\alpha\dot{\alpha}} := \partial^R_{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}} \quad \text{and} \quad \nabla^i_{\dot{\alpha}} := \partial^i_{\dot{\alpha}} + \mathcal{A}^i_{\dot{\alpha}} \, . \qquad \text{(VII.200)}$$

The equations (VII.199) are the constraint equations for $\mathcal{N}$-extended super SDYM theory.

§17 Comments on the real case. Although there is a diffeomorphism between the correspondence space $\mathcal{F}^{5|2\mathcal{N}}_R$ and $\mathcal{P}^{3|\mathcal{N}}_\varepsilon$ (up to the subtleties arising in the Kleinian case $\varepsilon = +1$), the linear systems (VII.198) and (VII.191) do not coincide here. Instead, we have

$$\begin{aligned} (\bar{V}^+_\alpha + \mathcal{A}^+_\alpha)\psi_+ &= 0 \, , & (\bar{V}^+_\alpha + \hat{\mathcal{A}}^+_\alpha)\hat{\psi}_+ &= 0 \, , \\ \partial_{\bar{\lambda}_+}\psi_+ &= 0 \, , & (\partial_{\bar{\lambda}_+} + \hat{\mathcal{A}}_{\bar{\lambda}_+})\hat{\psi}_+ &= 0 \, , \\ (\bar{V}^i_+ + \mathcal{A}^i_+)\psi_+ &= 0 \, , & \bar{V}^i_+\hat{\psi}_+ &= 0 \, , \end{aligned} \qquad \text{(VII.201)}$$

where the left-hand side is again (VII.198) and the right-hand side is (VII.191), written in components $\hat{\mathcal{A}}^+_\alpha := \bar{V}^+_\alpha \lrcorner \hat{\mathcal{A}}^{0,1}$ and $\hat{\mathcal{A}}^i_+ := \bar{V}^i_+ \lrcorner \hat{\mathcal{A}}^{0,1} = 0$. Thus, we can schematically write for the gauge transformations between the trivializations $\psi_\pm$ and $\hat{\psi}_\pm$

$$(\hat{\mathcal{A}}^\pm_\alpha \neq 0, \hat{\mathcal{A}}_{\bar{\lambda}_\pm} \neq 0, \hat{\mathcal{A}}^i_\pm = 0) \xrightarrow{\varphi} (\mathcal{A}^\pm_\alpha \neq 0, \mathcal{A}_{\bar{\lambda}_\pm} = 0, \mathcal{A}^i_\pm \neq 0) \, . \qquad \text{(VII.202)}$$

The main difference between the two gauges is that one can write down an action for the one with $\hat{\mathcal{A}}^{0,1}$ in the case $\mathcal{N} = 4$, while this is never possible for the other gauge potential.

§18 Super hCS theory. In the following, we will discuss holomorphic Chern-Simons theory using the components $\hat{\mathcal{A}}^+_\alpha := \bar{V}^+_\alpha \lrcorner \hat{\mathcal{A}}^{0,1}$ introduced above. The action (VII.194) is rewritten as

$$S_{\text{hCS}} := \int_{\mathscr{P}^{3|\mathcal{N}}_\varepsilon} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \wedge \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \bar{E}^1 \wedge \bar{E}^2 \, \Omega^\eta \, \mathrm{tr}\, \varepsilon^{abc} \left(\mathcal{A}_a V_b \mathcal{A}_c + \tfrac{2}{3} \mathcal{A}_a \mathcal{A}_b \mathcal{A}_c \right) \, . \qquad \text{(VII.203)}$$

Recall that we assumed in (VII.194) that $\mathcal{A}^i_\pm = 0$. The corresponding equations of motion read then e.g. on $\hat{\mathcal{U}}_+$ as

$$\bar{V}^+_\alpha \hat{\mathcal{A}}^+_\beta - \bar{V}^+_\beta \hat{\mathcal{A}}^+_\alpha + [\hat{\mathcal{A}}^+_\alpha, \hat{\mathcal{A}}^+_\beta] = 0 \, , \qquad \text{(VII.204a)}$$

$$\partial_{\bar{\lambda}_+} \hat{\mathcal{A}}^+_\alpha - \bar{V}^+_\alpha \hat{\mathcal{A}}_{\bar{\lambda}_+} + [\hat{\mathcal{A}}_{\bar{\lambda}_+}, \hat{\mathcal{A}}^+_\alpha] = 0 \qquad \text{(VII.204b)}$$

and very similarly on $\hat{\mathcal{U}}_-$. Here, $\hat{\mathcal{A}}^+_\alpha$ and $\hat{\mathcal{A}}_{\bar{\lambda}_+}$ are functions of $(x^{\alpha\dot{\alpha}}_R, \lambda_+, \bar{\lambda}_+, \eta^+_i)$. These equations are equivalent to the equations of self-dual $\mathcal{N}$-extended SYM theory on $\mathbb{R}^4$. As already mentioned, the most interesting case is $\mathcal{N}$=4 since the supertwistor space $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold and one can derive equations (VII.192) or (VII.204a), (VII.204b) from the manifestly Lorentz invariant action (VII.194) [297, 264]. For this reason, we mostly concentrate on the equivalence with self-dual SYM for the case $\mathcal{N}$=4.

§19 Field expansion for super hCS theory. Recall that $\hat{\mathcal{A}}_\alpha$ and $\hat{\mathcal{A}}_{\bar\lambda}$ are sections of the bundles $\mathcal{O}(1)\otimes\mathbb{C}^2$ and $\bar{\mathcal{O}}(-2)$ over $\mathbb{C}P^1$ since the vector fields $\bar{V}_\alpha$ and $\partial_{\bar\lambda_\pm}$ take values in $\mathcal{O}(1)$ and the holomorphic cotangent bundle of $\mathbb{C}P^1$ is $\mathcal{O}(-2)$. Together with the fact that the η_i^+s take values in the bundle $\Pi\mathcal{O}(1)$, this fixes the dependence of $\hat{\mathcal{A}}_\alpha^\pm$ and $\hat{\mathcal{A}}_{\bar\lambda_\pm}$ on $\lambda_\pm^{\dot\alpha}$ and $\hat\lambda_\pm^{\dot\alpha}$. In the case $\mathcal{N}$=4, this dependence can be brought to the form

$$\begin{aligned}\hat{\mathcal{A}}_\alpha^+ &= \lambda_+^{\dot\alpha} A_{\alpha\dot\alpha}(x_R) + \eta_i^+ \chi_\alpha^i(x_R) + \gamma_+ \tfrac{1}{2!}\eta_i^+\eta_j^+ \hat\lambda_+^{\dot\alpha} \phi_{\alpha\dot\alpha}^{ij}(x_R) + \qquad \text{(VII.205a)}\\ &\quad + \gamma_+^2 \tfrac{1}{3!}\eta_i^+\eta_j^+\eta_k^+ \hat\lambda_+^{\dot\alpha}\hat\lambda_+^{\dot\beta} \tilde\chi_{\alpha\dot\alpha\dot\beta}^{ijk}(x_R) + \gamma_+^3 \tfrac{1}{4!}\eta_i^+\eta_j^+\eta_k^+\eta_l^+ \hat\lambda_+^{\dot\alpha}\hat\lambda_+^{\dot\beta}\hat\lambda_+^{\dot\gamma} G_{\alpha\dot\alpha\dot\beta\dot\gamma}^{ijkl}(x_R)\ ,\\ \hat{\mathcal{A}}_{\bar\lambda_+} &= \gamma_+^2 \tfrac{1}{2!}\eta_i^+\eta_j^+ \phi^{ij}(x_R) + \gamma_+^3 \tfrac{1}{3!}\eta_i^+\eta_j^+\eta_k^+ \hat\lambda_+^{\dot\alpha} \tilde\chi_{\dot\alpha}^{ijk}(x_R) + \qquad \text{(VII.205b)}\\ &\quad + \gamma_+^4 \tfrac{1}{4!}\eta_i^+\eta_j^+\eta_k^+\eta_l^+ \hat\lambda_+^{\dot\alpha}\hat\lambda_+^{\dot\beta} G_{\dot\alpha\dot\beta}^{ijkl}(x_R)\ ,\end{aligned}$$

and there are similar expressions for $\hat{\mathcal{A}}_\alpha^-, \hat{\mathcal{A}}_{\bar\lambda_-}$. Here, $(A_{\alpha\dot\alpha}\,,\ \chi_\alpha^i\,,\ \phi^{ij}\,,\ \tilde\chi_{\dot\alpha i})$ is the ordinary field content of $\mathcal{N}$=4 super Yang-Mills theory and the field $G_{\dot\alpha\dot\beta}$ is the auxiliary field arising in the $\mathcal{N}$=4 self-dual case, as discussed in section IV.2.3. It follows from (VII.204b)-(VII.205b) that[18]

$$\phi_{\alpha\dot\alpha}^{ij} = -\nabla_{\alpha\dot\alpha}\phi^{ij}\ , \quad \tilde\chi_{\alpha\dot\alpha\dot\beta}^{ijk} = -\tfrac{1}{4}\nabla_{\alpha(\dot\alpha}\tilde\chi_{\dot\beta)}^{ijk} \quad \text{and} \quad G_{\alpha\dot\alpha\dot\beta\dot\gamma}^{ijkl} = -\tfrac{1}{18}\nabla_{\alpha(\dot\alpha}G_{\dot\beta\dot\gamma)}^{ijkl}\ , \qquad \text{(VII.206)}$$

i.e. these fields do not contain additional degrees of freedom. The expansion (VII.205a), (VII.205b) together with the field equations (VII.204a), (VII.204b) reproduces exactly the super SDYM equations (IV.62).

§20 The cases $\mathcal{N} < 4$. Since the η_i^+s are Graßmann variables and thus nilpotent, the expansion (VII.205) for $\mathcal{N} < 4$ will only have terms up to order $\mathcal{N}$ in the η_i^+s. This exactly reduces the expansion to the appropriate field content for $\mathcal{N}$-extended super SDYM theory:

$$\begin{aligned}\mathcal{N} &= 0 \quad A_{\alpha\dot\alpha}\\ \mathcal{N} &= 1 \quad A_{\alpha\dot\alpha},\ \chi_\alpha^i \quad \text{with } i = 1\\ \mathcal{N} &= 2 \quad A_{\alpha\dot\alpha},\ \chi_\alpha^i,\ \phi^{[ij]} \quad \text{with } i,j = 1,2 \qquad \text{(VII.207)}\\ \mathcal{N} &= 3 \quad A_{\alpha\dot\alpha},\ \chi_\alpha^i,\ \phi^{[ij]},\ \chi_{\dot\alpha}^{[ijk]} \quad \text{with } i,j,\ldots = 1,2,3\\ \mathcal{N} &= 4 \quad A_{\alpha\dot\alpha},\ \chi_\alpha^i,\ \phi^{[ij]},\ \chi_{\dot\alpha}^{[ijk]},\ G_{\dot\alpha\dot\beta}^{[ijkl]} \quad \text{with } i,j,\ldots = 1,\ldots,4\ .\end{aligned}$$

One should note that the antisymmetrization $[\cdot]$ leads to a different number of fields depending on the range of i. For example, in the case $\mathcal{N}$=2, there is only one real scalar ϕ^{12}, while for $\mathcal{N}$=4 there exist six real scalars. Inserting such a truncated expansion for $\mathcal{N}$<4 into the field equations (VII.204a) and (VII.204b), we obtain the first $\mathcal{N}$+1 equations of (IV.62), which is the appropriate set of equations for $\mathcal{N}$<4 super SDYM theory.

One should stress, however, that this expansion can only be written down in the real case due to the identification of the vector fields on $\mathcal{P}^{3|\mathcal{N}}$ with those along the projection π_2. This is in contrast to the superfield expansion of the gauge potentials participating in the constraint equations for $\mathcal{N}$-extended supersymmetric SDYM theory, which also holds in the complex case.

[18]Here, $(\cdot)$ denotes again symmetrization, i.e. $(\dot\alpha\dot\beta) = \dot\alpha\dot\beta + \dot\beta\dot\alpha$.

§21 The Penrose-Ward transform for dual supertwistors. The discussion of the Penrose-Ward transform over the dual supertwistor space $\mathcal{P}_*^{3|\mathcal{N}}$ is completely analogous to the above discussion, so we refrain from going in any detail. To make the transition to dual twistor space, one simply has to replace everywhere all λ and η by μ and θ, respectively, as well as dualize the spinor indices $\alpha \to \dot{\alpha}$, $\dot{\alpha} \to \alpha$ etc. and change the upper R-symmetry indices to lower ones and vice versa.

§22 Čech cohomology over supermanifolds. Note that in performing the Penrose-Ward transform, we have heavily relied on both the Čech and the Dolbeault description of holomorphic vector bundles. Recall that if the patches $\mathcal{U}_a$ of the covering $\mathfrak{U}$ are Stein manifolds, one can show that the first Čech cohomology sets are independent of the covering $\mathfrak{U}$ and depend only on the manifold M, e.g. $H^1(\mathfrak{U},\mathfrak{S}) = H^1(M,\mathfrak{S})$. Since the covering of the body of $\mathcal{P}^{3|\mathcal{N}}$ is obviously unaffected by the extension to an infinitesimal neighborhood,[19] we can assume that H^1 is also independent of the covering for supermanifolds.

§23 Summary. We have described a one-to-one correspondence between gauge equivalence classes of solutions to the $\mathcal{N}$-extended SDYM equations on $(\mathbb{R}^4, g)$ with $g = \mathrm{diag}(-\varepsilon, -\varepsilon, +1, +1)$ and equivalence classes of holomorphic vector bundles $\mathcal{E}$ over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ such that the bundles $\mathcal{E}$ are holomorphically trivial on each projective line $\mathbb{C}P^1_{x_R,\eta}$ in $\mathcal{P}^{3|\mathcal{N}}$. In other words, there is a bijection between the moduli spaces of hCS theory on $\mathcal{P}^{3|\mathcal{N}}$ and the one of self-dual $\mathcal{N}$-extended SDYM theory on $(\mathbb{R}^4, g)$. It is assumed that appropriate reality conditions are imposed. The Penrose-Ward transform and its inverse are defined by the formulæ (VII.205). In fact, these formulæ relate solutions of the equations of motion of hCS theory on $\mathcal{P}^{3|\mathcal{N}}$ to those of self-dual $\mathcal{N}$-extended SYM theory on $(\mathbb{R}^4, g)$. One can also write integral formulæ of type (VII.114) but we refrain from doing this.

VII.5 Penrose-Ward transform using exotic supermanifolds

VII.5.1 Motivation for considering exotic supermanifolds

The Calabi-Yau property, i.e. vanishing of the first Chern class or equivalently the existence of a globally well-defined holomorphic volume form, is essential for defining the B-model on a certain space. Consider the space $\mathcal{P}^{3|4}$ as introduced in the last section. Since the volume element Ω which is locally given by $\Omega_\pm := \pm \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \mathrm{d}\lambda_\pm \mathrm{d}\eta^\pm_1 \dots \mathrm{d}\eta^\pm_4$ is globally defined and holomorphic, $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold. Other spaces which have a twistorial $\mathcal{O}(1)\oplus\mathcal{O}(1)$ body and are still Calabi-Yau supermanifolds are, e.g., the weighted projective spaces[20] $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with (p,q) equal to $(1,3)$, $(2,2)$ and $(4,0)$ as considered in [231]. The topological B-model on these manifolds was shown to be equivalent to $\mathcal{N}=4$ SDYM theory with a truncated field content. Additionally in the cases $(2,2)$ and $(4,0)$, the parities of some fields are changed.

An obvious idea to obtain even more Calabi-Yau supermanifolds directly from $\mathcal{P}^{3|4}$ is to combine several fermionic variables into a single one,[21] e.g. to consider coordinates

[19] An infinitesimal neighborhood cannot be covered partially.

[20] In fact, one rather considers their open subspaces $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)\backslash W\mathbb{C}P^{1|2}(1,1|p,q)$.

[21] A similar situation has been considered in [174], where all the fermionic variables where combined into a single even nilpotent one.

($\zeta_1 := \eta_1$, $\zeta_2 := \eta_2\eta_3\eta_4$). In an analogous situation for bosonic variables, one could always at least locally find additional coordinates complementing the reduced set to a set describing the full space. Fixing the complementing coordinates to certain values then means that one considers a subvariety of the full space. However, as there is no inverse of Graßmann variables, the situation here is different. Instead of taking a subspace, we rather restrict the algebra of functions (and similarly the set of differential operators) by demanding a certain dependence on the Graßmann variables. One can indeed find complementing sets of functions to restore the full algebra of functions on $\mathcal{P}^{3|4}$. Underlining the argument that we do not consider a subspace of $\mathcal{P}^{3|4}$ is the observation that we still have to integrate over the full space $\mathcal{P}^{3|4}$: $\int \mathrm{d}\zeta_1\mathrm{d}\zeta_2 = \int \mathrm{d}\eta_1 \dots \mathrm{d}\eta_4$. This picture has a slight similarity to the definition of the body of a supermanifold as given in [72, 56].

Possible inequivalent groupings of the Graßmann coordinates of $\mathcal{P}^{3|4}$ are the previously given example ($\zeta_1 := \eta_1, \zeta_2 := \eta_2\eta_3\eta_4$) as well as ($\zeta_1 = \eta_1, \zeta_2 = \eta_2, \zeta_3 = \eta_3\eta_4$), ($\zeta_1 = \eta_1\eta_2$, $\zeta_2 = \eta_3\eta_4$), and ($\zeta_1 = \eta_1\eta_2\eta_3\eta_4$). They correspond to exotic supermanifolds of dimension $(3 \oplus 0|2)$, $(3 \oplus 1|2)$, $(3 \oplus 2|0)$, and $(3 \oplus 1|0)$, respectively. Considering hCS theory on them, one finds that the first one is equivalent to the case $W\mathbb{C}P^{3|2}(1,1,1,1|1,3)$ which was already discussed in [231]. The case $(3 \oplus 2|0)$ will be similar to the case $W\mathbb{C}P^{3|2}(1,1,1,1|2,2)$, but with a field content of partially different parity. The case $(3 \oplus 1|2)$ is a mixture easily derived from combining the full case $\mathcal{P}^{3|4}$ with the case $(3 \oplus 2|0)$. We restrict ourselves in the following to the cases $(3 \oplus 2|0)$ and $(3 \oplus 1|0)$.

Instead of considering independent twistor correspondences between fattened complex manifolds and the moduli space of relative deformations of the embedded $\mathbb{C}P^1$, we will focus on *reductions* of the correspondence between $\mathcal{P}^{3|4}$ and $\mathbb{C}^{4|8}$. This formulation allows for a more direct identification of the remaining subsectors of $\mathcal{N} = 4$ self-dual Yang-Mills theory and can in a sense be understood as a fermionic dimensional reduction.

VII.5.2 The twistor space $\mathcal{P}^{3\oplus 2|0}$

§1 Definition of $\mathcal{P}^{3\oplus 2|0}$. The starting point of our discussion is the supertwistor space $\mathcal{P}^{3|4} = (\mathcal{P}^3, \mathcal{O}_{[4]})$. Consider the differential operators

$$\mathcal{D}^{i1}_\pm \;:=\; \eta^\pm_1\eta^\pm_2\frac{\partial}{\partial\eta^\pm_i} \quad\text{and}\quad \mathcal{D}^{i2}_\pm \;:=\; \eta^\pm_3\eta^\pm_4\frac{\partial}{\partial\eta^\pm_{i+2}} \quad\text{for}\quad i \;=\; 1,2 \;, \qquad \text{(VII.208)}$$

which are maps $\mathcal{O}_{[4]} \to \mathcal{O}_{[4]}$. The space $\mathcal{P}^3$ together with the structure sheaf

$$\mathcal{O}_{(1,2)} \;:=\; \bigcap_{i,j=1,2} \ker\mathcal{D}^{ij}_+ \;=\; \bigcap_{i,j=1,2} \ker\mathcal{D}^{ij}_- \;, \qquad \text{(VII.209)}$$

which is a reduction of $\mathcal{O}_{[4]}$, is the fattened complex manifold $\mathcal{P}^{3\oplus 2|0}$, covered by two patches $\hat{\mathcal{U}}_+$ and $\hat{\mathcal{U}}_-$ and described by local coordinates ($z^\alpha_\pm$, $\lambda_\pm$, $e^\pm_1 := \eta^\pm_1\eta^\pm_2$, $e^\pm_2 := \eta^\pm_3\eta^\pm_4$). The two even nilpotent coordinates $e^\pm_i$ are each sections of the line bundle $\mathcal{O}(2)$ with the identification $(e^\pm_i)^2 \sim 0$.

§2 Derivatives on $\mathcal{P}^{3\oplus 2|0}$. As pointed out before, the coordinates $e^\pm_i$ do not allow for a complementing set of coordinates, and therefore it is not possible to use Leibniz calculus in the transition from the η-coordinates on $(\mathcal{P}^3, \mathcal{O}_{[4]})$ to the e-coordinates on $(\mathcal{P}^3, \mathcal{O}_{(1,2)})$.

Instead, from the observation that

$$\eta_2^\pm\frac{\partial}{\partial e_1^\pm} = \left.\frac{\partial}{\partial\eta_1^\pm}\right|_{\mathcal{O}_{(1,2)}} , \quad \eta_1^\pm\frac{\partial}{\partial e_1^\pm} = -\left.\frac{\partial}{\partial\eta_2^\pm}\right|_{\mathcal{O}_{(1,2)}} ,$$
$$\eta_4^\pm\frac{\partial}{\partial e_2^\pm} = \left.\frac{\partial}{\partial\eta_3^\pm}\right|_{\mathcal{O}_{(1,2)}} , \quad \eta_3^\pm\frac{\partial}{\partial e_2^\pm} = -\left.\frac{\partial}{\partial\eta_4^\pm}\right|_{\mathcal{O}_{(1,2)}} , \tag{VII.210}$$

one directly obtains the following identities on $(\mathcal{P}^3,\mathcal{O}_{(1,2)})$:

$$\frac{\partial}{\partial e_1^\pm} = \frac{\partial}{\partial\eta_2^\pm}\frac{\partial}{\partial\eta_1^\pm} \quad\text{and}\quad \frac{\partial}{\partial e_2^\pm} = \frac{\partial}{\partial\eta_4^\pm}\frac{\partial}{\partial\eta_3^\pm} . \tag{VII.211}$$

Equations (VII.210) are easily derived by considering an arbitrary section f of $\mathcal{O}_{(1,2)}$:

$$f = a^0 + a^1 e_1 + a^2 e_2 + a^{12} e_1 e_2 = a^0 + a^1\eta_1\eta_2 + a^2\eta_3\eta_4 + a^{12}\eta_1\eta_2\eta_3\eta_4 , \tag{VII.212}$$

where we suppressed the $\pm$ labels for convenience. Acting, e.g., by $\frac{\partial}{\partial\eta_1}$ on f, we see that this equals an action of $\eta_2\frac{\partial}{\partial e_1}$. It is then also obvious that we can make the formal identification (VII.211) on $(\mathcal{P}^3,\mathcal{O}_{(1,2)})$. Still, a few more comments on (VII.211) are in order. These differential operators clearly map $\mathcal{O}_{(1,2)}\to\mathcal{O}_{(1,2)}$ and fulfill

$$\frac{\partial}{\partial e_i^\pm} e_j^\pm = \delta_j^i . \tag{VII.213}$$

Note, however, that they do not quite satisfy the Leibniz rule, e.g.:

$$1 = \frac{\partial}{\partial e_1^\pm} e_1^\pm = \frac{\partial}{\partial e_1^\pm}(\eta_1^\pm\eta_2^\pm) \neq \left(\frac{\partial}{\partial e_1^\pm}\eta_1^\pm\right)\eta_2^\pm + \eta_1^\pm\left(\frac{\partial}{\partial e_1^\pm}\eta_2^\pm\right) = 0 . \tag{VII.214}$$

This does not affect the fattened complex manifold $\mathcal{P}^{3\oplus 2|0}$ at all, but it imposes an obvious constraint on the formal manipulation of expressions involving the e-coordinates rewritten in terms of the η-coordinates.

For the cotangent space, we have the identification $\mathrm{d}e_1^\pm = \mathrm{d}\eta_2^\pm\mathrm{d}\eta_1^\pm$ and $\mathrm{d}e_2^\pm = \mathrm{d}\eta_4^\pm\mathrm{d}\eta_3^\pm$ and similarly to above, one has to take care in formal manipulations, as integration is equivalent to differentiation.

§3 Moduli space of sections. As discussed in section VII.4, §3, holomorphic sections of the bundle $\mathcal{P}^{3|4}\to\mathbb{C}P^1$ are described by moduli which are elements of the space $\mathbb{C}^{4|8} = (\mathbb{C}^4,\mathcal{O}_{[8]})$. After the above reduction, holomorphic sections of the bundle $\mathcal{P}^{3\oplus 2|0}\to\mathbb{C}P^1$ are defined by the equations

$$z_\pm^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm \quad\text{and}\quad e_i^\pm = e_i^{\dot\alpha\dot\beta}\lambda_{\dot\alpha}^\pm\lambda_{\dot\beta}^\pm . \tag{VII.215}$$

While the Graßmann algebra of the coordinates $\eta_k^\pm$ of $\mathcal{P}^{3|4}$ immediately imposed a Graßmann algebra on the moduli $\eta_k^{\dot\alpha}\in\mathbb{C}^{0|8}$, the situation here is more subtle. We have[22] $e_1^{(\dot\alpha\dot\beta)} = \eta_1^{(\dot\alpha}\eta_2^{\dot\beta)}$ and from this, we already note that $(e_1^{\dot 1\dot 2})^2\neq 0$ but only $(e_1^{\dot 1\dot 2})^3 = 0$. Thus, the moduli space is a fattening of order 1 in $e_1^{\dot 1\dot 1}$ and $e_1^{\dot 2\dot 2}$, but a fattening of order 2 in $e_1^{\dot 1\dot 2}$ which analogously holds for $e_2^{\dot\alpha\dot\beta}$. Furthermore, we have the additional identities

$$e_i^{\dot 1\dot 2}e_i^{\dot 1\dot 2} = -\tfrac{1}{2}e_i^{\dot 1\dot 1}e_i^{\dot 2\dot 2} \quad\text{and}\quad e_i^{\dot 1\dot 2}e_i^{\dot 2\dot 2} = e_i^{\dot 1\dot 2}e_i^{\dot 1\dot 1} = 0 . \tag{VII.216}$$

[22] The brackets $(\cdot)$ and $[\cdot]$ denote symmetrization and antisymmetrization, respectively, of the enclosed indices with appropriate weight.

Additional conditions which appear when working with fattened complex manifolds are not unusual and similar problems were encountered, e.g., in the discussion of fattened ambitwistor spaces in [90].

More formally, one can introduce the differential operators

$$\mathcal{D}^{1c} = (\eta_1^{\dot\alpha}\partial^1_{\dot\alpha} - \eta_2^{\dot\alpha}\partial^2_{\dot\alpha}) , \quad \mathcal{D}^{2c} = (\eta_3^{\dot\alpha}\partial^3_{\dot\alpha} - \eta_4^{\dot\alpha}\partial^4_{\dot\alpha}) , \tag{VII.217}$$

$$\mathcal{D}^{1s} = (\partial^2_1\partial^1_2 - \partial^2_2\partial^1_1) , \quad \mathcal{D}^{2s} = (\partial^4_1\partial^3_2 - \partial^4_2\partial^3_1) \tag{VII.218}$$

which map $\mathcal{O}_{[8]} \to \mathcal{O}_{[8]}$, and consider the overlap of kernels

$$\mathcal{O}_{(1;2,6)} := \bigcap_{i=1,2} \left(\ker(\mathcal{D}^{ic}) \cap \ker(\mathcal{D}^{is})\right) . \tag{VII.219}$$

The space $\mathbb{C}^4$ together with the structure sheaf $\mathcal{O}_{(1;2,6)}$, which is a reduction of $\mathcal{O}_{[8]}$, is exactly the moduli space described above, i.e. a fattened complex manifold $\mathbb{C}^{4\oplus 6|0}$ on which the coordinates $e_i^{\dot\alpha\dot\beta}$ satisfy the additional constrains (VII.216).

§4 The double fibration. Altogether, we have the following reduction of the full double fibration (VII.162) for $\mathcal{N} = 4$:

$$\begin{array}{ccc} & (\mathbb{C}^4 \times \mathbb{C}P^1, \mathcal{O}_{[8]} \otimes \mathcal{O}_{\mathbb{C}P^1}) & \\ & {}^{\pi_2}\swarrow \quad \searrow{}^{\pi_1} & \\ (\mathcal{P}^3, \mathcal{O}_{[4]}) & & (\mathbb{C}^4, \mathcal{O}_{[8]}) \end{array} \longrightarrow \begin{array}{ccc} & (\mathbb{C}^4 \times \mathbb{C}P^1, \mathcal{O}_{(1;2,6)} \otimes \mathcal{O}_{\mathbb{C}P^1}) & \\ & {}^{\pi_2}\swarrow \quad \searrow{}^{\pi_1} & \\ (\mathcal{P}^3, \mathcal{O}_{(1,2)}) & & (\mathbb{C}^4, \mathcal{O}_{(1;2,6)}) \end{array} \tag{VII.220}$$

where $\mathcal{O}_{\mathbb{C}P^1}$ is the structure sheaf of the Riemann sphere $\mathbb{C}P^1$. The tangent spaces along the leaves of the projection π_2 are spanned by the vector fields

$$\begin{aligned} \bar V^\pm_\alpha &= \lambda^{\dot\alpha}_\pm \partial_{\alpha\dot\alpha} , & \bar V^\pm_\alpha &= \lambda^{\dot\alpha}_\pm \partial_{\alpha\dot\alpha} , \\ \bar V^k_\pm &= \lambda^{\dot\alpha}_\pm \frac{\partial}{\partial\eta^{\dot\alpha}_k} , & \bar V^i_{\dot\beta\pm} &= \lambda^{\dot\alpha}_\pm \frac{\partial}{\partial e_i^{(\dot\alpha\dot\beta)}} \end{aligned} \tag{VII.221}$$

in the left and right double fibration in (VII.220), where $k = 1, \ldots, 4$. Note that similarly to (VII.210), we have the identities

$$\begin{aligned} \eta_2^{\dot\alpha}\frac{\partial}{\partial e_1^{(\dot\alpha\dot\beta)}} &= \left.\frac{\partial}{\partial\eta_1^{\dot\beta}}\right|_{\mathcal{O}_{(1;2,6)}} , & \eta_1^{\dot\alpha}\frac{\partial}{\partial e_1^{(\dot\alpha\dot\beta)}} &= -\left.\frac{\partial}{\partial\eta_2^{\dot\beta}}\right|_{\mathcal{O}_{(1;2,6)}} , \\ \eta_4^{\dot\alpha}\frac{\partial}{\partial e_2^{(\dot\alpha\dot\beta)}} &= \left.\frac{\partial}{\partial\eta_3^{\dot\beta}}\right|_{\mathcal{O}_{(1;2,6)}} , & \eta_3^{\dot\alpha}\frac{\partial}{\partial e_2^{(\dot\alpha\dot\beta)}} &= -\left.\frac{\partial}{\partial\eta_4^{\dot\beta}}\right|_{\mathcal{O}_{(1;2,6)}} , \end{aligned} \tag{VII.222}$$

and it follows, e.g., that

$$\bar V^1_\pm\big|_{\mathcal{O}_{(1;2,6)}} = \eta_2^{\dot\alpha}\bar V^1_{\dot\alpha\pm} \quad \text{and} \quad \bar V^2_\pm\big|_{\mathcal{O}_{(1;2,6)}} = -\eta_1^{\dot\alpha}\bar V^1_{\dot\alpha\pm} . \tag{VII.223}$$

§5 Holomorphic Chern-Simons theory on $\mathcal{P}^{3\oplus 2|0}$. The topological B-model on $\mathcal{P}^{3\oplus 2|0} = (\mathcal{P}^3, \mathcal{O}_{(1,2)})$ is equivalent to hCS theory on $\mathcal{P}^{3\oplus 2|0}$ since a reduction of the structure sheaf does not affect the arguments used for this equivalence in [293, 297]. Consider a trivial rank n complex vector bundle[23] $\mathcal{E}$ over $\mathcal{P}^{3\oplus 2|0}$ with a connection $\hat{\mathcal{A}}$. The action for hCS theory on this space reads

$$S = \int_{\mathscr{P}^{3\oplus 2|0}} \Omega^{3\oplus 2|0} \wedge \operatorname{tr}\left(\hat{\mathcal{A}}^{0,1} \wedge \bar\partial\hat{\mathcal{A}}^{0,1} + \tfrac{2}{3}\hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1}\right) , \tag{VII.224}$$

[23]Note that the components of sections of ordinary vector bundles over a supermanifold are superfunctions. The same holds for the components of connections and transition functions.

where $\mathscr{P}^{3\oplus 2|0}$ is the subspace of $\mathcal{P}^{3\oplus 2|0}$ for which $\bar{e}_i^{\pm} = 0$, $\hat{\mathcal{A}}^{0,1}$ is the (0,1)-part of $\hat{\mathcal{A}}$ and $\Omega^{3\oplus 2|0}$ is the holomorphic volume form, e.g. $\Omega_+^{3\oplus 2|0} = \mathrm{d}z_+^1 \wedge \mathrm{d}z_+^2 \wedge \mathrm{d}\lambda_+ \mathrm{d}e_1^+ \mathrm{d}e_2^+$. The equations of motion read $\bar\partial \hat{\mathcal{A}}^{0,1} + \hat{\mathcal{A}}^{0,1} \wedge \hat{\mathcal{A}}^{0,1} = 0$ and solutions define a holomorphic structure $\bar\partial_{\hat{\mathcal{A}}}$ on $\mathcal{E}$. Given such a solution $\hat{\mathcal{A}}^{0,1}$, one can locally write $\hat{\mathcal{A}}^{0,1}|_{\hat{\mathcal{U}}_\pm} = \hat\psi_\pm \bar\partial \hat\psi_\pm^{-1}$ with regular matrix-valued functions $\hat\psi_\pm$ smooth on the patches $\hat{\mathcal{U}}_\pm$ and from the gluing condition $\hat\psi_+ \bar\partial \hat\psi_+^{-1} = \hat\psi_- \bar\partial \hat\psi_-^{-1}$ on the overlap $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$, one obtains $\bar\partial(\hat\psi_+^{-1}\hat\psi_-) = 0$. Thus, $f_{+-} := \hat\psi_+^{-1}\hat\psi_-$ defines a transition function for a holomorphic vector bundle $\tilde{\mathcal{E}}$, which is (smoothly) equivalent to $\mathcal{E}$.

§6 The linear system on the correspondence space. Consider now the pull-back of the bundle $\tilde{\mathcal{E}}$ along π_2 in (VII.220) to the space $\mathbb{C}^4 \times \mathbb{C}P^1$, i.e. the holomorphic vector bundle $\pi_2^*\tilde{\mathcal{E}}$ with transition function $\pi_2^* f_{+-}$ satisfying $\bar V_\alpha^{\pm}(\pi_2^* f_{+-}) = \bar V_\pm^k (\pi_2^* f_{+-}) = 0$. Let us suppose that the vector bundle $\pi_2^*\tilde{\mathcal{E}}$ becomes holomorphically trivial[24] when restricted to sections $\mathbb{C}P^1_{x,e} \hookrightarrow \mathcal{P}^{3|4}$. This implies that there is a splitting $\pi_2^* f_{+-} = \hat\psi_+^{-1}\hat\psi_-$, where $\hat\psi_\pm$ are group-valued functions which are holomorphic in the moduli $(x^{\alpha\dot\alpha}, \eta_k^{\dot\alpha})$ and $\lambda_\pm$. From the condition $\bar V_\alpha^{\pm}(\pi_2^* f_{+-}) = \bar V_\pm^k (\pi_2^* f_{+-}) = 0$ we obtain, e.g. on $\hat{\mathcal{U}}_+$

$$
\begin{aligned}
\hat\psi_+ \bar V_\alpha^+ \hat\psi_+^{-1} &= \hat\psi_- \bar V_\alpha^+ \hat\psi_-^{-1} =: \lambda_+^{\dot\alpha} \mathcal{A}_{\alpha\dot\alpha} =: \mathcal{A}_\alpha^+ \ ,\\
\hat\psi_+ \bar V_+^k \hat\psi_+^{-1} &= \hat\psi_- \bar V_+^k \hat\psi_-^{-1} =: \lambda_+^{\dot\alpha} \mathcal{A}_{\dot\alpha}^k =: \mathcal{A}_+^k \ ,\\
\hat\psi_+ \partial_{\bar\lambda_+} \hat\psi_+^{-1} &= \hat\psi_- \partial_{\bar\lambda_+} \hat\psi_-^{-1} =: \mathcal{A}_{\bar\lambda_+} = 0 \ ,\\
\hat\psi_+ \partial_{\bar x^{\alpha\dot\alpha}} \hat\psi_+^{-1} &= \hat\psi_- \partial_{\bar x^{\alpha\dot\alpha}} \hat\psi_-^{-1} = 0 \ .
\end{aligned}
\tag{VII.225}
$$

Considering the reduced structure sheaves, we can rewrite the second line of (VII.225), e.g. for $k = 1$ as

$$
\eta_2^{\dot\beta} \hat\psi_+ \bar V^1_{\dot\beta +} \hat\psi_+^{-1} = \eta_2^{\dot\beta} \hat\psi_- \bar V^1_{\dot\beta +} \hat\psi_-^{-1} =: \eta_2^{\dot\beta} \lambda_+^{\dot\alpha} \mathcal{A}^1_{\dot\alpha\dot\beta} \ , \tag{VII.226}
$$

which yields $\eta_2^{\dot\beta} \mathcal{A}^1_{\dot\alpha\dot\beta} = \mathcal{A}^1_{\dot\alpha}$. From this equation (and similar ones for other values of k) and the well-known superfield expansion of $\mathcal{A}^k_{\dot\alpha}$ (see e.g. [71]), one can now construct the superfield expansion of $\mathcal{A}^i_{\dot\alpha\dot\beta}$ by dropping all the terms, which are not in the kernel of the differential operators $\mathcal{D}^{jc}$ and $\mathcal{D}^{js}$. This will give rise to a bosonic subsector of $\mathcal{N} = 4$ SDYM theory.

§7 Compatibility conditions. To be more explicit, we can also use (VII.226) and introduce the covariant derivative $\nabla_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha} + [\mathcal{A}_{\alpha\dot\alpha}, \cdot]$ and the first order differential operator $\nabla^i_{\dot\alpha\dot\beta} := \partial^i_{\dot\alpha\dot\beta} + [\mathcal{A}^i_{\dot\alpha\dot\beta}, \cdot]$, which allow us to rewrite the compatibility conditions of the linear system behind (VII.225), (VII.226) for the reduced structure sheaf as

$$
\begin{gathered}
[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] = 0 \ , \qquad \eta_m^{\dot\gamma} \left([\nabla^i_{\dot\alpha\dot\gamma}, \nabla_{\beta\dot\beta}] + [\nabla^i_{\dot\beta\dot\gamma}, \nabla_{\beta\dot\alpha}] \right) = 0 \ ,\\
\eta_m^{\dot\gamma} \eta_n^{\dot\delta} \left([\nabla^i_{\dot\alpha\dot\gamma}, \nabla^j_{\dot\beta\dot\delta}] + [\nabla^i_{\dot\beta\dot\gamma}, \nabla^j_{\dot\alpha\dot\delta}] \right) = 0 \ ,
\end{gathered}
\tag{VII.227}
$$

where $m = 2i-1, 2i$ and $n = 2j-1, 2j$. Note that $\nabla^i_{\dot\alpha\dot\beta}$ is no true covariant derivative, as $\partial^i_{\dot\alpha\dot\beta}$ and $\mathcal{A}^i_{\dot\alpha\dot\beta}$ do not have the same symmetry properties in the indices. Nevertheless, the differential operators $\nabla_{\alpha\dot\alpha}$ and $\nabla^i_{\dot\alpha\dot\beta}$ satisfy the Bianchi identities on $(\mathbb{C}^4, \mathcal{O}_{(1;2,6)})$.

[24]This assumption is crucial for the Penrose-Ward transform and reduces the space of possible $\hat{\mathcal{A}}^{0,1}$ to an open subspace around $\hat{\mathcal{A}}^{0,1} = 0$.

§8 Constraint equations. By eliminating all λ-dependence, we have implicitly performed the push-forward of $\mathcal{A}$ along π_1 onto $(\mathbb{C}^4, \mathcal{O}_{(1;2,6)})$. Let us define further tensor superfields, which could roughly be seen as extensions of the supercurvature fields and which capture the solutions to the above equations:

$$\begin{aligned} [\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] \; &=: \; \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}_{\alpha\beta} \;, \quad [\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla_{\beta\dot{\beta}}] \; =: \; \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}^i_{\beta\dot{\gamma}} \;, \\ [\nabla^i_{\dot{\alpha}\dot{\gamma}}, \nabla^j_{\dot{\beta}\dot{\delta}}] \; &=: \; \varepsilon_{\dot{\alpha}\dot{\beta}} \mathcal{F}^{ij}_{\dot{\gamma}\dot{\delta}} \;, \end{aligned} \tag{VII.228}$$

where $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{(\alpha\beta)}$ and $\mathcal{F}^{ij}_{\dot{\gamma}\dot{\delta}} = \mathcal{F}^{(ij)}_{(\dot{\gamma}\dot{\delta})} + \mathcal{F}^{[ij]}_{[\dot{\gamma}\dot{\delta}]}$. Note, however, that we introduced too many of these components. Considering the third equation in (VII.227), one notes that for $i = j$, the terms symmetric in $\dot{\gamma}, \dot{\delta}$ vanish trivially. This means that the components $\mathcal{F}^{ii}_{(\dot{\gamma}\dot{\delta})}$ are in fact superfluous and we can ignore them in the following discussion. The second and third equations in (VII.228) can be contracted with $\varepsilon^{\dot{\alpha}\dot{\gamma}}$ and $\varepsilon^{\dot{\beta}\dot{\delta}}$, respectively, which yields

$$-2\nabla_{\beta\dot{\beta}} \mathcal{A}^i_{[\dot{1}\dot{2}]} \; = \; \mathcal{F}^i_{\beta\dot{\beta}} \quad \text{and} \quad -2\nabla^i_{\dot{\alpha}\dot{\gamma}} \mathcal{A}^j_{[\dot{1}\dot{2}]} \; = \; \mathcal{F}^{ij}_{\dot{\gamma}\dot{\alpha}} \;. \tag{VII.229}$$

Furthermore, using Bianchi identities, one obtains immediately the following equations:

$$\nabla^{\alpha\dot{\beta}} \mathcal{F}^i_{\alpha\dot{\gamma}} \; = \; 0 \quad \text{and} \quad \nabla_{\alpha\dot{\alpha}} \mathcal{F}^{ij}_{\dot{\beta}\dot{\gamma}} \; = \; \nabla^i_{\dot{\alpha}\dot{\beta}} \mathcal{F}^j_{\alpha\dot{\gamma}} \;. \tag{VII.230}$$

Due to self-duality, the first equation is in fact equivalent to $\nabla^{\alpha\dot{\beta}} \nabla_{\alpha\dot{\beta}} \mathcal{A}^i_{[\dot{1}\dot{2}]} = 0$, as is easily seen by performing all the spinor index sums. From the second equation, one obtains the field equation $\nabla_\alpha{}^{\dot{\beta}} \mathcal{F}^{(12)}_{\dot{\beta}\dot{\gamma}} = -2[\mathcal{A}^{(1}_{[\dot{1}\dot{2}]}, \nabla_{\alpha\dot{\gamma}} \mathcal{A}^{2)}_{[\dot{1}\dot{2}]}]$ after contracting with $\varepsilon^{\dot{\alpha}\dot{\beta}}$.

§9 The superfield expansion. To analyze the actual field content of this theory, we choose *transverse gauge* as in section IV.2.3, §21, i.e. we demand

$$\eta^{\dot{\alpha}}_k \mathcal{A}^k_{\dot{\alpha}} \; = \; 0 \;. \tag{VII.231}$$

Recall that this choice reduces the group of gauge transformations to ordinary, group-valued functions on the body of $\mathbb{C}^{4|8}$. By using the identities $\eta^{\dot{\beta}}_2 \mathcal{A}^1_{\dot{\alpha}\dot{\beta}} = \mathcal{A}^1_{\dot{\alpha}}$ etc., one sees that the above transverse gauge is equivalent to the transverse gauge for the reduced structure sheaf:

$$e^{\dot{\alpha}\dot{\beta}}_i \mathcal{A}^i_{\dot{\alpha}\dot{\beta}} \; = \; \eta^{(\dot{\alpha}}_1 \eta^{\dot{\beta})}_2 \mathcal{A}^1_{\dot{\alpha}\dot{\beta}} + \eta^{(\dot{\alpha}}_3 \eta^{\dot{\beta})}_4 \mathcal{A}^2_{\dot{\alpha}\dot{\beta}} \; = \; 0 \;. \tag{VII.232}$$

In the expansion in the es, the lowest components of $\mathcal{F}_{\alpha\beta}$, $\mathcal{A}^i_{[\dot{1}\dot{2}]}$ and $\mathcal{F}^{(12)}_{(\dot{\alpha}\dot{\beta})}$ are the self-dual field strength $f_{\alpha\beta}$, two complex scalars ϕ^i and the auxiliary field $G_{\dot{\alpha}\dot{\beta}}$, respectively. The two scalars ϕ^i can be seen as remainders of the six scalars contained in the $\mathcal{N} = 4$ SDYM multiplet, which will become even clearer in the real case. The remaining components $\mathcal{A}^i_{(\dot{\alpha}\dot{\beta})}$ vanish to zeroth order in the es due to the choice of transverse gauge. The field $\mathcal{F}^i_{\alpha\dot{\alpha}}$ does not contain any new physical degrees of freedom, as seen from the first equation in (VII.229), but it is a composite field. The same holds for $\mathcal{F}^{[12]}_{[\dot{\gamma}\dot{\alpha}]}$ as easily seen by contracting the second equation in (VII.229) by $\varepsilon^{\dot{\gamma}\dot{\alpha}}$: $\mathcal{F}^{[12]}_{[\dot{1}\dot{2}]} = -2[\mathcal{A}^1_{[\dot{1}\dot{2}]}, \mathcal{A}^2_{[\dot{1}\dot{2}]}]$.

§10 Equations of motion. The superfield equations of motion (VII.230) are in fact equivalent to the equations

$$f_{\dot{\alpha}\dot{\beta}} \; = \; 0 \;, \quad \square\phi^i \; = \; 0 \;, \quad \varepsilon^{\dot{\gamma}\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} G_{\dot{\gamma}\dot{\delta}} + 2[\phi^{(1}, \nabla_{\alpha\dot{\delta}} \phi^{2)}] \; = \; 0 \;. \tag{VII.233}$$

To lowest order in the es, the equations obviously match. Higher orders in the es can be verified by defining the Euler operator (cf. section IV.2.3, §21) $\mathcal{D} := e^{\dot{\alpha}\dot{\beta}}_i \nabla^i_{(\dot{\alpha}\dot{\beta})} = e^{\dot{\alpha}\dot{\beta}}_i \partial^i_{(\dot{\alpha}\dot{\beta})}$ and applying $\mathcal{D}$ on the superfields and equations of motion which then turn out to be satisfied if the equations (VII.233) are fulfilled.

VII.5.3 The twistor space $\mathcal{P}^{3\oplus 1|0}$

§11 Definition of $\mathcal{P}^{3\oplus 1|0}$. The discussion for $\mathcal{P}^{3\oplus 1|0}$ follows the same lines as for $\mathcal{P}^{3\oplus 2|0}$ and is even simpler. Consider again the supertwistor space $\mathcal{P}^{3|4} = (\mathcal{P}^3, \mathcal{O}_{[4]})$. This time, let us introduce the following differential operators:

$$\tilde{\mathcal{D}}^{kl}_\pm \;:=\; \eta^\pm_k \frac{\partial}{\partial \eta^\pm_l} \quad \text{for} \quad k,l \;=\; 1,\ldots,4 \;, \tag{VII.234}$$

which are maps $\mathcal{O}_{[4]} \to \mathcal{O}_{[4]}$. The space $\mathcal{P}^3$ together with the extended structure sheaf[25]

$$\mathcal{O}_{(1,1)} \;:=\; \bigcap_{k\neq l} \ker \tilde{\mathcal{D}}^{kl}_+ \;=\; \bigcap_{k\neq l} \ker \tilde{\mathcal{D}}^{kl}_- \;, \tag{VII.235}$$

which is a reduction of $\mathcal{O}_{[4]}$, is an order one thickening of $\mathcal{P}^3$, which we denote by $\mathcal{P}^{3\oplus 1|0}$. This manifold can be covered by two patches $\hat{\mathcal{U}}_+$ and $\hat{\mathcal{U}}_-$ on which we define the coordinates $(z^\alpha_\pm,\, \lambda_\pm,\, e^\pm := \eta^\pm_1\eta^\pm_2\eta^\pm_3\eta^\pm_4)$. The even nilpotent coordinate $e^\pm$ is a section of the line bundle $\mathcal{O}(4)$ with the identification $(e^\pm)^2 \sim 0$.

§12 Derivatives on $\mathcal{P}^{3\oplus 1|0}$. Similarly to the case $\mathcal{P}^{3\oplus 2|0}$, we have the following identities:

$$\begin{aligned} \eta^\pm_2\eta^\pm_3\eta^\pm_4 \frac{\partial}{\partial e^\pm} \;&=\; \left.\frac{\partial}{\partial \eta^\pm_1}\right|_{\mathcal{O}_{(1,1)}} \;, & \eta^\pm_1\eta^\pm_3\eta^\pm_4 \frac{\partial}{\partial e^\pm} \;&=\; -\left.\frac{\partial}{\partial \eta^\pm_2}\right|_{\mathcal{O}_{(1,1)}} \;,\\ \eta^\pm_1\eta^\pm_2\eta^\pm_4 \frac{\partial}{\partial e^\pm} \;&=\; \left.\frac{\partial}{\partial \eta^\pm_3}\right|_{\mathcal{O}_{(1,1)}} \;, & \eta^\pm_1\eta^\pm_2\eta^\pm_3 \frac{\partial}{\partial e^\pm} \;&=\; -\left.\frac{\partial}{\partial \eta^\pm_4}\right|_{\mathcal{O}_{(1,1)}} \end{aligned} \tag{VII.236}$$

which lead to the formal identifications

$$\frac{\partial}{\partial e^\pm} \;=\; \frac{\partial}{\partial \eta^\pm_4}\frac{\partial}{\partial \eta^\pm_3}\frac{\partial}{\partial \eta^\pm_2}\frac{\partial}{\partial \eta^\pm_1} \quad \text{and} \quad \mathrm{d}e^\pm \;=\; \mathrm{d}\eta^\pm_4\mathrm{d}\eta^\pm_3\mathrm{d}\eta^\pm_2\mathrm{d}\eta^\pm_1 \;=\; \Omega^\eta_\pm \;, \tag{VII.237}$$

but again with a restriction of the Leibniz rule in formal manipulations of expressions written in the η-coordinates as discussed in §2.

§13 Moduli space and double fibration. The holomorphic sections of the bundle $\mathcal{P}^{3\oplus 1|0} \to \mathbb{C}P^1$ are defined by the equations

$$z^{\dot\alpha}_\pm \;=\; x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha} \quad \text{and} \quad e^\pm \;=\; e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\lambda^\pm_{\dot\gamma}\lambda^\pm_{\dot\delta} \;. \tag{VII.238}$$

From the obvious identification $e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} = \eta_1^{(\dot\alpha}\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta)}$ we see that a product $e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}e^{(\dot\mu\dot\nu\dot\rho\dot\sigma)}$ will vanish, unless the number of indices equal to $\dot 1$ is the same as the number of indices equal to $\dot 2$. In this case, we have additionally the identity

$$\sum_p (-1)^{n_p} e^{p_1} e^{p_2} \;=\; 0 \;, \tag{VII.239}$$

where p is a permutation of $\dot 1\dot 1\dot 1\dot 1\dot 2\dot 2\dot 2\dot 2$, p_1 and p_2 are the first and second four indices of p, respectively, and n_p is the number of exchanges of a $\dot 1$ and a $\dot 2$ between p_1 and p_2, e.g. $n_{\dot 1\dot 1\dot 1\dot 2\dot 1\dot 2\dot 2\dot 2} = 1$.

The more formal treatment is much simpler. We introduce the differential operators

$$\begin{aligned} \tilde{\mathcal{D}}^{klc} \;&=\; \left(\eta^{\dot\alpha}_l \partial^l_{\dot\alpha} - \eta^{\dot\alpha}_k \partial^k_{\dot\alpha}\right) \quad \text{without summation over } k \text{ and } l \;,\\ \tilde{\mathcal{D}}^{kls} \;&=\; \left(\partial^k_{\dot 1}\partial^l_{\dot 2} - \partial^k_{\dot 2}\partial^l_{\dot 1}\right) \;, \end{aligned} \tag{VII.240}$$

[25] The same reduction can be obtained by imposing integral constraints [174].

which map $\mathcal{O}_{[8]} \to \mathcal{O}_{[8]}$. Then the space $\mathbb{C}^4$ with the extended structure sheaf $\mathcal{O}_{(1;2,5)}$ obtained by reducing $\mathcal{O}_{[8]}$ to the overlap of kernels

$$\mathcal{O}_{(1;2,5)} \;:=\; \bigcap_{k\neq l} \left(\ker \tilde{\mathcal{D}}^{klc} \cap \ker \tilde{\mathcal{D}}^{kls}\right) \tag{VII.241}$$

is the moduli space described above. Thus, we have the following reduction of the full double fibration (VII.162) for $\mathcal{N}=4$:

$$\begin{array}{ccc} (\mathbb{C}^4\times\mathbb{C}P^1, \mathcal{O}_{[8]}\otimes\mathcal{O}_{\mathbb{C}P^1}) & & (\mathbb{C}^4\times\mathbb{C}P^1, \mathcal{O}_{(1;2,5)}\otimes\mathcal{O}_{\mathbb{C}P^1}) \\ \pi_2\swarrow \quad \searrow\pi_1 & \longrightarrow & \pi_2\swarrow \quad \searrow\pi_1 \\ (\mathcal{P}^3,\mathcal{O}_{[4]}) \qquad (\mathbb{C}^4,\mathcal{O}_{[8]}) & & (\mathcal{P}^3,\mathcal{O}_{(1,1)}) \qquad (\mathbb{C}^4,\mathcal{O}_{(1;2,5)}) \end{array} \tag{VII.242}$$

where $\mathcal{O}_{\mathbb{C}P^1}$ is again the structure sheaf of the Riemann sphere $\mathbb{C}P^1$. The tangent spaces along the leaves of the projection π_2 are spanned by the vector fields

$$\begin{aligned} \bar{V}^{\pm}_{\alpha} &= \lambda^{\dot\alpha}_{\pm}\partial_{\alpha\dot\alpha}\ , & \bar{V}^{\pm}_{\alpha} &= \lambda^{\dot\alpha}_{\pm}\partial_{\alpha\dot\alpha}\ ,\\ \bar{V}^{k}_{\pm} &= \lambda^{\dot\alpha}_{\pm}\frac{\partial}{\partial\eta^{\dot\alpha}_{k}}\ , & \bar{V}^{\pm}_{\dot\beta\dot\gamma\dot\delta} &= \lambda^{\dot\alpha}_{\pm}\partial_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} \end{aligned} \tag{VII.243}$$

in the left and right double fibration in (VII.242), where $k=1,\ldots,4$. The further identities

$$\begin{aligned} \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\frac{\partial}{\partial e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} &= \left.\frac{\partial}{\partial\eta^{\dot\alpha}_1}\right|_{\mathcal{O}_{(1;2,5)}}\ , & \eta^{\dot\beta}_1\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\frac{\partial}{\partial e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} &= -\left.\frac{\partial}{\partial\eta^{\dot\alpha}_2}\right|_{\mathcal{O}_{(1;2,5)}}\ ,\\ \eta^{\dot\beta}_1\eta^{\dot\gamma}_2\eta^{\dot\delta}_4\frac{\partial}{\partial e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} &= \left.\frac{\partial}{\partial\eta^{\dot\alpha}_3}\right|_{\mathcal{O}_{(1;2,5)}}\ , & \eta^{\dot\beta}_1\eta^{\dot\gamma}_2\eta^{\dot\delta}_3\frac{\partial}{\partial e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}} &= -\left.\frac{\partial}{\partial\eta^{\dot\alpha}_4}\right|_{\mathcal{O}_{(1;2,5)}} \end{aligned} \tag{VII.244}$$

are easily derived and from them it follows that e.g.

$$\bar{V}^1_{\pm}\big|_{\mathcal{O}_{(1;2,5)}} = \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\bar{V}^{\pm}_{\dot\beta\dot\gamma\dot\delta} \quad\text{and}\quad \bar{V}^2_{\pm}\big|_{\mathcal{O}_{(1;2,5)}} = -\eta^{\dot\beta}_1\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\bar{V}^{\pm}_{\dot\beta\dot\gamma\dot\delta}\ . \tag{VII.245}$$

§14 hCS theory on $\mathcal{P}^{3\oplus1|0}$ and linear system. The topological B-model on $\mathcal{P}^{3\oplus1|0}$ is equivalent to hCS theory on $\mathcal{P}^{3\oplus1|0}$ and introducing a trivial rank n complex vector bundle $\mathcal{E}$ over $\mathcal{P}^{3\oplus1|0}$ with a connection $\hat{\mathcal{A}}$, the action reads

$$S = \int_{\mathscr{P}^{3\oplus1|0}} \Omega^{3\oplus1|0}\wedge \operatorname{tr}\left(\hat{\mathcal{A}}^{0,1}\wedge\bar\partial\hat{\mathcal{A}}^{0,1} + \tfrac{2}{3}\hat{\mathcal{A}}^{0,1}\wedge\hat{\mathcal{A}}^{0,1}\wedge\hat{\mathcal{A}}^{0,1}\right)\ , \tag{VII.246}$$

with $\mathscr{P}^{3\oplus1|0}$ being the chiral subspace for which $\bar{e}^{\pm}=0$ and $\hat{\mathcal{A}}^{0,1}$ the $(0,1)$-part of $\hat{\mathcal{A}}$. The holomorphic volume form $\Omega^{3\oplus1|0}$ can be defined, e.g. on $\hat{\mathcal{U}}_+$, as $\Omega^{3\oplus1|0}_{+} = \mathrm{d}z^1_+\wedge\mathrm{d}z^2_+\wedge \mathrm{d}\lambda_+\mathrm{d}e^+$. Following exactly the same steps as in the case $\mathcal{P}^{3\oplus2|0}$, we again obtain the equations

$$\begin{aligned} \psi_+\bar{V}^{+}_{\alpha}\psi_+^{-1} &= \psi_-\bar{V}^{+}_{\alpha}\psi_-^{-1} =: \lambda^{\dot\alpha}_+\mathcal{A}_{\alpha\dot\alpha} =: \mathcal{A}^+_{\alpha}\ ,\\ \psi_+\bar{V}^{k}_{+}\psi_+^{-1} &= \psi_-\bar{V}^{k}_{+}\psi_-^{-1} =: \lambda^{\dot\alpha}_+\mathcal{A}^k_{\dot\alpha} =: \mathcal{A}^k_{+}\ ,\\ \psi_+\partial_{\bar\lambda_+}\psi_+^{-1} &= \psi_-\partial_{\bar\lambda_+}\psi_-^{-1} =: \mathcal{A}_{\bar\lambda_+} = 0\ ,\\ \psi_+\partial_{\bar{x}^{\alpha\dot\alpha}}\psi_+^{-1} &= \psi_-\partial_{\bar{x}^{\alpha\dot\alpha}}\psi_-^{-1} = 0\ . \end{aligned} \tag{VII.247}$$

and by considering the reduced structure sheaves, we can rewrite the second line this time as

$$\begin{aligned} \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\,\psi_+\bar{V}^{+}_{\dot\beta\dot\gamma\dot\delta}\psi_+^{-1} = \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\,\psi_-\bar{V}^{+}_{\dot\beta\dot\gamma\dot\delta}\psi_-^{-1} &=: \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\,\lambda^{\dot\alpha}_+\mathcal{A}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\ ,\\ &=: \eta^{\dot\beta}_2\eta^{\dot\gamma}_3\eta^{\dot\delta}_4\,\mathcal{A}^{+}_{\dot\beta\dot\gamma\dot\delta} \end{aligned} \tag{VII.248}$$

for $k = 1$ which yields $\eta_2^{\dot\beta}\eta_3^{\dot\gamma}\eta_4^{\dot\delta}\mathcal{A}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = \mathcal{A}^1_{\dot\alpha}$. Similar formulæ are obtained for the other values of k, with which one can determine the superfield expansion of $\mathcal{A}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ again from the superfield expansion of $\mathcal{A}^k_{\dot\alpha}$ by dropping the terms which are not in the kernel of the differential operators $\tilde{\mathcal{D}}^{klc}$ and $\tilde{\mathcal{D}}^{kls}$ for $k \neq l$.

§15 Compatibility conditions. Analogously to the case $\mathcal{P}^{3\oplus 2|0}$, one can rewrite the linear system behind (VII.247), (VII.248) for the reduced structure sheaf. For this, we define the covariant derivative $\nabla_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha} + [\mathcal{A}_{\alpha\dot\alpha}, \cdot]$ and the first order differential operator $\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta} := \partial_{\dot\alpha\dot\beta\dot\gamma\dot\delta} + [\mathcal{A}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \cdot]$. Then we have

$$\begin{aligned} [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] &= 0\ , \\ \eta_k^{\dot\nu}\eta_m^{\dot\rho}\eta_n^{\dot\sigma}\left([\nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\alpha}] + [\nabla_{\dot\alpha\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\mu}]\right) &= 0\ , \\ \eta_r^{\dot\beta}\eta_s^{\dot\gamma}\eta_t^{\dot\delta}\eta_k^{\dot\nu}\eta_m^{\dot\rho}\eta_n^{\dot\sigma}\Big([\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}] + [\nabla_{\dot\mu\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\alpha\dot\nu\dot\rho\dot\sigma}]\Big) &= 0\ , \end{aligned} \tag{VII.249}$$

where (rst) and (kmn) are each a triple of pairwise different integers between 1 and 4. Again, in these equations the push-forward $\pi_{1*}\mathcal{A}$ is already implied and solutions to (VII.249) are captured by the following extensions of the supercurvature fields:

$$\begin{aligned} [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] &=: \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}\ , \\ [\nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}, \nabla_{\alpha\dot\alpha}] &=: \varepsilon_{\dot\alpha\dot\mu}\mathcal{F}_{\alpha\dot\nu\dot\rho\dot\sigma}\ , \\ [\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}, \nabla_{\dot\mu\dot\nu\dot\rho\dot\sigma}] &=: \varepsilon_{\dot\alpha\dot\mu}\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma}\ , \end{aligned}$$

where $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{(\alpha\beta)}$, $\mathcal{F}_{\alpha\dot\nu\dot\rho\dot\sigma} = \mathcal{F}_{\alpha(\dot\nu\dot\rho\dot\sigma)}$ and $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma} = \mathcal{F}_{(\dot\beta\dot\gamma\dot\delta)(\dot\nu\dot\rho\dot\sigma)}$ is symmetric under exchange of $(\dot\beta\dot\gamma\dot\delta) \leftrightarrow (\dot\nu\dot\rho\dot\sigma)$. Consider now the third equation of (VII.249). Note that the triples (rst) and (kmn) will have two numbers in common, while exactly one is different. Without loss of generality, let $r \neq k$, $s = m$ and $t = n$. Then one easily sees that the terms symmetric in $\dot\beta$, $\dot\nu$ vanish trivially. This means that the field components $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\nu\dot\rho\dot\sigma}$ which are symmetric in $\dot\beta$, $\dot\nu$ are again unconstrained additional fields, which do not represent any of the fields in the $\mathcal{N} = 4$ SDYM multiplet and we put them to zero, analogously to $\mathcal{F}^{ii}_{(\dot\gamma\dot\delta)}$ in the case $\mathcal{P}^{3\oplus 2|0}$.

§16 Derivation of the superfield expansion. The second equation in (VII.250) can be contracted with $\varepsilon^{\dot\mu\dot\nu}$ which yields $2\nabla_{\alpha\dot\alpha}\mathcal{A}_{[\dot1\dot2]\dot\rho\dot\sigma} = \mathcal{F}_{\alpha\dot\alpha\dot\rho\dot\sigma}$ and further contracting this equation with $\varepsilon^{\dot\alpha\dot\rho}$ we have $\nabla_\alpha{}^{\dot\alpha}\mathcal{A}_{[\dot1\dot2]\dot\alpha\dot\sigma} = 0$. After contracting the third equation with $\varepsilon^{\dot\mu\dot\nu}$, one obtains

$$-2\nabla_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\mathcal{A}_{[\dot1\dot2]\dot\rho\dot\sigma} = \mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\alpha\dot\rho\dot\sigma}\ . \tag{VII.250}$$

The transversal gauge condition $\eta_k^{\dot\alpha}\mathcal{A}^k_{\dot\alpha} = 0$ is on $\mathcal{O}_{(1;2,5)}$ equivalent to the condition

$$e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\mathcal{A}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0\ , \tag{VII.251}$$

as expected analogously to the case $\mathcal{P}^{3\oplus 2|0}$. To lowest order in $e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$, $\mathcal{F}_{\alpha\beta}$ can be identified with the self-dual field strength $f_{\alpha\beta}$ and $\mathcal{A}_{[\dot1\dot2]\dot\alpha\dot\beta}$ with the auxiliary field $G_{\dot\alpha\dot\beta}$. The remaining components of $\mathcal{F}_{\dot\beta\dot\gamma\dot\delta\dot\alpha\dot\rho\dot\sigma}$, i.e. those antisymmetric in $[\dot\alpha\dot\beta]$, are composite fields and do not contain any additional degrees of freedom which is easily seen by considering equation (VII.250).

§17 Equations of motion. Applying the Euler operator in transverse gauge $\mathcal{D} := e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\nabla_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)} = e^{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}\partial_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$, one can show that the lowest order field equations are equivalent to the full superfield equations of motion. Thus, (VII.249) is equivalent to

$$f_{\dot\alpha\dot\beta} = 0 \quad \text{and} \quad \nabla^{\alpha\dot\alpha}G_{\dot\alpha\dot\beta} = 0 \,. \tag{VII.252}$$

Altogether, we found the compatibility condition for a linear system encoding purely bosonic SDYM theory *including* the auxiliary field $G_{\dot\alpha\dot\beta}$.

VII.5.4 Fattened real manifolds

The field content of hCS theory on $\mathcal{P}^{3\oplus 2|0}$ and $\mathcal{P}^{3\oplus 1|0}$ becomes even more transparent after imposing a reality condition on these spaces. One can directly derive appropriate real structures from the one on $\mathcal{P}^{3|4}$, having in mind the picture of combining the Graßmann coordinates of $\mathcal{P}^{3|4}$ to the even nilpotent coordinates of $\mathcal{P}^{3\oplus 2|0}$ and $\mathcal{P}^{3\oplus 1|0}$. The real structure on $\mathcal{P}^{3|4}$ is discussed in detail in sections VII.3.1 and VII.4.1.

§18 Real structures. Recall the action of the two antilinear involutions τ_ε with $\varepsilon = \pm 1$ on the coordinates $(z^1_\pm, z^2_\pm, z^3_\pm)$:

$$\tau_\varepsilon(z^1_+, z^2_+, z^3_+) = \left(\frac{\bar z^2_+}{\bar z^3_+}, \frac{\varepsilon\bar z^1_+}{\bar z^3_+}, \frac{\varepsilon}{\bar z^3_+}\right) \quad \text{and} \quad \tau_\varepsilon(z^1_-, z^2_-, z^3_-) = \left(\frac{\varepsilon\bar z^2_-}{\bar z^3_-}, \frac{\bar z^1_-}{\bar z^3_-}, \frac{\varepsilon}{\bar z^3_-}\right) .$$

On $\mathcal{P}^{3\oplus 2|0}$, we have additionally

$$\tau_\varepsilon(e^1_+, e^2_+) = \left(\frac{\bar e^1_+}{(\bar z^3_+)^2}, \frac{\bar e^2_+}{(\bar z^3_+)^2}\right) \quad \text{and} \quad \tau_\varepsilon(e^1_-, e^2_-) = \left(\frac{\bar e^1_-}{(\bar z^3_-)^2}, \frac{\bar e^2_-}{(\bar z^3_-)^2}\right) , \tag{VII.253}$$

and on $\mathcal{P}^{3\oplus 1|0}$, it is

$$\tau_\varepsilon(e_+) = \frac{\bar e_+}{(\bar z^3_+)^4} \quad \text{and} \quad \tau_\varepsilon(e_-) = \frac{\bar e_-}{(\bar z^3_-)^4} \,. \tag{VII.254}$$

Recall that in the formulation of the twistor correspondence, the coordinates $z^3_\pm$ are usually kept complex for convenience sake. We do the same while on all other coordinates, we impose the condition $\tau_\varepsilon(\cdot) = \cdot$. On the body of the moduli space, this will lead to a Euclidean metric $(+,+,+,+)$ for $\varepsilon = -1$ and a Kleinian metric $(+,+,-,-)$ for $\varepsilon = +1$.

§19 Real superfield expansion. Recall that together with the identification (VII.68)

$$\frac{\partial}{\partial \bar z^1_+} = \gamma_+ \bar V^+_2 \quad \text{and} \quad \frac{\partial}{\partial \bar z^2_+} = \varepsilon\gamma_+ \bar V^+_1 \,, \tag{VII.255}$$

we can rewrite the hCS equations of motion, e.g. on $\hat{\mathcal{U}}_+$, as

$$\begin{aligned} \bar V^+_\alpha \mathcal{A}^+_\beta - \bar V^+_\beta \mathcal{A}^+_\alpha + [\mathcal{A}^+_\alpha, \mathcal{A}^+_\beta] &= 0 \,, \\ \partial_{\bar\lambda_+}\mathcal{A}^+_\alpha - \bar V^+_\alpha \mathcal{A}_{\bar\lambda_+} + [\mathcal{A}_{\bar\lambda_+}, \mathcal{A}^+_\alpha] &= 0 \,, \end{aligned} \tag{VII.256}$$

where the components of the gauge potential are defined via the contractions $\mathcal{A}^\pm_\alpha := \bar V^\pm_\alpha \lrcorner \hat{\mathcal{A}}^{0,1}$, $\mathcal{A}_{\bar\lambda_\pm} := \partial_{\bar\lambda_\pm} \lrcorner \hat{\mathcal{A}}^{0,1}$, and we assumed a gauge for which $\mathcal{A}^\pm_i := \partial_{\bar e^\pm_i} \lrcorner \hat{\mathcal{A}}^{0,1} = 0$, see also (VII.204). On the space $\mathcal{P}^{3\oplus 2|0}$ together with the field expansion

$$\mathcal{A}^+_\alpha = \lambda^{\dot\alpha}_+ A_{\alpha\dot\alpha} + \gamma_+ e^+_i \hat\lambda^{\dot\alpha}_+ \phi^i_{\alpha\dot\alpha} + \gamma^3_+ e^+_1 e^+_2 \hat\lambda^{\dot\alpha}_+ \hat\lambda^{\dot\beta}_+ \hat\lambda^{\dot\gamma}_+ G_{\alpha\dot\alpha\dot\beta\dot\gamma} \,, \tag{VII.257a}$$

$$\mathcal{A}_{\bar\lambda_+} = \gamma^2_+ e^+_i \phi^i - 2\varepsilon\gamma^4_+ e^+_1 e^+_2 \hat\lambda^{\dot\alpha}_+ \hat\lambda^{\dot\beta}_+ G_{\dot\alpha\dot\beta} \,, \tag{VII.257b}$$

the system of equations (VII.256) is equivalent to (VII.227). Note that similarly to the expansion (VII.205), the expansion (VII.257) is determined by the geometry of $\mathcal{P}^{3\oplus 2|0}$. Furthermore, one can identify $\phi^i_{\alpha\dot\alpha} = -\frac{1}{2}\mathcal{F}^i_{\alpha\dot\alpha}$ and $G_{\alpha\dot\alpha\dot\beta\dot\gamma} = \frac{1}{6}\nabla^{(1}_{\dot\alpha(\dot\beta}\mathcal{F}^{2)}_{\alpha\dot\gamma)}$. On $\mathcal{P}^{3\oplus 1|0}$, we can use

$$\mathcal{A}^+_\alpha \;=\; \lambda^{\dot\alpha}_+ A_{\alpha\dot\alpha} + \gamma^3_+ e^+ \hat\lambda^{\dot\alpha}_+ \hat\lambda^{\dot\beta}_+ \hat\lambda^{\dot\gamma}_+ G_{\alpha\dot\alpha\dot\beta\dot\gamma}\;, \tag{VII.258a}$$

$$\mathcal{A}_{\bar\lambda_+} \;=\; \gamma^4_+ e^+ \hat\lambda^{\dot\alpha}_+ \hat\lambda^{\dot\beta}_+ G_{\dot\alpha\dot\beta}\;. \tag{VII.258b}$$

to have (VII.256) equivalent with (VII.249) and $G_{\alpha\dot\alpha\dot\beta\dot\gamma} = \frac{1}{6}\mathcal{F}_{\alpha\dot\alpha\dot\beta\dot\gamma}$.

For compactness of the discussion, we refrain from explicitly writing down all the reality conditions imposed on the component fields and refer to section VII.3.2 for further details.

§20 Actions. One can reconstruct two action functionals, from which the equations of motion for the two cases arise. With our field normalizations, they read

$$S_{\mathcal{P}^{3\oplus 2|0}} \;=\; \int \mathrm{d}^4x\;\; \mathrm{tr}\,\left(G^{\dot\alpha\dot\beta} f_{\dot\alpha\dot\beta} - \phi^{(1}\Box\phi^{2)}\right)\;, \tag{VII.259}$$

$$S_{\mathcal{P}^{3\oplus 1|0}} \;=\; \int \mathrm{d}^4x\;\; \mathrm{tr}\,\left(G^{\dot\alpha\dot\beta} f_{\dot\alpha\dot\beta}\right)\;. \tag{VII.260}$$

The action $S_{\mathcal{P}^{3\oplus 1|0}}$ has first been proposed in [60].

VII.6 Penrose-Ward transform for mini-supertwistor spaces

It is well-known that the Bogomolny monopole equations are obtained from the four-dimensional self-dual Yang-Mills equations by the dimensional reduction $\mathbb{R}^4 \to \mathbb{R}^3$ and that there is a twistor space, the so-called *mini-twistor space* [126] $\mathcal{P}^2 := \mathcal{O}(2) \to \mathbb{C}P^1$, upon which a Penrose-Ward transform for the dimensionally reduced situation can be constructed. In this section, we will discuss the corresponding superextension, the mini-supertwistor space [63, 229], which will lead to a Penrose-Ward transform between certain holomorphic vector bundles and the supersymmetric Bogomolny monopole equations.

VII.6.1 The mini-supertwistor spaces

In the following, we will constrain the discussion for convenience to the real case $\varepsilon = -1$ with Euclidean signature. The Kleinian signature will require some adjustments, similarly to the ones in the case of Kleinian twistor spaces discussed in section VII.3.1.

§1 Definition by dimensional reduction. We start from the supertwistor space $\mathcal{P}^{3|\mathcal{N}}_{-1}$ with coordinates as defined in section VII.4.1. Let $\mathscr{G}$ be the Abelian group generated by the action of the vector field $\mathscr{T}_2 = \frac{\partial}{\partial x^2}$. This group is the real part of the holomorphic action of the complex group $\mathscr{G}_\mathbb{C} \cong \mathbb{C}$. In other words, we have

$$\begin{aligned}\mathscr{T}_2 \;&=\; \frac{\partial}{\partial x^2} \;=\; \frac{\partial z^a_+}{\partial x^2}\frac{\partial}{\partial z^a_+} + \frac{\partial \bar z^a_+}{\partial x^1}\frac{\partial}{\partial \bar z^a_+}\\ &=\; \left(-\frac{\partial}{\partial z^2_+} + z^3_+\frac{\partial}{\partial z^1_+}\right) + \left(-\frac{\partial}{\partial \bar z^2_+} + \bar z^3_+\frac{\partial}{\partial \bar z^1_+}\right) \;=:\; \mathscr{T}'_+ + \bar{\mathscr{T}}'_+\end{aligned} \tag{VII.261}$$

in the coordinates (z^a_+, η^+_i) on $\hat{\mathcal{U}}_+$, where

$$\mathscr{T}'_+ \;:=\; \mathscr{T}'|_{\hat{\mathcal{U}}_+} \;=\; -\frac{\partial}{\partial z^2_+} + z^3_+\frac{\partial}{\partial z^1_+} \tag{VII.262}$$

is the holomorphic part of the vector field $\mathscr{T}_2$ on $\hat{\mathcal{U}}_+$. Similarly, we obtain

$$\mathscr{T}_2 \;=\; \mathscr{T}'_- + \bar{\mathscr{T}}'_- \quad \text{with} \quad \mathscr{T}'_- \;:=\; \mathscr{T}'|_{\hat{\mathcal{U}}_-} \;=\; -z^3_- \frac{\partial}{\partial z^2_-} + \frac{\partial}{\partial z^1_-} \tag{VII.263}$$

on $\hat{\mathcal{U}}_-$ and $\mathscr{T}'_+ = \mathscr{T}'_-$ on $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. Holomorphic functions f on $\mathcal{P}^{3|4}_{-1}$ thus satisfy

$$\mathscr{T}_2 f(z^a_\pm, \eta^\pm_i) \;=\; \mathscr{T}' f(z^a_\pm, \eta^\pm_i) \tag{VII.264}$$

and therefore $\mathscr{T}'$-invariant holomorphic functions on $\mathcal{P}^{3|4}_{-1}$ can be considered as "free" holomorphic functions on a reduced space $\mathcal{P}^{2|\mathcal{N}}_{-1} \cong \mathcal{P}^{3|\mathcal{N}}_{-1}/\mathscr{G}_\mathbb{C}$ obtained as the quotient space of $\mathcal{P}^{3|4}_{-1}$ by the action of the complex Abelian group $\mathscr{G}_\mathbb{C}$ generated by $\mathscr{T}'$. For convenience, we will omit the subscript -1 on twistor spaces in the remainder of this section.

§2 Reduction diagram. Let us summarize the effect of this dimensional reductions on all the spaces involved in the double fibration by the following diagram:

$$\begin{array}{ccccc}
\mathcal{P}^{3|\mathcal{N}} & \cong & \mathbb{R}^{4|2\mathcal{N}} \times S^2 & \longrightarrow & \mathbb{R}^{4|2\mathcal{N}} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \mathbb{R}^{3|2\mathcal{N}} \times S^2 & & \\
 & \overset{\nu_2}{\swarrow} & & \overset{\nu_1}{\searrow} & \Big\downarrow \\
\mathcal{P}^{2|\mathcal{N}} & & & & \mathbb{R}^{3|2\mathcal{N}}
\end{array} \tag{VII.265}$$

Here, $\downarrow$ symbolizes projections generated by the action of the groups $\mathscr{G}$ or $\mathscr{G}_\mathbb{C}$ and ν_1 is the canonical projection. The projection ν_2 will be described in the next paragraphs.

§3 Local coordinates. The fibre coordinates on $\mathcal{P}^{2|\mathcal{N}}$ are

$$\begin{aligned} w^1_+ &:= -\mathrm{i}(z^1_+ + z^3_+ z^2_+)\ , \quad w^2_+ := z^3_+ \quad \text{and} \quad \eta^+_i \quad \text{on} \quad \hat{\mathcal{U}}_+\ , \\ w^1_- &:= -\mathrm{i}(z^2_- + z^3_- z^1_-)\ , \quad w^2_- := z^3_- \quad \text{and} \quad \eta^-_i \quad \text{on} \quad \hat{\mathcal{U}}_-\ , \end{aligned} \tag{VII.266}$$

since $w^1_\pm$ is constant along the $\mathscr{G}_\mathbb{C}$-orbits in $\mathcal{P}^{3|\mathcal{N}}$ and thus descend to the patches $\hat{\mathcal{V}}_\pm := \hat{\mathcal{U}}_\pm \cap \mathcal{P}^{2|\mathcal{N}}$ covering the mini-supertwistor space. On the overlap $\hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-$, we have

$$w^1_+ \;=\; \frac{1}{(w^2_-)^2} w^1_-\ , \quad w^2_+ \;=\; \frac{1}{w^2_-} \quad \text{and} \quad \eta^+_i \;=\; \frac{1}{w^2_-}\eta^-_i \tag{VII.267}$$

and thus the mini-supertwistor space coincides with the total space of a holomorphic vector bundle with typical fibre of dimension $1|\mathcal{N}$:

$$\mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^\mathcal{N} \;=\; \mathcal{P}^{2|\mathcal{N}} \tag{VII.268}$$

over the Riemann sphere $\mathbb{C}P^1$. In the case $\mathcal{N} = 4$, this space is a Calabi-Yau supermanifold [63] with a holomorphic volume form

$$\Omega|_{\hat{\mathcal{V}}_\pm} \;:=\; \pm \mathrm{d}w^1_\pm \wedge \mathrm{d}w^2_\pm \mathrm{d}\eta^\pm_1 \cdots \mathrm{d}\eta^\pm_4\ . \tag{VII.269}$$

As already mentioned, the body of the mini-supertwistor space $\mathcal{P}^{2|\mathcal{N}}$ is the mini-twistor space [126]

$$\mathcal{P}^2 \;\cong\; \mathcal{O}(2) \;\cong\; T^{1,0}\mathbb{C}P^1\ , \tag{VII.270}$$

where $T^{1,0}\mathbb{C}P^1$ denotes the holomorphic tangent bundle of the Riemann sphere $\mathbb{C}P^1$. Moreover, the space $\mathcal{P}^{2|4}$ can be considered as an open subset of the weighted projective space $W\mathbb{C}P^{2|4}(2,1,1|1,1,1,1)$.

§4 Real structure. Clearly, a real structure τ_{-1} on $\mathcal{P}^{2|\mathcal{N}}$ is induced from the one on $\mathcal{P}^{3|\mathcal{N}}$. On the local coordinates, τ_{-1} acts according to

$$\tau_{-1}(w^1_\pm, w^2_\pm, \eta^\pm_i) \;=\; \left(-\frac{\bar{w}^1_\pm}{(\bar{w}^2_\pm)^2}, -\frac{1}{\bar{w}^2_\pm}, \pm\frac{1}{\bar{w}^2_\pm} T_i{}^j \bar{\eta}^\pm_j\right) \;, \tag{VII.271}$$

where the matrix $T = (T_i{}^j)$ has already been defined in section III.4.2, §13.

§5 Incidence relations. From (VII.271), one sees that similarly to the case $\mathcal{P}^{3|\mathcal{N}}$, τ_{-1} has no fixed points in $\mathcal{P}^{2|\mathcal{N}}$ but leaves invariant projective lines $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{2|\mathcal{N}}$ defined by the equations

$$\begin{aligned} w^1_+ &= y - 2\lambda_+ x^1 - \lambda_+^2 \bar{y}\;, \quad \eta^+_i = \eta^{\dot{1}}_i + \lambda_+ \eta^{\dot{2}}_i \quad \text{with} \quad \lambda_+ = w^2_+ \in U_+\;,\\ w^1_- &= \lambda_-^2 y - 2\lambda_- x^1 - \bar{y}\;, \quad \eta^-_i = \lambda_- \eta^{\dot{1}}_i + \eta^{\dot{2}}_i \quad \text{with} \quad \lambda_- = w^2_- \in U_- \end{aligned} \tag{VII.272}$$

for fixed $(x,\eta) \in \mathbb{R}^{3|2\mathcal{N}}$. Here, $y = -(x^3 + \mathrm{i}x^4)$, $\bar{y} = -(x^3 - \mathrm{i}x^4)$ and x^1 are coordinates on $\mathbb{R}^3$.

We can use the coordinates $y^{\dot{\alpha}\dot{\beta}}$ which were introduced in (IV.77) with

$$y^{\dot{1}\dot{1}} = \bar{y}^{\dot{2}\dot{2}} = y \quad \text{and} \quad y^{\dot{1}\dot{2}} = \bar{y}^{\dot{1}\dot{2}} = -x^1 \tag{VII.273}$$

to rewrite (VII.272) concisely as

$$w^1_\pm = y^{\dot{\alpha}\dot{\beta}} \lambda^\pm_{\dot{\alpha}} \lambda^\pm_{\dot{\beta}}\;, \quad w^2_\pm = \lambda_\pm \quad \text{and} \quad \eta^\pm_i = \eta^{\dot{\alpha}}_i \lambda^\pm_{\dot{\alpha}}\;. \tag{VII.274}$$

In fact, the equations (VII.274) are the appropriate incidence relations for the further discussion. They naturally imply the double fibration

$$\begin{array}{ccc} & \mathcal{K}^{5|2\mathcal{N}} & \\ {\scriptstyle \nu_2} \swarrow & & \searrow {\scriptstyle \nu_1} \\ \mathcal{P}^{2|\mathcal{N}} & & \mathbb{R}^{3|2\mathcal{N}} \end{array} \tag{VII.275}$$

where $\mathcal{K}^{5|2\mathcal{N}} \cong \mathbb{R}^{3|2\mathcal{N}} \times S^2$, ν_1 is again the canonical projection onto $\mathbb{R}^{3|2\mathcal{N}}$ and the projection ν_2 is defined by the formula

$$\nu_2(x^a, \lambda_\pm, \eta^{\dot{\alpha}}_i) \;=\; \nu_2(y^{\dot{\alpha}\dot{\beta}}, \lambda^\pm_{\dot{\alpha}}, \eta^{\dot{\alpha}}_i) \;=\; (w^1_\pm, w^2_\pm, \eta^\pm_i)\;, \tag{VII.276}$$

where $a = 1, 2, 3$. We thus have again the one-to-one correspondences

$$\{\,\tau_{-1}\text{-invariant projective lines } \mathbb{C}P^1_{x,\eta} \text{ in } \mathcal{P}^{2|\mathcal{N}}\} \;\longleftrightarrow\; \{\,\text{points } (x,\eta) \text{ in } \mathbb{R}^{3|2\mathcal{N}}\}\;,$$
$$\{\,\text{points } p \text{ in } \mathcal{P}^{2|\mathcal{N}}\} \;\longleftrightarrow\; \{\,\text{oriented } (1|0)\text{-dimensional lines } \ell_p \text{ in } \mathbb{R}^{3|8}\}\;.$$

§6 Cauchy-Riemann structure on $\mathcal{K}^{5|2\mathcal{N}}$. Although the correspondence space $\mathcal{K}^{5|2\mathcal{N}}$ cannot be interpreted as a complex manifold due to its dimensionality, one can consider it as a Cauchy-Riemann (CR) manifold, see II.2.4, §41. There are now several possible CR structures on the body $\mathbb{R}^3 \times S^2$ of $\mathcal{K}^{5|2\mathcal{N}}$: One of them, which we denote by $\bar{\mathscr{D}}_0$, is generated by the vector fields $\{\partial_{\bar{y}}, \partial_{\bar{\lambda}_\pm}\}$ and corresponds to the identification $\mathcal{K}^5_0 := (\mathbb{R}^3 \times S^2, \bar{\mathscr{D}}_0) \cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}P^1$. Another one, denoted by $\bar{\mathscr{D}}$, is spanned by the basis sections $\{\partial_{\bar{w}^1_\pm}, \partial_{\bar{w}^2_\pm}\}$ of the bundle $T^c(\mathbb{R}^3 \times S^2)$. Note that $\bar{\mathscr{D}}$ is indeed a CR structure as $\mathscr{D} \cap \bar{\mathscr{D}} = \{0\}$ and the distribution $\bar{\mathscr{D}}$ is integrable: $[\partial_{\bar{w}^1_\pm}, \partial_{\bar{w}^2_\pm}] = 0$. Therefore, the pair $(\mathbb{R}^3 \times S^2, \bar{\mathscr{D}}) =: \mathcal{K}^5$ is also a CR manifold. It is obvious that there is a diffeomorphism between the manifolds $\mathcal{K}^5$ and $\mathcal{K}^5_0$, but this is not a CR diffeomorphism since it does not respect the chosen CR structures. Note that a CR five-manifold generalizing the above manifold $\mathcal{K}^5$ can be constructed as a sphere bundle over an arbitrary three-manifold with conformal metric [169]. Following [169], we shall call $\mathcal{K}^5$ the *CR twistor space*.

Since the definition of CR structures naturally carries over to the case of supermanifolds (see e.g. [134]), we can straightforwardly define the CR supermanifold

$$\mathcal{K}^{5|2\mathcal{N}} \;:=\; (\mathbb{R}^{3|2\mathcal{N}} \times S^2, \hat{\bar{\mathscr{D}}}) \quad \text{with} \quad \hat{\bar{\mathscr{D}}} \;=\; \text{span}\left\{ \frac{\partial}{\partial \bar{w}^1_\pm}, \frac{\partial}{\partial \bar{w}^2_\pm}, \frac{\partial}{\partial \bar{\eta}^\pm_i} \right\} . \tag{VII.277}$$

A second interpretation of the space $\mathcal{K}^{5|2\mathcal{N}}$ as a CR supermanifold is $\mathcal{K}^{5|2\mathcal{N}}_0 := (\mathbb{R}^{3|2\mathcal{N}} \times S^2, \hat{\bar{\mathscr{D}}}_0)$ with the distribution $\hat{\bar{\mathscr{D}}}_0 = \text{span}\{\frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{\lambda}_\pm}, \frac{\partial}{\partial \bar{\eta}^1_i}\}$. In both cases, the CR structures are of rank $2|\mathcal{N}$.

§7 Coordinates on $\mathcal{K}^{5|2\mathcal{N}}$. Up to now, we have used the coordinates $(y^{\dot\alpha\dot\beta}, \lambda^\pm_{\dot\alpha}, \hat\lambda^\pm_{\dot\alpha}, \eta^{\dot\alpha}_i)$ on the two patches $\tilde{\mathcal{V}}_\pm$ covering the superspace $\mathbb{R}^{3|2\mathcal{N}} \times S^2$. More convenient for the distribution (VII.277) are, however, the coordinates (VII.272) together with

$$\begin{aligned} w^3_+ \;&:=\; \tfrac{1}{1+\lambda_+\bar\lambda_+}\left[\bar\lambda_+ y + (1-\lambda_+\bar\lambda_+)x^3 + \lambda_+\bar{y}\right] \quad \text{on } \tilde{\mathcal{V}}_+ ,\\ w^3_- \;&:=\; \tfrac{1}{1+\lambda_-\bar\lambda_-}\left[\lambda_- y + (\lambda_-\bar\lambda_- - 1)x^3 + \bar\lambda_-\bar{y}\right] \quad \text{on } \tilde{\mathcal{V}}_- . \end{aligned} \tag{VII.278}$$

and we can write more concisely

$$w^1_\pm \;=\; y^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta} , \quad w^2_\pm \;=\; \lambda_\pm , \quad w^3_\pm \;=\; -\mathrm{i}\gamma_\pm y^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta} \quad \text{and} \quad \eta^\pm_i \;=\; \eta^{\dot\alpha}_i\lambda^\pm_{\dot\alpha} . \tag{VII.279}$$

Note that all the coordinates are complex except for $w^3_\pm$, the latter being real.

§8 Projection onto $\mathcal{P}^{2|\mathcal{N}}$. The coordinates (VII.279) obviously imply that the mini-supertwistor space $\mathcal{P}^{2|\mathcal{N}}$ is a complex subsupermanifold of the CR supermanifold $\mathcal{K}^{5|2\mathcal{N}}$, as they yield a projection

$$\nu_2 \;:\; \mathcal{K}^{5|2\mathcal{N}} \;\to\; \mathcal{P}^{2|\mathcal{N}} . \tag{VII.280}$$

The typical fibres of this projection are real one-dimensional spaces $\ell \cong \mathbb{R}$ parameterized by the coordinates $w^3_\pm$ and the pull-back of the real structure τ_{-1} on $\mathcal{P}^{2|\mathcal{N}}$ to $\mathcal{K}^{5|2\mathcal{N}}$ reverses the orientation of each line ℓ, since $\tau_{-1}(w^3_\pm) = -w^3_\pm$.

The geometry of the fibration (VII.280) becomes clearer when noting that the body $\mathcal{K}^5$ of the supermanifold $\mathcal{K}^{5|2\mathcal{N}}$ can be seen as the sphere bundle

$$S(T\mathbb{R}^3) \;=\; \{(x,u) \in T\mathbb{R}^3 \,|\, \delta_{ab}u^a u^b \;=\; 1\} \;\cong\; \mathbb{R}^3 \times S^2 \tag{VII.281}$$

whose fibres at points $x \in \mathbb{R}^3$ are spheres of unit vectors in $T_x\mathbb{R}^3$ [126]. Since this bundle is trivial, its projection onto $\mathbb{R}^3$ in (VII.275) is obviously $\nu_1(x,u) = x$. Moreover, the complex two-dimensional mini-twistor space $\mathcal{P}^2$ can be described as the space of all oriented lines in $\mathbb{R}^3$. That is, any such line ℓ is defined by a unit vector u^a in the direction of ℓ and a shortest vector v^a from the origin in $\mathbb{R}^3$ to ℓ, and one can easily show [126] that

$$\mathcal{P}^2 \;=\; \{(v,u) \in T\mathbb{R}^3 \,|\, \delta_{ab}u^a v^b \;=\; 0 ,\; \delta_{ab}u^a u^b \;=\; 1\} \;\cong\; T^{1,0}\mathbb{C}P^1 \;\cong\; \mathcal{O}(2) . \tag{VII.282}$$

The fibres of the projection $\nu_2 : \mathcal{K}^5 \to \mathcal{P}^2$ are the orbits of the action of the group $\mathscr{G}' \cong \mathbb{R}$ on $\mathbb{R}^3 \times S^2$ given by the formula $(v^a, u^b) \mapsto (v^a + tu^a, u^b)$ for $t \in \mathbb{R}$ and

$$\mathcal{P}^2 \;\cong\; \mathbb{R}^3 \times S^2/\mathscr{G}' . \tag{VII.283}$$

Now $\mathcal{K}^5$ is a (real) hypersurface in the twistor space $\mathcal{P}^3$, and $\mathcal{P}^2$ is a complex two-dimensional submanifold of $\mathcal{K}^5$. Thus, we have

$$\mathcal{P}^2 \;\subset\; \mathcal{K}^5 \;\subset\; \mathcal{P}^3 \quad \text{and} \quad \mathcal{P}^{2|\mathcal{N}} \;\subset\; \mathcal{K}^{5|2\mathcal{N}} \;\subset\; \mathcal{P}^{3|\mathcal{N}} . \tag{VII.284}$$

§9 Vector fields on $\mathcal{K}^{5|2\mathcal{N}}$. The vector fields on $\mathcal{K}^{5|2\mathcal{N}}$ in the complex bosonic coordinates (VII.279) are related to those in the coordinates $(y, \bar{y}, x^3, \lambda_\pm, \bar{\lambda}_\pm)$ via the formulæ

$$\begin{aligned}
\frac{\partial}{\partial w^1_+} &= \gamma_+^2 \left(\frac{\partial}{\partial y} - \bar{\lambda}_+ \frac{\partial}{\partial x^3} - \bar{\lambda}_+^2 \frac{\partial}{\partial \bar{y}}\right) =: \gamma_+^2 W_1^+ \ ,\\
\frac{\partial}{\partial w^2_+} &= W_2^+ + 2\gamma_+^2 (x^3 + \lambda_+ \bar{y}) W_1^+ - \gamma_+^2 (\bar{y} - 2\bar{\lambda}_+ x^3 - \bar{\lambda}_+^2 y) W_3^+ - \gamma_+ \bar{\eta}_i^{\dot{1}} V_+^i \ ,\\
\frac{\partial}{\partial w^3_+} &= 2\gamma_+ \left(\lambda_+ \frac{\partial}{\partial y} + \bar{\lambda}_+ \frac{\partial}{\partial \bar{y}} + \frac{1}{2}(1 - \lambda_+ \bar{\lambda}_+) \frac{\partial}{\partial x^3}\right) =: W_3^+ \ ,
\end{aligned} \qquad \text{(VII.285a)}$$

as well as

$$\begin{aligned}
\frac{\partial}{\partial w^1_-} &= \gamma_-^2 \left(\bar{\lambda}_-^2 \frac{\partial}{\partial y} - \bar{\lambda}_- \frac{\partial}{\partial x^3} - \frac{\partial}{\partial \bar{y}}\right) =: \gamma_-^2 W_1^- \ ,\\
\frac{\partial}{\partial w^2_-} &= W_2^- + 2\gamma_-^2 (x^3 - \lambda_- y) W_1^- + \gamma_-^2 (\bar{\lambda}_-^2 \bar{y} - 2\bar{\lambda}_- x^3 - y) W_3^- + \gamma_- \bar{\eta}_i^{\dot{2}} V_-^i \ ,\\
\frac{\partial}{\partial w^3_-} &= 2\gamma_- \left(\bar{\lambda}_- \frac{\partial}{\partial y} + \lambda_- \frac{\partial}{\partial \bar{y}} + \frac{1}{2}(\lambda_- \bar{\lambda}_- - 1) \frac{\partial}{\partial x^3}\right) =: W_3^- \ ,
\end{aligned}$$

(VII.285b)

where[26] $W_2^\pm := \frac{\partial}{\partial \lambda_\pm}$. With these identifications, we can also use the vector fields $\bar{W}_1^\pm$, $\bar{W}_2^\pm$ and $\bar{V}_\pm^i$ to generate the CR structure $\hat{\bar{\mathscr{D}}}$. In the simplifying spinorial notation we have furthermore

$$\begin{aligned}
W_1^\pm &= \hat{\lambda}_\pm^{\dot{\alpha}} \hat{\lambda}_\pm^{\dot{\beta}} \partial_{(\dot{\alpha}\dot{\beta})} \ , \quad W_2^\pm = \partial_{\lambda_\pm} \ , \quad W_3^\pm = 2\gamma_\pm \hat{\lambda}_\pm^{\dot{\alpha}} \lambda_\pm^{\dot{\beta}} \partial_{(\dot{\alpha}\dot{\beta})} \ ,\\
V_\pm^i &= -\hat{\lambda}_\pm^{\dot{\alpha}} T_j{}^i \frac{\partial}{\partial \eta_j^{\dot{\alpha}}}
\end{aligned} \qquad \text{(VII.286a)}$$

as well as

$$\begin{aligned}
\bar{W}_1^\pm &= -\lambda_\pm^{\dot{\alpha}} \lambda_\pm^{\dot{\beta}} \partial_{(\dot{\alpha}\dot{\beta})} \ , \quad \bar{W}_2^\pm = \partial_{\bar{\lambda}_\pm} \ , \quad \bar{W}_3^\pm = W_3^\pm = 2\gamma_\pm \hat{\lambda}_\pm^{\dot{\alpha}} \lambda_\pm^{\dot{\beta}} \partial_{(\dot{\alpha}\dot{\beta})} \ ,\\
\bar{V}_\pm^i &= \lambda_\pm^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^{\dot{\alpha}}} \ .
\end{aligned} \qquad \text{(VII.286b)}$$

§10 Forms on $\mathcal{K}^{5|2\mathcal{N}}$. The formulæ for the forms dual to the vector fields (VII.286a) and (VII.286b) read

$$\begin{aligned}
\Theta_\pm^1 &:= \gamma_\pm^2 \lambda_{\dot{\alpha}}^\pm \lambda_{\dot{\beta}}^\pm \mathrm{d}y^{\dot{\alpha}\dot{\beta}} \ , \quad \Theta_\pm^2 := \mathrm{d}\lambda_\pm \ , \quad \Theta_\pm^3 := -\gamma_\pm \lambda_{\dot{\alpha}}^\pm \hat{\lambda}_{\dot{\beta}}^\pm \mathrm{d}y^{\dot{\alpha}\dot{\beta}} \ ,\\
E_i^\pm &:= \gamma_\pm \lambda_{\dot{\alpha}}^\pm T_i{}^j \mathrm{d}\eta_j^{\dot{\alpha}}
\end{aligned} \qquad \text{(VII.287a)}$$

and

$$\begin{aligned}
\bar{\Theta}_\pm^1 &= -\gamma_\pm^2 \hat{\lambda}_{\dot{\alpha}}^\pm \hat{\lambda}_{\dot{\beta}}^\pm \mathrm{d}y^{\dot{\alpha}\dot{\beta}} \ , \quad \bar{\Theta}_\pm^2 = \mathrm{d}\bar{\lambda}_\pm \ , \quad \bar{\Theta}_\pm^3 = \Theta_\pm^3 \ ,\\
\bar{E}_i^\pm &= -\gamma_\pm \hat{\lambda}_{\dot{\alpha}}^\pm \mathrm{d}\eta_i^{\dot{\alpha}} \ ,
\end{aligned} \qquad \text{(VII.287b)}$$

where $T_i{}^j$ has been given in section III.4.2, §13. The exterior derivative on $\mathcal{K}^{5|2\mathcal{N}}$ accordingly given by

$$\begin{aligned}
\mathrm{d}|_{\tilde{\mathcal{V}}_\pm} &= \mathrm{d}w_\pm^1 \frac{\partial}{\partial w_\pm^1} + \mathrm{d}w_\pm^2 \frac{\partial}{\partial w_\pm^2} + \mathrm{d}\bar{w}_\pm^1 \frac{\partial}{\partial \bar{w}_\pm^1} + \mathrm{d}\bar{w}_\pm^2 \frac{\partial}{\partial \bar{w}_\pm^2} + \mathrm{d}w_\pm^3 \frac{\partial}{\partial w_\pm^3} + \dots\\
&= \Theta_\pm^1 W_1^\pm + \Theta_\pm^2 W_2^\pm + \bar{\Theta}_\pm^1 \bar{W}_1^\pm + \bar{\Theta}_\pm^2 \bar{W}_2^\pm + \Theta_\pm^3 W_3^\pm + E_i^\pm V_\pm^i + \bar{E}_i^\pm \bar{V}_\pm^i \ ,
\end{aligned} \qquad \text{(VII.288)}$$

[26]Note that the vector field $W_3^\pm$ is real.

where the dots stand for derivatives with respect to $\eta_i^\pm$ and $\bar\eta_i^\pm$. Note again that $\Theta_\pm^3$ and $W_3^\pm$ are both real.[27]

VII.6.2 Partially holomorphic Chern-Simons theory

§11 Outline. In the following, we will discuss a generalization of Chern-Simons theory on the correspondence space $\mathcal{K}^{5|2\mathcal{N}}$, which we call *partially holomorphic Chern-Simons theory* or phCS theory for short. Roughly speaking, this theory is a mixture of Chern-Simons and holomorphic Chern-Simons theory on the CR supertwistor space $\mathcal{K}^{5|2\mathcal{N}}$ which has one real and two complex bosonic dimensions. Eventually, we will find a one-to-one correspondence between the moduli space of solutions to the equations of motion of phCS theory on $\mathcal{K}^{5|2\mathcal{N}}$ and the moduli space of solutions to $\mathcal{N}$-extended supersymmetric Bogomolny equations on $\mathbb{R}^3$, quite similar to the correspondence between hCS theory on the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ and $\mathcal{N}$-extended supersymmetric SDYM theory in four dimensions.

§12 The integrable distribution $\mathcal{T}$ on $\mathcal{K}^{5|2\mathcal{N}}$. Combining the vector fields $\bar W_1^\pm$, $\bar W_2^\pm$, $\bar V_\pm^i$ from the CR structure $\hat{\bar{\mathscr{D}}}$ with the vector field $\bar W_3^\pm$ yields an integrable distribution, which we denote by $\mathcal{T}$. The distribution $\mathcal{T}$ is integrable since we have $[\bar W_2^\pm, \bar W_3^\pm] = \pm 2\gamma_\pm^2 \bar W_1^\pm$ and all other commutators vanish. Also, $\mathscr{V} := \mathcal{T} \cap \bar{\mathcal{T}}$ is a real one-dimensional and hence integrable. Note that $\mathscr{V}$ is spanned by the vector fields $\bar W_3^\pm$ over the patches $\tilde{\mathcal{V}}_\pm \subset \mathcal{K}^{5|2\mathcal{N}}$. Furthermore, the mini-supertwistor space $\mathcal{P}^{2|\mathcal{N}}$ is a subsupermanifold of $\mathcal{K}^{5|2\mathcal{N}}$ transversal to the leaves of $\mathscr{V} = \mathcal{T} \cap \bar{\mathcal{T}}$ and $\mathcal{T}|_{\mathcal{P}^{2|\mathcal{N}}} = \hat{\bar{\mathscr{D}}}$. Thus, we have an integrable distribution $\mathcal{T} = \hat{\bar{\mathscr{D}}} \oplus \mathscr{V}$ on the CR supertwistor space $\mathcal{K}^{5|2\mathcal{N}}$ and we will denote by $\mathcal{T}_b$ its bosonic part generated by the vector fields $\bar W_1^\pm$, $\bar W_2^\pm$ and $\bar W_3^\pm$.

§13 Field equations of phCS theory. Let $\mathcal{E}$ be a trivial rank n complex vector bundle over $\mathcal{K}^{5|2\mathcal{N}}$ and $\hat{\mathcal{A}}_\mathcal{T}$ a $\mathcal{T}$-connection on $\mathcal{E}$. We define the field equations of partial holomorphic Chern-Simons theory to be

$$\mathrm{d}_\mathcal{T} \hat{\mathcal{A}}_\mathcal{T} + \hat{\mathcal{A}}_\mathcal{T} \wedge \hat{\mathcal{A}}_\mathcal{T} \;=\; 0 \;, \tag{VII.289}$$

In the nonholonomic basis $\{\bar W_a^\pm, \bar V_\pm^i\}$ of the distribution $\mathcal{T}$ over $\tilde{\mathcal{V}}_\pm \subset \mathcal{K}^{5|2\mathcal{N}}$, these equations read as

$$\bar W_1^\pm \hat{\mathcal{A}}_2^\pm - \bar W_2^\pm \hat{\mathcal{A}}_1^\pm + [\hat{\mathcal{A}}_1^\pm, \hat{\mathcal{A}}_2^\pm] \;=\; 0 \;, \tag{VII.290a}$$
$$\bar W_2^\pm \hat{\mathcal{A}}_3^\pm - \bar W_3^\pm \hat{\mathcal{A}}_2^\pm + [\hat{\mathcal{A}}_2^\pm, \hat{\mathcal{A}}_3^\pm] \mp 2\gamma_\pm^2 \hat{\mathcal{A}}_1^\pm \;=\; 0 \;, \tag{VII.290b}$$
$$\bar W_1^\pm \hat{\mathcal{A}}_3^\pm - \bar W_3^\pm \hat{\mathcal{A}}_1^\pm + [\hat{\mathcal{A}}_1^\pm, \hat{\mathcal{A}}_3^\pm] \;=\; 0 \;, \tag{VII.290c}$$

where the components $\hat{\mathcal{A}}_a^\pm$ are defined via the contractions $\hat{\mathcal{A}}_a^\pm := \bar W_a^\pm \lrcorner \hat{\mathcal{A}}_\mathcal{T}$. Analogously to the case of super holomorphic Chern-Simons theory on $\mathcal{P}^{3|\mathcal{N}}$, we assume that

$$\bar V_\pm^i \lrcorner \hat{\mathcal{A}}_\mathcal{T} \;=\; 0 \quad \text{and} \quad \bar V_\pm^i (\hat{\mathcal{A}}_a^\pm) \;=\; 0 \;. \tag{VII.291}$$

§14 Action functional. When restricting to the case $\mathcal{N} = 4$, we can write down an action functional for phCS theory. As it was noted in [63], the $\mathcal{N} = 4$ mini-supertwistor space is a Calabi-Yau supermanifold and thus, there is a holomorphic volume form Ω on

[27] To homogenize the notation later on, we shall also use $\bar W_3^\pm$ and $\partial_{\bar w_3^\pm}$ instead of $W_3^\pm$ and $\partial_{w_3^\pm}$, respectively.

$\mathcal{P}^{2|4}$. Moreover, the pull-back $\tilde{\Omega}$ of this form to $\mathcal{K}^{5|8}$ is globally defined and we obtain locally on the patches $\tilde{\mathcal{V}}_\pm \subset \mathcal{K}^{5|8}$

$$\tilde{\Omega}|_{\tilde{\mathcal{V}}_\pm} \;=\; \pm \mathrm{d}w^1_\pm \wedge \mathrm{d}w^2_\pm \mathrm{d}\eta_1^\pm \cdots \mathrm{d}\eta_4^\pm \;. \tag{VII.292}$$

Together with the assumptions in (VII.291), we can write down the CS-type action functional

$$S_{\text{phCS}} \;=\; \int_{\mathscr{K}^{5|8}} \tilde{\Omega} \wedge \operatorname{tr}\left(\hat{\mathcal{A}}_{\mathcal{T}} \wedge \mathrm{d}_{\mathcal{T}} \hat{\mathcal{A}}_{\mathcal{T}} + \tfrac{2}{3} \hat{\mathcal{A}}_{\mathcal{T}} \wedge \hat{\mathcal{A}}_{\mathcal{T}} \wedge \hat{\mathcal{A}}_{\mathcal{T}}\right) \;, \tag{VII.293}$$

where

$$\mathrm{d}_{\mathcal{T}}|_{\tilde{\mathcal{V}}_\pm} \;=\; \mathrm{d}\bar{w}^a_\pm \frac{\partial}{\partial \bar{w}^a_\pm} + \mathrm{d}\bar{\eta}_i^\pm \frac{\partial}{\partial \bar{\eta}_i^\pm} \tag{VII.294}$$

is the $\mathcal{T}$-differential on $\mathcal{K}^{5|8}$ and $\mathscr{K}^{5|8}$ is the chiral subspace of $\mathcal{K}^{5|8}$ for which $\bar{\eta}^i = 0$. The action functional (VII.293) reproduces the phCS equations of motion (VII.289).

§15 Supersymmetric Bogomolny equations. The equations of motion of the phCS theory defined above are equivalent to the supersymmetric Bogomolny equations (IV.85). To show this, we will give the explicit field expansion similar to (VII.205) necessary to cast the equations (VII.290) into the form (IV.85). As before in (VII.205), we will only consider the case $\mathcal{N} = 4$, from which all other cases $\mathcal{N} < 4$ can be derived by truncation of the field expansion. First, note that due to (VII.286b), we have

$$\bar{W}_1^+ \;=\; \lambda_+^2 \bar{W}_1^- \;,\quad \bar{W}_2^+ \;=\; -\bar{\lambda}_+^{-2} \bar{W}_2^- \quad \text{and} \quad \gamma_+^{-1} \bar{W}_3^+ \;=\; \lambda_+ \bar{\lambda}_+ \left(\gamma_-^{-1} \bar{W}_3^-\right) \tag{VII.295}$$

and therefore $\hat{\mathcal{A}}_1^\pm$, $\hat{\mathcal{A}}_2^\pm$ and $\gamma_\pm^{-1} \hat{\mathcal{A}}_3^\pm$ take values in the bundles $\mathcal{O}(2)$, $\bar{\mathcal{O}}(-2)$ and $\mathcal{O}(1) \otimes \bar{\mathcal{O}}(1)$, respectively. Together with the definitions (VII.291) of $\hat{\mathcal{A}}_a^\pm$ and (VII.286) of $\bar{W}_a^\pm$ as well as the fact that the $\eta_i^\pm$ are nilpotent and $\mathcal{O}(1)$-valued, this determines the dependence of $\hat{\mathcal{A}}_a^\pm$ on $\eta_i^\pm$, $\lambda_\pm$ and $\bar{\lambda}_\pm$ to be

$$\hat{\mathcal{A}}_1^\pm \;=\; -\lambda_\pm^{\dot{\alpha}} \mathcal{B}_{\dot{\alpha}}^\pm \quad \text{and} \quad \hat{\mathcal{A}}_3^\pm \;=\; 2\gamma_\pm \hat{\lambda}_\pm^{\dot{\alpha}} \mathcal{B}_{\dot{\alpha}}^\pm \tag{VII.296}$$

with the abbreviations

$$\begin{aligned} \mathcal{B}_{\dot{\alpha}}^\pm \;:=&\; \lambda_\pm^{\dot{\beta}} B_{\dot{\alpha}\dot{\beta}} + \mathrm{i}\eta_i^\pm \chi^i_{\dot{\alpha}} + \tfrac{1}{2!}\gamma_\pm \eta_i^\pm \eta_j^\pm \hat{\lambda}_\pm^{\dot{\beta}} \phi^{ij}_{\dot{\alpha}\dot{\beta}} + \tfrac{1}{3!}\gamma_\pm^2 \eta_i^\pm \eta_j^\pm \eta_k^\pm \hat{\lambda}_\pm^{\dot{\beta}} \hat{\lambda}_\pm^{\dot{\gamma}} \tilde{\chi}^{ijk}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \\ &+ \tfrac{1}{4!}\gamma_\pm^3 \eta_i^\pm \eta_j^\pm \eta_k^\pm \eta_l^\pm \hat{\lambda}_\pm^{\dot{\beta}} \hat{\lambda}_\pm^{\dot{\gamma}} \hat{\lambda}_\pm^{\dot{\delta}} G^{ijkl}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \end{aligned} \tag{VII.297a}$$

$$\begin{aligned} \hat{\mathcal{A}}_2^\pm \;=&\; \pm\Big(\tfrac{1}{2!}\gamma_\pm^2 \eta_i^\pm \eta_j^\pm \phi^{ij} + \tfrac{1}{3!}\gamma_\pm^3 \eta_i^\pm \eta_j^\pm \eta_k^\pm \hat{\lambda}_\pm^{\dot{\alpha}} \tilde{\chi}^{ijk}_{\dot{\alpha}} \\ &+ \tfrac{1}{4!}\gamma_\pm^4 \eta_i^\pm \eta_j^\pm \eta_k^\pm \eta_l^\pm \hat{\lambda}_\pm^{\dot{\alpha}} \hat{\lambda}_\pm^{\dot{\beta}} G^{ijkl}_{\dot{\alpha}\dot{\beta}}\Big) \;. \end{aligned} \tag{VII.297b}$$

Note that in this expansion, all fields $B_{\dot{\alpha}\dot{\beta}}, \chi^i_{\dot{\alpha}}, \ldots$ depend only on the coordinates $(y^{\dot{\alpha}\dot{\beta}}) \in \mathbb{R}^3$. Substituting (VII.297) into (VII.290a) and (VII.290b), we obtain the equations

$$\begin{gathered} \phi^{ij}_{\dot{\alpha}\dot{\beta}} \;=\; -\left(\partial_{(\dot{\alpha}\dot{\beta})}\phi^{ij} + [B_{\dot{\alpha}\dot{\beta}}, \phi^{ij}]\right) \;,\quad \tilde{\chi}^{ijk}_{\dot{\alpha}(\dot{\beta}\dot{\gamma})} \;=\; -\tfrac{1}{2}\left(\partial_{(\dot{\alpha}(\dot{\beta})}\tilde{\chi}^{ijk}_{\dot{\gamma})} + [B_{\dot{\alpha}(\dot{\beta}}, \tilde{\chi}^{ijk}_{\dot{\gamma})}]\right) \;, \\ G^{ijkl}_{\dot{\alpha}(\dot{\beta}\dot{\gamma}\dot{\delta})} \;=\; -\tfrac{1}{3}\left(\partial_{(\dot{\alpha}(\dot{\beta})} G^{ijkl}_{\dot{\gamma}\dot{\delta})} + [B_{\dot{\alpha}(\dot{\beta}}, G^{ijkl}_{\dot{\gamma}\dot{\delta})}]\right) \end{gathered} \tag{VII.298}$$

showing that $\phi^{ij}_{\dot{\alpha}\dot{\beta}}$, $\tilde{\chi}^{ijk}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ and $G^{ijkl}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ are composite fields, which do not describe independent degrees of freedom. Furthermore, the field $B_{\dot{\alpha}\dot{\beta}}$ can be decomposed into its symmetric part, denoted by $A_{\dot{\alpha}\dot{\beta}} = A_{(\dot{\alpha}\dot{\beta})}$, and its antisymmetric part, proportional to Φ, such that

$$B_{\dot{\alpha}\dot{\beta}} \;=\; A_{\dot{\alpha}\dot{\beta}} - \tfrac{\mathrm{i}}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\Phi \;. \tag{VII.299}$$

Defining additionally

$$\tilde{\chi}_{i\dot{\alpha}} \; := \; \tfrac{1}{3!}\varepsilon_{ijkl}\tilde{\chi}^{jkl}_{\dot{\alpha}} \quad \text{and} \quad G_{\dot{\alpha}\dot{\beta}} \; := \; \tfrac{1}{4!}\varepsilon_{ijkl}G^{ijkl}_{\dot{\alpha}\dot{\beta}} \;, \tag{VII.300}$$

we have thus recovered the field content of the $\mathcal{N}=4$ super Bogomolny equations together with the appropriate field equations (IV.85). Up to a constant, the action functional of the super Bogomolny model (IV.87) can be obtained from the action functional of phCS theory (VII.293) by substituting the above given expansion and integrating over the sphere $\mathbb{C}P^1_{x,\eta} \subset \mathcal{P}^{2|4}$.

§16 The linear systems. To improve our understanding of the vector bundle $\mathcal{E}$ over $\mathcal{K}^{5|2\mathcal{N}}$, let us consider the linear system underlying the equations (VII.290). This system reads

$$\begin{aligned} (\bar{W}^{\pm}_a + \hat{\mathcal{A}}^{\pm}_a)\hat{\psi}_{\pm} \; &= \; 0 \;, \\ \bar{V}^i_{\pm}\hat{\psi}_{\pm} \; &= \; 0 \;, \end{aligned} \tag{VII.301}$$

and has indeed (VII.290) as its compatibility conditions. Using the splitting $A_{\mathcal{T}}|_{\tilde{\mathcal{V}}_{\pm}} = \psi_{\pm}\mathrm{d}_{\mathcal{T}}\psi_{\pm}^{-1}$, we can switch now to the Čech description of an equivalent vector bundle $\tilde{\mathcal{E}}$ with transition function $f_{+-} = \psi_+^{-1}\psi_-$. Similarly to the description of the vector bundles involved in the Penrose-Ward transform over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$, we can find a gauge transformation generated by the globally defined group valued function φ, which acts by $\psi \mapsto \hat{\psi}_{\pm} = \varphi^{-1}\psi_{\pm}$ and leads to

$$\begin{aligned} \hat{\mathcal{A}}^{\pm}_1 \;\mapsto\; \mathcal{A}^{\pm}_1 \;&=\; \varphi^{-1}\hat{\mathcal{A}}^{\pm}_1\varphi + \varphi^{-1}\bar{W}^{\pm}_1\varphi \;=\; \psi_{\pm}\bar{W}^{\pm}_1\psi_{\pm}^{-1} \;, \\ \hat{\mathcal{A}}^{\pm}_2 \;\mapsto\; \mathcal{A}^{\pm}_2 \;&=\; \varphi^{-1}\hat{\mathcal{A}}^{\pm}_2\varphi + \varphi^{-1}\bar{W}^{\pm}_2\varphi \;=\; \psi_{\pm}\bar{W}^{\pm}_2\psi_{\pm}^{-1} \;=\; 0 \;, \\ \hat{\mathcal{A}}^{\pm}_3 \;\mapsto\; \mathcal{A}^{\pm}_3 \;&=\; \varphi^{-1}\hat{\mathcal{A}}^{\pm}_3\varphi + \varphi^{-1}\bar{W}^{\pm}_3\varphi \;=\; \psi_{\pm}\bar{W}^{\pm}_3\psi_{\pm}^{-1} \;, \\ 0 \;=\; \hat{\mathcal{A}}^i_{\pm} \;:=\; \hat{\psi}_{\pm}\bar{V}^i_{\pm}\hat{\psi}_{\pm}^{-1} \;\mapsto\; \mathcal{A}^i_{\pm} \;&=\; \varphi^{-1}\bar{V}^i_{\pm}\varphi \;=\; \psi_{\pm}\bar{V}^i_{\pm}\psi_{\pm}^{-1} \;, \end{aligned} \tag{VII.302}$$

while $f_{+-} \;=\; \hat{\psi}_+^{-1}\hat{\psi}_- = \psi_+^{-1}\psi_-$ remains invariant. In this new gauge, one generically has $\mathcal{A}^i_{\pm} \neq 0$ and the new gauge potential fits into the following linear system of differential equations:

$$(\bar{W}^{\pm}_1 + \mathcal{A}^{\pm}_1)\psi_{\pm} \;=\; 0 \;, \tag{VII.303a}$$

$$\bar{W}^{\pm}_2\psi_{\pm} \;=\; 0 \;, \tag{VII.303b}$$

$$(\bar{W}^{\pm}_3 + \mathcal{A}^{\pm}_3)\psi_{\pm} \;=\; 0 \;, \tag{VII.303c}$$

$$(\bar{V}^i_{\pm} + \mathcal{A}^i_{\pm})\psi_{\pm} \;=\; 0 \;, \tag{VII.303d}$$

which is gauge equivalent to the system (VII.301).

Due to the holomorphy of $\psi_{\pm}$ in $\lambda_{\pm}$ and the condition $\mathcal{A}^+_{\mathcal{I}} = \mathcal{A}^-_{\mathcal{I}}$ on $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, the components $\mathcal{A}^{\pm}_1$, $\gamma_{\pm}^{-1}\mathcal{A}^{\pm}_3$ and $\mathcal{A}^i_{\pm}$ must take the form

$$\mathcal{A}^{\pm}_1 \;=\; -\lambda^{\dot{\alpha}}_{\pm}\lambda^{\dot{\beta}}_{\pm}\mathcal{B}_{\dot{\alpha}\dot{\beta}} \;, \quad \gamma_{\pm}^{-1}\mathcal{A}^{\pm}_3 \;=\; 2\hat{\lambda}^{\dot{\alpha}}_{\pm}\lambda^{\dot{\beta}}_{\pm}\mathcal{B}_{\dot{\alpha}\dot{\beta}} \quad \text{and} \quad \mathcal{A}^i_{\pm} \;=\; \lambda^{\dot{\alpha}}_{\pm}\hat{\mathcal{A}}^i_{\dot{\alpha}} \;, \tag{VII.304}$$

with λ-independent superfields $\mathcal{B}_{\dot{\alpha}\dot{\beta}} := \hat{\mathcal{A}}_{\dot{\alpha}\dot{\beta}} - \frac{\mathrm{i}}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\Phi$ and $\hat{\mathcal{A}}^i_{\dot{\alpha}}$. After introducing the first-order differential operators $\nabla_{\dot{\alpha}\dot{\beta}} := \partial_{(\dot{\alpha}\dot{\beta})} + \mathcal{B}_{\dot{\alpha}\dot{\beta}}$ and $D^i_{\dot{\alpha}} = \frac{\partial}{\partial\eta^{\dot{\alpha}}_i} + \hat{\mathcal{A}}^i_{\dot{\alpha}} =: \partial^i_{\dot{\alpha}} + \hat{\mathcal{A}}^i_{\dot{\alpha}}$, we can write the compatibility conditions of the linear system (VII.303) as

$$\begin{aligned} [\nabla_{\dot{\alpha}\dot{\gamma}}, \nabla_{\dot{\beta}\dot{\delta}}] + [\nabla_{\dot{\alpha}\dot{\delta}}, \nabla_{\dot{\beta}\dot{\gamma}}] \;=\; 0 \;, \quad [D^i_{\dot{\alpha}}, \nabla_{\dot{\beta}\dot{\gamma}}] + [D^i_{\dot{\gamma}}, \nabla_{\dot{\beta}\dot{\alpha}}] \;=\; 0 \;, \\ \{D^i_{\dot{\alpha}}, D^j_{\dot{\beta}}\} + \{D^i_{\dot{\beta}}, D^j_{\dot{\alpha}}\} \;=\; 0 \;. \end{aligned} \tag{VII.305}$$

These equations are the constraint equations of the super Bogomolny model, see also (IV.88).

§17 The one-to-one correspondence. Summarizing, we have described a bijection between the moduli space $\mathcal{M}_{\mathrm{phCS}}$ of certain solutions to the field equations (VII.290) of phCS theory and the moduli space $\mathcal{M}_{\mathrm{sB}}$ of solutions to the supersymmetric Bogomolny equations (IV.85). The moduli spaces are obtained from the respective solution spaces by factorizing with respect to the action of the corresponding groups of gauge transformations.

VII.6.3 Holomorphic BF theory

So far, we defined a Chern-Simons type theory corresponding to the super Bogomolny model, but this model was constructed on the correspondence space $\mathcal{K}^{5|2\mathcal{N}}$, after interpreting it as a partially holomorphic manifold. This is somewhat unusual, and one is naturally led to ask whether there is an equivalent model on the mini-supertwistor space. In fact there is, and we will define it in this section. For simplicity, we will restrict all our considerations from now on to the case $\mathcal{N}=4$.

§18 Holomorphic BF theory on $\mathcal{P}^{2|4}$. Consider the mini-supertwistor space $\mathcal{P}^{2|4}$ together with a topologically trivial holomorphic vector bundle $\mathcal{E}$ of rank n over M. Let $\hat{\mathcal{A}}^{0,1}$ be the $(0,1)$-part of its connection one-form $\hat{\mathcal{A}}$, which we assume to satisfy the conditions $\bar{V}^i_\pm \lrcorner \hat{\mathcal{A}}^{0,1}=0$ and $\bar{V}^i_\pm(\partial_{\bar{w}^{1,2}_\pm}\lrcorner\hat{\mathcal{A}}^{0,1})=0$. Recall that $\mathcal{P}^{2|4}$ is a Calabi-Yau supermanifold and thus comes with the holomorphic volume form Ω which is defined in (VII.269). Hence, we can define a holomorphic BF (hBF) type theory (cf. [227, 142, 21]) on $\mathcal{P}^{2|4}$ with the action

$$S_{\mathrm{hBF}} \;=\; \int_{\mathscr{P}^{2|4}} \Omega\wedge\,\mathrm{tr}\,\{B(\bar{\partial}\hat{\mathcal{A}}^{0,1}+\hat{\mathcal{A}}^{0,1}\wedge\hat{\mathcal{A}}^{0,1})\} \;=\; \int_{\mathscr{P}^{2|4}} \Omega\wedge\,\mathrm{tr}\,\{B\mathcal{F}^{0,2}\}\;, \qquad \text{(VII.306)}$$

where B is a scalar field in the adjoint representation of the gauge group $\mathsf{GL}(n,\mathbb{C})$, $\bar{\partial}$ is the antiholomorphic part of the exterior derivative on $\mathcal{P}^{2|4}$ and $\mathcal{F}^{0,2}$ the $(0,2)$ part of the curvature two-form. The space $\mathscr{P}^{2|4}$ is the subsupermanifold of $\mathcal{P}^{2|4}$ constrained[28] by $\bar{\eta}^\pm_i=0$.

§19 Equations of motion. The corresponding equations of motion of hBF theory are readily derived to be

$$\bar{\partial}\hat{\mathcal{A}}^{0,1}+\hat{\mathcal{A}}^{0,1}\wedge\hat{\mathcal{A}}^{0,1} \;=\; 0\;, \qquad \text{(VII.307a)}$$

$$\bar{\partial}B+[\hat{\mathcal{A}}^{0,1},B] \;=\; 0\;. \qquad \text{(VII.307b)}$$

Furthermore, both these equations as well as the Lagrangian in (VII.306) can be obtained from the equations (VII.289) and the Lagrangian in (VII.293), respectively, by imposing the condition $\partial_{\bar{w}^3_\pm}\hat{\mathcal{A}}_{\bar{w}^a_\pm}=0$ and identifying

$$\hat{\mathcal{A}}^{0,1}|_{\hat{\mathcal{V}}_\pm} \;=\; \mathrm{d}\bar{w}^1_\pm\hat{\mathcal{A}}_{\bar{w}^1_\pm}+\mathrm{d}\bar{w}^2_\pm\hat{\mathcal{A}}_{\bar{w}^2_\pm} \quad\text{and}\quad B^\pm \;:=\; B|_{\hat{\mathcal{V}}_\pm} \;=\; \hat{\mathcal{A}}_{\bar{w}^3_\pm}\;. \qquad \text{(VII.308)}$$

On $\mathcal{P}^{2|4}$, $\hat{\mathcal{A}}_{\bar{w}^3_\pm}$ behaves as a scalar and thus, (VII.307) can be obtained from (VII.289) by demanding invariance of all fields under the action of the group $\mathscr{G}'$.

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[28]In string theory, one would regard $\mathscr{P}^{2|4}$ as the worldvolume of a stack of n not quite space-filling D3-branes.

§20 Interpretation of the B-field. By construction, $B = \{B^{\pm}\}$ is a $\mathfrak{gl}(n, \mathbb{C})$-valued function generating trivial infinitesimal gauge transformations of $\hat{\mathcal{A}}^{0,1}$ and therefore it does not contain any physical degrees of freedom. To understand this statement, let us look at the infinitesimal level of gauge transformations of $\hat{\mathcal{A}}^{0,1}$, which take the form

$$\delta\mathcal{A}^{0,1} \;=\; \bar{\partial}B + [\hat{\mathcal{A}}^{0,1}, B] \tag{VII.309}$$

with $B \in H^0(\mathcal{P}^{2|4}, \mathsf{End}E)$. Such a field B solving moreover (VII.307b) generates holomorphic transformations such that $\delta\hat{\mathcal{A}}^{0,1} \;=\; 0$. Their finite version is

$$\tilde{\mathcal{A}}^{0,1} \;=\; \varphi\hat{\mathcal{A}}^{0,1}\varphi^{-1} + \varphi\bar{\partial}\varphi^{-1} \;=\; \hat{\mathcal{A}}^{0,1}\ , \tag{VII.310}$$

and for a solution $(\hat{\mathcal{A}}^{0,1}, B)$ to equations (VII.307) of the form

$$\mathcal{A}^{0,1}|_{\hat{\mathcal{V}}_\pm} \;=\; \tilde{\psi}_\pm\bar{\partial}\tilde{\psi}_\pm^{-1} \quad\text{and}\quad B^\pm \;=\; \tilde{\psi}_\pm B_0^\pm\tilde{\psi}_\pm^{-1}\ , \tag{VII.311}$$

such a φ takes the form

$$\varphi_\pm \;=\; \tilde{\psi}_\pm \mathrm{e}^{B_0^\pm}\tilde{\psi}_\pm^{-1} \quad\text{with}\quad \varphi_+ \;=\; \varphi_- \quad\text{on}\quad \hat{\mathcal{V}}_+ \cap \hat{\mathcal{V}}_-\ . \tag{VII.312}$$

§21 Full equivalences. Altogether, we arrive at the conclusion that the moduli space of solutions to hBF theory given by the action (VII.306) is bijective to the moduli space of solutions to the phCS-equations, and therefore we can sum up the discussion up to now with the diagram

phCS theory on $\mathcal{K}^{5|8}$
$\swarrow\!\!\nearrow$ $\qquad\qquad$ $\nwarrow\!\!\searrow$ $\qquad$ (VII.313)
hBF theory on $\mathcal{P}^{2|4}$ $\longleftrightarrow$ supersymmetric Bogomolny model on $\mathbb{R}^3$

describing equivalent theories defined on different spaces. Here, it is again implied that the appropriate subsets of the solution spaces to phCS and hBF theories are considered.

VII.7 Superambitwistors and mini-superambitwistors

VII.7.1 The superambitwistor space

Recall that in the construction of the ambitwistor space in section VII.3.3, we "glued together" both the self-dual and anti-self-dual subsectors of Yang-Mills theory to obtain the full theory. It is now possible to define a super-extension of this construction, which sheds more light on the rôle played by the third order thickening.

§1 Definition. For the definition of the superambitwistor space, we take a supertwistor space $\mathcal{P}^{3|3}$ with coordinates $(z_\pm^\alpha, z_\pm^3, \eta_i^\pm)$ together with a "dual" copy[29] $\mathcal{P}_*^{3|3}$ with coordinates $(u_\pm^{\dot\alpha}, u_\pm^{\dot 3}, \theta_\pm^i)$. The dual supertwistor space is considered as a holomorphic supervector bundle over the Riemann sphere $\mathbb{C}P_*^1$ covered by the patches $U_\pm^*$ with the standard local coordinates $\mu_\pm = u_\pm^{\dot 3}$. For convenience, we again introduce the spinorial notation $(\mu_{\dot\alpha}^+) = (1, \mu_+)^T$ and $(\mu_{\dot\alpha}^-) = (\mu_-, 1)^T$. The two patches covering $\mathcal{P}_*^{3|3}$ will be denoted by

[29]The word "dual" refers again to the spinor indices and *not* to the line bundles underlying $\mathcal{P}^{3|3}$.

$\mathcal{U}^*_\pm := \mathcal{P}^{3|3}_*|_{U^*_\pm}$ and the product space $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ of the two supertwistor spaces is thus covered by the four patches

$$\mathcal{U}_{(1)} := \mathcal{U}_+ \times \mathcal{U}^*_+ \ , \quad \mathcal{U}_{(2)} := \mathcal{U}_- \times \mathcal{U}^*_+ \ , \quad \mathcal{U}_{(3)} := \mathcal{U}_+ \times \mathcal{U}^*_- \ , \quad \mathcal{U}_{(4)} := \mathcal{U}_- \times \mathcal{U}^*_- \ , \tag{VII.314}$$

on which we have the coordinates $(z^\alpha_{(a)}, z^3_{(a)}, \eta^{(a)}_i; u^{\dot\alpha}_{(a)}, u^{\dot 3}_{(a)}, \theta^i_{(a)})$. This space is furthermore a rank $4|6$ supervector bundle over the space $\mathbb{C}P^1 \times \mathbb{C}P^1_*$. The global sections of this bundle are parameterized by elements of $\mathbb{C}^{4|6} \times \mathbb{C}^{4|6}_*$ in the following way:

$$z^\alpha_{(a)} = x^{\alpha\dot\alpha}_R \lambda^{(a)}_{\dot\alpha} \ , \quad \eta^{(a)}_i = \eta^{\dot\alpha}_i \lambda^{(a)}_{\dot\alpha} \ ; \quad u^{\dot\alpha}_{(a)} = x^{\alpha\dot\alpha}_L \mu^{(a)}_\alpha \ , \quad \theta^i_{(a)} = \theta^{\alpha i} \mu^{(a)}_\alpha \ . \tag{VII.315}$$

The superambitwistor space is now the subspace $\mathcal{L}^{5|6} \subset \mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ obtained from the *quadric condition* (the "gluing condition")

$$\kappa_{(a)} := z^\alpha_{(a)} \mu^{(a)}_\alpha - u^{\dot\alpha}_{(a)} \lambda^{(a)}_{\dot\alpha} + 2\theta^i_{(a)} \eta^{(a)}_i = 0 \ . \tag{VII.316}$$

In the following, we will denote the restrictions of $\mathcal{U}_{(a)}$ to $\mathcal{L}^{5|6}$ by $\hat{\mathcal{U}}_{(a)}$.

§2 Moduli space and the double fibration. Due to the quadric condition (VII.316), the bosonic moduli are not independent on $\mathcal{L}^{5|6}$, but one rather has the relation

$$x^{\alpha\dot\alpha}_R = x^{\alpha\dot\alpha} - \theta^{\alpha i} \eta^{\dot\alpha}_i \quad \text{and} \quad x^{\alpha\dot\alpha}_L = x^{\alpha\dot\alpha} + \theta^{\alpha i} \eta^{\dot\alpha}_i \ . \tag{VII.317}$$

The moduli $(x^{\alpha\dot\alpha}_R)$ and $(x^{\alpha\dot\alpha}_L)$ are therefore indeed anti-chiral and chiral coordinates on the (complex) superspace $\mathbb{C}^{4|12}$ and with this identification, one can establish the following double fibration using equations (VII.315):

$$\begin{array}{ccc} & \mathcal{F}^{6|12} & \\ {\scriptstyle \pi_2} \swarrow & & \searrow {\scriptstyle \pi_1} \\ \mathcal{L}^{5|6} & & \mathbb{C}^{4|12} \end{array} \tag{VII.318}$$

where $\mathcal{F}^{6|12} \cong \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ and π_1 is the trivial projection. Thus, one has the correspondences

$$\begin{aligned} \big\{ \text{subspaces } (\mathbb{C}P^1 \times \mathbb{C}P^1_*)_{x,\eta,\theta} \text{ in } \mathcal{L}^{5|6} \big\} \ &\longleftrightarrow \ \big\{ \text{points } (x,\eta,\theta) \text{ in } \mathbb{C}^{4|12} \big\} \ , \\ \big\{ \text{points } p \text{ in } \mathcal{L}^{5|6} \big\} \ &\longleftrightarrow \ \big\{ \text{null superlines in } \mathbb{C}^{4|12} \big\} \ . \end{aligned} \tag{VII.319}$$

The above-mentioned null superlines are intersections of α-superplanes and dual β-superplanes. Given a solution $(\hat{x}^{\alpha\dot\alpha}, \hat\eta^{\dot\alpha}_i, \hat\theta^{\alpha i})$ to the incidence relations (VII.315) for a fixed point p in $\mathcal{L}^{5|6}$, the set of points on such a null superline takes the form

$$\{(x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i, \theta^{\alpha i})\} \ \text{with} \ x^{\alpha\dot\alpha} = \hat{x}^{\alpha\dot\alpha} + t\mu^\alpha_{(a)} \lambda^{\dot\alpha}_{(a)} \ , \quad \eta^{\dot\alpha}_i = \hat\eta^{\dot\alpha}_i + \varepsilon_i \lambda^{\dot\alpha}_{(a)} \ , \quad \theta^{\alpha i} = \hat\theta^{\alpha i} + \tilde\varepsilon^i \mu^\alpha_{(a)} \ .$$

Here, t is an arbitrary complex number and ε_i and $\tilde\varepsilon^i$ are both three-vectors with Graßmann-odd components. The coordinates $\lambda^{\dot\alpha}_{(a)}$ and $\mu^\alpha_{(a)}$ are chosen from arbitrary patches on which they are both well-defined. Note that these null superlines are in fact of dimension $1|6$.

§3 Vector fields. The space $\mathcal{F}^{6|12}$ is covered by four patches $\tilde{\mathcal{U}}_{(a)} := \pi_2^{-1}(\hat{\mathcal{U}}_{(a)})$ and the tangent spaces to the $1|6$-dimensional leaves of the fibration $\pi_2 : \mathcal{F}^{6|12} \to \mathcal{L}^{5|6}$ from (VII.318) are spanned by the holomorphic vector fields

$$W^{(a)} := \mu^\alpha_{(a)} \lambda^{\dot\alpha}_{(a)} \partial_{\alpha\dot\alpha} \ , \quad D^i_{(a)} = \lambda^{\dot\alpha}_{(a)} D^i_{\dot\alpha} \quad \text{and} \quad D^{(a)}_i = \mu^\alpha_{(a)} D_{\alpha i} \ , \tag{VII.320}$$

where $D_{\alpha i}$ and $D^i_{\dot\alpha}$ are the superderivatives defined by

$$D_{\alpha i} \ := \ \frac{\partial}{\partial\theta^{\alpha i}} + \eta_i^{\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}} \quad \text{and} \quad D^i_{\dot\alpha} \ := \ \frac{\partial}{\partial\eta_i^{\dot\alpha}} + \theta^{\alpha i}\frac{\partial}{\partial x^{\alpha\dot\alpha}}\ . \qquad \text{(VII.321)}$$

Recall that there is a one-to-one correspondence between isomorphism classes of vector bundles and locally free sheaves and therefore the superambitwistor space $\mathcal{L}^{5|6}$ corresponds in a natural way to the sheaf $\mathscr{L}^{5|6}$ of holomorphic sections of the bundle $\mathcal{L}^{5|6} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$.

§4 $\mathcal{L}^{5|6}$ as a Calabi-Yau supermanifold. Just as the space $\mathcal{P}^{3|4}$, the superambitwistor space $\mathcal{L}^{5|6}$ is a Calabi-Yau supermanifold. To prove this, note that it is sufficient to show that the tangent bundle of the body $\mathcal{L}^5$ of $\mathcal{L}^{5|6}$ has first Chern number 6, which is then cancelled by the contribution of -6 from the (unconstrained) fermionic tangent directions. Consider therefore the map

$$\kappa : (z^\alpha_{(a)}, \eta_i^{(a)}, \lambda_{\dot\alpha}^{(a)}, u^{\dot\alpha}_{(a)}, \theta^i_{(a)}, \mu_\alpha^{(a)}) \ \mapsto \ (\kappa_{(a)}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_\alpha)\ , \qquad \text{(VII.322)}$$

where $\kappa_{(a)}$ has been defined in (VII.316). This map is a vector bundle morphism and gives rise to the short exact sequence

$$0 \ \longrightarrow\ \mathcal{L}^5 \ \longrightarrow\ \mathcal{O}(1,0)\oplus\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(0,1) \ \stackrel{\kappa}{\longrightarrow}\ \mathcal{O}(1,1) \ \longrightarrow\ 0\ , \qquad \text{(VII.323)}$$

where $\mathcal{O}(m,n)$ is a line bundle over the base $\mathbb{C}P^1\times\mathbb{C}P^1_*$ having first Chern numbers m and n with respect to the two $\mathbb{C}P^1$s in the base. The first and second Chern classes of the bundles in this sequence are elements of $H^2(\mathbb{C}P^1\times\mathbb{C}P^1,\mathbb{Z})\cong\mathbb{Z}\times\mathbb{Z}$ and $H^4(\mathbb{C}P^1\times\mathbb{C}P^1,\mathbb{Z})\cong\mathbb{Z}$, respectively. Let us denote the elements of $H^2(\mathbb{C}P^1\times\mathbb{C}P^1,\mathbb{Z})$ by ih_1+jh_2 and the elements of $H^4(\mathbb{C}P^1\times\mathbb{C}P^1,\mathbb{Z})$ by kh_1h_2 with $i,j,k\in\mathbb{Z}$. (That is, h_1, h_2 and h_1h_2 are the generators of the respective cohomology groups.) Then the short exact sequence (VII.323) together with the Whitney product formula yields

$$(1+h_1)(1+h_1)(1+h_2)(1+h_2) \ = \ (1+\alpha_1h_1+\alpha_2h_2+\beta h_1h_2)(1+h_1+h_2)\ , \qquad \text{(VII.324)}$$

where $\alpha_1+\alpha_2$ and β are the first and second Chern numbers of $\mathcal{L}^5$ considered as a holomorphic vector bundle over $\mathbb{C}P^1\times\mathbb{C}P^1_*$. It follows that $c_1=2$ (and $c_2=4$), and taking into account the contribution of the tangent space to the base[30] $\mathbb{C}P^1\times\mathbb{C}P^1_*$, we conclude that the tangent space to $\mathcal{L}^5$ has first Chern number 6.

Since $\mathcal{L}^{5|6}$ is a Calabi-Yau supermanifold, this space can be used as a target space for the topological B-model. However, it is still unclear what the corresponding gauge theory action will look like. The most obvious guess would be some holomorphic BF-type theory, see section IV.3.3, §12, with B a "Lagrange multiplier (0,3)-form".

§5 Reality conditions on the superambitwistor space. Recall that there is a real structure which leads to Kleinian signature on the body of the moduli space $\mathbb{R}^{4|2\mathcal{N}}$ of real holomorphic sections of the fibration π_2 in (VII.162). Furthermore, if $\mathcal{N}$ is even, one can define a second real structure which yields Euclidean signature. Above, we saw that the superambitwistor space $\mathcal{L}^{5|6}$ originates from two copies of $\mathcal{P}^{3|3}$ and therefore, we cannot impose the Euclidean reality condition. However, besides the real structure leading to a Kleinian signature, one can additionally impose a reality condition for which we obtain a Minkowski metric on the body of $\mathbb{R}^{4|4\mathcal{N}}$. In the following, we will focus on the latter.

[30] Recall that $T^{1,0}\mathbb{C}P^1\cong\mathcal{O}(2)$.

Consider the anti-linear involution τ_M which acts on the coordinates of $\mathcal{L}^{5|6}$ according to

$$\tau_M(z^\alpha_\pm, \lambda^\pm_{\dot\alpha}, \eta^\pm_i; u^{\dot\alpha}, \mu^\pm_\alpha, \theta^i_\pm) \;:=\; \left(-\overline{u^{\dot\alpha}_\pm}, \overline{\mu^\pm_\alpha}, \overline{\theta^i_\pm}; -\overline{z^\alpha_\pm}, \overline{\lambda^\pm_{\dot\alpha}}, \overline{\eta^\pm_i}\right) \;. \tag{VII.325}$$

Sections of the bundle $\mathcal{L}^{5|6} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$ which are τ_M-real are thus parameterized by moduli satisfying

$$x^{\alpha\dot\beta} \;=\; -\overline{x^{\dot\beta\alpha}} \quad\text{and}\quad \eta^{\dot\alpha}_i \;=\; \overline{\theta^{\alpha i}} \;. \tag{VII.326}$$

We can extract furthermore the contained real coordinates via the identification

$$\begin{aligned} x^{1\dot1} &= -\mathrm{i}x^0 - \mathrm{i}x^3 \;, & x^{1\dot2} &= -\mathrm{i}x^1 - x^2 \;, \\ x^{2\dot1} &= -\mathrm{i}x^1 + x^2 \;, & x^{2\dot2} &= -\mathrm{i}x^0 + \mathrm{i}x^3 \;, \end{aligned} \tag{VII.327}$$

and obtain a metric of signature $(3,1)$ on $\mathbb{R}^4$ from $\mathrm{d}s^2 := \det(\mathrm{d}x^{\alpha\dot\alpha})$. In this section, we will always adopt this convention, even in the complexified Euclidean situation.

VII.7.2 The Penrose-Ward transform on the superambitwistor space

§6 The holomorphic vector bundle $\mathcal{E}$. Let $\mathcal{E}$ be a topologically trivial holomorphic vector bundle of rank n over $\mathcal{L}^{5|6}$ which becomes holomorphically trivial when restricted to any subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x,\eta,\theta} \hookrightarrow \mathcal{L}^{5|6}$. Due to the equivalence of the Čech and the Dolbeault descriptions of holomorphic vector bundles, we can describe $\mathcal{E}$ either by holomorphic transition functions $\{f_{ab}\}$ or by a holomorphic structure $\bar\partial_{\hat{\mathcal{A}}} = \bar\partial + \hat{\mathcal{A}}$: Starting from a transition function f_{ab}, there is a splitting

$$f_{ab} \;=\; \hat\psi_a^{-1}\hat\psi_b \;, \tag{VII.328}$$

where the $\hat\psi_a$ are smooth $\mathsf{GL}(n,\mathbb{C})$-valued functions[31] on $\mathcal{U}_{(a)}$, since the bundle $\mathcal{E}$ is topologically trivial. This splitting allows us to switch to the holomorphic structure $\bar\partial + \hat{\mathcal{A}}$ with $\hat{\mathcal{A}} = \hat\psi\bar\partial\hat\psi^{-1}$, which describes a trivial vector bundle $\hat{\mathcal{E}} \cong \mathcal{E}$. Note that the additional condition of holomorphic triviality of $\mathcal{E}$ on subspaces $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x,\eta,\theta}$ will restrict the explicit form of $\hat{\mathcal{A}}$.

§7 Relation to $\mathcal{N} = 3$ SYM theory. Back at the bundle $\mathcal{E}$, consider its pull-back $\pi_2^*\mathcal{E}$ with transition functions $\{\pi_2^* f_{ab}\}$, which are constant along the fibres of $\pi_2 : \mathcal{F}^{6|12} \to \mathcal{L}^{5|6}$:

$$W^{(a)}\pi_2^* f_{ab} \;=\; D^i_{(a)}\pi_2^* f_{ab} \;=\; D^{(a)}_i \pi_2^* f_{ab} \;=\; 0 \;, \tag{VII.329}$$

The additional assumption of holomorphic triviality upon reduction onto a subspace allows for a splitting

$$\pi_2^* f_{ab} \;=\; \psi_a^{-1}\psi_b \tag{VII.330}$$

into $\mathsf{GL}(n,\mathbb{C})$-valued functions $\{\psi_a\}$ which are holomorphic on $\tilde{\mathcal{U}}_{(a)}$: Evidently, there is such a splitting holomorphic in the coordinates $\lambda_{(a)}$ and $\mu_{(a)}$ on $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{x,\eta,\theta}$, since $\mathcal{E}$ becomes holomorphically trivial when restricted to these spaces. Furthermore, these subspaces are holomorphically parameterized by the moduli $(x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i, \theta^{\alpha i})$, and thus the splitting (VII.330) is holomorphic in all the coordinates of $\mathcal{F}^{6|12}$. Due to (VII.329), we have on the intersections $\tilde{\mathcal{U}}_{(a)} \cap \tilde{\mathcal{U}}_{(b)}$

$$\psi_a D^i_{(a)}\psi_a^{-1} \;=\; \psi_b D^i_{(a)}\psi_b^{-1} \;=:\; \lambda^{\dot\alpha}_{(a)}\mathcal{A}^i_{\dot\alpha} \;, \tag{VII.331a}$$

$$\psi_a D^{(a)}_i\psi_a^{-1} \;=\; \psi_b D^{(a)}_i\psi_b^{-1} \;=:\; \mu^\alpha_{(a)}\mathcal{A}_{\alpha i} \;, \tag{VII.331b}$$

$$\psi_a W^{(a)}\psi_a^{-1} \;=\; \psi_b W^{(a)}\psi_b^{-1} \;=:\; \mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)}\mathcal{A}_{\alpha\dot\alpha} \;, \tag{VII.331c}$$

[31] In fact, the collection $\{\hat\psi_a\}$ forms a Čech 0-cochain.

where $\mathcal{A}^i_{\dot\alpha}$, $\mathcal{A}_{\alpha i}$ and $\mathcal{A}_{\alpha\dot\alpha}$ are independent of $\mu_{(a)}$ and $\lambda_{(a)}$. The introduced components of the supergauge potential $\mathcal{A}$ fit into the linear system

$$\mu^\alpha_{(a)}\lambda^{\dot\alpha}_{(a)}(\partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha})\psi_a \;=\; 0\;, \tag{VII.332a}$$

$$\lambda^{\dot\alpha}_{(a)}(D^i_{\dot\alpha}+\mathcal{A}^i_{\dot\alpha})\psi_a \;=\; 0\;, \tag{VII.332b}$$

$$\mu^\alpha_{(a)}(D_{\alpha i}+\mathcal{A}_{\alpha i})\psi_a \;=\; 0\;, \tag{VII.332c}$$

whose compatibility conditions are

$$\begin{gathered}\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}+\{\nabla^i_{\dot\beta},\nabla^j_{\dot\alpha}\} \;=\; 0\;,\quad \{\nabla_{\alpha i},\nabla_{\beta j}\}+\{\nabla_{\beta i},\nabla_{\alpha j}\} \;=\; 0\;,\\ \{\nabla_{\alpha i},\nabla^j_{\dot\alpha}\}-2\delta^j_i\nabla_{\alpha\dot\alpha} \;=\; 0\;.\end{gathered} \tag{VII.333}$$

Here, we used the obvious shorthand notations $\nabla^i_{\dot\alpha}:=D^i_{\dot\alpha}+\mathcal{A}^i_{\dot\alpha}$, $\nabla_{\alpha i}:=D_{\alpha i}+\mathcal{A}_{\alpha i}$, and $\nabla_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha}$. Equations (VII.333) are well known to be equivalent to the equations of motion of $\mathcal{N}=3$ SYM theory on[32] $\mathbb{C}^4$ [290], and therefore (up to a reality condition) also to $\mathcal{N}=4$ SYM theory on $\mathbb{C}^4$.

We thus showed that there is a correspondence between certain holomorphic structures on $\mathcal{L}^{5|6}$, holomorphic vector bundles over $\mathcal{L}^{5|6}$ which become holomorphically trivial when restricted to certain subspaces and solutions to the $\mathcal{N}=4$ SYM equations on $\mathbb{C}^4$. The redundancy in each set of objects is modded out by considering gauge equivalence classes and holomorphic equivalence classes of vector bundles, which renders the above correspondences one-to-one.

VII.7.3 The mini-superambitwistor space $\mathcal{L}^{4|6}$

In this section, we define and examine the mini-superambitwistor space $\mathcal{L}^{4|6}$, which we will use to build a Penrose-Ward transform leading to solutions to $\mathcal{N}=8$ SYM theory in three dimensions. We will first give an abstract definition of $\mathcal{L}^{4|6}$ by a short exact sequence, and present more heuristic ways of obtaining the mini-superambitwistor space later.

§8 Abstract definition of the mini-superambitwistor space. The starting point is the product space $\mathcal{P}^{2|3}\times\mathcal{P}^{2|3}_*$ of two copies of the $\mathcal{N}=3$ mini-supertwistor space. In analogy to the space $\mathcal{P}^{3|3}\times\mathcal{P}^{3|3}_*$, we have coordinates

$$\left(w^1_{(a)},\; w^2_{(a)} \;=\; \lambda_{(a)},\; \eta^{(a)}_i;\; v^1_{(a)},\; v^2_{(a)} \;=\; \mu_{(a)},\; \theta^i_{(a)}\right) \tag{VII.334}$$

on the patches $\mathcal{V}_{(a)}$ which are unions of $\mathcal{V}_\pm$ and $\mathcal{V}^*_\pm$:

$$\mathcal{V}_{(1)} \;:=\; \mathcal{V}_+\times\mathcal{V}^*_+\;,\quad \mathcal{V}_{(2)} \;:=\; \mathcal{V}_-\times\mathcal{V}^*_+\;,\quad \mathcal{V}_{(3)} \;:=\; \mathcal{V}_+\times\mathcal{V}^*_-\;,\quad \mathcal{V}_{(4)} \;:=\; \mathcal{V}_-\times\mathcal{V}^*_-\;. \tag{VII.335}$$

For convenience, let us introduce the subspace $\mathbb{C}P^1_\Delta$ of the base of the fibration $\mathcal{P}^{2|3}\times\mathcal{P}^{2|3}_*\to\mathbb{C}P^1\times\mathbb{C}P^1_*$ as

$$\mathbb{C}P^1_\Delta \;:=\; \mathrm{diag}(\mathbb{C}P^1\times\mathbb{C}P^1_*) \;=\; \{(\mu_\pm,\lambda_\pm)\in\mathbb{C}P^1\times\mathbb{C}P^1_*\,|\,\mu_\pm=\lambda_\pm\}\;. \tag{VII.336}$$

Consider now the map $\xi:\mathcal{P}^{2|3}\times\mathcal{P}^{2|3}_*\to\mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$ which is defined by

$$\xi:\left(w^1_{(a)},w^2_{(a)},\eta^{(a)}_i;v^1_{(a)},v^2_{(a)},\theta^i_{(a)}\right)\;\mapsto\;\begin{cases}\left(w^1_\pm-v^1_\pm+2\theta^i_\pm\eta^\pm_i,\;w^2_\pm\right) & \text{for } w^2_\pm \;=\; v^2_\pm\\ \left(0,\;w^2_{(a)}\right) & \text{else}\end{cases}\;, \tag{VII.337}$$

[32] Note that most of our considerations concern the complexified case.

where $\mathcal{O}_{\mathbb{C}P^1_\Delta}(2)$ is the line bundle $\mathcal{O}(2)$ over $\mathbb{C}P^1_\Delta$. In this definition, we used the fact that a point for which $w^2_\pm = v^2_\pm$ holds, is located at least on one of the patches $\mathcal{V}_{(1)}$ and $\mathcal{V}_{(4)}$. Note in particular that the map ξ is a morphism of vector bundles. Therefore, we can define a space $\mathcal{L}^{4|6}$ via the short exact sequence

$$0 \longrightarrow \mathcal{L}^{4|6} \longrightarrow \mathcal{P}^{2|3} \times \mathcal{P}^{2|3}_* \stackrel{\xi}{\longrightarrow} \mathcal{O}_{\mathbb{C}P^1_\Delta}(2) \longrightarrow 0 . \qquad \text{(VII.338)}$$

We shall call this space the *mini-superambitwistor space* and denote the restrictions of the patches $\mathcal{V}_{(a)}$ to $\mathcal{L}^{4|6}$ by $\hat{\mathcal{V}}_{(a)}$.

§9 $\mathcal{L}^{4|6}$ is not a vector bundle. An important consequence of this definition is that the sheaf $\mathscr{L}^{4|6}$ of holomorphic sections of $\mathcal{L}^{4|6}$ is *not* a locally free sheaf, because over any open neighborhood of $\mathbb{C}P^1_\Delta$, it is impossible to write $\mathcal{L}^{4|6}$ as a direct sum of line bundles. This is simply due to the fact that the stalks over $\mathbb{C}P^1_\Delta$ are isomorphic to the stalks of $\mathscr{O}_{\mathbb{C}P^1_\Delta}(2)$, while the stalks over $(\mathbb{C}P^1 \times \mathbb{C}P^1_*)\backslash\mathbb{C}P^1_\Delta$ are isomorphic to the stalks of $\mathscr{O}_{\mathbb{C}P^1\times\mathbb{C}P^1_*}(2,2)$.

It immediately follows that the space $\mathcal{L}^{4|6}$ is not a vector bundle. However, one can easily see that $p : \mathcal{L}^{4|6} \to \mathbb{C}P^1 \times \mathbb{C}P^1_* =: B$ is a fibration since the necessary homotopy lifting property is inherited from the one on $\mathcal{L}^{5|6}$. Given a commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{h} & \mathcal{L}^{4|6} \\ \downarrow r & & \downarrow p \\ X \times [0,1] & \xrightarrow{h_t} & B \end{array} \qquad \text{(VII.339)}$$

the homotopy lifting property demands a map $g : X \times [0,1] \to \mathcal{L}^{4|6}$, which turns the commutative square diagram into two commutative triangle diagrams. One can now always lift the map h to a map $\hat{h} : X \times \{0\} \to \mathcal{L}^{5|6}$ and since $\mathcal{L}^{5|6}$ is a vector bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ and thus a fibration, there is a map $\hat{g} : X \times [0,1] \to \mathcal{L}^{5|6}$ which leads to two commutative triangle diagrams in the square diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{h}} & \mathcal{L}^{5|6} \\ \downarrow \hat{r} & & \downarrow \hat{p} \\ X \times [0,1] & \xrightarrow{h_t} & B \end{array} \qquad \text{(VII.340)}$$

The function g we are looking for is then constructed by composition: $g = \pi \circ \hat{g}$, where π is the natural projection $\pi : \mathcal{L}^{5|6} \to \mathcal{L}^{4|6}$ with $p \circ \pi = \hat{p}$.

The fact that the space $\mathcal{L}^{4|6}$ is neither a supermanifold nor a supervector bundle over B seems at first slightly disturbing. However, once one is aware of this new aspect, it does not cause any deep difficulties as far as the twistor correspondence and the Penrose-Ward transform are concerned.

§10 The mini-superambitwistor space by dimensional reduction. To obtain a clearer picture of the fibration $\mathcal{L}^{4|6}$ and its sections, let us now consider the dimensional reduction of the space $\mathcal{L}^{5|6}$. We will first reduce the product space $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ and then impose the appropriate reduced quadric condition. For the first step, we want to eliminate in both $\mathcal{P}^{3|3}$ and $\mathcal{P}^{3|3}_*$ the dependence on the bosonic modulus x^2. Thus we should factorize by

$$\mathscr{T}_{(a)} = \begin{cases} \frac{\partial}{\partial z_+^2} - z_+^3 \frac{\partial}{\partial z_+^1} & \text{on } \mathcal{U}_{(1)} \\ z_-^3 \frac{\partial}{\partial z_-^2} - \frac{\partial}{\partial z_-^1} & \text{on } \mathcal{U}_{(2)} \\ \frac{\partial}{\partial z_+^2} - z_+^3 \frac{\partial}{\partial z_+^1} & \text{on } \mathcal{U}_{(3)} \\ z_-^3 \frac{\partial}{\partial z_-^2} - \frac{\partial}{\partial z_-^1} & \text{on } \mathcal{U}_{(4)} \end{cases} \quad \text{and} \quad \mathscr{T}^*_{(a)} = \begin{cases} \frac{\partial}{\partial u_+^{\dot 2}} - u_+^{\dot 3} \frac{\partial}{\partial u_+^{\dot 1}} & \text{on } \mathcal{U}_{(1)} \\ \frac{\partial}{\partial u_+^{\dot 2}} - u_+^{\dot 3} \frac{\partial}{\partial u_+^{\dot 1}} & \text{on } \mathcal{U}_{(2)} \\ u_-^{\dot 3} \frac{\partial}{\partial u_-^{\dot 2}} - \frac{\partial}{\partial u_-^{\dot 1}} & \text{on } \mathcal{U}_{(3)} \\ u_-^{\dot 3} \frac{\partial}{\partial u_-^{\dot 2}} - \frac{\partial}{\partial u_-^{\dot 1}} & \text{on } \mathcal{U}_{(4)} \end{cases}, \tag{VII.341}$$

which leads us to the orbit space

$$\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} = (\mathcal{P}^{3|3}/\mathcal{G}) \times (\mathcal{P}_*^{3|3}/\mathcal{G}^*) , \tag{VII.342}$$

where $\mathcal{G}$ and $\mathcal{G}^*$ are the Abelian groups generated by $\mathscr{T}$ and $\mathscr{T}^*$, respectively. Recall that the coordinates we use on this space have been defined in (VII.334). The global sections of the bundle $\mathcal{P}^{2|4} \times \mathcal{P}_*^{2|4} \to \mathbb{C}P^1 \times \mathbb{C}P^1_*$ are captured by the parameterization

$$w^1_{(a)} = y^{\dot\alpha\dot\beta}\lambda^{(a)}_{\dot\alpha}\lambda^{(a)}_{\dot\beta} , \quad v^1_{(a)} = y_*^{\dot\alpha\dot\beta}\mu^{(a)}_{\dot\alpha}\mu^{(a)}_{\dot\beta} , \quad \theta^i_{(a)} = \theta^{\dot\alpha i}\mu^{(a)}_{\dot\alpha} , \quad \eta^{(a)}_i = \eta_i^{\dot\alpha}\lambda^{(a)}_{\dot\alpha} , \tag{VII.343}$$

where we relabel the indices of $\mu^{(a)}_\alpha \to \mu^{(a)}_{\dot\alpha}$ and the moduli $y_*^{\alpha\beta} \to y_*^{\dot\alpha\dot\beta}$, $\theta^{i\alpha} \to \theta^{i\dot\alpha}$, since there is no distinction between left- and right-handed spinors on $\mathbb{R}^3$ or its complexification $\mathbb{C}^3$.

The next step is obviously to impose the quadric condition, gluing together the self-dual and anti-self-dual parts. Note that when acting with $\mathscr{T}$ and $\mathscr{T}^*$ on $\kappa_{(a)}$ as given in (VII.316), we obtain

$$\begin{aligned} \mathscr{T}_{(1)}\kappa_{(1)} &= \mathscr{T}^*_{(1)}\kappa_{(1)} = (\mu_+ - \lambda_+) , \quad & \mathscr{T}_{(2)}\kappa_{(2)} &= \mathscr{T}^*_{(2)}\kappa_{(2)} = (\lambda_-\mu_+ - 1) , \\ \mathscr{T}_{(3)}\kappa_{(3)} &= \mathscr{T}^*_{(3)}\kappa_{(3)} = (1 - \lambda_+\mu_-) , \quad & \mathscr{T}_{(4)}\kappa_{(4)} &= \mathscr{T}^*_{(4)}\kappa_{(4)} = (\lambda_- - \mu_-) . \end{aligned} \tag{VII.344}$$

This implies that the orbits generated by $\mathscr{T}$ and $\mathscr{T}^*$ become orthogonal to the orbits of $\frac{\partial}{\partial\kappa}$ only at $\mu_\pm = \lambda_\pm$. Therefore, it is sufficient to impose the quadric condition $\kappa_{(a)} = 0$ at $\mu_\pm = \lambda_\pm$, after which this condition will automatically be satisfied at the remaining values of $\mu_\pm$ and $\lambda_\pm$. Altogether, we are simply left with

$$\left(w^1_\pm - v^1_\pm + 2\theta^i_\pm\eta_i^\pm\right)\big|_{\lambda_\pm=\mu_\pm} = 0 , \tag{VII.345}$$

and the subset of $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3}$ which satisfies this condition is obviously identical to the mini-superambitwistor space $\mathcal{L}^{4|6}$ defined above.

The condition (VII.345) naturally fixes the parameterization of global sections of the fibration $\mathcal{L}^{4|6}$ by giving a relation between the moduli used in (VII.343). This relation is completely analogous to (VII.317) and reads

$$y^{\dot\alpha\dot\beta} = y_0^{\dot\alpha\dot\beta} - \theta^{(\dot\alpha i}\eta_i^{\dot\beta)} \quad \text{and} \quad y_*^{\dot\alpha\dot\beta} = y_0^{\dot\alpha\dot\beta} + \theta^{(\dot\alpha i}\eta_i^{\dot\beta)} . \tag{VII.346}$$

We clearly see that this parameterization arises from (VII.317) by dimensional reduction from $\mathbb{C}^4 \to \mathbb{C}^3$. Thus indeed, imposing the condition (VII.345) only at $\lambda_\pm = \mu_\pm$ is the dimensionally reduced analogue of imposing the condition (VII.316) on $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$.

§11 Comments on further ways of constructing $\mathcal{L}^{4|6}$. Although the construction presented above seems most natural, one can imagine other approaches of defining the space $\mathcal{L}^{4|6}$. Completely evident is a second way, which uses the description of $\mathcal{L}^{5|6}$ in terms of coordinates on $\mathcal{F}^{6|12}$. Here, one factorizes the correspondence space $\mathcal{F}^{6|12}$ by the groups generated by the vector field $\mathscr{T}_2 = \mathscr{T}_2^*$ and obtains the correspondence space $\mathcal{K}^{5|12} \cong \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ together with equation (VII.346). A subsequent projection π_2 from the dimensionally reduced correspondence space $\mathcal{K}^{5|12}$ then yields the mini-superambitwistor space $\mathcal{L}^{4|6}$ as defined above.

Furthermore, one can factorize $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}_*$ only by $\mathcal{G}$ to eliminate the dependence on one modulus. This will lead to $\mathcal{P}^{2|3} \times \mathcal{P}^{3|3}_*$ and following the above discussion of imposing the quadric condition on the appropriate subspace, one arrives again at (VII.345) and the space $\mathcal{L}^{4|6}$. Here, the quadric condition already implies the remaining factorization of $\mathcal{P}^{2|3} \times \mathcal{P}^{3|3}_*$ by $\mathcal{G}^*$.

Eventually, one could anticipate the identification of moduli in (VII.346) and therefore want to factorize by the group generated by the combination $\mathscr{T} + \mathscr{T}^*$. Acting with this sum on $\kappa_{(a)}$ will produce the sum of the results given in (VII.344), and the subsequent discussion of the quadric condition follows the one presented above.

§12 Double fibration. Knowing the parameterization of global sections of the mini-superambitwistor space fibred over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$ as defined in (VII.346), we can establish a double fibration, similarly to all the other twistor spaces we encountered so far. Even more instructive is the following diagram, in which the dimensional reduction of the involved spaces becomes evident:

$$
\begin{array}{ccccc}
 & & \mathcal{F}^{6|12} & & \\
 & \overset{\pi_2}{\swarrow} & \downarrow & \overset{\pi_1}{\searrow} & \\
\mathcal{L}^{5|6} & & & & \mathbb{C}^{4|12} \\
\downarrow & & \mathcal{K}^{5|12} & & \downarrow \\
 & \overset{\nu_2}{\swarrow} & & \overset{\nu_1}{\searrow} & \\
\mathcal{L}^{4|6} & & & & \mathbb{C}^{3|12}
\end{array}
\qquad \text{(VII.347)}
$$

The upper half is just the double fibration for the quadric (VII.318), while the lower half corresponds to the dimensionally reduced case. The reduction of $\mathbb{C}^{4|12}$ to $\mathbb{C}^{3|12}$ is obviously done by factorizing with respect to the group generated by $\mathscr{T}_2$. The same is true for the reduction of $\mathcal{F}^{6|12} \cong \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$ to $\mathcal{K}^{5|12} \cong \mathbb{C}^{3|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1_*$. The reduction from $\mathcal{L}^{5|6}$ to $\mathcal{L}^{4|6}$ was given above and the projection ν_2 from $\mathcal{K}^{5|12}$ onto $\mathcal{L}^{4|6}$ is defined by equations (VII.343). The four patches covering $\mathcal{F}^{6|12}$ will be denoted by $\tilde{\mathcal{V}}_{(a)} := \nu_2^{-1}(\hat{\mathcal{V}}_{(a)})$.

The double fibration defined by the projections ν_1 and ν_2 yields the following twistor correspondences:

$$
\begin{aligned}
\left\{ \text{subspaces } (\mathbb{C}P^1 \times \mathbb{C}P^1)_{y_0,\eta,\theta} \text{ in } \mathcal{L}^{4|6} \right\} &\longleftrightarrow \left\{ \text{points } (y_0, \eta, \theta) \text{ in } \mathbb{C}^{3|12} \right\} , \\
\left\{ \text{generic points } p \text{ in } \mathcal{L}^{4|6} \right\} &\longleftrightarrow \left\{ \text{null superlines in } \mathbb{C}^{3|12} \right\} , \\
\left\{ \text{points } p \text{ in } \mathcal{L}^{4|6} \text{ with } \lambda_\pm \ = \ \mu_\pm \right\} &\longleftrightarrow \left\{ \text{superplanes in } \mathbb{C}^{3|12} \right\} .
\end{aligned}
\qquad \text{(VII.348)}
$$

The null superlines and the superplanes in $\mathbb{C}^{3|12}$ are defined as the sets $\{(y^{\dot\alpha\dot\beta}, \eta_i^{\dot\alpha}, \theta^{\dot\alpha i})\}$ with

$$
\begin{aligned}
y^{\dot\alpha\dot\beta} &= \hat{y}^{\dot\alpha\dot\beta} + t\lambda^{(\dot\alpha}_{(a)}\mu^{\dot\beta)}_{(a)} , \quad & \eta_i^{\dot\alpha} &= \hat{\eta}_i^{\dot\alpha} + \varepsilon_i \lambda^{\dot\alpha}_{(a)} , \quad & \theta^{\dot\alpha i} &= \hat{\theta}^{\dot\alpha i} + \tilde{\varepsilon}^i \mu^{\dot\alpha}_{(a)} , \\
y^{\dot\alpha\dot\beta} &= \hat{y}^{\dot\alpha\dot\beta} + \kappa^{(\dot\alpha}\lambda^{\dot\beta)}_{(a)} , \quad & \eta_i^{\dot\alpha} &= \hat{\eta}_i^{\dot\alpha} + \varepsilon_i \lambda^{\dot\alpha}_{(a)} , \quad & \theta^{\dot\alpha i} &= \hat{\theta}^{\dot\alpha i} + \tilde{\varepsilon}^i \lambda^{\dot\alpha}_{(a)} ,
\end{aligned}
$$

where t, $\kappa^{\dot\alpha}$, ε_i and $\tilde\varepsilon^i$ are an arbitrary complex number, a complex commuting two-spinor and two three-vectors with Graßmann-odd components, respectively. Note that in the last line, $\lambda_\pm^{\dot\alpha}=\mu_\pm^{\dot\alpha}$, and we could also have written

$$\{(y^{\dot\alpha\dot\beta},\eta_i^{\dot\alpha},\theta^{\dot\alpha i})\}\quad\text{with}\quad y^{\dot\alpha\dot\beta}\;=\;\hat y^{\dot\alpha\dot\beta}+\kappa^{(\dot\alpha}\mu_{(a)}^{\dot\beta)}\ ,\quad \eta_i^{\dot\alpha}\;=\;\hat\eta_i^{\dot\alpha}+\varepsilon_i\mu_{(a)}^{\dot\alpha}\ ,\quad \theta^{\dot\alpha i}\;=\;\hat\theta^{\dot\alpha i}+\tilde\varepsilon^i\mu_{(a)}^{\dot\alpha}\ .$$

The vector fields spanning the tangent spaces to the leaves of the fibration ν_2 are for generic values of $\mu_\pm$ and $\lambda_\pm$ given by

$$\begin{aligned}
W^{(a)} &:= \mu_{(a)}^{\dot\alpha}\lambda_{(a)}^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\ ,\\
\tilde D_{(a)}^i &:= \lambda_{(a)}^{\dot\beta}\tilde D_{\dot\beta}^i \;:=\; \lambda_{(a)}^{\dot\beta}\left(\frac{\partial}{\partial\eta_i^{\dot\beta}}+\theta^{\dot\alpha i}\partial_{(\dot\alpha\dot\beta)}\right)\ ,\\
D_i^{(a)} &:= \mu_{(a)}^{\dot\alpha}D_{\dot\alpha i} \;:=\; \mu_{(a)}^{\dot\alpha}\left(\frac{\partial}{\partial\theta^{\dot\alpha i}}+\eta_i^{\dot\beta}\partial_{(\dot\alpha\dot\beta)}\right)\ ,
\end{aligned}\tag{VII.349}$$

where the derivatives $\partial_{(\dot\alpha\dot\beta)}$ have been defined in (IV.77) and (IV.78). At $\mu_\pm=\lambda_\pm$, however, the fibres of the fibration $\mathcal{L}^{4|6}$ over $\mathbb{C}P^1\times\mathbb{C}P^1_*$ loose one bosonic dimension. As the space $\mathcal{K}^{5|12}$ is a manifold, this means that this dimension has to become tangent to the projection ν_2. In fact, one finds that over $\mathbb{C}P^1_\Delta$, besides the vector fields given in (VII.349), also the vector fields

$$\tilde W_{\dot\beta}^\pm \;=\; \mu_\pm^{\dot\alpha}\partial_{(\dot\alpha\dot\beta)} \;=\; \lambda_\pm^{\dot\alpha}\partial_{(\dot\alpha\dot\beta)}\tag{VII.350}$$

annihilate the coordinates on $\mathcal{L}^{4|6}$. Therefore, the leaves to the projection $\nu_2:\mathcal{K}^{5|12}\to\mathcal{L}^{4|6}$ are of dimension $2|6$ for $\mu_\pm=\lambda_\pm$ and of dimension $1|6$ everywhere else.

§13 Real structure on $\mathcal{L}^{4|6}$. Quite evidently, a real structure on $\mathcal{L}^{4|6}$ is inherited from the one on $\mathcal{L}^{5|6}$, and we obtain directly from (VII.325) the action of τ_M on $\mathcal{P}^{2|4}\times\mathcal{P}^{2|4}_*$, which is given by

$$\tau_M(w_\pm^1,\lambda_{\dot\alpha}^\pm,\eta_i^\pm;v_\pm^1,\mu_{\dot\alpha}^\pm,\theta_\pm^i)\;:=\;\left(-\overline{v_\pm^1},\overline{\mu_{\dot\alpha}^\pm},\overline{\theta_\pm^i};-\overline{w_\pm^1},\overline{\lambda_{\dot\alpha}^\pm},\overline{\eta_i^\pm}\right)\ .\tag{VII.351}$$

This action descends in an obvious manner to $\mathcal{L}^{4|6}$, which leads to a real structure on the moduli space $\mathbb{C}^{3|12}$ via the double fibration (VII.347). Thus, we have as the resulting reality condition

$$y_0^{\dot\alpha\dot\beta}\;=\;-\overline{y_0^{\dot\beta\dot\alpha}}\quad\text{and}\quad\eta_i^{\dot\alpha}\;=\;\overline{\theta^{\dot\alpha i}}\ ,\tag{VII.352}$$

and the identification of the bosonic moduli $y^{\dot\alpha\dot\beta}$ with the coordinates on $\mathbb{R}^3$ reads as

$$y_0^{\dot1\dot1}\;=\;-\mathrm{i}x^0-\mathrm{i}x^3\ ,\quad y_0^{\dot1\dot2}\;=\;y_0^{\dot2\dot1}\;=\;-\mathrm{i}x^1\ ,\quad y_0^{\dot2\dot2}\;=\;-\mathrm{i}x^0+\mathrm{i}x^3\ .\tag{VII.353}$$

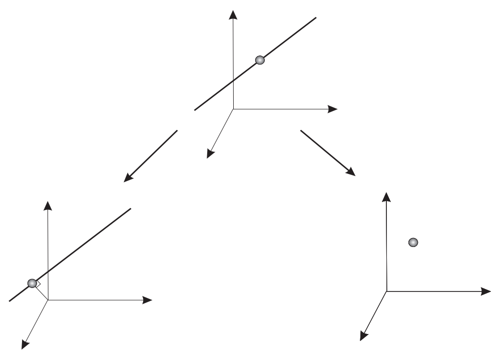
The reality condition $\tau_M(\cdot)=\cdot$ is indeed fully compatible with the condition (VII.345) which reduces $\mathcal{P}^{2|4}\times\mathcal{P}^{2|4}_*$ to $\mathcal{L}^{4|6}$. The base space $\mathbb{C}P^1\times\mathbb{C}P^1_*$ of the fibration $\mathcal{L}^{4|6}$ is reduced to a single sphere S^2 with real coordinates $\frac12(\lambda_\pm+\mu_\pm)=\frac12(\lambda_\pm+\bar\lambda_\pm)$ and $\frac{1}{2\mathrm{i}}(\lambda_\pm-\mu_\pm)=\frac{1}{2\mathrm{i}}(\lambda_\pm-\bar\lambda_\pm)$, while the diagonal $\mathbb{C}P^1_\Delta$ is reduced to a circle S^1_Δ parameterized by the real coordinates $\frac12(\lambda_\pm+\bar\lambda_\pm)$. The τ_M-real sections of $\mathcal{L}^{4|6}$ have to satisfy $w_\pm^1=\tau_M(w_\pm^1)=\bar v_\pm^1$. Thus, the fibres of the fibration $\mathcal{L}^{4|6}\to\mathbb{C}P^1\times\mathbb{C}P^1_*$, which are of complex dimension $2|6$ over generic points in the base and complex dimension $1|6$ over $\mathbb{C}P^1_\Delta$, are reduced to fibres of real dimension $2|6$ and $1|6$, respectively. In particular, note that $\theta_\pm^i\eta_i^\pm=\bar\eta_i^\pm\bar\theta_\pm^i=-\bar\theta_\pm^i\bar\eta_i^\pm$ is purely imaginary and therefore the quadric condition (VII.345) together with the real structure τ_M implies that $w_\pm^1=\bar v_\pm^1=\bar w_\pm^1+2\bar\theta_\pm^i\bar\eta_i^\pm$ for $\lambda_\pm=\mu_\pm=\bar\lambda_\pm$. Thus, the body $\mathring{w}_\pm^1$ of $w_\pm^1$ is purely real and we have $w_\pm^1\;=\;\mathring{w}_\pm^1-\theta_\pm^i\eta_i^\pm$ and $v_\pm^1\;=\;\mathring{w}_\pm^1+\theta_\pm^i\eta_i^\pm$ on the diagonal S^1_Δ.

§14 Interpretation of the involved real geometries. For the best-known twistor correspondences, i.e. the correspondence (VII.45), its dual and the correspondence (VII.129), there is a nice description in terms of flag manifolds, see e.g. the diagrams (VII.74), (VII.75) and (VII.132) as well as the discussion in [284]. For the spaces involved in the twistor correspondences including mini-twistor spaces, one has a similarly nice interpretation after restricting to the real situation. For simplicity, we reduce our considerations to the bodies[33] of the involved geometries, as the extension to corresponding supermanifolds is quite straightforward.

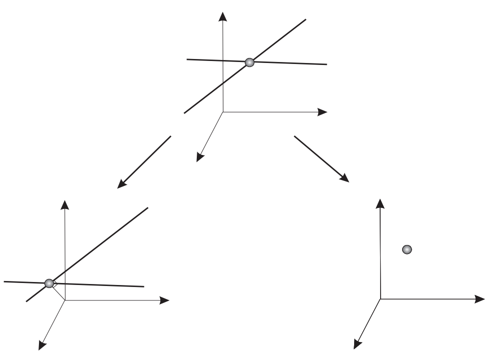
Let us first discuss the double fibration for the mini-twistor space, cf. (VII.275),

$$\begin{array}{ccc} & \mathcal{K}^{5|2\mathcal{N}} & \\ \nu_2 \swarrow & & \searrow \nu_1 \\ \mathcal{P}^{2|\mathcal{N}} & & \mathbb{C}^{3|2\mathcal{N}} \end{array} \qquad \text{(VII.354)}$$

and assume that we have imposed a suitable reality condition on the sections of $\mathcal{P}^{2|\mathcal{N}} \to \mathbb{C}P^1$, the details of which are not important. We follow again the usual discussion of the real case and leave the coordinates on the sphere complex.



As correspondence space on top of the double fibration, we have thus the space $\mathbb{R}^3 \times S^2$, which we can understand as the set of oriented lines[34] in $\mathbb{R}^3$ with one marked point. Clearly, the point of such a line is given by an element of $\mathbb{R}^3$, and the direction of this line in $\mathbb{R}^3$ is parameterized by a point on S^2. The mini-twistor space $\mathcal{P}^2 \cong \mathcal{O}(2)$ now is simply the space of all lines in $\mathbb{R}^3$ [126]. Similarly to the case of flag manifolds, the projections ν_1 and ν_2 in (VII.354) become therefore obvious. For ν_1, simply drop the line and keep the marked point. For ν_2, drop the marked point and keep the line. Equivalently, we can understand ν_2 as moving the marked point on the line to its shortest possible distance from the origin. This leads to the space $TS^2 \cong \mathcal{O}(2)$, where the S^2 parameterizes again the direction of the line, which can subsequently be still moved orthogonally to this direction, and this freedom is parameterized by the tangent planes to S^2, which are isomorphic to $\mathbb{R}^2$.



Now in the case of the fibration which is included in (VII.347), we impose the reality condition (VII.351) on the fibre coordinates of $\mathcal{L}^4$. In the real case, the correspondence space $\mathcal{K}^5$ becomes the space $\mathbb{R}^3 \times S^2 \times S^2$ and this is the space of two oriented lines in $\mathbb{R}^3$ intersecting in a point. More precisely, this is the space of two oriented lines in $\mathbb{R}^3$ each with one marked point, for which the two marked points coincide. The projections ν_1 and ν_2 in (VII.347) are then interpreted as follows. For ν_1, simply drop the two lines and keep the marked point. For ν_2, fix one line and move the marked point (the intersection point) together with the second line to its shortest distance to the origin.

[33] i.e. drop the fermionic directions

[34] not only the ones through the origin

Thus, the space $\mathcal{L}^4$ is the space of configurations in $\mathbb{R}^3$, in which a line has a common point with another line at its shortest distance to the origin.

Let us summarize all the above findings in the following table:

Space	Relation to $\mathbb{R}^3$
$\mathbb{R}^3$	marked points in $\mathbb{R}^3$
$\mathbb{R}^3 \times S^2$	oriented lines with a marked point in $\mathbb{R}^3$
$\mathcal{P}^2 \cong \mathcal{O}(2)$	oriented lines in $\mathbb{R}^3$ (with a marked point at shortest distance to the origin.)
$\mathbb{R}^3 \times S^2 \times S^2$	two oriented lines with a common marked point in $\mathbb{R}^3$
$\mathcal{L}^4$	two oriented lines with a common marked point at shortest distance from one of the lines to the origin in $\mathbb{R}^3$

§15 Remarks concerning a topological B-model on $\mathcal{L}^{4|6}$. The space $\mathcal{L}^{4|6}$ is not well-suited as a target space for a topological B-model since it is not a (Calabi-Yau) manifold. However, one clearly expects that it is possible to define an analogous model since, if we assume that the conjecture in [202] is correct, such a model should simply be the mirror of the mini-twistor string theory considered in [63]. This model would furthermore yield some holomorphic Chern-Simons type equations of motion. The latter equations would then define holomorphic $\mathcal{L}^{4|6}$-bundles by an analogue of a holomorphic structure. These bundles will be introduced in section 4.3 and in our discussion, they substitute the holomorphic vector bundles.

Interestingly, the space $\mathcal{L}^{4|6}$ has a property which comes close to vanishing of a first Chern class. Recall that for any complex vector bundle, its Chern classes are Poincaré dual to the degeneracy cycles of certain sets of sections (this is a Gauß-Bonnet formula). More precisely, to calculate the first Chern class of a rank r vector bundle, one considers r generic sections and arranges them into an $r \times r$ matrix L. The degeneracy loci on the base space are then given by the zero locus of $\det(L)$. Clearly, this calculation can be translated directly to $\mathcal{L}^{4|6}$.

We will now show that $\mathcal{L}^{4|6}$ and $\mathcal{L}^{5|6}$ have equivalent degeneracy loci, i.e. they are equal up to a principal divisor, which, if we were speaking of ordinary vector bundles, would not affect the first Chern class. Our discussion simplifies considerably if we restrict our attention to the bodies of the two supertwistor spaces and put all the fermionic coordinates to zero. Instead of the ambitwistor spaces, it is also easier to consider the vector bundles $\mathcal{P}^3 \times \mathcal{P}^3_*$ and $\mathcal{P}^2 \times \mathcal{P}^2_*$ over $\mathbb{C}P^1 \times \mathbb{C}P^1_*$, respectively, with the appropriately restricted sets of sections. Furthermore, we will stick to our inhomogeneous coordinates and perform the calculation only on the patch $\mathcal{U}_{(1)}$, but all this directly translates into homogeneous, patch-independent coordinates. The matrices to be considered are

$$L_{\mathcal{L}^5} = \begin{pmatrix} x_1^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_2^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_3^{1\dot\alpha}\lambda^+_{\dot\alpha} & x_4^{1\dot\alpha}\lambda^+_{\dot\alpha} \\ x_1^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_2^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_3^{2\dot\alpha}\lambda^+_{\dot\alpha} & x_4^{2\dot\alpha}\lambda^+_{\dot\alpha} \\ x_1^{\alpha 1}\mu^+_{\alpha} & x_2^{\alpha 1}\mu^+_{\alpha} & x_3^{\alpha 1}\mu^+_{\alpha} & x_4^{\alpha 1}\mu^+_{\alpha} \\ x_1^{\alpha 2}\mu^+_{\alpha} & x_2^{\alpha 2}\mu^+_{\alpha} & x_3^{\alpha 2}\mu^+_{\alpha} & x_4^{\alpha 2}\mu^+_{\alpha} \end{pmatrix} , \quad L_{\mathcal{L}^4} = \begin{pmatrix} y_1^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta} & y_2^{\dot\alpha\dot\beta}\lambda^+_{\dot\alpha}\lambda^+_{\dot\beta} \\ y_1^{\dot\alpha\dot\beta}\mu^+_{\dot\alpha}\mu^+_{\dot\beta} & y_2^{\dot\alpha\dot\beta}\mu^+_{\dot\alpha}\mu^+_{\dot\beta} \end{pmatrix} ,$$

and one computes the degeneracy loci for generic moduli to be given by the equations

$$(\lambda_+ - \mu_+)^2 = 0 \quad \text{and} \quad (\lambda_+ - \mu_+)(\lambda_+ - \varrho_+) = 0 \qquad \text{(VII.355)}$$

on the bases of $\mathcal{L}^5$ and $\mathcal{L}^4$, respectively. Here, ϱ_+ is a rational function of μ_+ and therefore it is obvious that both degeneracy cycles are equivalent.

When dealing with degenerated twistor spaces, one usually retreats to the correspondence space endowed with some additional symmetry conditions [195]. It is conceivable that a similar procedure will help to define the topological B-model in our case. Also, defining a suitable blow-up of $\mathcal{L}^{4|6}$ over $\mathbb{C}P^1_\Delta$ could be the starting point for finding an appropriate action.

VII.7.4 The Penrose-Ward transform using mini-ambitwistor spaces

§16 $\mathcal{L}^{4|6}$-bundles. Because the mini-superambitwistor space is only a fibration and not a manifold, there is no notion of holomorphic vector bundles over $\mathcal{L}^{4|6}$. However, our space is close enough to a manifold to translate all the necessary terms in a simple manner.

Let us fix the covering $\mathfrak{U}$ of the total space of the fibration $\mathcal{L}^{4|6}$ to be given by the patches $\mathcal{V}_{(a)}$ introduced above. Furthermore, define $\mathfrak{S}$ to be the sheaf of smooth $\mathsf{GL}(n,\mathbb{C})$-valued functions on $\mathcal{L}^{4|6}$ and $\mathfrak{H}$ to be its subsheaf consisting of holomorphic $\mathsf{GL}(n,\mathbb{C})$-valued functions on $\mathcal{L}^{4|6}$, i.e. smooth and holomorphic functions which depend only on the coordinates given in (VII.343) and $\lambda_{(a)},\mu_{(a)}$.

We define a *complex $\mathcal{L}^{4|6}$-bundle* of rank n by a Čech 1-cocycle $\{f_{ab}\}\in Z^1(\mathfrak{U},\mathfrak{S})$ on $\mathcal{L}^{4|6}$ in full analogy with transition functions defining ordinary vector bundles, see section II.2.3. If the 1-cocycle is an element of $Z^1(\mathfrak{U},\mathfrak{H})$, we speak of a *holomorphic $\mathcal{L}^{4|6}$-bundle.* Two $\mathcal{L}^{4|6}$-bundles given by Čech 1-cocycles $\{f_{ab}\}$ and $\{f'_{ab}\}$ are called *topologically equivalent* (*holomorphically equivalent*) if there is a Čech 0-cochain $\{\psi_a\}\in C^0(\mathfrak{U},\mathfrak{S})$ (a Čech 0-cochain $\{\psi_a\}\in C^0(\mathfrak{U},\mathfrak{H})$) such that $f_{ab}=\psi_a^{-1}f'_{ab}\psi_b$. An $\mathcal{L}^{4|6}$-bundle is called *trivial* (*holomorphically trivial*) if it is topologically equivalent (holomorphically equivalent) to the trivial $\mathcal{L}^{4|6}$-bundle given by $\{f_{ab}\}=\{\mathbb{1}_{ab}\}$.

In the corresponding discussion of Čech cohomology on ordinary manifolds, one can achieve independence of the covering if the patches of the covering are all Stein manifolds. An analogous argument should be also applicable here, but for our purposes, it is enough to restrict to the covering $\mathfrak{U}$.

Besides the Čech description, it is also possible to introduce an equivalent Dolbeault description, which will, however, demand an extended notion of Dolbeault cohomology classes.

§17 The Penrose-Ward transform. With the double fibration contained in (VII.347), it is not hard to establish the corresponding Penrose-Ward transform, which is essentially a dimensional reduction of the four-dimensional case presented in section 4.1.

On $\mathcal{L}^{4|6}$, we start from a trivial rank n holomorphic $\mathcal{L}^{4|6}$-bundle defined by a 1-cocycle $\{f_{ab}\}$ which becomes a holomorphically trivial vector bundle upon restriction to any subspace $(\mathbb{C}P^1\times\mathbb{C}P^1)_{y,\eta,\theta}\hookrightarrow\mathcal{L}^{4|6}$. The pull-back of the $\mathcal{L}^{4|6}$-bundle along ν_2 is the vector bundle $\tilde{\mathcal{E}}$ with transition functions $\{\nu_2^*f_{ab}\}$ satisfying by definition

$$W^{(a)}\nu_2^*f_{ab} \;=\; \tilde{D}^i_{(a)}\nu_2^*f_{ab} \;=\; D^{(a)}_i\nu_2^*f_{ab} \;=\; 0\;, \tag{VII.356}$$

at generic points of $\mathcal{L}^{4|6}$ and for $\lambda_\pm=\mu_\pm$, we have

$$\tilde{W}^{(a)}_{\dot\alpha}\nu_2^*f_{ab} \;=\; \tilde{D}^i_{(a)}\nu_2^*f_{ab} \;=\; D^{(a)}_i\nu_2^*f_{ab} \;=\; 0\;. \tag{VII.357}$$

Restricting the bundle $\tilde{\mathcal{E}}$ to a subspace $(\mathbb{C}P^1\times\mathbb{C}P^1)_{y,\eta,\theta}\hookrightarrow\mathcal{L}^{4|6}\subset\mathcal{F}^{5|12}$ yields a splitting of the transition function $\nu_2^*f_{ab}$

$$\nu_2^*f_{ab} \;=\; \psi_a^{-1}\psi_b\;, \tag{VII.358}$$

where $\{\psi_a\}$ are again $\mathsf{GL}(n,\mathbb{C})$-valued functions on $\tilde{\mathcal{V}}_{(a)}$ which are holomorphic. From this splitting together with (VII.356), one obtains at generic points of $\mathcal{L}^{4|6}$ (we will discuss the situation over $\mathbb{C}P^1_\Delta$ shortly) that

$$\begin{aligned} \psi_a \tilde{D}^i_{(a)} \psi_a^{-1} &= \psi_b \tilde{D}^i_{(a)} \psi_b^{-1} =: \lambda^{\dot\alpha}_{(a)} \tilde{\mathcal{A}}^i_{\dot\alpha} , \\ \psi_a D^{(a)}_i \psi_a^{-1} &= \psi_b D^{(a)}_i \psi_b^{-1} =: \mu^{\dot\alpha}_{(a)} \mathcal{A}_{\dot\alpha i} , \\ \psi_a W^{(a)} \psi_a^{-1} &= \psi_b W^{(a)} \psi_b^{-1} =: \mu^{\dot\alpha}_{(a)} \lambda^{\dot\beta}_{(a)} \mathcal{B}_{\dot\alpha\dot\beta} , \end{aligned} \tag{VII.359}$$

where $\mathcal{B}_{\dot\alpha\dot\beta}$ is a superfield which decomposes into a gauge potential and a Higgs field Φ:

$$\mathcal{B}_{\dot\alpha\dot\beta} := \mathcal{A}_{(\dot\alpha\dot\beta)} + \tfrac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\Phi . \tag{VII.360}$$

The zeroth order component in the superfield expansion of Φ will be the seventh real scalar joining the six scalars of $\mathcal{N}=4$ SYM in four dimensions, which are the zeroth component of the superfield Φ_{ij} defined in

$$\{D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i}, D_{\dot\beta j} + \mathcal{A}_{\dot\beta j}\} =: -2\varepsilon_{\dot\alpha\dot\beta}\Phi_{ij} . \tag{VII.361}$$

Thus, as mentioned above, the $\mathsf{Spin}(7)$ R-symmetry group of $\mathcal{N}=8$ SYM theory in three dimensions will not be manifest in this description.

The equations (VII.359) are equivalent to the linear system

$$\begin{aligned} \mu^{\dot\alpha}_{(a)} \lambda^{\dot\beta}_{(a)} (\partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta})\psi_a &= 0 , \\ \lambda^{\dot\alpha}_{(a)} (\tilde{D}^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha})\psi_a &= 0 , \\ \mu^{\dot\alpha}_{(a)} (D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i})\psi_a &= 0 . \end{aligned} \tag{VII.362}$$

To discuss the corresponding compatibility conditions, we introduce the following differential operators:

$$\begin{aligned} \tilde\nabla^i_{\dot\alpha} := \tilde{D}^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha} , \quad \nabla_{\dot\alpha i} := D_{\dot\alpha i} + \mathcal{A}_{\dot\alpha i} , \\ \nabla_{\dot\alpha\dot\beta} := \partial_{(\dot\alpha\dot\beta)} + \mathcal{B}_{\dot\alpha\dot\beta} . \end{aligned} \tag{VII.363}$$

We thus arrive at

$$\begin{aligned} \{\tilde\nabla^i_{\dot\alpha}, \tilde\nabla^j_{\dot\beta}\} + \{\tilde\nabla^i_{\dot\beta}, \tilde\nabla^j_{\dot\alpha}\} = 0 , \quad \{\nabla_{\dot\alpha i}, \nabla_{\dot\beta j}\} + \{\nabla_{\dot\beta i}, \nabla_{\dot\alpha j}\} = 0 , \\ \{\nabla_{\dot\alpha i}, \tilde\nabla^j_{\dot\beta}\} - 2\delta^j_i \nabla_{\dot\alpha\dot\beta} = 0 , \end{aligned} \tag{VII.364}$$

and one clearly sees that equations (VII.364) are indeed equations (VII.333) after a dimensional reduction $\mathbb{C}^4 \to \mathbb{C}^3$ and defining $\Phi := A_2$. (Recall that we are reducing the coordinates by x^2.) As it is well known, the supersymmetry (and the R-symmetry) of $\mathcal{N}=4$ SYM theory are enlarged by this dimensional reduction and we therefore obtained indeed $\mathcal{N}=8$ SYM theory on $\mathbb{C}^3$.

Let us now examine how the special case $\lambda_\pm = \mu_\pm$ fits into the picture. One immediately notes that a transition function $\nu_2^* f_{ab}$, which satisfies (VII.356) is of the form

$$f_{ab} = f_{ab}(y^{\dot\alpha\dot\beta}\lambda^{(a)}_{\dot\alpha}\lambda^{(a)}_{\dot\beta}, y^{\dot\alpha\dot\beta}\mu^{(a)}_{\dot\alpha}\mu^{(a)}_{\dot\beta}, \lambda^{(a)}_{\dot\alpha}, \mu^{(a)}_{\dot\alpha}) , \tag{VII.365}$$

and thus the condition (VII.357) is obviously fulfilled for $\lambda_\pm = \mu_\pm$. This implies in particular that for $\lambda_\pm = \mu_\pm$, nothing peculiar happens, and it suffices to consider the linear system (VII.362).

Following the above analysis in a straightforward manner for $\lambda_\pm = \mu_\pm$, one arrives at a linear system which contains singular operators on $\mathbb{C}P^1_\Delta$ and the compatibility conditions of this system cannot be pushed forward from the correspondence space $\mathcal{K}^{5|12}$ down to $\mathbb{C}^{3|12}$. As mentioned above, we can ignore this point, as it will be equivalent to considering the linear system (VII.362) over $\mathbb{C}P^1_\Delta$.

To sum up, we obtained a correspondence between holomorphic $\mathcal{L}^{4|6}$-bundles which become holomorphically trivial vector bundles upon reduction to any subspace $(\mathbb{C}P^1 \times \mathbb{C}P^1)_{y,\eta,\theta} \hookrightarrow \mathcal{L}^{4|6}$ and solutions to the three-dimensional $\mathcal{N} = 8$ SYM equations. As this correspondence arises by a dimensional reduction of a correspondence which is one-to-one, it is rather evident that also in this case, we have a bijection between both the holomorphic $\mathcal{L}^{4|6}$-bundles and the solutions after factorizing with respect to holomorphic equivalence and gauge equivalence, respectively.

§18 Yang-Mills-Higgs theory in three dimensions. One can translate the discussion of the ambitwistor space in VII.3.3 to the three-dimensional situation, giving rise to a Penrose-Ward transform between holomorphic $\mathcal{L}^4$ bundles and the Yang-Mills-Higgs equations. First of all, recall from section IV.2.5, §30 the appropriate Yang-Mills-Higgs equations obtained by dimensional reduction are

$$\nabla^{(\dot\alpha\dot\beta)} F_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)} \;=\; [\phi, \nabla_{(\dot\gamma\dot\delta)}\phi] \quad \text{and} \quad \triangle\phi \;:=\; \nabla^{(\dot\alpha\dot\beta)}\nabla_{(\dot\alpha\dot\beta)}\phi \;=\; 0\;, \tag{VII.366}$$

while the self-dual and anti-self-dual Yang-Mills equations correspond after the dimensional reduction to two Bogomolny equations which read

$$F_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)} \;=\; \varepsilon_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)(\dot\varepsilon\dot\zeta)}\nabla^{(\dot\varepsilon\dot\zeta)}\phi \quad \text{and} \quad F_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)} \;=\; -\varepsilon_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)(\dot\varepsilon\dot\zeta)}\nabla^{(\dot\varepsilon\dot\zeta)}\phi\;, \tag{VII.367}$$

respectively. Using the decomposition $F_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)} = \varepsilon_{\dot\alpha\dot\gamma} f_{\dot\beta\dot\delta} + \varepsilon_{\dot\beta\dot\delta} f_{\dot\alpha\dot\gamma}$, the above two equations can be simplified to

$$f_{\dot\alpha\dot\beta} \;=\; \tfrac{\mathrm{i}}{2}\nabla_{(\dot\alpha\dot\beta)}\phi \quad \text{and} \quad f_{\dot\alpha\dot\beta} \;=\; -\tfrac{\mathrm{i}}{2}\nabla_{(\dot\alpha\dot\beta)}\phi\;. \tag{VII.368}$$

Analogously to the four-dimensional case, we start from a vector bundle E over the space $\mathbb{C}^3 \times \mathbb{C}^3$ with coordinates $p^{(\dot\alpha\dot\beta)}$ and $q^{(\dot\alpha\dot\beta)}$; additionally we introduce the coordinates

$$y^{(\dot\alpha\dot\beta)} \;=\; \tfrac{1}{2}(p^{(\dot\alpha\dot\beta)} + q^{(\dot\alpha\dot\beta)}) \quad \text{and} \quad h^{(\dot\alpha\dot\beta)} \;=\; \tfrac{1}{2}(p^{(\dot\alpha\dot\beta)} - q^{(\dot\alpha\dot\beta)}) \tag{VII.369}$$

and a gauge potential

$$A \;=\; A^p_{(\dot\alpha\dot\beta)}\mathrm{d}p^{(\dot\alpha\dot\beta)} + A^q_{(\dot\alpha\dot\beta)}\mathrm{d}q^{(\dot\alpha\dot\beta)} \;=\; A^y_{(\dot\alpha\dot\beta)}\mathrm{d}y^{(\dot\alpha\dot\beta)} + A^h_{(\dot\alpha\dot\beta)}\mathrm{d}h^{(\dot\alpha\dot\beta)} \tag{VII.370}$$

on E. The differential operators we will consider in the following are obtained from covariant derivatives by dimensional reduction and take, e.g., the shape

$$\nabla^y_{\dot\alpha\dot\beta} \;=\; \frac{\partial}{\partial y^{(\dot\alpha\dot\beta)}} + [A^y_{(\dot\alpha\dot\beta)} + \tfrac{\mathrm{i}}{2}\varepsilon_{\dot\alpha\dot\beta}\phi^y, \cdot\,]\;. \tag{VII.371}$$

We now claim that the Yang-Mills-Higgs equations (VII.366) are equivalent to the equations

$$\begin{aligned}
[\nabla^p_{\dot\alpha\dot\beta}, \nabla^p_{\dot\gamma\dot\delta}] \;&=\; *[\nabla^p_{\dot\alpha\dot\beta}, \nabla^p_{\dot\gamma\dot\delta}] + \mathcal{O}(h^2)\;,\\
[\nabla^q_{\dot\alpha\dot\beta}, \nabla^q_{\dot\gamma\dot\delta}] \;&=\; -*[\nabla^q_{\dot\alpha\dot\beta}, \nabla^q_{\dot\gamma\dot\delta}] + \mathcal{O}(h^2)\;,\\
[\nabla^p_{\dot\alpha\dot\beta}, \nabla^q_{\dot\gamma\dot\delta}] \;&=\; \mathcal{O}(h^2)\;,
\end{aligned} \tag{VII.372}$$

where we can use $*[\nabla^{p,q}_{\dot\alpha\dot\beta},\nabla^{p,q}_{\dot\gamma\dot\delta}]=\varepsilon_{\dot\alpha\dot\beta\dot\gamma\dot\delta\dot\varepsilon\dot\zeta}\nabla^{\dot\varepsilon\dot\zeta}_{p,q}\phi^{p,q}$. These equations can be simplified in the coordinates (y,h) to equations similar to (VII.151), which are solved by the field expansion

$$\begin{aligned}
A^h_{(\dot\alpha\dot\beta)} &= -\tfrac{1}{2}F^{y,0}_{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)}h^{(\dot\gamma\dot\delta)}-\tfrac{1}{3}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\varepsilon_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}(\nabla^{(\dot\sigma\dot\tau)}_{y,0}\phi)h^{(\dot\varepsilon\dot\zeta)}\ ,\\
\phi^h &= \tfrac{1}{2}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\phi^{y,0}h^{(\dot\gamma\dot\delta)}+\tfrac{1}{6}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}\varepsilon^{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}F^{y,0}_{(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}h_{(\dot\alpha\dot\beta)}\ ,\\
A^y_{(\dot\alpha\dot\beta)} &= A^{y,0}_{(\dot\alpha\dot\beta)}-\varepsilon_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}(\nabla^{(\dot\sigma\dot\tau)}_{y,0}\phi^{y,0})h^{(\dot\varepsilon\dot\zeta)}-\tfrac{1}{2}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}(F^{y,0}_{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)})h^{(\dot\varepsilon\dot\zeta)}\ ,\\
\phi^y &= \phi^{y,0}+\tfrac{1}{2}\varepsilon^{(\dot\alpha\dot\beta)(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}F^{y,0}_{(\dot\varepsilon\dot\zeta)(\dot\sigma\dot\tau)}h_{(\dot\alpha\dot\beta)}+\tfrac{1}{2}h^{(\dot\gamma\dot\delta)}\nabla^{y,0}_{(\dot\gamma\dot\delta)}(\nabla_{(\dot\alpha\dot\beta)}\phi^{y,0})h^{(\dot\alpha\dot\beta)}\ ,
\end{aligned}\qquad\text{(VII.373)}$$

if and only if the Yang-Mills-Higgs equations (VII.366) are satisfied.

Thus, solutions to the Yang-Mills-Higgs equations (VII.366) correspond to solutions to equations (VII.372) on $\mathbb{C}^3\times\mathbb{C}^3$. Recall that solutions to the first two equations of (VII.372) correspond in the twistor description to holomorphic vector bundles over $\mathcal{P}^2\times\mathcal{P}^2_*$. Furthermore, the expansion of the gauge potential (VII.373) is an expansion in a second order infinitesimal neighborhood of diag$(\mathbb{C}^3\times\mathbb{C}^3)$. As we saw in the construction of the mini-superambitwistor space $\mathcal{L}^{4|6}$, the diagonal for which $h^{(\dot\alpha\dot\beta)}=0$ corresponds to $\mathcal{L}^4\subset\mathcal{P}^2\times\mathcal{P}^2_*$. The neighborhoods of this diagonal will then correspond to *sub-thickenings* of $\mathcal{L}^4$ inside $\mathcal{P}^2\times\mathcal{P}^2_*$, i.e. for $\mu_\pm=\lambda_\pm$, we have the additional nilpotent coordinate ξ. In other words, the sub-thickening of $\mathcal{L}^4$ in $\mathcal{P}^2\times\mathcal{P}^2_*$ is obtained by turning one of the fiber coordinates of $\mathcal{P}^2\times\mathcal{P}^2$ over $\mathbb{C}P^1_\Delta$ into a nilpotent even coordinate (in a suitable basis). Then we can finally state the following:

Gauge equivalence classes of solutions to the three-dimensional Yang-Mills-Higgs equations are in one-to-one correspondence with gauge equivalence classes of holomorphic $\mathcal{L}^4$-bundles over a third order sub-thickening of $\mathcal{L}^4$, which become holomorphically trivial vector bundles when restricted to a $\mathbb{C}P^1\times\mathbb{C}P^1$ holomorphically embedded into $\mathcal{L}^4$.

VII.8 Solution generating techniques

In this section, we will discuss solution generating techniques which are related to the twistorial description of field theories.

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instantons [13] reduces the self-duality equations to a simple set of matrix equations. This construction has been shown to be complete, i.e. all instanton solutions can be obtained by this algorithm. The original idea was to find an instanton bundle over $\mathcal{P}^3$ (a topologically trivial holomorphic vector bundle, which becomes holomorphically trivial upon restriction to any $\mathbb{C}P^1_x\subset\mathcal{P}^3$) from a so-called monad. Nevertheless, a very nice interpretation in terms of D-brane configurations has been found later on [295, 84, 85], see also [83, 274]. Furthermore, supersymmetric extensions of the ADHM construction have been proposed [256, 282].

The corresponding reduction to the three-dimensional Bogomolny equations is given by the Nahm construction [200] with a D-brane interpretation developed in [75]. A corresponding superextension was proposed in [176], and we will present this extension in section VII.8.4.

We will present further solution generating techniques in section VIII.3.2.

VII.8.1 The ADHM construction from monads

In discussing the ADHM construction from monads, we follow essentially the presentations in [284] and [94]. The technique of obtaining vector bundles from monads stems originally from Horrocks [132], see also [12].

§1 Monads. A *monad* $\mathfrak{M}$ over a manifold M is a triple of vector bundles A, B, C over M, which fits into the sequence of vector bundles

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \ , \tag{VII.374}$$

and thus the linear maps α and β satisfy $\beta\alpha = 0$. The vector bundle $E = \ker\beta/\mathrm{im}\alpha$ is called the *cohomology of the monad.*

The rank and the total Chern class of the cohomology E of the monad $\mathfrak{M}$ constructed above can be derived from the corresponding data of the triple A, B, C via the formulæ

$$\begin{aligned} \mathrm{rk}E &= \mathrm{rk}B - \mathrm{rk}A - \mathrm{rk}C \ , \\ c(E) &= c(B)c(A)^{-1}c(C)^{-1} \ . \end{aligned} \tag{VII.375}$$

§2 Annihilator. The annihilator $U^0 \subset V$ of a subspace U of a symplectic vector space[35] V is given by those vectors $v \in V$, which vanish upon pairing with any element of U and applying the symplectic form:

$$U^0 \ := \ \{v \in V | \omega(v, u) = 0 \text{ for all } u \in U\} \ . \tag{VII.376}$$

§3 The instanton monad. Let us now construct a monad $\mathfrak{M}$, which yields an instanton bundle as its cohomology. For simplicity, we will restrict ourselves to the gauge group $\mathsf{SU}(2)$, but via embeddings, it is possible to generalize this discussion to gauge groups $\mathsf{SU}(n)$, $\mathsf{SO}(n)$ and $\mathsf{Sp}(n)$.

Note that by introducing a symplectic form ω on a vector bundle B, B can be identified with its own dual. Furthermore, A will be dual to C and α to β. Thus, we can reduce the data defining our monad $\mathfrak{M}$ to A, (B, ω) and α. For our construction, we choose A to be $\mathcal{O}^k(-1)$ over $\mathbb{C}P^3$ and B is the trivial bundle $\mathbb{C}^{2k+2} \to \mathbb{C}P^3$.

It now remains to specify α. For this, take two complex vector spaces $V = \mathbb{C}^{2k+2}$ and $W = \mathbb{C}^k$ with a symplectic form ω on V, on both of which we have antilinear maps τ, with $\tau_W^2 = \mathbb{1}$ (a complex conjugation) and $\tau_V^2 = -\mathbb{1}$ (induced from the real structure on $\mathbb{C}P^3$), the latter being compatible with the symplectic form ω: $\omega(\tau v_1, \tau v_2) = \overline{\omega(v_1, v_2)}$ and the induced Hermitian form $h(v_1, v_2) = \omega(v_1, \tau v_2)$ for $v_{1,2} \in V$ shall be positive definit. Additionally, we assume a map

$$\alpha : W \ \to \ V \quad \text{with} \quad \alpha \ = \ A_i Z^i \ = \ A^{\dot\alpha}\lambda_{\dot\alpha} + A_\alpha \omega^\alpha \ , \tag{VII.377}$$

where $(Z^i) := (\omega^\alpha, \lambda_{\dot\alpha})$ will become the homogeneous coordinates on the twistor space $\mathbb{C}P^3$ and $A^{\dot\alpha}, A_\alpha$ are constant linear maps from W to V. The map α satisfies the compatibility condition $\tau\alpha(Z)w = \alpha(\tau Z)\tau w$ with the maps τ. Since α is linear in Z, we can also see it as a homomorphism of vector bundles $\alpha : W(-1) \to V \times \mathbb{C}P^3$, where $W(-1) = W \otimes \mathcal{O}_{\mathbb{C}P^3}(-1)$. From the map $\alpha^\vee : V^\vee = V \to W^\vee$, we obtain the monad $\mathfrak{M}$

$$W(-1) \xrightarrow{\alpha} V \times \mathbb{C}P^3 \xrightarrow{\alpha^\vee} W^\vee(1) \ , \tag{VII.378}$$

[35] A symplectic vector space is a vector space equipped with a symplectic form ω. That is, ω is a nondegenerate, skew-symmetric bilinear form.

where $W^\vee(1)$ is the space $W \otimes (\mathcal{O}_{\mathbb{C}P^3}(-1))^\vee$.

We impose now two additional conditions on the linear map α. First, the space $U_Z := \alpha W$ is of dimension k and second, for all $Z \neq 0$, U_Z is a subset of U_Z^0. The latter condition is automatically satisfied for $k = 1$. For $k > 1$, this amounts to the matrix equation $\alpha^T \omega \alpha = 0$. The instanton bundle over[36] $\mathbb{C}P^3$ is then given by the resulting cohomology

$$E_Z \;:=\; U_Z^0/U_Z \tag{VII.379}$$

of $\mathfrak{M}$. Since both U_Z^0 and U_Z are independent of the scaling, we have $E_Z = E_{tZ}$ and therefore the family of all E_Z is indeed a vector bundle E over $\mathbb{C}P^3$. In particular, since $\dim U_Z = k$ and $\dim U_Z^0 = k+2$, we have $\dim E_Z = 2$, which is the desired result for an $\mathsf{SU}(2)$-instanton bundle. The symplectic form ω on V induces a symplectic form ω on E_Z, which renders the latter bundles structure group to $\mathsf{SL}(2,\mathbb{C})$.

One can verify that the bundle E constructed in this manner is in fact an instanton bundle [12], and via the Penrose-Ward transform, one obtains the corresponding self-dual gauge potential.

§4 The picture over the moduli space. Instead of constructing the vector spaces V and W over the twistor space $\mathbb{C}P^3$ fibered over S^4, we can discuss them directly over the space S^4. To this end, define V and W as before and choose τ to be the complex conjugation on W. The symplectic form ω on V is given by a skew-symmetric tri-band matrix of dimension $(2k+2)\times(2k+2)$ with entries ± 1. The reality condition $\tau\alpha(Z)w = \alpha(\tau Z)\tau w$ on the map α can now be restated in the following way: Let us denote the components of the matrix B by $B^{\dot\alpha}_{i,j}$. Then for fixed values of m,n, the 2×2-matrix $B^{\dot\alpha}_{2m+\dot\beta-1,n}$ should be a quaternion. Applying the same argument to C, we arrive at a representation of the map α in terms of a $(k+1)\times k$-dimensional matrix of quaternions

$$\Delta \;=\; A - Cx\ . \tag{VII.380}$$

The remaining condition that $\alpha(Z^i)W$ should be of dimension k for $Z^i \neq 0$ amounts to the fact that $\bar\Delta(x)\Delta(x)$ is nonsingular and real for each x, where $\bar\Delta$ is the conjugate transpose of Δ. This condition is equivalent to the so-called ADHM equations, which will arise in the following section. One can easily "supersymmetrize" the above considerations, by considering a supertwistor space $\mathbb{C}P^{3|\mathcal{N}}$ and adding appropriate linear terms to (VII.377) and (VII.380).

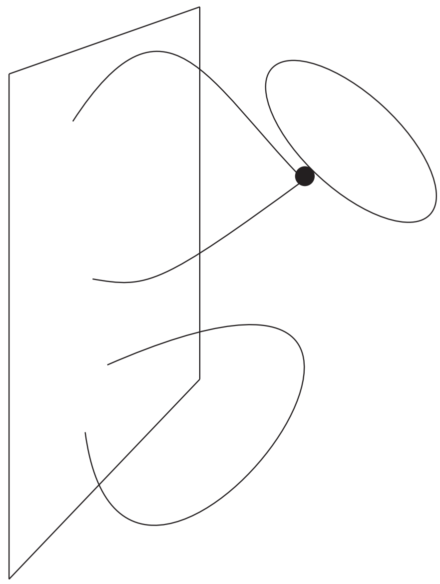
VII.8.2 The ADHM construction in the context of D-branes

§5 The D5-D9-brane system. As stated in the introduction to this section, the ADHM algorithm for constructing instanton solutions has found a nice interpretation in the context of string theory. We start from a configuration of k D5-branes bound to a stack of n D9-branes, which – upon dimensional reduction – will eventually yield a configuration of k D(-1)-branes inside a stack of n D3-branes.

§6 D5-D5 strings. From the perspective of the D5-branes, the $\mathcal{N}=2$ supersymmetry of type IIB superstring theory is broken down to $\mathcal{N}=(1,1)$ on the six-dimensional worldvolume of the D5-brane, which is BPS. The fields in the ten-dimensional Yang-Mills multiplet are rearranged into an $\mathcal{N}=2$ vector multiplet $(\phi_a, A_{\alpha\dot\alpha}, \chi^i_\alpha, \bar\mu^{\dot\alpha}_i)$, where

[36] Usually in this discussion, one considers the twistor space $\mathbb{C}P^3$ of S^4, and imposes the restrictions only for $Z^i \neq 0$.

the indices $i = 1, \ldots, 4$, $a = 1, \ldots 6$ and $\alpha, \dot\alpha = 1, 2$ label the representations of the Lorentz group $\mathsf{SO}(5,1) \sim \mathsf{SU}(4)$ and the R-symmetry group $\mathsf{SO}(4) \sim \mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$, respectively. Thus, ϕ and A denote bosons, while χ and $\bar\mu$ refer to fermionic fields.



Note that the presence of the D9-branes will further break supersymmetry down to $\mathcal{N} = (0,1)$ and therefore the above multiplet splits into the vector multiplet $(\phi_a, \bar\mu_i^{\dot\alpha})$ and the hypermultiplet $(A_{\alpha\dot\alpha}, \chi^i_\alpha)$. In the following, we will discuss the field theory on the D5-branes in the language of $\mathcal{N} = (0,1)$ supersymmetry.

Let us now consider the vacuum moduli space of this theory which is called the Higgs branch. This is the sector of the theory, where the D-field, i.e. the auxiliary field for the $\mathcal{N} = (0,1)$ vector multiplet, vanishes[37]. Therefore, we can restrict our analysis in the following to a few terms of the action. From the Yang-Mills part describing the vector multiplet, we have the contribution $4\pi^2\alpha'^2 \int \mathrm{d}^6x \,\mathrm{tr}_k \frac{1}{2} D^2_{\mu\nu}$, where we also introduce the notation $D_{\mu\nu} = \mathrm{tr}_2(\vec\sigma\bar\sigma_{\mu\nu}) \cdot \vec D$. The hypermultiplet leads to an additional contribution of $\int \mathrm{d}^6x \,\mathrm{tr}_k \mathrm{i}\vec D \cdot \vec\sigma^{\dot\alpha}{}_{\dot\beta} \bar A^{\alpha\dot\beta} A_{\alpha\dot\alpha}$. Note that we will use a bar instead of the dagger to simplify notation. However, this bar must not be confused with complex conjugation.

§7 D5-D9 strings. It remains to include the contributions from open strings having one end on a D5-brane and the other one on a D9-brane. These additional degrees of freedom are aware of both branes and therefore form hypermultiplets under $\mathcal{N} = (0,1)$ supersymmetry. One of the hypermultiplets is in the $(\bar{\mathbf{k}}, \mathbf{n})$ representation of $\mathsf{U}(k) \times \mathsf{U}(n)$, while the other one transforms as $(\mathbf{k}, \bar{\mathbf{n}})$. We denote them by $(w_{\dot\alpha}, \psi^i)$ and $(\bar w^{\dot\alpha}, \bar\psi^i)$, where $w_{\dot\alpha}$ and $\bar w^{\dot\alpha}$ and ψ^i and $\bar\psi^i$ denote four complex scalars and eight Weyl spinors, respectively. The contribution to the D-terms is similar to the hypermultiplet considered above: $\int \mathrm{d}^6x \,\mathrm{tr}_k \mathrm{i}\vec D \cdot \vec\sigma^{\dot\alpha}{}_{\dot\beta} \bar w^{\dot\beta} w_{\dot\alpha}$.

§8 The D-flatness condition. Collecting all the (algebraic) contributions of the D-field to the action and varying them yields the equations of motion

$$\alpha'^2 \vec D = \frac{\mathrm{i}}{16\pi^2} \vec\sigma^{\dot\alpha}{}_{\dot\beta} (\bar w^{\dot\beta} w_{\dot\alpha} + \bar A^{\alpha\dot\beta} A_{\alpha\dot\alpha}) \ . \tag{VII.381}$$

After performing the dimensional reduction of the D5-brane to a D(-1)-brane, the condition that $\vec D$ vanishes is precisely equivalent to the ADHM constraints.

§9 The zero-dimensional Dirac operator. Spelling out all possible indices on our fields, we have $A_{\alpha\dot\alpha pq}$ and $w_{up\dot\alpha}$, where $p, q = 1, \ldots, k$ denote indices of the representation $\mathbf{k}$ of the gauge group $\mathsf{U}(k)$ while $u = 1, \ldots, n$ belongs to the $\mathbf{n}$ of $\mathsf{U}(n)$. Let us introduce the new combinations of indices $r = u \oplus p \otimes \alpha = 1, \ldots, n + 2k$ together with the matrices

$$(a_{rq\dot\alpha}) = \begin{pmatrix} w_{uq\dot\alpha} \\ A_{\alpha\dot\alpha pq} \end{pmatrix} , \quad (\bar a_q^{\dot\alpha r}) = \begin{pmatrix} \bar w^{\dot\alpha}_{qu} & A^{\alpha\dot\alpha}_{pq} \end{pmatrix} \quad \text{and} \quad (b^\beta_{rq}) = \begin{pmatrix} 0 \\ \delta_\alpha{}^\beta \delta_{pq} \end{pmatrix} , \tag{VII.382}$$

[37] This is often referred to as the D-flatness condition.

which are of dimension $(n+2k)\times 2k$, $2k\times(n+2k)$ and $(n+2k)\times 2k$, respectively. Now we are ready to define a $(n+2k)\times 2k$ dimensional matrix, the zero-dimensional Dirac operator of the ADHM construction, which reads

$$\Delta_{rp\dot\alpha}(x) \;=\; a_{rp\dot\alpha}+b^{\alpha}_{rp}x_{\alpha\dot\alpha}\ , \tag{VII.383}$$

and we put $\bar\Delta^{\dot\alpha r}_p := (\Delta_{rp\dot\alpha})^*$. Written in the new components (VII.382), the ADHM constraints amounting to the D-flatness condition read $\vec\sigma^{\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta}a_{\dot\alpha})=0$, or, more explicitly,

$$\bar a_{\dot\alpha}a_{\dot\beta}+\bar a_{\dot\beta}a_{\dot\alpha} \;=\; 0\ , \tag{VII.384}$$

where we defined as usual $\bar a_{\dot\alpha}=\varepsilon_{\dot\alpha\dot\beta}\bar a^{\dot\beta}$. All further conditions, which are sometimes also summarized under ADHM constraints, are automatically satisfied due to our choice of b^{α}_{rp} and the reality properties of our fields.

§10 Construction of solutions. The kernel of the zero-dimensional Dirac operator is generally of dimension n, as this is the difference between its numbers of rows and columns. It is spanned by vectors, which can be arranged to a complex matrix U_{ru} which satisfies

$$\bar\Delta^{\dot\alpha r}_p U_{ru} \;=\; 0\ . \tag{VII.385}$$

Upon demanding that the frame U_{ru} is orthonormal, i.e. that $\bar U^r_u U_{rv} = \delta_{uv}$, we can construct a self-dual $\mathsf{SU}(n)$-instanton configuration from

$$(\mathscr{A}_{\alpha\dot\alpha})_{uv} \;=\; \bar U^r_u\partial_{\alpha\dot\alpha}U_{rv}\ . \tag{VII.386}$$

Usually, one furthermore introduces the auxiliary matrix f via

$$f \;=\; 2(\bar w^{\dot\alpha}w_{\dot\alpha}+(A_{\alpha\dot\alpha}+x_{\alpha\dot\alpha}\otimes\mathbb{1}_k)^2)^{-1}\ , \tag{VII.387}$$

which fits in the factorization condition $\bar\Delta^{\dot\alpha r}_p\Delta_{rq\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}(f^{-1})_{pq}$. Note that the latter condition is equivalent to the ADHM constraints (VII.384) arising from (VII.381). The matrix f allows for an easy computation of the field strength

$$\mathscr{F}_{\mu\nu} \;=\; 4\bar U b\sigma_{\mu\nu}f\bar b U \tag{VII.388}$$

and the instanton number

$$-\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \mathrm{tr}_{\,n}\mathscr{F}^2_{\mu\nu} \;=\; \frac{1}{16\pi^2}\int \mathrm{d}^4x\ \Box^2\,\mathrm{tr}_{\,n}\log f\ . \tag{VII.389}$$

Note that the self-duality of $\mathscr{F}_{\mu\nu}$ in (VII.388) is evident from the self-duality property of $\sigma_{\mu\nu}$.

VII.8.3 Super ADHM construction and super D-branes

§11 Superspace formulation of SYM theories. First, recall from section IV.2 that one can formulate the equations of motion of $\mathcal{N}=4$ super Yang-Mills theory and self-dual Yang-Mills theory with arbitrary $\mathcal{N}$ both in terms of ordinary fields on $\mathbb{R}^4$ (or its complexification $\mathbb{C}^4$) and in terms of superfields on certain superspaces having $\mathbb{R}^4$ as their body. For $\mathcal{N}=4$ SYM theory, the appropriate superspace is $\mathbb{R}^{4|16}$ (or $\mathbb{C}^{4|16}$), while for $\mathcal{N}$-extended SDYM theory, one has to use $\mathbb{R}^{4|2\mathcal{N}}$ (or $\mathbb{C}^{4|2\mathcal{N}}$). One can find an Euler operator, which easily shows the equivalence of the superfield formulation with the formulation in terms of ordinary fields.

§12 Superbrane system. For the super ADHM construction, let us consider k D5|8-branes inside n D9|8-branes. To describe this scenario, it is only natural to extend the fields arising from the strings in this configuration to superfields on $\mathbb{C}^{10|8}$ and the appropriate subspaces, respectively. In particular, we will extend the fields $w_{\dot\alpha}$ and $A_{\alpha\dot\alpha}$ entering into the D-flatness condition in the purely bosonic setup to superfields living on $\mathbb{C}^{6|8}$. However, since supersymmetry is broken down to four copies of $\mathcal{N}=1$ due to the presence of the two stacks of D-branes, these superfields can only be linear in the Graßmann variables. From the discussion in [120], we can then even state what the superfield expansion should look like:

$$w_{\dot\alpha} \;=\; \overset{\circ}{w}_{\dot\alpha} + \psi^i\eta_{i\dot\alpha} \quad\text{and}\quad A_{\alpha\dot\alpha} \;=\; \overset{\circ}{A}_{\alpha\dot\alpha} + \chi^i_\alpha\eta_{i\dot\alpha}\;. \tag{VII.390}$$

§13 Super ADHM equations. The D-flatness condition we arrive at after following the above discussion of the field theories involved in the D-brane configurations reads again

$$\alpha'^2\vec{D} \;=\; \frac{\mathrm{i}}{16\pi^2}\vec{\sigma}^{\dot\alpha}{}_{\dot\beta}(\bar{w}^{\dot\beta}w_{\dot\alpha} + \bar{A}^{\alpha\dot\beta}A_{\alpha\dot\alpha}) \;=\; 0\;, \tag{VII.391}$$

but here, all the fields are true superfields. After performing the dimensional reduction of the D9|8-D5|8-brane configuration to one containing D3|8- and D(-1|8)-branes, and arranging the resulting field content according to (VII.382), we can construct the zero-dimensional super Dirac operator

$$\Delta_{ri\dot\alpha} \;=\; a_{ri\dot\alpha} + b^\alpha_{ri}x^R_{\alpha\dot\alpha} \;=\; \overset{\circ}{a}_{ri\dot\alpha} + b^\alpha_{ri}x^R_{\alpha\dot\alpha} + c^j_{ri}\eta_{j\dot\alpha}\;, \tag{VII.392}$$

where $(x_R^{\alpha\dot\alpha},\eta_i^{\dot\alpha})$ are coordinates on the (anti-)chiral superspace $\mathbb{C}^{4|8}$. That is, from the point of view of the full superspace $\mathbb{C}^{4|16}$ with coordinates $(x^{\alpha\dot\alpha},\theta^{i\alpha},\eta_i^{\dot\alpha})$, we have $x_R^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta_i^{\dot\alpha}$. The ADHM constraints are now turned into the super ADHM constraints, which were discussed in [256] for the first time, see also [7] for a related recent discussion.

Explicitly, these super constraints (VII.384) read here

$$\overset{\circ}{\bar{a}}_{\dot\alpha}\overset{\circ}{a}_{\dot\beta} + \overset{\circ}{\bar{a}}_{\dot\beta}\overset{\circ}{a}_{\dot\alpha} \;=\; 0\;,\quad \overset{\circ}{\bar{a}}_{\dot\alpha}c_i - \bar{c}_i\overset{\circ}{a}_{\dot\alpha} \;=\; 0\;,\quad \bar{c}_ic_j - \bar{c}_jc_i \;=\; 0\;. \tag{VII.393}$$

The additional sign in the equations involving c_i arises from ordering and extracting the Graßmann variables $\eta_i^{\dot\alpha}$ as well as the definition $\overline{c_i\eta_i^{\dot\alpha}} = \eta_i^{\dot\alpha}\bar{c}_i = -\bar{c}_i\eta_i^{\dot\alpha}$.

§14 Construction of solutions. As proven in [256, 280], this super ADHM construction gives rise to solutions to the $\mathcal{N}=4$ supersymmetrically extended self-dual Yang-Mills equations in the form of the super gauge potentials

$$\mathscr{A}_{\alpha\dot\alpha} \;=\; \bar{U}\partial_{\alpha\dot\alpha}U \quad\text{and}\quad \mathscr{A}^i_{\dot\alpha} \;=\; \bar{U}D^i_{\dot\alpha}U\;, \tag{VII.394}$$

where U and $\bar{U}$ are again zero modes of $\bar\Delta$ and Δ and furthermore satisfy $\bar{U}U = \mathbb{1}$. That is, the super gauge potentials in (VII.394) satisfy the constraint equations of $\mathcal{N}=4$ self-dual Yang-Mills theory (IV.64).

One might be tempted to generalize the Dirac operator in (VII.392) to higher orders in the Graßmann variables, but this is unnatural both from the point of view of broken supersymmetry due to the presence of D-branes and from the construction of instanton bundles via monads (the original idea which gave rise to the ADHM construction). Besides this, higher powers of Graßmann variables will render the super ADHM equations insufficient for producing solutions to the self-dual Yang-Mills equations[38]. Note also

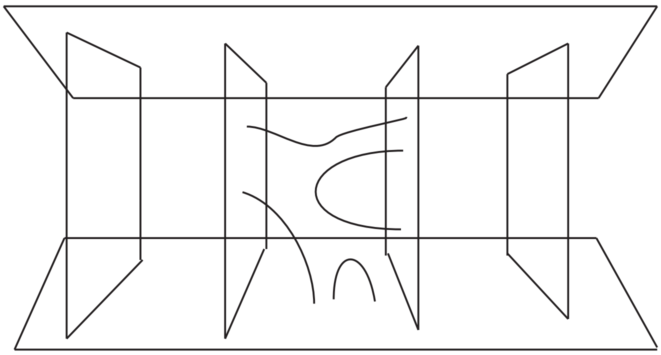
[38] In [82], a Dirac operator with higher powers is mentioned, but it is not used to obtain solutions in the way we do.

that this construction leads to solutions of the $\mathcal{N} = 4$ SDYM equations, for which the Higgs fields tend to zero as $x \to \infty$. Since the Higgs-fields describe the motion of the D3-brane in the ambient ten-dimensional space, this merely amounts to a choice of coordinates: The axes of the remaining six directions go through both "ends" of the stack of D3-branes at infinity. For a discussion of the construction of solutions which do not tend to zero but to a constant value $\sim \sigma^3$ see [82] and references therein.

The fact that solutions to the $\mathcal{N} = 4$ SDYM equations in general do not satisfy the $\mathcal{N} = 4$ SYM equations does not spoil our interpretation of such solutions as D(-1|8)-branes, since in our picture, $\mathcal{N} = 4$ supersymmetry is broken down to four copies of $\mathcal{N} = 1$ supersymmetry. Note furthermore that $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 4$ SDYM theory can be seen as different weak coupling limits of *one* underlying field theory [297].

VII.8.4 The D-brane interpretation of the Nahm construction

Before presenting its super extension, let us briefly recollect the ordinary Nahm construction [200] starting from its D-brane interpretation [75] and [124], see also [274]. For simplicity, we restrict ourselves to the case of $\mathsf{SU}(2)$-monopoles, but a generalization of our discussion to gauge groups of higher rank is possible and rather straightforward.



§15 The D3-D1-brane system. We start in ten-dimensional type IIB superstring theory with a pair of D3-branes extended in the directions $1, 2, 3$ and located at $x^4 = \pm 1$, $x^M = 0$ for $M > 4$. Consider now a bound state of these D3-branes with k D1-branes extending along the x^4-axis and ending on the D3-branes. As in the case of the ADHM construction, we can look at this configuration from two different points of view.

From the perspective of the D3-branes, the effective field theory on their worldvolume is $\mathcal{N} = 4$ super Yang-Mills theory. The D1-branes bound to the D3-branes and ending on them impose a BPS condition, which amounts to the Bogomolny equations in three dimensions

$$D_a \Phi \;=\; \tfrac{1}{2}\varepsilon_{abc} F_{bc} \;, \qquad \text{(VII.395)}$$

where $a, b, c = 1, 2, 3$. The end of the D1-branes act as magnetic charges in the worldvolume of the D3-branes, and they can therefore be understood as magnetic monopoles [51], whose field configuration (Φ, A^a) satisfy the Bogomolny equations. These monopoles are static solutions of the underlying Born-Infeld action.

From the perspective of the D1-branes, the effective field theory is first $\mathcal{N} = (8, 8)$ super Yang-Mills theory in two dimensions, but supersymmetry is broken by the presence of the two D3-branes to $\mathcal{N} = (4, 4)$. As before, one can write down the corresponding D-terms [274] and impose a D-flatness condition:

$$D = \frac{\partial X^a}{\partial x^4} + [A_4, X^a] - \tfrac{1}{2}\varepsilon_{abc}[X^b, X^c] + R \;=\; 0 \;, \qquad \text{(VII.396)}$$

where the X^a are the scalar fields corresponding to the directions in which the D3-branes extend. The R-term is proportional to $\delta(x^4 \pm 1)$ and allow for the D1-branes to end on the D3-branes. They are related to the so-called Nahm boundary conditions, which we do not discuss. The theory we thus found is simply self-dual Yang-Mills theory, reduced to one dimension.

§16 Nahm equations. By imposing *temporal gauge* $A_4 = 0$, we arrive at the Nahm equations

$$\frac{\partial X^a}{\partial s} - \tfrac{1}{2}\varepsilon_{abc}[X^b, X^c] \;=\; 0 \quad \text{for} \quad -1 \;<\; s \;<\; 1\ , \tag{VII.397}$$

where we substituted $s = x^4$. From solutions to these (integrable) equations, we can construct the one-dimensional Dirac operator

$$\Delta^{\dot\alpha\dot\beta} \;=\; (\mathbb{1}_2)^{\dot\alpha\dot\beta} \otimes \frac{\partial}{\partial s} + \sigma_a^{(\dot\alpha\dot\beta)}(x^a - X^a)\ . \tag{VII.398}$$

The equations (VII.397) are analogously to the ADHM equations the condition for $\bar\Delta\Delta$ to commute with the Pauli matrices, or equivalently, to have an inverse f:

$$\bar\Delta\Delta \;=\; \mathbb{1}_2 \otimes f^{-1}\ . \tag{VII.399}$$

§17 Construction of solutions. The normalized zero modes U of the Dirac operator $\bar\Delta$ satisfying

$$\bar\Delta(s)U \;=\; 0\ , \qquad \int_{-1}^{1} \mathrm{d}s\ \bar U(s)U(s) \;=\; \mathbb{1} \tag{VII.400}$$

then give rise to solutions to the Bogomolny equations (VII.395) via the definitions

$$\Phi(x,t) \;=\; \int_{-1}^{1} \mathrm{d}s\ \bar U(s) s U(s) \quad \text{and} \quad \mathscr{A}_a(x,t) \;=\; \int_{-1}^{1} \mathrm{d}s\ \bar U(s)\partial_a U(s)\ . \tag{VII.401}$$

The verification of this statement is straightforward when using the identity

$$U(s)\bar U(s') \;=\; \delta(s-s') - \overrightarrow{\Delta}(s) f(s,s') \overleftarrow{\bar\Delta}(s')\ . \tag{VII.402}$$

Note that all the fields considered above stem from D1-D1 strings. The remaining D1-D3 strings are responsible for imposing the BPS condition and the Nahm boundary conditions for the X^a at $s = \pm 1$.

§18 Super Nahm construction. The superextension of the Nahm construction is obtained analogously to the superextension of the ADHM construction by extending the Dirac operator (VII.398) according to

$$\Delta^{\dot\alpha\dot\beta} \;=\; (\mathbb{1}_2)^{\dot\alpha\dot\beta} \otimes \frac{\partial}{\partial s} + \sigma_a^{(\dot\alpha\dot\beta)}(x^a - X^a) + (\eta_i^{(\dot\alpha}\chi^{\dot\beta)i})\ . \tag{VII.403}$$

The fields $\chi^{\dot\alpha i}$ are Weyl-spinors and arise from the D1-D1 strings. (More explicitly, consider a bound state of D7-D5-branes, which dimensionally reduces to our D3-D1-brane system. The spinor $\chi^{\dot\alpha i}$ is the spinor χ^i_α we encountered before when discussing the $\mathcal{N} = (0,1)$ hypermultiplet on the D5-brane.)

Chapter VIII

Matrix Models

It is essentially three matrix models which received the most attention from string theorists during the last years. First, there is the Hermitian matrix model, which appeared in the early nineties in the context of two-dimensional gravity and $c = 1$ non-critical string theory, see [103] and references therein. It experienced a renaissance in 2002 by the work of Dijkgraaf and Vafa [78]. Furthermore, there are the two matrix models which are related to dimensional reductions of ten-dimensional super Yang-Mills theory, the BFSS matrix model [20] and the IKKT matrix model [140], see also [5]. The latter two are conjectured to yield non-perturbative and in particular background independent definitions of M-theory and type IIB superstring theory, respectively. The same aim underlies the work of Smolin [262], in which the simplest possible matrix model, the cubic matrix model (CMM), was proposed as a fundamental theory.

In this chapter, we will furthermore present the results of [176], in which two pairs of matrix models were constructed in the context of twistor string theory.

VIII.1 Matrix models obtained from SYM theory

For the comparison with the twistor matrix models presented later, let us review some aspects of the BFSS and the IKKT matrix models. The motivation for both these models was to find a non-perturbative definition of string theory and M-theory, respectively.

VIII.1.1 The BFSS matrix model

In their famous paper [20], Banks, Fishler, Shenkar and Susskind conjectured that $\mathcal{M}$-theory in the infinite momentum frame (IMF, see e.g. [34]) is exactly described by large N supersymmetric matrix quantum mechanics.

§1 Matrix quantum mechanics. The Lagrangian for matrix quantum mechanics with Minkowski time is given by

$$\mathcal{L} \;=\; \mathrm{tr}\,\left(\tfrac{1}{2}\dot{\Phi}^2 - U(\Phi)\right)\;, \tag{VIII.1}$$

where Φ is a Hermitian $N \times N$ matrix. This Lagrangian is invariant under time-independent $\mathsf{SU}(N)$ rotations. To calculate further, it is useful to decompose Φ into eigenvalues and angular degrees of freedom by using $\Phi(t) = \Omega^\dagger(t)\Lambda(t)\Omega(t)$, where $\Lambda(t)$ is a diagonal matrix with the eigenvalues of $\Phi(t)$ as its entries and $\Omega(t) \in \mathsf{SU}(N)$. We can furthermore rewrite $\mathrm{tr}\,\dot{\Phi}^2 = \mathrm{tr}\,\dot{\Lambda}^2 + \mathrm{tr}\,[\Lambda, \dot{\Omega}\Omega^\dagger]^2$ and decompose $\dot{\Omega}\Omega^\dagger$ using symmetric, antisymmetric and diagonal generators with coefficients $\dot{\alpha}_{ij}$, $\dot{\beta}_{ij}$ and $\dot{\alpha}_i$, respectively. After performing the trace, the Lagrangian reads as

$$\mathcal{L} \;=\; \sum_i \left(\tfrac{1}{2}\dot{\lambda}_i^2 + U(\lambda_i)\right) + \tfrac{1}{2}\sum_{i<j}(\lambda_i - \lambda_j)^2(\dot{\alpha}_{ij}^2 + \dot{\beta}_{ij}^2)\;. \tag{VIII.2}$$

The integration measure of the path integral is transformed to $\mathscr{D}\Phi = \mathscr{D}\Omega\Delta^2(\Lambda)\Pi_i \mathrm{d}\lambda_i$, where $\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the so-called *Vandermonde determinant*. The corresponding Hamiltonian reads as

$$\mathcal{H} \;=\; -\tfrac{1}{2\beta^2\Delta(\Lambda)}\sum_i \frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2}\Delta(\Lambda) + \sum_i U(\lambda_i) + \sum_{i<j}\frac{\Pi_{ij}^2 + \tilde{\Pi}_{ij}^2}{(\lambda_i - \lambda_j)^2}\;, \tag{VIII.3}$$

where Π_{ij} and $\tilde{\Pi}_{ij}$ are the momenta conjugate to α_{ij} and β_{ij}. For more details, see [153].

§2 BFSS action. The action of the BFSS model, describing N D0-branes, can be obtained by dimensional reduction of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $\mathsf{U}(N)$ to $0 + 1$ dimensions in temporal gauge $A_0 = 0$:

$$S = \tfrac{1}{2g}\int \mathrm{d}t\, \left[\mathrm{tr}\, \dot{X}^i \dot{X}^i + 2\theta^T\dot{\theta} + \frac{1}{2}\,\mathrm{tr}\,[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right]. \tag{VIII.4}$$

Here, the X^i are nine Hermitian $N \times N$ matrices and θ is a Majorana-Weyl spinor. Note that the bosonic part of the BFSS Lagrangian is a matrix quantum mechanics Lagrangian. Putting all the fermions to zero, one obtains the bosonic equations of motion

$$\ddot{X}^i \;=\; -[[X^i, X^j], X^j]\;. \tag{VIII.5}$$

Restricting to the special class of classical (vacuum) solutions which satisfy $[X^i, X^j] = 0$, the matrices are simultaneously diagonalizable and for gauge group $\mathsf{U}(N)$ we can interpret such solutions as a stack of N D0-branes, whose positions in the normal directions are given by the eigenvalues of the X^i.

The remaining classical solutions to (VIII.5) do not annihilate the positive-definite potential term and are thus no vacuum solutions. They break supersymmetry, and in particular, correspond to D0-branes, whose worldvolumes are smeared out in the normal directions.

§3 The BFSS model on a circle. To describe D0-branes in a spacetime, which has been compactified in one direction normal to the worldvolume of the D0-branes, we consider infinitely many copies of a D0-brane configuration and mod out the lattice symmetry group afterwards. A good reference here is [272].

To describe the copies of the D0-brane configuration, we extend the $N \times N$-dimensional matrices X^i to $\infty \times \infty$-dimensional matrices X^i_{nm} which are divided into $N \times N$-dimensional blocks, specified by the indices $n, m \in \mathbb{Z}$. We furthermore impose the condition $X_{mn} = -X^\dagger_{nm}$ on the blocks. The new Lagrangian then reads

$$\mathcal{L} \;=\; \frac{1}{2g}\left[\mathrm{tr}\, \dot{X}^i_{mn}\dot{X}^i_{nm} + \frac{1}{2}\,\mathrm{tr}\,(X^i_{mq}X^j_{qn} - X^j_{mq}X^i_{qn})(X^i_{nr}X^j_{rm} - X^j_{nr}X^i_{rm})\right]. \tag{VIII.6}$$

The periodicity condition from compactifying the X^1-direction on a circle with radius R translates into the following conditions on the matrices X^i_{mn}:

$$X^i_{mn} \;=\; X^i_{(m-1)(n-1)}, \qquad i > 1 \tag{VIII.7}$$

$$X^1_{mn} \;=\; X^1_{(m-1)(n-1)}, \qquad m \neq n \tag{VIII.8}$$

$$X^1_{mm} \;=\; X^1_{(m-1)(m-1)} + 2\pi R\mathbb{1}. \tag{VIII.9}$$

The first equation renders all blocks on diagonals equal for $i > 1$, the second equation does the same for some of the diagonals of X^1. The third equation shifts subsequent blocks on the principal diagonal for $i = 1$ by an amount of $2\pi R$, the circumference of the circle. Anti-Hermiticity of the X^i_{mn} implies furthermore $(X^i_n)^\dagger = -X^i_{-n}$. We thus arrive at

$$X^1 \;=\; \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & X^1_0 - 2\pi R\mathbb{1} & X^1_1 & X^1_2 & \cdots \\ \cdots & X^1_{-1} & X^1_0 & X^1_1 & \cdots \\ \cdots & X^1_{-2} & X^1_{-1} & X^1_0 + 2\pi R\mathbb{1} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{VIII.10}$$

Rewriting the Lagrangian (VIII.6) in terms of X^i_n gives the description of D0-branes moving in a compactified spacetime. Expansions around the classical vacuum $[X^i, X^j] = 0$ lead to the expected mass terms proportional to the distance of the branes plus the winding contribution $2\pi Rn$, similarly to our discussion of T-duality in section V.2.3.

Note that this matrix quantum mechanics is automatically a second quantized formalism as it allows for an arbitrary number of D0-branes. This analysis can also be easily generalized to more than one compact dimension [271].

§4 Reconstruction of spatial dimensions. Let us consider the following correspondence:

$$\phi(\hat{x}) \;=\; \sum_n \hat{\phi}_n \mathrm{e}^{\mathrm{i}n\hat{x}/\hat{R}} \;\leftrightarrow\; \begin{pmatrix} \vdots \\ \hat{\phi}_{-1} \\ \hat{\phi}_0 \\ \hat{\phi}_1 \\ \vdots \end{pmatrix} \tag{VIII.11}$$

where $\hat{R} = \frac{1}{2\pi R}$. Then X^1 is a matrix representation of the covariant derivative along the compactified direction:

$$(\mathrm{i}\hat{\partial}_1 + A_1(\hat{x}))\phi(\hat{x}) \;\leftrightarrow\; X^1 \vec{\hat{\phi}} \tag{VIII.12}$$

where $A^1(\hat{x})$ is a gauge potential whose Fourier modes are identified with X^1_n:

$$A^1(\hat{x}) \;=\; \sum_n A^1_n \mathrm{e}^{\mathrm{i}n\hat{x}/\hat{R}} \;=\; \sum_n X^1_n \mathrm{e}^{\mathrm{i}n\hat{x}/\hat{R}}. \tag{VIII.13}$$

Here, the derivative leads to the inhomogeneous terms $\sim 2\pi n\mathbb{1}$ and the gauge field gives rise to the remaining components.

To return to the dual space, we can consider an analogous Fourier decomposition of $Y^i(\hat{x}) = \sum_n X^i_n \mathrm{e}^{\mathrm{i}n\hat{x}/\hat{R}}$, with which we can rewrite the BFSS Lagrangian as

$$\mathcal{L} \;=\; \int \frac{\mathrm{d}x^1}{2\pi R}\frac{1}{2g}\left[\mathrm{tr}\,\dot{Y}^i\dot{Y}^i + +\,\mathrm{tr}\,\dot{A}^1\dot{A}^1 - \,\mathrm{tr}\,(\partial_1 Y^i - \mathrm{i}[A^1, Y^i])^2 + \frac{1}{2}\,\mathrm{tr}\,[Y^i, Y^j]^2\right] .$$

By integrating over x^1, we obtain again the Lagrangian (VIII.6).

This result corresponds to T-duality in the underlying string theory and describes k D1-branes wrapped around a compact circle of radius $R' = \frac{1}{2\pi R}$. Using this construction, we can reduce the infinite-dimensional matrices of the model to finite dimensional ones by introducing an additional integral.

Altogether, we have identified the degrees of freedom of a compact $\mathsf{U}(\infty)$ matrix model with the degrees of freedom of a $\mathsf{U}(N)$ gauge potential on a circle $\hat{S}^1$ with a radius dual to R. In this manner, one can successively reconstruct all spatial dimensions.

VIII.1.2 The IKKT matrix model

The IKKT model was proposed in [140] by Ishibashi, Kawai, Kitazawa and Tsuchiya. It is closely related to the BFSS model, but while the latter is conjectured to give rise to a description of M-theory, the former should capture aspects of the type IIB superstring.

§5 Poisson brackets. A super Poisson structure has already been introduced in section III.2.1, §8. Here, we want to be more explicit and consider a two-dimensional Riemann surface Σ, i.e. the worldsheet of a string. On Σ, we define

$$\{X, Y\} \;:=\; \tfrac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X \partial_b Y \ , \tag{VIII.14}$$

where $\sqrt{g}$ is the usual factor containing the determinant of the worldsheet metric and ε^{ab} is the antisymmetric tensor in two dimensions.

§6 Schild-type action. Using the above Poisson brackets, we can write down the *Schild action*, which has been shown to be equivalent[1] to the usual Green-Schwarz action (V.29) of the type IIB superstring in the Nambu-Goto form. It reads as

$$S_{\text{Schild}} \;=\; \int \mathrm{d}^2\sigma\ \sqrt{g}\left(\alpha\left(\tfrac{1}{4}\{X^\mu, X^\nu\}^2 - \tfrac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu\{X^\mu, \psi\}\right) + \beta\right) \ . \tag{VIII.15}$$

§7 $\mathcal{N} = 2$ supersymmetry. The Schild action is invariant under an $\mathcal{N} = 2$ (worldsheet) supersymmetry similarly to the Green-Schwarz action (V.29). Here, the symmetry algebra is

$$\begin{aligned} \delta^1\psi \;&=\; -\tfrac{1}{2}\sigma_{\mu\nu}\Gamma^{\mu\nu}\varepsilon \ , \qquad & \delta^1 X^\mu \;&=\; \mathrm{i}\bar{\varepsilon}\Gamma^\mu\psi \ ,\\ \delta^2\psi \;&=\; \xi \ , & \delta^2 X^\mu \;&=\; 0 \ , \end{aligned} \tag{VIII.16}$$

where

$$\sigma_{\mu\nu} \;=\; \varepsilon^{ab}\partial_a X_\mu \partial_b X\nu \ . \tag{VIII.17}$$

§8 Quantization. From the Schild action (VIII.15), one can construct a matrix model by the following (quantization) prescription:

$$\{\cdot,\cdot\} \;\rightarrow\; -\mathrm{i}[\cdot,\cdot] \quad \text{and} \quad \int \mathrm{d}^2\sigma\ \sqrt{g} \;\rightarrow\; \mathrm{tr} \ . \tag{VIII.18}$$

This is consistent, and crucial properties of the trace like

$$\mathrm{tr}\,[X, Y] \;=\; 0 \quad \text{and} \quad \mathrm{tr}\,(X[Y, Z]) \;=\; \mathrm{tr}\,(Z[X, Y]) \tag{VIII.19}$$

are also fulfilled after performing the inverse transition to (VIII.18).

§9 The matrix model. Applying the rules (VIII.18) to the Schild action (VIII.15), we arrive at

$$S_{\text{IKKT}} \;=\; \alpha\left(-\frac{1}{4}\,\mathrm{tr}\,[A_\mu, A_\nu]^2 - \frac{1}{2}\,\mathrm{tr}\,\left(\bar{\psi}\gamma^\mu[A_\mu, \psi]\right)\right) + \beta\,\mathrm{tr}\,\mathbb{1} \ . \tag{VIII.20}$$

Here, A_μ and ψ are bosonic and fermionic anti-Hermitian matrices, respectively. Except for the term containing β, this action can also be obtained by dimensional reduction of ten-dimensional SYM theory to a point.

There is a remnant of the gauge symmetry, which is given by the adjoint action of the gauge group on the matrices:

$$A_\mu \;\mapsto\; gA_\mu g^{-1} \quad \text{and} \quad \psi \;\mapsto\; g\psi g^{-1} \ . \tag{VIII.21}$$

The $\mathcal{N} = 2$ supersymmetry (VIII.16) is directly translated into

$$\begin{aligned} \delta^1\psi \;&=\; \tfrac{\mathrm{i}}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\varepsilon \ , \qquad & \delta^1 A_\mu \;&=\; \mathrm{i}\bar{\varepsilon}\Gamma_\mu\psi \ ,\\ \delta^2\psi \;&=\; \xi \ , & \delta^2 A_\mu \;&=\; 0 \ . \end{aligned} \tag{VIII.22}$$

[1] This equivalence is demonstrated by using the equation for $\sqrt{g}$ in the Schild action.

§10 Equations of motion. The equations of motion for the bosonic part for $\psi = 0$ are

$$[A_\mu, [A_\mu, A_\nu]] \;=\; 0. \qquad \text{(VIII.23)}$$

One can interpret classical vacuum solutions, which are given by matrices A_μ satisfying $[A_\mu, A_\nu] = 0$, in terms of D(-1)-branes, similar to the interpretation of such solutions in the BFSS model in terms of D0-branes: The dimension of the fundamental representation space of the gauge group underlying the IKKT model (formally infinity) is the number of D(-1)-branes in the picture and the diagonal entries of the matrices A_μ, after simultaneous diagonalization, correspond to their positions in ten-dimensional spacetime.

VIII.2 Further matrix models

VIII.2.1 Dijkgraaf-Vafa dualities and the Hermitian matrix model

§1 Preliminary remarks. Four years ago, Dijkgraaf and Vafa showed in three papers [78, 79, 80] that one can compute the effective superpotential in certain supersymmetric gauge theories by performing purely perturbative calculations in matrix models. While these computations were motivated by string theory dualities, a proof was given in [80] using purely field theoretical methods.

§2 Chain of dualities. The chain of dualities used in string theory to obtain a connection between supersymmetric gauge theories and Hermitian matrix models is presented e.g. in [80]. One starts from a $\mathcal{N} = 2$ supersymmetric gauge theory geometrically engineered within type IIB superstring theory. By adding a tree-level superpotential, which translates into a deformation of the Calabi-Yau geometry used in the geometric engineering, one breaks the supersymmetry of the gauge theory further down to $\mathcal{N} = 1$. The engineered open string description can now be related to a closed string description via a large N duality, in which the Calabi-Yau geometry undergoes a geometric transition (cf. the conifold transition in section II.3.3, §16) and the D-branes are turned into certain 3-form fluxes H. In this description, the effective superpotential is just given by the integral $\int_X H \wedge \Omega$, where X is the new Calabi-Yau geometry. Introducing a basis of homology cycles, one can rewrite this integral in terms of a prepotential $\mathcal{F}_0$ determined by the Calabi-Yau geometry. It is furthermore known that the computation of $\mathcal{F}_0$ reduces to a calculation within closed topological string theory, which in turn is connected to an open topological string theory via essentially the same large N duality, we used for switching between the open and closed ten-dimensional strings of the geometric engineering picture above. The open topological string reduced now not only to a holomorphic Chern-Simons theory, but under certain conditions, only zero-modes survive and we arrive at a Hermitian matrix model, which is completely soluble.

In the subsequently discussed variations of this gauge theory/matrix model correspondence, several matrix models appeared, which shall be briefly introduced in the following.

§3 The Hermitian matrix model. The partition function of the Hermitian matrix model is given by

$$Z = \frac{1}{\mathrm{vol}(G)} \int \mathrm{d}\Phi \; \mathrm{e}^{-\frac{1}{g_s} \operatorname{tr} W(\Phi)} \;, \qquad \text{(VIII.24)}$$

where Φ is a Hermitian $N \times N$ matrix, $W(\Phi)$ is a polynomial in Φ and $\mathrm{vol}(G)$ is the volume of the gauge group. One can switch to an eigenvalue formulation analogously to the above discussed case of matrix quantum mechanics. Here, the partition functions reads

$$Z = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \; \Delta(\lambda)^2 \mathrm{e}^{-\frac{1}{g_s}\sum_{i=1}^{N} W(\lambda_i)} \;, \tag{VIII.25}$$

where $\Delta(\lambda)$ is the usual Vandermonde determinant. The effective action is found via exponentiating the contribution of the Vandermonde determinant, which yields

$$S_{\text{eff}} \;=\; \frac{1}{g_s}\sum_{i=1}^{N} W(\lambda_i) - 2\sum_{i<j}\log(\lambda_i - \lambda_j)\;. \tag{VIII.26}$$

This action describes N repelling eigenvalues moving in the potential $W(\lambda)$ and the corresponding equations of motion read as

$$\frac{\partial S_{\text{eff}}}{\partial \lambda_i} \;=\; 0 \;=\; \frac{1}{g_s}W'(\lambda_i) - 2\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\;. \tag{VIII.27}$$

§4 The unitary matrix model. The partition function of the unitary matrix model is formally the same as the one of the Hermitian matrix model,

$$Z = \frac{1}{\text{vol}(\mathsf{U}(N))}\int_{\mathsf{U}(N)} \mathrm{d}U \; \mathrm{e}^{-\frac{1}{g_s}\operatorname{tr} W(U)} \;, \tag{VIII.28}$$

but now the matrices U are unitary $N \times N$ matrices. The eigenvalue formulation of the partition function reads now

$$Z = \int \prod_{k=1}^{N} \mathrm{d}\alpha_k \prod_{i<j}\sin^2\left(\frac{\alpha_i - \alpha_j}{2}\right) \mathrm{e}^{-\frac{1}{g_s}\sum_{i=1}^{N} W(\alpha_k)} \;, \tag{VIII.29}$$

which is due to the fact that the Vandermonde determinant became periodic, as we identified $U = \exp(\mathrm{i}\Phi)$ where Φ is a Hermitian matrix. After regularizing the expression, one obtains the sine-term.

The corresponding equations of motion of the effective action are

$$W'(\alpha_i) - 2g_s\sum_{j\neq i}\cot\left(\frac{\alpha_i - \alpha_j}{2}\right) \;=\; 0\;, \tag{VIII.30}$$

and this system describes a Dyson gas of eigenvalues moving on the unit circle.

The so-called Gross-Witten model is the unitary matrix model with potential

$$W(U) \;=\; \frac{\varepsilon}{2g_s}\operatorname{tr}(U + U^{-1}) \;=\; \frac{\varepsilon}{g_s}\cos(\Phi)\;. \tag{VIII.31}$$

§5 The holomorphic matrix model. To introduce the holomorphic matrix model, which is proposed as the true matrix model underlying the Dijkgraaf-Vafa conjecture and a cure for several of its problems in [168], we need the following definitions:

Let $\mathsf{Mat}(N, \mathbb{C})$ be the set of complex $N \times N$ matrices and $p_M(\lambda) = \det(\lambda\mathbb{1} - M)$ the characteristic polynomial of the matrix M. Let $\mathcal{M}$ be the subset of matrices in $\mathsf{Mat}(N, \mathbb{C})$ with distinct roots. This implies that elements of $\mathcal{M}$ are diagonalizable. The set of eigenvalues of a matrix $M \in \mathcal{M}$ is called the spectrum of M: $\sigma(M) = \{\lambda_1, \ldots, \lambda_N\}$. Choose a submanifold $\Gamma \subset \mathcal{M}$ with $\dim_{\mathbb{R}}(\Gamma) = \dim_{\mathbb{C}}(\mathcal{M}) = N^2$ and define furthermore the symplectic form on $\mathrm{M}_N(\mathbb{C})$ as $w = \wedge_{i,j}\mathrm{d}M_{ij}$, where the indices are taken in lexicographic order to fix the sign.

Given a potential $W(t) = \sum_m t_m z^m$, $t_m \in \mathbb{C}$, the partition function for the holomorphic matrix model is defined as

$$Z_N(\Gamma, t) \;:=\; \mathcal{N} \int_\Gamma w\mathrm{e}^{-N \operatorname{tr} W(M)} \;. \qquad \text{(VIII.32)}$$

For its eigenvalue representation, we choose an open curve $\gamma : \mathbb{R} \to \mathbb{C}$ without self-intersections and a corresponding subset of matrices $\Gamma(\gamma) := \{M \in \mathcal{M} | \sigma(M) \subset \gamma\}$. After integrating out a group volume, giving rise to a new normalization constant $\mathcal{N}'$ we arrive at

$$Z_N(\Gamma(\gamma), t) \;=\; \mathcal{N}' \int_\gamma \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j) \mathrm{e}^{-N \sum_{j=1}^N W(\lambda_i)}. \qquad \text{(VIII.33)}$$

VIII.2.2 Cubic matrix models and Chern-Simons theory

§6 Action and equations of motion. In [262, 261, 180], the simplest nontrivial matrix model has been examined and proposes as a fundamental theory. The field content consists of a single field A, which is a matrix with Lie-algebra valued entries. For convenience, we decompose A according to $A = A_a \tau^a$, where $A_a \in \mathsf{Mat}(N, \mathbb{R})$ and the τ^a are generators of some (super-)Lie algebra[2] $\mathfrak{g}$. Its action is given by:

$$S[A] \;=\; \operatorname{tr}_{\mathsf{Mat}(N,\mathbb{R})} A^\beta_\alpha [A^\gamma_\beta, A^\alpha_\gamma] \;=\; \operatorname{tr}_\mathfrak{g} A^j_i [A^k_j, A^i_k] \;=\; \tfrac{1}{3} \phi^{abc} \operatorname{tr}_{\mathsf{Mat}(N,\mathbb{R})} A_a [A_b, A_c] \;, \qquad \text{(VIII.34)}$$

where we introduced the usual structure constants $\phi^{abc} = \frac{3}{2} \operatorname{tr}_\mathfrak{g} \tau^a [\tau^b, \tau^c]$ of the Lie algebra $\mathfrak{g}$. The indices α, β, γ belong to the representation of $\mathfrak{g}$, while the indices i, j, k are indices for matrices in $\mathsf{Mat}(N, \mathbb{R})$.

This action has two symmetries: one by adjoint action of elements of the group generated by $\mathfrak{g}$, the other one by adjoint action of the group $\mathsf{GL}(N, \mathbb{R})$.

The equations of motion, defining the classical solutions, are

$$\phi^{abc} [X_b, X_c] \;=\; 0 \;. \qquad \text{(VIII.35)}$$

By choosing different backgrounds (i.e., compactifications etc.) and different Lie algebras $\mathfrak{g}$ one can obtain several standard matrix models as BFSS or IKKT, as we will briefly discuss in §8.

§7 Example $\mathfrak{g} = \mathfrak{su}(2)$. Let us consider the example $\mathfrak{g} = \mathfrak{su}(2)$, which reproduces in the large matrix limit and after triple toroidal compactification Chern-Simons field theory on a 3-torus. The structure constants are given by $\phi^{abc} = \mathrm{i}\varepsilon^{abc}$. We expand around a classical solution $X_i, i \in \{1, 2, 3\}$ of (VIII.35):

$$S_X[A] \;:=\; S[X + A] - S[A] \;=\; \mathrm{i}\varepsilon^{abc} \left(\operatorname{tr} A_a [X_b, A_c] + \frac{1}{3} \operatorname{tr} A_a [A_b, A_c] \right) \;. \qquad \text{(VIII.36)}$$

Now we use the usual trick of toroidal compactification described in section VIII.1.1, §4 to compactify this system on the three-torus. For this, the large N limit is taken in a special manner. We split the remaining $N \times N$ matrices in $M \times M$ blocks in such a way that

$$\operatorname{tr}_{\mathsf{Mat}(N,\mathbb{R})} A^i_a \;=\; \sum_{k=-n}^{n} \operatorname{tr}_{\mathsf{Mat}(M,\mathbb{R})} \tilde{A}^i(k) \qquad \text{(VIII.37)}$$

[2]For a super Lie algebra, the commutator and the trace have to be replaced by the supercommutator and the supertrace, respectively, in the following discussion.

and $T = (2n+1)l_{Pl}$ is constant in the limiting process, $l_{Pl} \to 0$ being an arbitrary length scale. Next, we identify X_i with the covariant derivative on the dual space and construct new functions $a(x)$ as:

$$X_i \;=\; \hat{\partial}_i + a_i, \qquad a_i(x) \;:=\; \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\hat{x}/\hat{R}} \tilde{A}^i(k) \;. \tag{VIII.38}$$

In the dual picture, the infinite trace becomes a finite trace and an integral over the torus, as discussed above. So we eventually obtain:

$$S = \varepsilon^{ijk} \int_{T^3} \frac{\mathrm{d}^3 x}{(2\pi R')^3} \operatorname{tr}\left(a_i \partial_j a_k + \tfrac{4}{3} a_i [a_j, a_k]\right) \tag{VIII.39}$$

This can be regarded as the action for $\mathsf{U}(M)$ or $\mathsf{GL}(M, \mathbb{R})$ Chern-Simons theory on a 3-torus.

An interesting point about this example is that, after performing BRST quantization [180], the one-loop effective action contains the quadric interaction term of the IKKT model.

§8 Cubic matrix model and the IKKT and BFSS models. It was claimed in [15] that the IKKT model is naturally contained in the CMM (VIII.34) with algebra $\mathfrak{g} = \mathfrak{osp}(1|32)$. Furthermore, the field content and the $\mathcal{N} = 2$ SUSY of the IKKT model was identified with the field content and the supersymmetries of the CMM. This identification is two-to-one, i.e. the CMM contains twice the IKKT in distinct "chiral" sectors.

The CMM was also considered in [16], but in this case, a connection to the BFSS model was found. Here, one starts from an embedding of the $\mathsf{SO}(10,1)$-Poincaré algebra in $\mathfrak{osp}(1|32)$, and switches to the IMF leaving $\mathsf{SO}(9)$ as a symmetry group. After integrating out some fields, the effective action is identical to the BFSS action.

VIII.3 Matrix models from twistor string theory

VIII.3.1 Construction of the matrix models

§1 Preliminary remarks. In this section, we construct four different matrix models. We start with dimensionally reducing $\mathcal{N} = 4$ SDYM theory to a point, which yields the first matrix model. The matrices here are just finite-dimensional matrices from the Lie algebra of the gauge group $\mathsf{U}(n)$. The second matrix model we consider results from a dimensional reduction of hCS theory on $\mathcal{P}^{3|4}_\varepsilon$ to a subspace $\mathcal{P}^{1|4}_\varepsilon \subset \mathcal{P}^{3|4}_\varepsilon$. We obtain a form of matrix quantum mechanics with a complex "time". This matrix model is linked by a Penrose-Ward transform to the first matrix model.

By considering again $\mathcal{N} = 4$ SDYM theory, but on noncommutative spacetime, we obtain a third matrix model. Here, we have finite-dimensional matrices with operator entries which can be realized as infinite-dimensional matrices acting on the tensor product of the gauge algebra representation space and the Fock space. The fourth and last matrix model is obtained by rendering the fibre coordinates in the vector bundle $\mathcal{P}^{3|4}_\varepsilon \to \mathbb{C}P^{1|4}$ noncommutative. In the operator formulation, this again yields a matrix model with infinite-dimensional matrices and there is also a Penrose-Ward transform which renders the two noncommutative matrix models equivalent.

In a certain limit, in which the ranks of the gauge groups $\mathsf{U}(n)$ and $\mathsf{GL}(n,\mathbb{C})$ of the SDYM and the hCS matrix model tend to infinity, one expects them to become equivalent to the respective matrix models obtained from noncommutativity.

Dimensional reductions of holomorphic Chern-Simons theory on purely bosonic local Calabi-Yau manifolds have been studied recently in [39, 38], where also D-brane interpretations of the models were given.

§2 Matrix model of $\mathcal{N}=4$ SDYM theory. We start from the Lagrangian in the action (IV.63) of $\mathcal{N}=4$ supersymmetric self-dual Yang-Mills theory in four dimension with gauge group $\mathsf{U}(n)$. One can dimensionally reduce this theory to a point by assuming that all the fields are independent of $x\in\mathbb{R}^4$. This yields the matrix model action

$$\begin{aligned} S_{\mathrm{SDYMMM}} \;=\; \mathrm{tr}\Bigg(& G^{\dot\alpha\dot\beta}\left(-\tfrac{1}{2}\varepsilon^{\alpha\beta}[A_{\alpha\dot\alpha},A_{\beta\dot\beta}]\right)+\tfrac{\varepsilon}{2}\varepsilon_{ijkl}\tilde\chi^{\dot\alpha ijk}[A_{\alpha\dot\alpha},\chi^{\alpha l}] \\ & +\tfrac{\varepsilon}{4}\varepsilon_{ijkl}\phi^{ij}[A_{\alpha\dot\alpha},[A^{\alpha\dot\alpha},\phi^{kl}]]+\varepsilon_{ijkl}\phi^{ij}\chi^{\alpha k}\chi^{l}_{\alpha}\Bigg)\;, \end{aligned}\tag{VIII.40}$$

which is invariant under the adjoint action of the gauge group $\mathsf{U}(n)$ on all the fields. This symmetry is the remnant of gauge invariance. The corresponding equations of motion read

$$\begin{aligned} \varepsilon^{\alpha\beta}[A_{\alpha\dot\alpha},A_{\beta\dot\beta}] &= 0\;,\\ [A_{\alpha\dot\alpha},\chi^{\alpha i}] &= 0\;,\\ \tfrac12[A^{\alpha\dot\alpha},[A_{\alpha\dot\alpha},\phi^{ij}]] &= -\tfrac{\varepsilon}{2}\{\chi^{\alpha i},\chi^{j}_{\alpha}\}\;,\\ [A_{\alpha\dot\alpha},\tilde\chi^{\dot\alpha ijk}] &= +2\varepsilon\,[\phi^{ij},\chi^{k}_{\alpha},]\;,\\ \varepsilon^{\dot\alpha\dot\gamma}[A_{\alpha\dot\alpha},G^{ijkl}_{\dot\gamma\dot\delta}] &= +\varepsilon\{\chi^{i}_{\alpha},\tilde\chi^{jkl}_{\dot\delta}\}-\varepsilon\,[\phi^{ij},[A_{\alpha\dot\delta},\phi^{kl}]]\;. \end{aligned}\tag{VIII.41}$$

These equations can certainly also be obtained by dimensionally reducing equations (IV.62) to a point. On the other hand, the equations of motion of $\mathcal{N}=4$ SDYM theory are equivalent to the constraint equations (IV.64) which are defined on the superspace $\mathbb{R}^{4|8}$. Therefore, (VIII.41) are equivalent to the equations

$$[\tilde{\mathcal{A}}_{\alpha\dot\alpha},\tilde{\mathcal{A}}_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}\;,\quad \nabla^i_{\dot\alpha}\tilde{\mathcal{A}}_{\beta\dot\beta} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^i_{\beta}\;,\quad \{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^{ij} \tag{VIII.42}$$

obtained from (IV.64) by dimensional reduction to the supermanifold[3] $\mathbb{R}^{0|8}$.

Recall that the IKKT matrix model [140] can be obtained by dimensionally reducing $\mathcal{N}=1$ SYM theory in ten dimensions or $\mathcal{N}=4$ SYM in four dimensions to a point. In this sense, the above matrix model is the self-dual analogue of the IKKT matrix model.

§3 Matrix model from hCS theory. So far, we have constructed a matrix model for $\mathcal{N}=4$ SDYM theory, the latter being defined on the space $(\mathbb{R}^{4|8},g_\varepsilon)$ with $\varepsilon=-1$ corresponding to Euclidean signature and $\varepsilon=+1$ corresponding to Kleinian signature of the metric on $\mathbb{R}^4$. The next step is evidently to ask what theory corresponds to the matrix model introduced above on the twistor space side.

Recall that for two signatures on $\mathbb{R}^4$ we use the supertwistor spaces

$$\mathcal{P}^{3|4}_{\varepsilon}\;\cong\;\Sigma^1_\varepsilon\times\mathbb{R}^{4|8} \tag{VIII.43}$$

where

$$\Sigma^1_{-1}\;:=\;\mathbb{C}P^1\quad\text{and}\quad\Sigma^1_{+1}\;:=\;H^2 \tag{VIII.44}$$

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[3]Following the usual nomenclature of superlines and superplanes, this would be a "superpoint".

and the two-sheeted hyperboloid H^2 is considered as a complex space. As was discussed in section 3.1, the equations of motion (VIII.41) of the matrix model (VIII.40) can be obtained from the constraint equations (IV.64) by reducing the space $\mathbb{R}^{4|8}$ to the supermanifold $\mathbb{R}^{0|8}$ and expanding the superfields contained in (VIII.42) in the Graßmann variables $\eta_i^{\dot\alpha}$. On the twistor space side, this reduction yields the orbit spaces

$$\Sigma_\varepsilon^1 \times \mathbb{R}^{0|8} \;=\; \mathcal{P}_\varepsilon^{3|4}/\mathscr{G}\ , \tag{VIII.45}$$

where $\mathscr{G}$ is the group of translations generated by the bosonic vector fields $\frac{\partial}{\partial x^{\alpha\dot\alpha}}$. Equivalently, one can define the spaces $\mathcal{P}_\varepsilon^{1|4}$ as the orbit spaces

$$\mathcal{P}_\varepsilon^{1|4} \;:=\; \mathcal{P}_\varepsilon^{3|4}/\mathscr{G}^{1,0}\ , \tag{VIII.46}$$

where $\mathscr{G}^{1,0}$ is the complex Abelian group generated by the vector fields $\frac{\partial}{\partial z_\pm^\alpha}$. These spaces with $\varepsilon = \pm1$ are covered by the two patches $U_\pm^\varepsilon \cong \mathbb{C}^{1|4}$ and they are obviously diffeomorphic to the spaces (VIII.45), i.e.

$$\mathcal{P}_\varepsilon^{1|4} \;\cong\; \Sigma_\varepsilon^1 \times \mathbb{R}^{0|8} \tag{VIII.47}$$

due to the diffeomorphism (VIII.43). In the coordinates $(z_\pm^3, \eta_i^\pm)$ on $\mathcal{P}_\varepsilon^{1|4}$ and $(\lambda_\pm, \eta_i^{\dot\alpha})$ on $\Sigma_\varepsilon^1 \times \mathbb{R}^{0|8}$, the diffeomorphism is defined e.g. by the formulæ

$$\begin{aligned}
\eta_1^{\dot 1} &= \frac{\eta_1^+ - z_+^3\bar\eta_2^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_1^- - \bar\eta_2^-}{1+z_-^3\bar z_-^3}\ , \quad \eta_2^{\dot 1} = \frac{\eta_2^+ + z_+^3\bar\eta_1^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_2^- + \bar\eta_1^-}{1+z_-^3\bar z_-^3}\ ,\\
\eta_3^{\dot 1} &= \frac{\eta_3^+ - z_+^3\bar\eta_4^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_3^- - \bar\eta_4^-}{1+z_-^3\bar z_-^3}\ , \quad \eta_4^{\dot 1} = \frac{\eta_4^+ + z_+^3\bar\eta_3^+}{1+z_+^3\bar z_+^3} = \frac{\bar z_-^3\eta_4^- + \bar\eta_3^-}{1+z_-^3\bar z_-^3}\ ,
\end{aligned} \tag{VIII.48}$$

in the Euclidean case $\varepsilon = -1$. Thus, we have a dimensionally reduced twistor correspondence between the spaces $\mathcal{P}_\varepsilon^{1|4}$ and $\mathbb{R}^{0|8}$

$$\begin{array}{ccc}
 & \mathbb{R}^{0|8}\times\Sigma_\varepsilon^1 & \\
 {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\
\mathcal{P}_\varepsilon^{1|4} & & \mathbb{R}^{0|8}
\end{array} \tag{VIII.49}$$

where the map π_2 is the diffeomorphism (VIII.47). It follows from (VIII.49) that we have a correspondence between points $\eta \in \mathbb{R}^{0|8}$ and subspaces $\mathbb{C}P_\eta^1$ of $\mathcal{P}_\varepsilon^{1|4}$.

Holomorphic Chern-Simons theory on $\mathcal{P}_\varepsilon^{3|4}$ with the action (VII.194) is defined by a gauge potential $\mathcal{A}^{0,1}$ taking values in the Lie algebra of $\mathsf{GL}(n,\mathbb{C})$ and constrained by the equations $\bar V_\pm^i(\bar V_a^\pm \lrcorner \mathcal{A}^{0,1}) = 0$, $\bar V_\pm^i \lrcorner \mathcal{A}^{0,1} = 0$ for $a = 1,2,3$. After reduction to $\mathcal{P}_\varepsilon^{1|4}$, $\mathcal{A}^{0,1}$ splits into a gauge potential and two complex scalar fields taking values in the normal bundle $\mathbb{C}^2 \otimes \mathcal{O}(1)$ to the space $\mathcal{P}_\varepsilon^{1|4} \hookrightarrow \mathcal{P}_\varepsilon^{3|4}$. In components, we have

$$\mathcal{A}_{\Sigma\pm}^{0,1} \;=\; \mathrm{d}\bar\lambda_\pm \mathcal{A}_{\bar\lambda_\pm} \quad \text{and} \quad \mathcal{A}_\alpha^\pm \;\mapsto\; \mathcal{X}_\alpha^\pm \quad \text{on} \quad U_\pm^\varepsilon\ , \tag{VIII.50}$$

where both $\mathcal{X}_\alpha^\pm$ and $\mathcal{A}_{\bar\lambda_\pm}$ are Lie algebra valued superfunctions on the subspaces $U_\pm^\varepsilon$ of $\mathcal{P}_\varepsilon^{1|4}$. The integral over the chiral subspace $\mathscr{P}_\varepsilon^{3|4} \subset \mathcal{P}_\varepsilon^{3|4}$ should be evidently substituted by an integral over the chiral subspace $\mathscr{P}_\varepsilon^{1|4} \subset \mathcal{P}_\varepsilon^{1|4}$. This dimensional reduction of the bosonic coordinates becomes even clearer with the help of the identity

$$\mathrm{d}\lambda_\pm \wedge \mathrm{d}\bar\lambda_\pm \wedge \mathrm{d}z_\pm^1 \wedge \mathrm{d}z_\pm^2 \wedge \bar E_\pm^1 \wedge \bar E_\pm^2 \;=\; \mathrm{d}\lambda_\pm \wedge \mathrm{d}\bar\lambda_\pm \wedge \mathrm{d}x^{1\dot1} \wedge \mathrm{d}x^{1\dot2} \wedge \mathrm{d}x^{2\dot1} \wedge \mathrm{d}x^{2\dot2}\ . \tag{VIII.51}$$

Altogether, the dimensionally reduced action reads

$$S_{\rm hCS,red} \;:=\; \int_{\mathscr{P}^{1|4}_\varepsilon} \omega\wedge\,\mathrm{tr}\,\varepsilon^{\alpha\beta}\mathcal{X}_\alpha\left(\bar\partial\mathcal{X}_\beta+\left[\mathcal{A}^{0,1}_\Sigma,\mathcal{X}_\beta\right]\right)\;, \tag{VIII.52}$$

where the form ω is a restriction of the form Ω from (VII.186) and has components

$$\omega_\pm \;:=\; \Omega|_{U^\varepsilon_\pm} \;=\; \pm\mathrm{d}\lambda_\pm\mathrm{d}\eta^\pm_1\dots\mathrm{d}\eta^\pm_4 \tag{VIII.53}$$

and thus takes values in the bundle $\mathcal{O}(-2)$. Note furthermore that $\bar\partial$ here is the Dolbeault operator on Σ^1_ε and the integral in (VIII.52) is well-defined since the $\mathcal{X}_\alpha$ take values in the bundles $\mathcal{O}(1)$. The corresponding equations of motion are given by

$$[\mathcal{X}_1,\mathcal{X}_2] \;=\; 0\;, \tag{VIII.54a}$$
$$\bar\partial\mathcal{X}_\alpha+[\mathcal{A}^{0,1}_\Sigma,\mathcal{X}_\alpha] \;=\; 0\;. \tag{VIII.54b}$$

The gauge symmetry is obviously reduced to the transformations

$$\mathcal{X}_\alpha\;\mapsto\;\varphi^{-1}\mathcal{X}_\alpha\varphi\quad\text{and}\quad\mathcal{A}^{0,1}_\Sigma\;\mapsto\;\varphi^{-1}\mathcal{A}^{0,1}_\Sigma\varphi+\varphi^{-1}\bar\partial\varphi\;, \tag{VIII.55}$$

where φ is a smooth $\mathsf{GL}(n,\mathbb{C})$-valued function on $\mathcal{P}^{1|4}_\varepsilon$. The matrix model given by (VIII.52) and the field equations (VIII.54) can be understood as matrix quantum mechanics with a complex "time" $\lambda\in\Sigma^1_\varepsilon$.

Both the matrix models obtained by dimensional reductions of $\mathcal{N}=4$ supersymmetric SDYM theory and hCS theory are (classically) equivalent. This follows from the dimensional reduction of the formulæ (VII.205) defining the Penrose-Ward transform. The reduced superfield expansion is fixed by the geometry of $\mathcal{P}^{1|4}_\varepsilon$ and reads explicitly as

$$\begin{aligned}\mathcal{X}^+_\alpha \;&=\; \lambda^{\dot\alpha}_+A_{\alpha\dot\alpha}+\eta^+_i\chi^i_\alpha+\gamma_+\tfrac{1}{2!}\,\eta^+_i\eta^+_j\,\hat\lambda^{\dot\alpha}_+\,\phi^{ij}_{\alpha\dot\alpha}+ \\ &\quad+\gamma^2_+\,\tfrac{1}{3!}\,\eta^+_i\eta^+_j\eta^+_k\,\hat\lambda^{\dot\alpha}_+\,\hat\lambda^{\dot\beta}_+\,\tilde\chi^{ijk}_{\alpha\dot\alpha\dot\beta}+\gamma^3_+\,\tfrac{1}{4!}\,\eta^+_i\eta^+_j\eta^+_k\eta^+_l\,\hat\lambda^{\dot\alpha}_+\,\hat\lambda^{\dot\beta}_+\,\hat\lambda^{\dot\gamma}_+\,G^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma}\;,\end{aligned} \tag{VIII.56a}$$

$$\begin{aligned}\mathcal{A}_{\bar\lambda_+} \;&=\; \gamma^2_+\eta^+_i\eta^+_j\,\phi^{ij}-\gamma^3_+\eta^+_i\eta^+_j\eta^+_k\,\hat\lambda^{\dot\alpha}_+\,\tilde\chi^{ijk}_{\dot\alpha}+ \\ &\quad+2\gamma^4_+\eta^+_i\eta^+_j\eta^+_k\eta^+_l\,\hat\lambda^{\dot\alpha}_+\,\hat\lambda^{\dot\beta}_+G^{ijkl}_{\dot\alpha\dot\beta}\;,\end{aligned} \tag{VIII.56b}$$

where all component fields are independent of $x\in\mathbb{R}^4$. One can substitute this expansion into the action (VIII.52) and after a subsequent integration over $\mathcal{P}^{1|4}_\varepsilon$, one obtains the action (VIII.40) up to a constant multiplier, which is the volume[4] of $\Sigma^1_\pm$.

§4 Noncommutative star product in spinor notation. Noncommutative field theories have received much attention recently, as they were found to arise in string theory in the presence of D-branes and a constant NS B-field background [255, 87, 268].

There are two completely equivalent ways of introducing a noncommutative deformation of classical field theory: a star-product formulation and an operator formalism. In the first approach, one simply deforms the ordinary product of classical fields (or their components) to the noncommutative star product which reads in spinor notation as

$$(f\star g)(x)\;:=\;f(x)\exp\left(\tfrac{\mathrm{i}}{2}\overleftarrow{\partial_{\alpha\dot\alpha}}\theta^{\alpha\dot\alpha\beta\dot\beta}\overrightarrow{\partial_{\beta\dot\beta}}\right)g(x) \tag{VIII.57}$$

[4]In the Kleinian case, this volume is naïvely infinite, but one can regularize it by utilizing a suitable partition of unity.

with $\theta^{\alpha\dot\alpha\beta\dot\beta}=-\theta^{\beta\dot\beta\alpha\dot\alpha}$ and in particular

$$x^{\alpha\dot\alpha}\star x^{\beta\dot\beta}-x^{\beta\dot\beta}\star x^{\alpha\dot\alpha} \;=\; \mathrm{i}\theta^{\alpha\dot\alpha\beta\dot\beta}\ . \tag{VIII.58}$$

In the following, we restrict ourselves to the case of a self-dual ($\kappa=1$) or an anti-self-dual ($\kappa=-1$) tensor $\theta^{\alpha\dot\alpha\beta\dot\beta}$ and choose coordinates such that

$$\theta^{1\dot1 2\dot2} \;=\; -\theta^{2\dot2 1\dot1} \;=\; -2\mathrm{i}\kappa\varepsilon\theta \quad\text{and}\quad \theta^{1\dot2 2\dot1} \;=\; -\theta^{2\dot1 1\dot2} \;=\; 2\mathrm{i}\varepsilon\theta\ . \tag{VIII.59}$$

The formulation of noncommutative $\mathcal{N}=4$ SDYM theory on $(\mathbb{R}^4_\theta, g_\varepsilon)$ is now achieved by replacing all products in the action (IV.63) by star products. For example, the noncommutative field strength will read

$$F_{\alpha\dot\alpha\beta\dot\beta} \;=\; \partial_{\alpha\dot\alpha}A_{\beta\dot\beta}-\partial_{\beta\dot\beta}A_{\alpha\dot\alpha}+A_{\alpha\dot\alpha}\star A_{\beta\dot\beta}-A_{\beta\dot\beta}\star A_{\alpha\dot\alpha}\ . \tag{VIII.60}$$

§5 Operator formalism. For the matrix reformulation of our model, it is necessary to switch to the operator formalism, which trades the star product for operator-valued coordinates $\hat x^{\alpha\dot\alpha}$ satisfying

$$[\hat x^{\alpha\dot\alpha},\hat x^{\beta\dot\beta}] \;=\; \mathrm{i}\theta^{\alpha\dot\alpha\beta\dot\beta}\ . \tag{VIII.61}$$

This defines the noncommutative space $\mathbb{R}^4_\theta$ and on this space, derivatives are inner derivations of the Heisenberg algebra (VIII.61):

$$\begin{aligned}\frac{\partial}{\partial \hat x^{1\dot1}}f &:= -\frac{1}{2\kappa\varepsilon\theta}[\hat x^{2\dot2},f]\ , &\qquad \frac{\partial}{\partial \hat x^{2\dot2}}f &:= +\frac{1}{2\kappa\varepsilon\theta}[\hat x^{1\dot1},f]\ ,\\ \frac{\partial}{\partial \hat x^{1\dot2}}f &:= +\frac{1}{2\varepsilon\theta}[\hat x^{2\dot1},f]\ , &\qquad \frac{\partial}{\partial \hat x^{1\dot2}}f &:= -\frac{1}{2\varepsilon\theta}[\hat x^{1\dot2},f]\ .\end{aligned} \tag{VIII.62}$$

The obvious representation space for the algebra (VIII.61) is the two-oscillator Fock space $\mathcal{H}$ which is created from a vacuum state $|0,0\rangle$. This vacuum state is annihilated by the operators

$$\begin{aligned}\hat a_1 &= \mathrm{i}\left(\frac{1-\varepsilon}{2}\hat x^{2\dot1}+\frac{1+\varepsilon}{2}\hat x^{1\dot2}\right)\ ,\\ \hat a_2 &= -\mathrm{i}\left(\frac{1-\kappa\varepsilon}{2}\hat x^{2\dot2}+\frac{1+\kappa\varepsilon}{2}\hat x^{1\dot1}\right)\end{aligned} \tag{VIII.63}$$

and all other states of $\mathcal{H}$ are obtained by acting with the corresponding creation operators on $|0,0\rangle$. Thus, coordinates as well as fields are to be regarded as operators in $\mathcal{H}$.

Via the Moyal-Weyl map [255, 87, 268], any function $\Phi(x)$ in the star-product formulation can be related to an operator-valued function $\hat\Phi(\hat x)$ acting in $\mathcal{H}$. This yields the operator equivalent of star multiplication and integration

$$f\star g \;\mapsto\; \hat f\hat g \quad\text{and}\quad \int \mathrm{d}^4x\ f \;\mapsto\; (2\pi\theta)^2\,\mathrm{tr}_{\mathcal{H}}\hat f\ , \tag{VIII.64}$$

where $\mathrm{tr}_{\mathcal{H}}$ signifies the trace over the Fock space $\mathcal{H}$.

We now have all the ingredients for defining noncommutative $\mathcal{N}=4$ super SDYM theory in the operator formalism. Starting point is the analogue of the covariant derivatives which are given by the formulæ

$$\begin{aligned}\hat X_{1\dot1} &= -\frac{1}{2\kappa\varepsilon\theta}\hat x^{2\dot2}\otimes\mathbb{1}_n+\hat A_{1\dot1}\ , &\quad \hat X_{2\dot2} &= \frac{1}{2\kappa\varepsilon\theta}\hat x^{1\dot1}\otimes\mathbb{1}_n+\hat A_{2\dot2}\ ,\\ \hat X_{1\dot2} &= \frac{1}{2\varepsilon\theta}\hat x^{2\dot1}\otimes\mathbb{1}_n+\hat A_{1\dot2}\ , &\quad \hat X_{2\dot1} &= -\frac{1}{2\varepsilon\theta}\hat x^{1\dot2}\otimes\mathbb{1}_n+\hat A_{2\dot1}\ .\end{aligned}$$

These operators act on the tensor product of the Fock space $\mathcal{H}$ and the representation space of the Lie algebra of the gauge group $\mathsf{U}(n)$. The operator-valued field strength has then the form

$$\hat{F}_{\alpha\dot{\alpha}\beta\dot{\beta}} \;=\; [\hat{X}_{\alpha\dot{\alpha}},\hat{X}_{\beta\dot{\beta}}] + \mathrm{i}\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}\otimes \mathbb{1}_n\ , \tag{VIII.65}$$

where the tensor $\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}$ has components

$$\theta_{1\dot{1}2\dot{2}} \;=\; -\theta_{2\dot{2}1\dot{1}} \;=\; \mathrm{i}\frac{\kappa\varepsilon}{2\theta}\ ,\quad \theta_{1\dot{2}2\dot{1}} \;=\; -\theta_{2\dot{1}1\dot{2}} \;=\; -\mathrm{i}\frac{\varepsilon}{2\theta}\ , \tag{VIII.66}$$

Recall that noncommutativity restricts the set of allowed gauge groups and we therefore had to choose to work with $\mathsf{U}(n)$ instead of $\mathsf{SU}(n)$.

The action of noncommutative SDYM theory on $(\mathbb{R}^4_\theta, g_\varepsilon)$ reads

$$\begin{aligned} S^{\mathcal{N}=4}_{\mathrm{ncSDYM}} \;=\;& \operatorname{tr}_{\mathcal{H}}\operatorname{tr}\Bigg(-\tfrac{1}{2}\varepsilon^{\alpha\beta}\hat{G}^{\dot{\alpha}\dot{\beta}}\Big([\hat{X}_{\alpha\dot{\alpha}},\hat{X}_{\beta\dot{\beta}}] + \mathrm{i}\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}\otimes \mathbb{1}_n\Big) \\ &+\tfrac{\varepsilon}{2}\varepsilon_{ijkl}\hat{\tilde{\chi}}^{\dot{\alpha}ijk}[\hat{X}_{\alpha\dot{\alpha}},\hat{\chi}^{\alpha l}] + \tfrac{\varepsilon}{2}\varepsilon_{ijkl}\hat{\phi}^{ij}[\hat{X}_{\alpha\dot{\alpha}},[\hat{X}^{\alpha\dot{\alpha}},\hat{\phi}^{kl}]] + \varepsilon_{ijkl}\hat{\phi}^{ij}\hat{\chi}^{\alpha k}\hat{\chi}^{l}_{\alpha}\Bigg)\ . \end{aligned} \tag{VIII.67}$$

For $\kappa=+1$, the term containing $\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}$ vanishes after performing the index sums. Note furthermore that in the limit of $n\to\infty$ for the gauge group $\mathsf{U}(n)$, one can render the ordinary $\mathcal{N}=4$ SDYM matrix model (VIII.40) equivalent to noncommutative $\mathcal{N}=4$ SDYM theory defined by the action (VIII.67). This is based on the fact that there is an isomorphism of spaces $\mathbb{C}^\infty\cong\mathcal{H}$ and $\mathbb{C}^n\otimes\mathcal{H}$.

§6 Noncommutative hCS theory. The natural question to ask at this point is whether one can translate the Penrose-Ward transform completely into the noncommutative situation and therefore obtain a holomorphic Chern-Simons theory on a noncommutative supertwistor space. For the Penrose-Ward transform in the purely bosonic case, the answer is positive (see e.g. [151, 270, 172, 173]).

In the supersymmetric case, by defining the correspondence space as $(\mathbb{R}^{4|8}_\theta, g_\varepsilon)\times\Sigma^1_\varepsilon$ with the coordinate algebra (VIII.61) and unchanged algebra of Graßmann coordinates, we arrive together with the incidence relations in (VII.161) at noncommutative coordinates[5] on the twistor space $\mathcal{P}^{3|4}_{\varepsilon,\theta}$ satisfying the relations

$$\begin{aligned} [\hat{z}^1_\pm,\hat{z}^2_\pm] \;&=\; 2(\kappa-1)\varepsilon\lambda_\pm\theta\ , & [\hat{\bar{z}}^1_\pm,\hat{\bar{z}}^2_\pm] \;&=\; -2(\kappa-1)\varepsilon\bar{\lambda}_\pm\theta\ ,\\ [\hat{z}^1_+,\hat{\bar{z}}^1_+] \;&=\; 2(\kappa\varepsilon-\lambda_+\bar{\lambda}_+)\theta\ , & [\hat{z}^1_-,\hat{\bar{z}}^1_-] \;&=\; 2(\kappa\varepsilon\lambda_-\bar{\lambda}_- - 1)\theta\ ,\\ [\hat{z}^2_+,\hat{\bar{z}}^2_+] \;&=\; 2(1-\varepsilon\kappa\lambda_+\bar{\lambda}_+)\theta\ , & [\hat{z}^2_-,\hat{\bar{z}}^2_-] \;&=\; 2(\lambda_-\bar{\lambda}_- - \varepsilon\kappa)\theta\ , \end{aligned} \tag{VIII.68}$$

with all other commutators vanishing. Here, we clearly see the advantage of choosing a self-dual deformation tensor $\kappa=+1$: the first line in (VIII.68) becomes trivial. We will restrict our considerations to this case[6] in the following.

Thus, we see that the coordinates z^α and $\bar{z}^\alpha$ are turned into sections $\hat{z}^\alpha$ and $\hat{\bar{z}}^\alpha$ of the bundle $\mathcal{O}(1)$ which are functions on $\mathcal{P}^{1|4}_\varepsilon$ and take values in the space of operators acting on the Fock space $\mathcal{H}$. The derivatives along the bosonic fibres of the fibration $\mathcal{P}^{3|4}_\varepsilon\to\mathcal{P}^{1|4}_\varepsilon$ are turned into inner derivatives of the algebra (VIII.68):

$$\frac{\partial}{\partial\hat{\bar{z}}^1_\pm}f \;=\; \frac{\varepsilon}{2\theta}\gamma_\pm\,[\hat{z}^1_\pm,f]\ ,\quad \frac{\partial}{\partial\hat{\bar{z}}^2_\pm}f \;=\; \frac{1}{2\theta}\gamma_\pm\,[\hat{z}^2_\pm,f]\ . \tag{VIII.69}$$

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[5]Observe that the coordinates on Σ^1_ε stay commutative.

[6]Recall, however, that the singularities of the moduli space of self-dual solutions are not resolved when choosing a self-dual deformation tensor.

Together with the identities (VII.181), we can furthermore derive

$$\hat{V}_1^{\pm} f \;=\; -\frac{\varepsilon}{2\theta}\,[\hat{z}^2_{\pm}, f] \quad \text{and} \quad \hat{V}_2^{\pm} f \;=\; -\frac{\varepsilon}{2\theta}\,[\hat{z}^1_{\pm}, f]\;. \tag{VIII.70}$$

The formulæ (VIII.70) allow us to define the noncommutatively deformed version of the hCS action (VII.203):

$$\begin{aligned} S_{\text{nchCS}} \;:=\; & \int_{\mathscr{P}_\varepsilon^{1|4}} \omega \wedge \operatorname{tr}_{\mathcal{H}} \otimes \operatorname{tr} \left\{ \left(\hat{\mathcal{A}}_2 \bar{\partial} \hat{\mathcal{A}}_1 - \hat{\mathcal{A}}_1 \bar{\partial} \hat{\mathcal{A}}_2 \right) + 2 \hat{\mathcal{A}}^{0,1}_{\Sigma} \left[\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2 \right] - \right. , \\ & -\frac{\varepsilon}{2\theta} \left. \left(\hat{\mathcal{A}}_1 \left[\hat{z}^1, \hat{\mathcal{A}}^{0,1}_{\Sigma} \right] - \hat{\mathcal{A}}^{0,1}_{\Sigma} \left[\hat{z}^1, \hat{\mathcal{A}}_1 \right] + \hat{\mathcal{A}}^{0,1}_{\Sigma} \left[\hat{z}^2, \hat{\mathcal{A}}_2 \right] - \hat{\mathcal{A}}_2 \left[\hat{z}^2, \hat{\mathcal{A}}^{0,1}_{\Sigma} \right] \right) \right\} \;, \end{aligned} \tag{VIII.71}$$

where $\mathscr{P}_\varepsilon^{1|4}$ is again the chiral subspace of $\mathcal{P}_\varepsilon^{1|4}$ for which $\bar{\eta}^i_{\pm} = 0$, ω is the form defined in (VIII.53) and $\operatorname{tr}_{\mathcal{H}}$ and tr denote the traces over the Fock space $\mathcal{H}$ and the representation space of $\mathfrak{gl}(n, \mathbb{C})$, respectively. The hats indicate that the components of the gauge potential $\hat{\mathcal{A}}^{0,1}$ are now operators with values in the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$.

We can simplify the above action by introducing the operators

$$\hat{\mathcal{X}}^1_{\pm} \;=\; -\frac{\varepsilon}{2\theta} \hat{z}^2_{\pm} \otimes \mathbb{1}_n + \hat{\mathcal{A}}^{\pm}_1 \quad \text{and} \quad \hat{\mathcal{X}}^2_{\pm} \;=\; -\frac{\varepsilon}{2\theta} \hat{z}^1_{\pm} \otimes \mathbb{1}_n + \hat{\mathcal{A}}^{\pm}_2 \tag{VIII.72}$$

which yields

$$S_{\text{nchCS}} \;=\; \int_{\mathscr{P}_\varepsilon^{1|4}} \omega \wedge \operatorname{tr}_{\mathcal{H}} \otimes \operatorname{tr} \varepsilon^{\alpha\beta} \hat{\mathcal{X}}_\alpha \left(\bar{\partial} \hat{\mathcal{X}}_\beta + [\hat{\mathcal{A}}^{0,1}_{\Sigma}, \hat{\mathcal{X}}_\beta] \right) \;, \tag{VIII.73}$$

where the $\hat{\mathcal{X}}_\alpha$ take values in the bundle $\mathcal{O}(1)$, so that the above integral is indeed well defined. Note that in the matrix model (VIII.52), we considered matrices taking values in the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, while the fields $\hat{\mathcal{X}}_\alpha$ and $\hat{\mathcal{A}}^{0,1}_{\Sigma}$ in the model (VIII.73) take values in $\mathfrak{gl}(n, \mathbb{C}) \otimes \mathsf{End}(\mathcal{H})$ and can be represented by infinite-dimensional matrices.

§7 String field theory. The form of the matrix model action given by (VIII.73) is identical to an action recently given as a cubic string field theory for open $\mathcal{N} = 2$ strings [174]. Let us comment on that point in more detail.

First of all, recall cubic string field theory action from section V.2.4 which reads as

$$S = \tfrac{1}{2} \int \left(\mathcal{A} \star Q \mathcal{A} + \tfrac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A} \right) \;. \tag{VIII.74}$$

To qualify as a string field, $\mathcal{A}$ is a functional of the embedding map Φ from the string parameter space to the string target space. For the case at hand, we take

$$\Phi : [0, \pi] \times G \;\to\; \mathcal{P}_\varepsilon^{3|4} \;, \tag{VIII.75}$$

where $\sigma \in [0, \pi]$ parameterizes the open string and $G \ni v$ provides the appropriate set of Graßmann variables on the worldsheet. Expanding $\Phi(\sigma, v) = \phi(\sigma) + v\psi(\sigma)$, this map embeds the $\mathcal{N} = 2$ spinning string into supertwistor space. Next, we recollect $\phi = (z^\alpha = x^{\alpha\dot{\alpha}} \lambda_{\dot{\alpha}}, \eta_i = \eta_i^{\dot{\alpha}} \lambda_{\dot{\alpha}}, \lambda, \bar{\lambda})$ and allow the string to vibrate only in the z^α-directions but keep the G-even zero modes of $(\eta_i, \lambda, \bar{\lambda})$, so that the string field depends on $\{z^\alpha(\sigma), \eta_i, \lambda, \bar{\lambda}; \psi^{\alpha\ddot{\alpha}}(\sigma)\}$ only [174]. Note that with ψ and η, we have two types of fermionic fields present, since we are implicitly working in the doubly supersymmetric description of superstrings [265], which we will briefly discuss in section 4.2. Therefore, the two fermionic fields are linked via a superembedding condition. We employ a suitable BRST operator $Q = \bar{D} + \bar{\partial}$, where $\bar{D} = \psi^{\alpha\ddot{1}} \lambda^{\dot{\alpha}} \partial_\sigma x_{\alpha\dot{\alpha}} \in \mathcal{O}(1)$ and $\bar{\partial} \in \mathcal{O}(0)$ are type

(0, 1) vector fields on the fibres and the base of $\mathcal{P}_\varepsilon^{3|4}$, respectively, and split the string field accordingly, $\mathcal{A} = \mathcal{A}_{\bar{D}} + \mathcal{A}_{\bar{\partial}}$. With a holomorphic integration measure on $\mathcal{P}_\varepsilon^{3|4}$, the Chern-Simons action (VII.194) projects to [174]

$$S \;=\; \int_{\mathscr{P}_\varepsilon^{1|4}} \omega \wedge \langle\, \mathrm{tr}\,(\mathcal{A}_{\bar{D}} \star \bar{\partial}\mathcal{A}_{\bar{D}} + 2\mathcal{A}_{\bar{D}} \star \bar{D}\mathcal{A}_{\bar{\partial}} + 2\mathcal{A}_{\bar{\partial}} \star \mathcal{A}_{\bar{D}} \star \mathcal{A}_{\bar{D}}\rangle\ . \qquad \text{(VIII.76)}$$

Note that the string fields $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\partial}}$ are fermionic, i.e. they behave in the action as if they were forms multiplied with the wedge product. Furthermore, the above-mentioned $\mathbb{Z}$-grading of all the ingredients of this action has to be adjusted appropriately. Giving an expansion in η_i for these string fields similar to the one in (VII.205a) and (VII.205b), one recovers the super string field theory proposed by Berkovits and Siegel [30]. Its zero modes describe self-dual $\mathcal{N} = 4$ SDYM theory.

By identifying $Q + \mathcal{A}_Q$ with $\hat{\mathcal{X}}$, $\bar{\partial}$ with $\bar{\partial}_{\bar{\lambda}}$ and $\mathcal{A}_{\bar{\partial}}$ with $\hat{\mathcal{A}}_\Sigma^{0,1}$ and adjusting the $\mathbb{Z}_2$-grading of the fields, one obtains the action[7] (VIII.73) from (VIII.76). Therefore, we can e.g. translate solution generating techniques which are at hand for our matrix model immediately to the string field theory (VIII.76).

VIII.3.2 Classical solutions to the noncommutative matrix model

Simple classical solutions of $\mathcal{N} = 4$ noncommutative SDYM theory can be obtained by considering only the helicity ± 1 part of the field content and putting all other fields to zero.[8] Solutions with all the fields being nontrivial can then be recovered by either acting with supersymmetry transformations on the previously obtained seed solutions or by a supersymmetric extension of the dressing method [23]. After considering the simple example of Abelian instantons, we will present two solution generating techniques inspired by twistor methods similar to the ones presented in [229].

§8 Abelian instantons. We start from the simplest case and consider Abelian instantons[9], i.e. let us choose the gauge group to be $\mathsf{U}(1)$. Contrary to commutative spaces, this gauge group allows for instantons in the noncommutative setting. In some cases, problematic singularities in the moduli space of instantons are resolved by noncommutativity, but this will not be the case for self-dual instantons on a space with self-dual deformation.

The equations of motion for helicity ± 1 derived from (VIII.73) read for this choice

$$\varepsilon^{\alpha\beta}[\hat{X}_{\alpha\dot{\alpha}}, \hat{X}_{\beta\dot{\beta}}] \;=\; 0 \quad \text{and} \quad [\hat{X}_{\alpha\dot{\alpha}}, \hat{G}^{\dot{\alpha}\dot{\gamma}}] \;=\; 0\ . \qquad \text{(VIII.77)}$$

The first equation can be reduced to components and is then spelled out as

$$\begin{aligned} [\hat{X}_{1\dot{1}}, \hat{X}_{2\dot{1}}] \;=\; [\hat{X}_{1\dot{2}}, \hat{X}_{2\dot{2}}] \;&=\; 0\ ,\\ [\hat{X}_{1\dot{1}}, \hat{X}_{2\dot{2}}] - [\hat{X}_{2\dot{1}}, \hat{X}_{1\dot{2}}] \;&=\; 0\ . \end{aligned} \qquad \text{(VIII.78)}$$

A trivial solution of these equations is the following:

$$\hat{X}_{1\dot{1}} \;=\; \mathrm{i}\hat{x}^{2\dot{2}}\ , \quad \hat{X}_{1\dot{2}} \;=\; \mathrm{i}\hat{x}^{2\dot{1}}\ , \quad \hat{X}_{2\dot{2}} \;=\; \mathrm{i}\hat{x}^{1\dot{1}}\ , \quad \hat{X}_{2\dot{1}} \;=\; \mathrm{i}\hat{x}^{1\dot{2}}\ , \qquad \text{(VIII.79)}$$

[7]Note that $\bar{\partial}Q + Q\bar{\partial} = 0$.

[8]This situation can also be obtained by taking holomorphic Chern-Simons theory on a thickening of ordinary twistor space as discussed in [244] as a starting point for constructing the matrix model.

[9]Instantons on noncommutative spaces were first discussed in [205], cf. also [137], appendix B.

where the factors of i were included to have anti-Hermitian gauge potentials satisfying

$$\hat{X}_{1\dot{2}} \;=\; \hat{X}^{\dagger}_{2\dot{1}} \quad \text{and} \quad \hat{X}_{2\dot{2}} \;=\; -\hat{X}^{\dagger}_{1\dot{1}} \;, \tag{VIII.80}$$

cf. equations (VII.113). These solutions can now be used as seed solutions for the dressing method. In the two-oscillator Fock space $\mathcal{H}$, we introduce the projector on the vacuum

$$P_0 \;=\; |0,0\rangle\langle 0,0| \;, \tag{VIII.81}$$

and two shift operators S and $S^\dagger$ satisfying

$$S^\dagger S \;=\; \mathbb{1} \;, \quad SS^\dagger \;=\; \mathbb{1} - P_0 \;, \quad P_0 S \;=\; S^\dagger P_0 \;=\; 0 \;. \tag{VIII.82}$$

Then a new solution of (VIII.78) is given by

$$\tilde{\hat{X}}_{\alpha\dot{\alpha}} \;=\; S\hat{X}_{\alpha\dot{\alpha}}S^\dagger \;, \tag{VIII.83}$$

where $\hat{X}_{\alpha\dot{\alpha}}$ is given in (VIII.79). In fact, one can show that this method generates all possible solutions if we allow for a more general projector P_0.

To obtain a seed solution to the second equation in (VIII.77), we consider this equation in components:

$$\begin{aligned}
[\hat{x}^{2\dot{2}}, \hat{G}^{\dot{1}\dot{1}}] + [\hat{x}^{2\dot{1}}, \hat{G}^{\dot{2}\dot{1}}] \;&=\; 0 \;, \quad [\hat{x}^{1\dot{2}}, \hat{G}^{\dot{1}\dot{1}}] + [\hat{x}^{1\dot{1}}, \hat{G}^{\dot{2}\dot{1}}] \;=\; 0 \;,\\
[\hat{x}^{1\dot{2}}, \hat{G}^{\dot{1}\dot{2}}] + [\hat{x}^{1\dot{1}}, \hat{G}^{\dot{2}\dot{2}}] \;&=\; 0 \;, \quad [\hat{x}^{2\dot{2}}, \hat{G}^{\dot{1}\dot{2}}] + [\hat{x}^{2\dot{1}}, \hat{G}^{\dot{2}\dot{2}}] \;=\; 0 \;.
\end{aligned}$$

Additionally, when constructing a solution, we have to guarantee $\hat{G}^{\dot{\alpha}\dot{\beta}} = \hat{G}^{\dot{\beta}\dot{\alpha}}$. One easily observes that the problem of finding solutions to (VIII.84) decomposes into finding solutions to the left two equations and solutions to the right two equations and we choose the trivial solutions

$$\begin{aligned}
\hat{G}^{\dot{1}\dot{1}} \;&=\; \hat{x}^{1\dot{1}} + c_1\hat{x}^{2\dot{2}} \;, & \hat{G}^{\dot{1}\dot{1}} \;&=\; \hat{x}^{2\dot{1}} + c_3\hat{x}^{1\dot{2}} \;,\\
\hat{G}^{\dot{1}\dot{2}} \;=\; \hat{G}^{\dot{2}\dot{1}} \;&=\; \hat{x}^{1\dot{2}} + \hat{x}^{2\dot{1}} \;, & \hat{G}^{\dot{1}\dot{2}} \;=\; \hat{G}^{\dot{2}\dot{1}} \;&=\; \hat{x}^{1\dot{1}} + \hat{x}^{2\dot{2}} \;,\\
\hat{G}^{\dot{2}\dot{2}} \;&=\; \hat{x}^{2\dot{2}} + c_2\hat{x}^{1\dot{1}} \;, & \hat{G}^{\dot{1}\dot{1}} \;&=\; \hat{x}^{1\dot{2}} + c_4\hat{x}^{2\dot{1}} \;,
\end{aligned} \tag{VIII.84}$$

respectively, where the c_i are arbitrary complex constants. An appropriate reality condition on the field $\hat{G}^{\dot{\alpha}\dot{\beta}}$ demands, however, that $c_1^* = -c_2$ and $c_3 = c_4^*$. Note that both solutions can be linearly combined.

With the help of the shift operators S and $S^\dagger$, one can construct the "dressed" solution

$$\tilde{\hat{G}}^{\dot{\alpha}\dot{\beta}} \;=\; S\hat{G}^{\dot{\alpha}\dot{\beta}}S^\dagger \;. \tag{VIII.85}$$

§9 Supersymmetric ADHM construction. Although the solutions obtained above by "educated guessing" can be made more interesting by supersymmetry transformations, the procedure to obtain further such solutions would be rather cumbersome. Therefore, we will turn our attention in the following to more sophisticated solution generating techniques.

Besides a supersymmetric extension of the dressing method [23], the most obvious candidate is the supersymmetric extension of the ADHM construction, see section VII.8.3. The self-dual gauge potential obtained from this construction will have components which are superfunctions, and the component expansion is the one given in (IV.70).

Although this method seems to be nicely suited for constructing solutions, it proves to be rather difficult to restrict to a nontrivial field content for the helicity ± 1 fields. An extension of well-known bosonic solutions, as e.g. the 't Hooft instanton, to solutions including a nontrivial field $G^{\dot{\alpha}\dot{\beta}}$ can be done in principle but requires an unreasonable effort in calculations. Therefore we discard this ansatz.

§10 Infinitesimal deformations. A nicer and more interesting way of systematically obtaining solutions for the auxiliary field $\hat{G}^{\dot\alpha\dot\beta}$ is found by observing that the equations of motion for $\hat{G}^{\dot\alpha\dot\beta}$ with vanishing spinors and scalar fields coincide with the linearized equations of motion[10] for $\delta\hat{G}^{\dot\alpha\dot\beta}$:

$$\nabla_{\alpha\dot\alpha}\delta\hat{G}^{\dot\alpha\dot\beta} \;=\; \nabla_{\alpha\dot\alpha}\hat{G}^{\dot\alpha\dot\beta} \;=\; 0\ . \qquad \text{(VIII.86)}$$

Thus, it is sufficient for our purposes to find solutions for $\delta\hat{G}^{\dot\alpha\dot\beta}$ and render them subsequently finite. This can be easily achieved by considering perturbations of the representant of the Čech cohomology class corresponding to a given holomorphic structure $\bar\partial + \mathcal{A}^{0,1}$ as we will show in the following.

For this, let us start from a gauge potential $\mathcal{A}^{0,1}$ and perform the gauge transformation (VII.94). The result is a gauge potential with components $(\mathcal{A}^\pm_\alpha, \mathcal{A}^i_\pm)$, which satisfy the constraint equation of supersymmetric self-dual Yang-Mills theory (IV.64). Let us recall that these constraint equation are the compatibility conditions of the linear system

$$(\bar V^+_\alpha + \mathcal{A}^+_\alpha)\psi_\pm \;=\; 0\ ,\quad \partial_{\bar\lambda_\pm}\psi_\pm \;=\; 0\ ,\quad (\bar V^i_+ + \mathcal{A}^i_+)\psi_\pm \;=\; 0\ , \qquad \text{(VIII.87)}$$

on $\tilde{\mathcal{U}}_+$ and a similar one on $\tilde{\mathcal{U}}_-$. Therefore, we can write

$$\begin{aligned} \mathcal{A}^+_\alpha &= \psi_\pm \bar V^+_\alpha \psi_\pm^{-1}\ ,\quad \mathcal{A}_{\bar\lambda_+} = \psi_\pm\partial_{\bar\lambda_+}\psi_\pm^{-1} = 0\ ,\quad \mathcal{A}^i_+ = \psi_\pm \bar V^i_+\psi_\pm^{-1}\ ,\\ \mathcal{A}^-_\alpha &= \psi_\pm \bar V^-_\alpha \psi_\pm^{-1}\ ,\quad \mathcal{A}_{\bar\lambda_-} = \psi_\pm\partial_{\bar\lambda_-}\psi_\pm^{-1} = 0\ ,\quad \mathcal{A}^i_- = \psi_\pm \bar V^i_-\psi_\pm^{-1}\ , \end{aligned} \qquad \text{(VIII.88)}$$

for some regular matrix valued functions $\psi_\pm$. On the overlap $\mathcal{U}_+\cap\mathcal{U}_-$, we have the gluing condition $\psi_+\bar D^\pm_I\psi_+^{-1} = \psi_-\bar D^\pm_I\psi_-^{-1}$, where $\bar D^\pm_I = (\bar V^\pm_\alpha, \partial_{\bar\lambda_\pm}, \bar V^i_\pm)$. This condition is equivalent to $\bar D_I(\psi_+^{-1}\psi_-) = 0$. Thus, we obtained an element $f_{+-} := \psi_+^{-1}\psi_-$ of the first Čech cohomology set, which contains the same information as the original gauge potential $\mathcal{A}^{0,1}$, which was an element of a Dolbeault cohomology group[11]. The function f_{+-} is the transition function of a holomorphic vector bundle over $\mathcal{P}^{3|4}_\varepsilon$, and, when obtained from an anti-Hermitian gauge potential, this function satisfies the reality condition

$$f_{+-}(x,\lambda_+,\eta) \;=\; (f_{+-}(\tau(x,\lambda_+,\eta)))^\dagger\ . \qquad \text{(VIII.89)}$$

An infinitesimal deformation[12] of $f_{+-}\mapsto f_{+-}+\delta f_{+-}$ leads to infinitesimal deformations of the functions $\psi_\pm$:

$$\begin{aligned} f_{+-}+\delta f_{+-} &= (\psi_+ + \delta\psi_+)^{-1}(\psi_- + \delta\psi_-) \;=\; (\psi_+^{-1} - \psi_+^{-1}\delta\psi_+\psi_+^{-1})(\psi_- + \delta\psi_-)\\ \Rightarrow\ \delta f_{+-} &= f_{+-}\psi_-^{-1}\delta\psi_- - \psi_+^{-1}\delta\psi_+ f_{+-}\ . \end{aligned} \qquad \text{(VIII.90)}$$

It is convenient to introduce the auxiliary function $\varphi_{+-} := \psi_+(\delta f_{+-})\psi_-^{-1}$, which may be written as the difference $\varphi_{+-} = \phi_+ - \phi_-$ of two regular matrix valued functions $\phi_\pm$, holomorphic in $\lambda_\pm$. Note, however, that there is a freedom of assigning sections constant in $\lambda_\pm$ to either ϕ_+ or ϕ_-, which corresponds to a (partial) choice of the gauge. We can now write $\delta\psi_\pm = -\phi_\pm\psi_\pm$ and from the functions $\phi_\pm$, we can reconstruct the variation of the components of the gauge potential:

$$\delta\mathcal{A}^+_\alpha \;=\; \delta\psi_\pm\bar V^+_\alpha\psi_\pm^{-1} + \psi_\pm\bar V^+_\alpha\delta(\psi_\pm^{-1}) \;=\; \nabla^+_\alpha\phi_\pm\ , \qquad \text{(VIII.91a)}$$

$$\delta\mathcal{A}^i_+ \;=\; \delta\psi_\pm\bar V^i_+\psi_\pm^{-1} + \psi_\pm\bar V^i_+\delta(\psi_\pm^{-1}) \;=\; \nabla^i_+\phi_\pm\ , \qquad \text{(VIII.91b)}$$

where we wrote $\nabla^+_\alpha = \bar V^+_\alpha + [\mathcal{A}^+_\alpha,\cdot]$ and $\nabla^i_+ = \bar V^i_+ + [\mathcal{A}^i_+,\cdot]$ for the covariant derivatives.

[10]The linearized equations of motion are obtained from the noncommutative version of (IV.62) by considering a finite gauge potential while all other fields in the supermultiplet are only infinitesimal.

[11]One should stress that the actual transition from the Dolbeault- to the Čech description can be done directly without using the gauge transformation (VII.94), which we included only for convenience sake.

[12]See also [298, 229].

§11 The ansatz and transverse gauge. As an ansatz to endow any bosonic self-dual gauge potential $A_{\alpha\dot\alpha}$ with a nontrivial auxiliary field $G^{\dot\alpha\dot\beta}$, we choose

$$\delta f_{+-} \;=\; \eta_1\eta_2\eta_3\eta_4 R(x,\lambda) \;=\; \eta_1\eta_2\eta_3\eta_4[K, f_{+-}]\ , \tag{VIII.92}$$

where $K \in \mathfrak{gl}(n,\mathbb{C})$. Using the commutator is important to guarantee that the splitting $f_{+-} = \psi_+^{-1}\psi_-$ persists after the deformation. Equations (VIII.91a) and (VIII.91b) will moreover lead to additional components in the η-expansion of the gauge potentials, which match the appearances of $G^{\dot\alpha\dot\beta}$ in (IV.70), i.e. this ansatz will give rise to terms of order η^4 in $\mathcal{A}^{\pm}_{\alpha}$ and of order η^3 in $\mathcal{A}^i_{\pm}$. Note at this point that although it produces a finite field, this deformation is infinitesimal, as it leads to nilpotent expressions for δf_{+-}, $\delta\psi_\pm$ and $\delta(\psi_\pm^{-1})$.

To recover the physical field content, we have to obtain the potentials $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ (which are extracted from $\mathcal{A}_\alpha$ and $\mathcal{A}^i$) in transverse gauge. Therefore, we carefully have to choose the splitting $\varphi_{+-} = \phi_+ - \phi_-$. First note that different splittings are given by

$$\varphi_{+-} \;=\; \phi_+ - \phi_- \;=\; (\phi_+ + \phi_0) - (\phi_- + \phi_0) \;=:\; \tilde\phi_+ - \tilde\phi_-\ , \tag{VIII.93}$$

and our task is to find a suitable function ϕ_0. With our choice of R, i.e. with R being homogeneous of order four in the $\eta^{\dot\alpha}_i$, and starting from $\nabla^i_+ = \bar V^i_+$ we can simply use

$$\phi_0 \;:=\; -\tfrac{1}{4}\eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha}\ . \tag{VIII.94}$$

Its additive contribution to the new potential $\tilde{\mathcal{A}}^i_{\dot\alpha}$ is $\partial^i_{\dot\alpha}\phi_0$ (as ϕ_0 is independent of $\lambda_\pm$) and therefore[13]

$$\eta^{\dot\alpha}_i\tilde{\mathcal{A}}^i_{\dot\alpha} \;=\; \eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha} + \eta^{\dot\alpha}_i\partial^i_{\dot\alpha}\phi_0 \;=\; \eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha} - \eta^{\dot\alpha}_i\mathcal{A}^i_{\dot\alpha} \;=\; 0\ . \tag{VIII.95}$$

This procedure gives obviously rise to additional terms of order η^4 in $\tilde{\mathcal{A}}^{\pm}_{\alpha}$ and of order η^3 in $\tilde{\mathcal{A}}^i_{\pm}$ only. Since our gauge potential is now in transverse gauge, we know for sure that it is of the form (IV.70), and thus one can consistently extract the physical field content by comparing the expansions. In particular, it is not necessary to worry about higher orders in η than the ones considered here, as the field content is already completely defined at third order in η, cf. (IV.70). With our ansatz, we therefore are certain to obtain a solution of the form ($A_{\alpha\dot\alpha} \neq 0, \chi^{\alpha i} = 0, \phi^{ij} = 0, \tilde\chi^{\dot\alpha ijk} = 0, G^{\dot\alpha\dot\beta} \neq 0$).

Note that the method described above directly translates to the noncommutative setting.

§12 Dressed Penrose-Ward transform. Recalling the dressing method, one is led to finding a solution to the equation $\partial_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta} = 0$ by a Penrose transform[14] and inserting the dressing factors accounting for a non-Abelian gauge group appropriately. Thus, let us start (in the commutative setting) from the gauge group $\mathsf{U}(N)$ and choose a gauge potential $A_{\alpha\dot\alpha} = 0$ as a seed solution for the dressing method. The equation of motion for the auxiliary field $G^{\dot\alpha\dot\beta}$ then reduces to $\nabla_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta} = \partial_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta}$, and the solutions to this equation are given by

$$G^{\dot\alpha\dot\beta} \;=\; \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\lambda_+^{\dot\alpha}\lambda_+^{\dot\beta} f(Z^q)\ , \tag{VIII.96}$$

[13] Note that $\mathcal{D} := \eta^{\dot\alpha}_i\partial^i_{\dot\alpha}$ satisfies $\mathcal{D}\mathcal{D} = h\mathcal{D}$, when acting on functions which are homogeneous in the $\eta^{\dot\alpha}_i$ of degree h.

[14] See also the discussion of the Penrose transform in section VII.1.3.

where $f(Z^q)$ is a Lie algebra valued function, which is of homogeneity -4 in the twistor coordinates $(Z^q) = (x^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \lambda_{\dot\alpha})$, cf. the discussion of elementary states in section VII.1.3, §12. Furthermore, $f(Z^q)$ has two distinct poles (without counting the multiplicities). The curve γ is chosen in such a way that it separates the poles from each other. That this is in fact a solution of the given equation is easily seen by pulling the derivative $\partial_{\alpha\dot\alpha}$ into the integral and noting that $\lambda_+^{\dot\alpha}\partial_{\alpha\dot\alpha}z^\beta = \lambda_+^{\dot\alpha}\lambda_{\dot\alpha}^+\delta_\alpha^\beta = 0$.

When dressing the seed solution of $A_{\alpha\dot\alpha}$ to a nontrivial solution $\lambda_+^{\dot\alpha}\tilde A_{\alpha\dot\alpha} = \psi_+\bar V_\alpha^+\psi_+^{-1} = \psi_-\bar V_\alpha^+\psi_-^{-1}$, we have to adapt equation (VIII.96) in the following way:

$$G^{\dot\alpha\dot\beta} = \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\lambda_+^{\dot\alpha}\lambda_+^{\dot\beta}\psi_+ f(Z^q)\psi_+^{-1}\ . \tag{VIII.97}$$

Again, this is a solution since differentiating under the integral yields

$$\begin{aligned}\partial_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta} &= \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\lambda_+^{\dot\beta}(\bar V_\alpha^+\psi_+)f(Z^q)\psi_+^{-1} + \psi_+ f(Z^q)(\bar V_\alpha^+\psi_+^{-1})\\ &= \oint_\gamma \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}}\lambda_+^{\dot\alpha}\lambda_+^{\dot\beta}[f(Z^q), A_{\alpha\dot\alpha}]\\ &= [G^{\dot\alpha\dot\beta}, A_{\alpha\dot\alpha}]\ .\end{aligned} \tag{VIII.98}$$

From the above calculation, one also sees that one is free to choose any combination of $\psi_\pm$ and $\psi_\pm^{-1}$ around the function $f(Z^q)$.

§13 Noncommutative example. Let us now turn to the noncommutative case, and complement the noncommutative BPST instanton with a solution for the auxiliary field. We start from the noncommutative BPST instanton on Euclidean spacetime ($\varepsilon = -1$) as described e.g. in[15] [133]. The gauge potential reads

$$\begin{aligned}\hat A_{1\dot 1} &= \begin{pmatrix} -\left(\frac{[-1]}{[0]}-1\right)\frac{\hat x^{2\dot 2}}{2\theta} & \frac{\mathrm{i}}{[0][-1]}\\ 0 & -\left(\frac{[2]}{[1]}-1\right)\frac{\hat x^{2\dot 2}}{2\theta}\end{pmatrix}\ , \quad \hat A_{2\dot 2} = -\hat A_{1\dot 1}^\dagger\ ,\\ \hat A_{2\dot 1} &= \begin{pmatrix} \frac{\hat x^{1\dot 2}}{2\theta}\left(\frac{[-1]}{[0]}-1\right) & 0\\ \frac{\hat x^{2\dot 2}}{[0][-1]} & \frac{\hat x^{1\dot 2}}{2\theta}\left(\frac{[2]}{[1]}-1\right)\end{pmatrix}\ , \qquad \hat A_{1\dot 2} = \hat A_{2\dot 1}^\dagger\ ,\end{aligned} \tag{VIII.99}$$

where we introduced the shorthand notation $[n] := \sqrt{\hat r^2 + \Lambda^2 + 2n\theta}$ with $\hat r^2 = \hat x^{1\dot 1}\hat x^{2\dot 2} - \hat x^{2\dot 1}\hat x^{1\dot 2}$. The parameter Λ can be identified with the size of the instanton. Moreover, we have the useful identities

$$\begin{aligned}\hat x^{1\dot 1}[n] &= [n-1]\hat x^{1\dot 1}\ , \quad \hat x^{1\dot 2}[n] = [n-1]\hat x^{1\dot 2}\ ,\\ \hat x^{2\dot 1}[n] &= [n+1]\hat x^{2\dot 1}\ , \quad \hat x^{2\dot 2}[n] = [n+1]\hat x^{2\dot 2}\end{aligned} \tag{VIII.100}$$

which can be verified by interpreting $\hat r^2$ as the number operator $\hat N$ plus a constant term, and then have it act on an arbitrary state, decomposed into eigenstates of $\hat N$.

[15] For doing the calculations in the following, we used the computer algebra software Mathematica together with the "Operator Linear Algebra Package" by Liu Zhao and the "Grassmann" package by Matthew Headrick.

The corresponding matrix-valued functions $\hat\psi_+$, $\hat\psi_-^{-1}$ and $\hat f_{+-}$ read

$$\begin{aligned}
\hat\psi_+ &= -\frac{1}{\Lambda}\begin{pmatrix}[-1] & 0\\ 0 & [+1]\end{pmatrix}\begin{pmatrix} -\hat x^{1\dot 2}\hat x^{2\dot 1}+\Lambda^2+\lambda_+\hat x^{1\dot 2}\hat x^{2\dot 2} & -\hat x^{1\dot 2}\hat x^{1\dot 1}+\lambda_+(\hat x^{1\dot 2})^2\\ \hat x^{2\dot 1}\hat x^{2\dot 2}-\lambda_+(\hat x^{2\dot 2})^2 & \hat x^{2\dot 2}\hat x^{1\dot 1}+\Lambda^2-\lambda_+\hat x^{1\dot 2}\hat x^{2\dot 2}\end{pmatrix},\\
\hat\psi_-^{-1} &= -\frac{1}{\Lambda}\begin{pmatrix}[-1] & 0\\ 0 & [+1]\end{pmatrix}\begin{pmatrix} -\hat x^{1\dot 2}\hat x^{2\dot 1}+\Lambda^2+\frac{\hat x^{1\dot 1}\hat x^{2\dot 1}}{\lambda_+} & -\hat x^{1\dot 1}\hat x^{1\dot 2}+\frac{(\hat x^{1\dot 1})^2}{\lambda_+}\\ \hat x^{2\dot 2}\hat x^{2\dot 1}-\frac{(\hat x^{2\dot 1})^2}{\lambda_+} & \hat x^{2\dot 2}\hat x^{1\dot 1}+\Lambda^2-\frac{\hat x^{1\dot 1}\hat x^{2\dot 1}}{\lambda_+}\end{pmatrix},\\
\hat f_{+-} &= \frac{1}{\Lambda^2}\begin{pmatrix} [1]^2-\lambda_+\hat x^{1\dot 2}\hat x^{2\dot 2}-\frac{\hat x^{2\dot 1}\hat x^{1\dot 1}}{\lambda_+}+2\hat x^{2\dot 1}\hat x^{1\dot 2} & -\frac{(\hat x^{1\dot 1})^2}{\lambda_+}-\lambda_+(\hat x^{1\dot 2})^2+2\hat x^{1\dot 1}\hat x^{1\dot 2}\\ \frac{(\hat x^{2\dot 1})^2}{\lambda_+}+\lambda_+(\hat x^{2\dot 2})^2-2\hat x^{2\dot 1}\hat x^{2\dot 2} & [-1]^2-2\hat x^{1\dot 1}\hat x^{2\dot 2}+\lambda_+\hat x^{1\dot 2}\hat x^{2\dot 2}+\frac{\hat x^{2\dot 1}\hat x^{1\dot 1}}{\lambda_+}\end{pmatrix}.
\end{aligned}$$

Additionally we use

$$\hat\psi_+^{-1} = \begin{pmatrix}\Lambda^2+\hat x^{1\dot 1}\hat x^{2\dot 2}-\lambda_+\hat x^{2\dot 2}\hat x^{1\dot 2} & -\lambda_+(\hat x^{1\dot 2})^2+\hat x^{1\dot 2}\hat x^{1\dot 1}\\ \lambda_+(\hat x^{2\dot 2})^2-\hat x^{2\dot 1}\hat x^{2\dot 2} & \Lambda^2-\hat x^{2\dot 1}\hat x^{1\dot 2}+\lambda\hat x^{2\dot 2}\hat x^{1\dot 2}\end{pmatrix}. \tag{VIII.101}$$

As an ansatz for $f(Z^q)$, we choose the simple form

$$\hat f(\hat z_+^1,\hat z_+^2,\lambda_+) = \frac{\hat z_+^1}{\lambda_+^2(1+\lambda_+)}\sigma^3 . \tag{VIII.102}$$

Note that it is not possible, to have the noncommutative coordinates $\hat z^\alpha$ appear in a holomorphic way in the denominator. This is due to the fact that the $\hat z^\alpha$ are operators on a Fock space, for which only infinite-dimensional representations exists. Therefore, the inverse of holomorphic functions of $\hat z^\alpha$ does not exist in general.

The singularities at $\lambda_+=0$ and $\lambda_+=-1$ are separated by a circle γ of radius $r<1$ around $\lambda_+=0$. Thus, the equations (VIII.96) reduce to

$$\begin{aligned}
\hat G^{\dot 1\dot 1} &= \langle\hat f(Z^q)\rangle_{-3}\ , & \hat G^{\dot 1\dot 2} &= -\langle\hat f(Z^q)\rangle_{-2}\ ,\\
\hat G^{\dot 2\dot 1} &= -\langle\hat f(Z^q)\rangle_{-2}\ , & \hat G^{\dot 2\dot 2} &= \langle\hat f(Z^q)\rangle_{-1}\ ,
\end{aligned} \tag{VIII.103}$$

where $\langle\cdot\rangle_n$ denotes the nth coefficient in a Laurent series. The undressed field $\hat G_0^{\dot\alpha\dot\beta}$ is then easily obtained and has as the only non-vanishing components $\hat G_0^{\dot\alpha\dot\beta}=\hat G_0^{\dot\alpha\dot\beta 3}\sigma^3$ with

$$\hat G_0^{\dot 1\dot 2 3} = \hat x^{1\dot 1} \quad\text{and}\quad \hat G_0^{\dot 2\dot 2 3} = (\hat x^{1\dot 1}-\hat x^{1\dot 2})\ . \tag{VIII.104}$$

The "dressed" solution on the other hand are determined by equation (VIII.97), which reduces to

$$\begin{aligned}
\hat G^{\dot 1\dot 1} &= \langle\hat\psi_+\hat f(Z^q)\hat\psi_+^{-1}\rangle_{-3}\ , & \hat G^{\dot 1\dot 2} &= -\langle\hat\psi_+\hat f(Z^q)\hat\psi_+^{-1}\rangle_{-2}\ ,\\
\hat G^{\dot 2\dot 1} &= -\langle\hat\psi_+\hat f(Z^q)\hat\psi_+^{-1}\rangle_{-2}\ , & \hat G^{\dot 2\dot 2} &= \langle\hat\psi_+\hat f(Z^q)\hat\psi_+^{-1}\rangle_{-1}\ .
\end{aligned} \tag{VIII.105}$$

Performing the calculation and decomposing the field according to $\hat G^{\dot\alpha\dot\beta}=\hat G^{\dot\alpha\dot\beta a}\sigma^a$, we arrive at the following expressions:

$$
\begin{aligned}
\hat{G}^{\dot{1}\dot{1}k} &= 0\\
\hat{G}^{\dot{1}\dot{2}0} &= -\,\theta(3\Lambda^2\hat{x}^{1\dot{1}} + 2(\hat{x}^{1\dot{1}})^2\hat{x}^{2\dot{2}} - 4\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}})\\
\hat{G}^{\dot{1}\dot{2}1} &= \Lambda^2\theta\hat{x}^{2\dot{1}} + (\Lambda^2-2\theta)(\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} + (\Lambda^2+4\theta)\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}} - \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2 + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^2\\
\hat{G}^{\dot{1}\dot{2}2} &= \mathrm{i}(\Lambda^2\theta\hat{x}^{2\dot{1}} - (\Lambda^2-2\theta)(\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} + (\Lambda^2+4\theta)\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}} + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2 + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^2)\\
\hat{G}^{\dot{1}\dot{2}3} &= \Lambda^2(\Lambda^2+\theta)\hat{x}^{1\dot{1}} + (\Lambda^2-2\theta)(\hat{x}^{1\dot{1}})^2\hat{x}^{2\dot{2}} - (\Lambda^2+4\theta)\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}} - 2\hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}}\\
\hat{G}^{\dot{2}\dot{2}0} &= \theta(4\theta\hat{x}^{1\dot{2}} - \Lambda^2(3\hat{x}^{1\dot{1}} + \hat{x}^{1\dot{2}}) - 2(\hat{x}^{1\dot{1}})^2\hat{x}^{2\dot{2}} + 4\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}} + 4\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}})\\
\hat{G}^{\dot{2}\dot{2}1} &= \theta(-8\theta\hat{x}^{2\dot{2}} + \Lambda^2(\hat{x}^{2\dot{1}} + \hat{x}^{2\dot{2}})) + (\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^3 - (\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2\hat{x}^{2\dot{2}} -\\
&\quad 2\theta(\hat{x}^{1\dot{1}}(\hat{x}^{1\dot{2}})^2 - \hat{x}^{1\dot{1}}(\hat{x}^{2\dot{2}})^2 + (\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} + (2\mathrm{i})\hat{x}^{2\dot{2}}(\hat{x}^{1\dot{1}} - \hat{x}^{2\dot{2}}) - 2\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}}) -\\
&\quad \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2 + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^2 + \Lambda^2(\hat{x}^{1\dot{1}}(\hat{x}^{2\dot{2}})^2 + (\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} + \hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}} - \hat{x}^{2\dot{1}}\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}})\\
\hat{G}^{\dot{2}\dot{2}2} &= (\mathrm{i})(\theta(-8\theta\hat{x}^{2\dot{2}} + \Lambda^2(\hat{x}^{2\dot{1}} + \hat{x}^{2\dot{2}})) + (\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^3 + (\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2\hat{x}^{2\dot{2}} + 2\theta(\hat{x}^{1\dot{1}}(\hat{x}^{1\dot{2}})^2 +\\
&\quad \hat{x}^{1\dot{1}}(\hat{x}^{2\dot{2}})^2 + (\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} - (2\mathrm{i})\hat{x}^{2\dot{2}}(\hat{x}^{1\dot{1}} - \hat{x}^{2\dot{2}}) + 2\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}}) +\\
&\quad \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{1\dot{2}})^2 + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2(\hat{x}^{2\dot{2}})^2 + \Lambda^2(\hat{x}^{1\dot{1}}(\hat{x}^{2\dot{2}})^2 - (\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}} + \hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{2\dot{2}} - \hat{x}^{2\dot{1}}\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}}))\\
\hat{G}^{\dot{2}\dot{2}3} &= \Lambda^4(\hat{x}^{1\dot{1}} - \hat{x}^{1\dot{2}}) - 4\theta^2\hat{x}^{1\dot{2}} + \Lambda^2\theta(\hat{x}^{1\dot{1}} + \hat{x}^{1\dot{2}}) + (\Lambda^2-2\theta)(\hat{x}^{1\dot{1}})^2\hat{x}^{2\dot{2}} + \Lambda^2\hat{x}^{2\dot{1}}(\hat{x}^{1\dot{2}})^2 -\\
&\quad (\Lambda^2+8\theta)\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}} - (\Lambda^2+4\theta)\hat{x}^{2\dot{1}}\hat{x}^{1\dot{1}}\hat{x}^{1\dot{2}} - 2((\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}}(\hat{x}^{2\dot{2}})^2 + \hat{x}^{2\dot{1}}(\hat{x}^{1\dot{1}})^2\hat{x}^{1\dot{2}}\hat{x}^{2\dot{2}})\ .
\end{aligned}
$$

§14 Reality conditions for $\hat{G}^{\dot{\alpha}\dot{\beta}}$. Solutions constructed from the two algorithms we discussed do not necessarily satisfy any reality condition. Imposing a reality condition a priori which would guarantee a real field $\hat{G}^{\dot{\alpha}\dot{\beta}}$ as an outcome of the constructions always complicates the calculations. Instead, it is possible to adjust the fields after all calculations are performed.

The easiest way to obtain the conditions which one should impose on $\hat{G}^{\dot{\alpha}\dot{\beta}}$ is the observation that this field appears in the form $\hat{G}^{\dot{\alpha}\dot{\beta}}\hat{f}_{\dot{\alpha}\dot{\beta}}$ in the Lagrangian. Thus the reality condition on $\hat{G}^{\dot{\alpha}\dot{\beta}}$ has to be the same as for $\hat{f}_{\dot{\alpha}\dot{\beta}}$ to ensure that the Lagrangian is either purely real or purely imaginary. From $\hat{A}_{1\dot{1}} = -\hat{A}^{\dagger}_{2\dot{2}}$ and $\hat{A}_{1\dot{2}} = \hat{A}^{\dagger}_{2\dot{1}}$ one concludes that $\hat{f}_{\dot{1}\dot{1}} = -\hat{f}^{\dagger}_{\dot{2}\dot{2}}$ and $\hat{f}_{\dot{1}\dot{2}} = \hat{f}^{\dagger}_{\dot{2}\dot{1}}$. Therefore, we impose the following conditions:

$$\hat{G}^{\dot{1}\dot{1}} \;=\; -\left(\hat{G}^{\dot{2}\dot{2}}\right)^{\dagger} \quad\text{and}\quad \hat{G}^{\dot{1}\dot{2}} \;=\; \left(\hat{G}^{\dot{2}\dot{1}}\right)^{\dagger}\ , \tag{VIII.106}$$

or, extracting an anti-Hermitian generator σ^a of the gauge group:

$$\hat{G}^{\dot{1}\dot{1}a} \;=\; \left(\hat{G}^{\dot{2}\dot{2}a}\right)^{\dagger} \quad\text{and}\quad \hat{G}^{\dot{1}\dot{2}a} \;=\; -\left(\hat{G}^{\dot{2}\dot{1}a}\right)^{\dagger}\ . \tag{VIII.107}$$

Given a complex solution $\hat{G}^{\dot{\alpha}\dot{\beta}}$, we can now construct a real solution $\hat{G}^{\dot{\alpha}\dot{\beta}}_r$ by

$$
\begin{aligned}
\hat{G}^{\dot{1}\dot{1}a}_r &:= \tfrac{1}{2}\left(\hat{G}^{\dot{1}\dot{1}a} + (\hat{G}^{\dot{2}\dot{2}a})^{\dagger}\right)\ ,\quad \hat{G}^{\dot{1}\dot{2}a}_r := \tfrac{1}{2}\left(\hat{G}^{\dot{1}\dot{2}a} - (\hat{G}^{\dot{1}\dot{2}a})^{\dagger}\right)\ ,\\
\hat{G}^{\dot{2}\dot{2}a}_r &:= \tfrac{1}{2}\left(\hat{G}^{\dot{2}\dot{2}a} + (\hat{G}^{\dot{1}\dot{1}a})^{\dagger}\right)\ ,
\end{aligned} \tag{VIII.108}
$$

which satisfies by construction the equations of motion, as one easily checks by plugging these linear combinations into the equations of motion.

VIII.3.3 String theory perspective

§15 Holomorphically embedded submanifolds and their normal bundles. Recall that the equations

$$z^{\alpha}_{\pm} \;=\; x^{\alpha\dot{\alpha}}\lambda^{\pm}_{\dot{\alpha}} \quad\text{and}\quad \eta^{\pm}_{i} \;=\; \eta^{\dot{\alpha}}_{i}\lambda^{\pm}_{\dot{\alpha}} \tag{VIII.109}$$

describe a holomorphic embedding of the space $\mathbb{C}P^1$ into the supertwistor space $\mathcal{P}^{3|4}$. That is, for fixed moduli $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$, equations (VIII.109) yield a projective line $\mathbb{C}P^1_{x,\eta}$ inside the supertwistor space. The normal bundle to any $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{P}^{3|4}$ is $\mathcal{N}^{2|4}=\mathbb{C}^2\otimes\mathcal{O}(1)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1)$ and we have

$$h^0(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4}) \;=\; \dim_{\mathbb{C}} H^0(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4}) \;=\; 4|8\ . \tag{VIII.110}$$

Furthermore, there are no obstructions to the deformation of the $\mathbb{C}P^{1|0}_{x,\eta}$ inside $\mathcal{P}^{3|4}$ since $h^1(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4})=0|0$.

On the other hand, one can fix only the even moduli $x^{\alpha\dot\alpha}$ and consider a holomorphic embedding $\mathbb{C}P^{1|4}_x\hookrightarrow\mathcal{P}^{3|4}$ defined by the equations

$$z^\alpha_\pm \;=\; x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha}\ . \tag{VIII.111}$$

Recall that the normal bundle to $\mathbb{C}P^{1|0}_x\hookrightarrow\mathcal{P}^{3|0}$ is the rank two vector bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$. In the supercase, the formal definition of the normal bundle by the short exact sequence

$$0\;\to\; T\mathbb{C}P^{1|4}\;\to\; T\mathcal{P}^{3|4}|_{\mathbb{C}P^{1|4}}\;\to\;\mathcal{N}^{2|0}\;\to\;0 \tag{VIII.112}$$

yields that $\mathcal{N}^{2|0}=T\mathcal{P}^{3|4}|_{\mathbb{C}P^{1|4}}/T\mathbb{C}P^{1|4}$ is a rank two holomorphic vector bundle over $\mathbb{C}P^{1|4}$ which is (in the real case) locally spanned by the vector fields $\gamma_\pm V^\pm_\alpha$, where $V^\pm_\alpha$ is the complex conjugate of $\bar V^\pm_\alpha$. A global section of $\mathcal{N}^{2|0}$ over $\mathcal{U}_\pm\cap\mathbb{C}P^{1|4}$ is of the form $s_\pm=T^\alpha_\pm\gamma_\pm V^\pm_\alpha$. Obviously, the transformation of the components $T^\alpha_\pm$ from patch to patch is given by $T^\alpha_+=\lambda_+T^\alpha_-$, i.e. $\mathcal{N}^{2|0}=\mathcal{O}(1)\oplus\mathcal{O}(1)$.

§16 Topological D-branes and the matrix models. The interpretation of the matrix model (VIII.52) is now rather straightforward. For gauge group $\mathsf{GL}(n,\mathbb{C})$, it describes a stack of n almost space-filling D(1|4)-branes, whose fermionic dimensions only extend in the holomorphic directions of the target space $\mathcal{P}^{3|4}_\varepsilon$. These D-branes furthermore wrap a $\mathbb{C}P^{1|4}_x\hookrightarrow\mathcal{P}^{3|4}_\varepsilon$.

We can use the expansion $\mathcal{X}_\alpha=\mathcal{X}^0_\alpha+\mathcal{X}^i_\alpha\eta_i+\mathcal{X}^{ij}_\alpha\eta_i\eta_j+\dots$ on any patch of $\mathbb{C}P^{1|4}$ to examine the equations of motion (VII.204a) more closely:

$$\begin{aligned}
[\mathcal{X}^0_1,\mathcal{X}^0_2] \;&=\; 0\ ,\\
[\mathcal{X}^i_1,\mathcal{X}^0_2]+[\mathcal{X}^0_1,\mathcal{X}^i_2] \;&=\; 0\ ,\\
\{\mathcal{X}^i_1,\mathcal{X}^j_2\}-\{\mathcal{X}^j_1,\mathcal{X}^i_2\}+[\mathcal{X}^{ij}_1,\mathcal{X}^0_2]+[\mathcal{X}^0_1,\mathcal{X}^{ij}_2] \;&=\; 0\ ,\\
&\dots
\end{aligned} \tag{VIII.113}$$

Clearly, the bodies $\mathcal{X}^0_\alpha$ of the Higgs fields can be diagonalized simultaneously, and the diagonal entries describe the position of the D(1|4)-brane in the normal directions of the ambient space $\mathcal{P}^{3|4}_\varepsilon$. In the fermionic directions, this commutation condition is relaxed and thus, the D-branes can be smeared out in these directions even in the classical case.

§17 Interpretation within $\mathcal{N}=2$ string theory. Recall from section V.2.5 that the critical $\mathcal{N}=2$ string has a four-dimensional target space and its open string effective field theory is self-dual Yang-Mills theory (or its noncommutative deformation [175] in the presence of a B-field). It has been argued [260] that, after extending the $\mathcal{N}=2$ string effective action in a natural way to recover Lorentz invariance, the effective field theory becomes the full $\mathcal{N}=4$ supersymmetrically extended SDYM theory, and we will adopt this point of view in the following.

D-branes within critical $\mathcal{N} = 2$ string theory were already discussed in V.4.5, §15; recall from there that the low-energy effective action on such a D-brane is SDYM theory dimensionally reduced to its worldvolume. Thus, we have a first interpretation of our matrix model (VIII.40) in terms of a stack of n D0- or D(0|8)-branes in $\mathcal{N} = 2$ string theory, and the topological D(1|4)-brane is the equivalent configuration in B-type topological string theory.

As usual, turning on a B-field background will give rise to noncommutative deformations of the ambient space, and therefore the matrix model (VIII.73) describes a stack of n D4-branes in $\mathcal{N} = 2$ string theory within such a background.

The moduli superspaces $\mathbb{R}^{4|8}_\theta$ and $\mathbb{R}^{0|8}$ for both the noncommutative and the ordinary matrix model can therefore be seen as *chiral* D(4|8)- and D(0|8)-branes, respectively, with $\mathcal{N} = 4$ self-dual Yang-Mills theory as the appropriate (chiral) low energy effective field theory.

VIII.3.4 SDYM matrix model and super ADHM construction

§18 The pure matrix model. While a solution to the $\mathcal{N} = 4$ SDYM equations with gauge group $\mathsf{U}(n)$ and second Chern number $c_2 = k$ describes a bound state of k D(-1|8)-branes with n D(3|8)-branes at low energies, the SDYM matrix model obtained by a dimensional reduction of this situation describes a bound state between $k + n$ D(-1|8)-branes. This implies that there is only one type of strings, i.e. those having both ends on the D(-1|8)-branes. In the ADHM construction, one can simply account for this fact by eliminating the field content which arose previously from the open strings having one endpoint on a D(-1|8)-brane and the other one on a D(3|8)-brane. That is, we put $w_{uq\dot{\alpha}}$ and ψ^i to zero.

In fact, the remaining ADHM constraints read

$$\vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{A}^{\alpha\dot{\beta}}A_{\alpha\dot{\alpha}}) \;=\; 0 \;, \tag{VIII.114}$$

and one can use the reality conditions together with the definition of the ordinary sigma matrices to show that these equations are equivalent to

$$\varepsilon^{\alpha\beta}[A_{\alpha\dot{\alpha}}, A_{\beta\dot{\beta}}] \;=\; 0 \;, \tag{VIII.115}$$

which are simply the matrix-SDYM equations (VIII.41) with fields with more than one R-symmetry index put to zero. The missing fermionic equations follow from this equation via the expansion (VII.390), the latter being determined by the expansion of superfields for both the full and the self-dual super Yang-Mills theories. Thus, we recovered the equations of motion of the matrix model (VIII.40) in the ADHM construction as expected from the interpretation via D-branes.

§19 Extension of the matrix model. It is now conceivable that the D3-D(-1)-brane[16] system explaining the ADHM construction can be carried over to the supertwistor space $\mathcal{P}^{3|4}$. That is we take a D1-D5-brane system and analyze it either via open D5-D5 strings with excitations corresponding in the holomorphic Chern-Simons theory to gauge configurations with non-trivial second Chern character or by looking at the D1-D1 and the D1-D5 strings. The latter point of view will give rise to a holomorphic Chern-Simons analogue of the ADHM configuration, as we will show in the following.

[16]For simplicity, let us suppress the fermionic dimensions of the D-branes in the following.

The action for the D1-D1 strings is evidently our hCS matrix model (VIII.52). To incorporate the D1-D5 strings, we can use an action proposed by Witten in [297][17]

$$\int \mathrm{d}\lambda\Omega^\eta \;\; \mathrm{tr}\,(\beta\bar{\partial}\alpha + \beta\mathcal{A}^{0,1}_\Sigma\alpha)\;, \tag{VIII.116}$$

where the fields α and β take values in the line bundles $\mathcal{O}(1)$ and they transform in the fundamental and antifundamental representation of the gauge group $\mathsf{GL}(n,\mathbb{C})$, respectively.

The equations of motion of the total matrix model which is the sum of (VIII.52) and (VIII.116) are then modified to

$$\begin{aligned} \bar{\partial}\mathcal{X}_\alpha + [\mathcal{A}^{0,1}_\Sigma, \mathcal{X}_\alpha] \;&=\; 0\;, \\ [\mathcal{X}_1, \mathcal{X}_2] + \alpha\beta \;&=\; 0\;, \\ \bar{\partial}\alpha + \mathcal{A}^{0,1}_\Sigma\alpha \;=\; 0 \quad \text{and} \quad \bar{\partial}\beta + \beta\mathcal{A}^{0,1}_\Sigma \;&=\; 0\;. \end{aligned} \tag{VIII.117}$$

Similarly to the Higgs fields $\mathcal{X}_\alpha$ and the gauge potential $\mathcal{A}^{0,1}_\Sigma$, we can give a general field expansion for β and $\alpha = \bar{\beta}$:

$$\begin{aligned} \beta_+ \;&=\; \lambda^{\dot\alpha}_+ w_{\dot\alpha} + \psi^i\eta^+_i + \gamma_+\tfrac{1}{2!}\eta^+_i\eta^+_j\hat{\lambda}^{\dot\alpha}_+\rho^{ij}_{\dot\alpha} + \gamma^2_+\tfrac{1}{3!}\eta^+_i\eta^+_j\eta^+_k\hat{\lambda}^{\dot\alpha}_+\hat{\lambda}^{\dot\beta}_+\sigma^{ijk}_{\dot\alpha\dot\beta} \\ &\quad + \gamma^3_+\tfrac{1}{4!}\eta^+_i\eta^+_j\eta^+_k\eta^+_l\hat{\lambda}^{\dot\alpha}_+\hat{\lambda}^{\dot\beta}_+\hat{\lambda}^{\dot\gamma}_+\tau^{ijkl}_{\dot\alpha\dot\beta\dot\gamma}\;, \\ \alpha_+ \;&=\; \lambda^{\dot\alpha}_+\varepsilon_{\dot\alpha\dot\beta}\bar{w}^{\dot\beta}_+ + \bar{\psi}^i\eta^+_i + \dots\;. \end{aligned} \tag{VIII.118}$$

Applying the equations of motion, one learns that the fields beyond linear order in the Graßmann variables are composite fields:

$$\rho^{ij}_{\dot\alpha} \;=\; w_{\dot\alpha}\phi^{ij}\;, \quad \sigma^{ijk}_{\dot\alpha\dot\beta} \;=\; \tfrac{1}{2}w_{(\dot\alpha}\tilde{\chi}^{ijk}_{\dot\beta)} \quad \text{and} \quad \tau^{ijkl}_{\dot\alpha\dot\beta\dot\gamma} \;=\; \tfrac{1}{3}w_{(\dot\alpha}G^{ijkl}_{\dot\beta\dot\gamma)}\;. \tag{VIII.119}$$

We intentionally denoted the zeroth order components of α and β by $\lambda_{\dot\alpha}\bar{w}^{\dot\alpha}$ and $\lambda^{\dot\alpha}w_{\dot\alpha}$, respectively, since this expansion together with the field equations (VIII.117) are indeed the (super) ADHM equations

$$\vec{\sigma}^{\dot\alpha}{}_{\dot\beta}(\bar{w}^{\dot\beta}w_{\dot\alpha} + A^{\alpha\dot\alpha}A_{\alpha\dot\beta}) \;=\; 0\;, \tag{VIII.120}$$

which are equivalent to the condition that $\bar{\Delta}\Delta = \mathbb{1}_2 \otimes f^{-1}$. Recall, however, that for superfields with components beyond linear order in the Graßmann fields, the super ADHM equations do *not* yield solutions to the supersymmetric self-dual Yang-Mills equations. Therefore, we additionally have to put these fields to zero in the Higgs fields $\mathcal{X}_\alpha$ and the gauge potential $\mathcal{A}^{0,1}_\Sigma$ (which automatically does the same for the fields α and β).

This procedure seems at first slightly ad-hoc, but again it becomes quite natural, when recalling that for the ADHM D-brane configuration, supersymmetry is broken from $\mathcal{N} = 4$ to four times $\mathcal{N} = 1$. Furthermore, the fields which are put to zero give rise to the potential terms in the action, and thus, we can regard putting these fields to zero as an additional "D-flatness condition" arising on the topological string side.

With this additional constraint, our matrix model (VIII.52) together with the extension (VIII.116) is equivalent to the ADHM equations and therefore it is in the same sense

[17] In fact, he uses this action to complement the hCS theory in such a way that it will give rise to full Yang-Mills theory on the moduli space. For this, he changes the parity of the fields α and β to be fermionic.

dual to holomorphic Chern-Simons theory on the full supertwistor space $\mathcal{P}^{3|4}$, in which the ADHM construction is dual to SDYM theory.

Summarizing, the D3-D(-1)-brane system can be mapped via an extended Penrose-Ward-transform to a D5-D1-brane system in topological string theory. The arising super SDYM theory on the D3-brane corresponds to hCS theory on the D5-brane, while the matrix model describing the effective action on the D(-1)-brane corresponds to our hCS matrix model on a topological D1-brane. The additional D3-D(-1) strings completing the picture from the perspective of the D(-1)-brane can be directly translated into additional D5-D1 strings on the topological side. The ADHM equations can furthermore be obtained from an extension of the hCS matrix model on the topological D1-brane with a restriction on the field content.

§20 D-branes in a nontrivial B-field background. Except for the remarks on the $\mathcal{N}=2$ string, we have not yet discussed the matrix model which we obtained from deforming the moduli space $\mathbb{R}^{4|8}$ to a noncommutative spacetime.

In general, noncommutativity is interpreted as the presence of a Kalb-Ramond B-field background in string theory. Thus, solutions to the noncommutative SDYM theory (VIII.67) on $\mathbb{R}^{4|8}_\theta$ are D(-1|8)-branes bound to a stack of space-filling D(3|8)-branes in the presence of a B-field background. This distinguishes the commutative from the noncommutative matrix model: The noncommutative matrix model is now dual to the ADHM equations, instead of being embedded like the commutative one.

The matrix model on holomorphic Chern-Simons theory describes analogously a topological almost space-filling D(5|4)-brane in the background of a B-field. Note that a noncommutative deformation of the target space $\mathcal{P}^{3|4}_\varepsilon$ does not yield any inconsistencies in the context of the topological B-model. Such deformations have been studied e.g. in [150] and [138], see also [166].

On the one hand, we found two pairs of matrix models, which are dual to each other (as the ADHM equations are dual to the SDYM equations). On the other hand, we expect both pairs to be directly equivalent to one another in a certain limit, in which the rank of the gauge group of the commutative matrix model tends to infinity. The implications of this observation might reveal some further interesting features.

VIII.3.5 Dimensional reductions related to the Nahm equations

After the discussion of the ADHM construction in the previous section, one is led to try to also translate the D-brane interpretation of the Nahm construction to some topological B-model on a Calabi-Yau supermanifold. This is in fact possible, but since the D-brane configuration is somewhat more involved, we will refrain from presenting many details. In the subsequent discussion, we strongly rely on results from [229] presented in section VII.6, where further details complementing our rather condensed presentation can be found. In this section, we will constrain our considerations to real structures yielding Euclidean signature, i.e. $\varepsilon=-1$.

§21 The superspaces $\mathcal{Q}^{3|4}$ and $\hat{\mathcal{Q}}^{3|4}$. We want to consider a holomorphic Chern-Simons theory which describes magnetic monopoles and their superextensions. For this, we start from the holomorphic vector bundle

$$\mathcal{Q}^{3|4} \;=\; \mathcal{O}(2)\oplus\mathcal{O}(0)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1) \qquad \text{(VIII.121)}$$

of rank 2|4 over the Riemann sphere $\mathbb{C}P^1$. This bundle is covered by two patches $\tilde{\mathcal{V}}_\pm$ on which we have the coordinates $\lambda_\pm=w_2^\pm$ on the base space and $w_1^\pm, w_3^\pm$ in the bosonic fibres. On the overlap $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, we have thus[18]

$$w^1_+ \;=\; (w^2_+)^2 w^1_- \;,\quad w^2_+ \;=\; \frac{1}{w^2_-}\;,\quad w^3_+ \;=\; w^3_- \;. \tag{VIII.122}$$

The coordinates on the fermionic fibres of $\mathcal{Q}^{3|4}$ are the same as the ones on $\mathcal{P}^{3|4}$, i.e. we have $\eta_i^\pm$ with $i = 1,\dots 4$, satisfying $\eta_i^+ = \lambda_+\eta_i^-$ on $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$. From the Chern classes of the involved line bundles, we clearly see that $\mathcal{Q}^{3|4}$ is a Calabi-Yau supermanifold.

Note that holomorphic sections of the vector bundle $\mathcal{Q}^{3|4}$ are parameterized by elements $(y^{(\dot\alpha\dot\beta)}, y^4, \eta_i^{\dot\alpha})$ of the moduli space $\mathbb{C}^{4|8}$ according to

$$w^1_\pm \;=\; y^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\alpha}\lambda^\pm_{\dot\beta}\;,\quad w^3_\pm \;=\; y^4\;,\quad \eta^\pm_i \;=\; \eta^{\dot\alpha}_i\lambda^\pm_{\dot\alpha} \quad\text{with}\quad \lambda_\pm \;=\; w^2_\pm\;. \tag{VIII.123}$$

Let us now deform and restrict the sections of $\mathcal{Q}^{3|4}$ by identifying the modulus y^4 with $-\gamma_\pm\lambda^\pm_{\dot\alpha}\hat\lambda^\pm_{\dot\beta}y^{\dot\alpha\dot\beta}$, where the coordinates $\hat\lambda_{\dot\alpha}$ were defined in (VII.51). We still have $w^3_+ = w^3_-$ on the overlap $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, but w^3 no longer describes a section of a holomorphic line bundle. It is rather a section of a smooth line bundle, which we denote by $\hat{\mathcal{O}}(0)$. This deformation moreover reduces the moduli space from $\mathbb{C}^{4|8}$ to $\mathbb{C}^{3|8}$. We will denote the resulting total bundle by $\hat{\mathcal{Q}}^{3|4}$.

§22 Field theories and dimensional reductions. First, we impose a reality condition on $\hat{\mathcal{Q}}^{3|4}$ which is (for the bosonic coordinates) given by

$$\tau(w^1_\pm, w^2_\pm) \;=\; \left(-\frac{\bar w^1_\pm}{(\bar w^2_\pm)^2}, -\frac{1}{\bar w^2_\pm}\right) \quad\text{and}\quad \tau(w^3_\pm) \;=\; \bar w^3_\pm\;, \tag{VIII.124}$$

cf. (VII.271), and keep as usual the coordinate $w^2_\pm$ on the base $\mathbb{C}P^1$ complex. Then $w^1_\pm$ remains complex, but $w^3_\pm$ becomes real. In the identification with the real moduli $(x^1, x^3, x^4) \in \mathbb{R}^3$, we find that

$$y^{\dot1\dot1} \;=\; -(x^3 + \mathrm{i}x^4) \;=\; -\bar y^{\dot2\dot2} \quad\text{and}\quad w^3_\pm \;=\; x^1 \;=\; -y^{\dot1\dot2}\;. \tag{VIII.125}$$

Thus, the space $\hat{\mathcal{Q}}^{3|4}$ reduces to a Cauchy-Riemann (CR) manifold[19], which we label by $\hat{\mathcal{Q}}^{3|4}_{-1} = \mathcal{K}^{5|8}$. This space has been extensively studied in [229], and it was found there that a partial holomorphic Chern-Simons theory obtained from a certain natural integrable distribution on $\mathcal{K}^{5|8}$ is equivalent to the supersymmetric Bogomolny model on $\mathbb{R}^3$. Furthermore, it is evident that the complexification of this partial holomorphic Chern-Simons theory is holomorphic Chern-Simons theory on our space $\hat{\mathcal{Q}}^{3|4}$. This theory describes holomorphic structures $\bar\partial_{\mathcal{A}}$ on a vector bundle $\mathcal{E}$ over $\hat{\mathcal{Q}}^{3|4}$, i.e. a gauge potential $\mathcal{A}^{0,1}$ satisfying $\bar\partial\mathcal{A}^{0,1} + \mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1} = 0$.

There are now three possibilities for (bosonic) dimensional reductions

$$\hat{\mathcal{Q}}^{3|4} \;=\; \mathcal{O}(2)\oplus\mathcal{O}(0)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1) \;\to\; \left\{\begin{array}{lcl} \mathcal{P}^{2|4} &:=& \mathcal{O}(2)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1)\\ \hat{\mathcal{Q}}^{2|4} &:=& \mathcal{O}(0)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1)\\ \mathbb{C}P^{1|4} &:=& \mathbb{C}^4\otimes\Pi\mathcal{O}(1)\end{array}\right. , \tag{VIII.126}$$

which we want to discuss in the following.

The dimensional reduction of the holomorphic Chern-Simons theory to the space $\mathcal{P}^{2|4}$ has been studied in [229]. It yields a holomorphic BF-theory, see section IV.3.3, §12,

[18] The labelling of coordinates is chosen to become as consistent as possible with [229].

[19] Roughly speaking, a CR manifold is a complex manifold with additional real directions.

where the scalar B-field originates as the component $\frac{\partial}{\partial \bar{w}^3_\pm} \lrcorner \mathcal{A}^{0,1}$ of the gauge potential $\mathcal{A}^{0,1}$ on $\mathcal{E} \to \hat{\mathcal{Q}}^{3|4}$. This theory is also equivalent to the above-mentioned super Bogomolny model on $\mathbb{R}^3$. It is furthermore the effective theory on a topological D3-brane and – via a Penrose-Ward transform – can be mapped to static BPS gauge configurations on a stack of D3-branes in type IIB superstring theory. These gauge configurations have been shown to amount to BPS D1-branes being suspended between the D3-branes and extending in their normal directions. Therefore, the holomorphic BF-theory is the topological analogue of the D3-brane point of view of the D3-D1-brane system.

From the above discussion, the field theory arising from the reduction to $\hat{\mathcal{Q}}^{2|4}$ is also evident. Note that considering this space is equivalent to considering $\hat{\mathcal{Q}}^{3|4}$ with the additional restriction $y^{\dot{1}\dot{1}} = y^{\dot{2}\dot{2}} = 0$. Therefore, we reduced the super Bogomolny model from $\mathbb{R}^3$ to $\mathbb{R}^1$, and we arrive at a (partially) holomorphic BF-theory, which is equivalent to self-dual Yang-Mills theory in one dimension. Since this theory yields precisely the gauge-covariant Nahm equations, we conclude that this is the D1-brane point of view of the D3-D1-brane system.

The last reduction proposed above is the one to $\mathbb{C}P^{1|4}$. This amounts to a reduction of the super Bogomolny model from $\mathbb{R}^3$ to a point, i.e. SDYM theory in zero dimensions. Thus, we arrive again at the matrix models (VIII.52) and (VIII.40) discussed previously. It is interesting to note that the matrix model cannot tell whether it originated from the space $\mathcal{P}^{3|4}$ or $\hat{\mathcal{Q}}^{3|4}$.

§23 The Nahm construction from topological D-branes. In the previous paragraph, we saw that both the physical D3-branes and the physical D1-branes correspond to topological D3-branes wrapping either the space $\mathcal{P}^{2|4} \subset \hat{\mathcal{Q}}^{3|4}$ or $\hat{\mathcal{Q}}^{2|4} \subset \hat{\mathcal{Q}}^{3|4}$. The bound system of D3-D1-branes therefore corresponds to a bound system of D3-D3-branes in the topological picture. The two D3-branes are separated by the same distance[20] as the physical ones in the normal direction $\mathcal{N}_{\mathcal{P}^{2|4}} \cong \mathcal{O}(2)$ in $\hat{\mathcal{Q}}^{3|4}$. It is important to stress, however, that since supersymmetry is broken twice by the D1- and the D3-branes, in the topological picture, we have to put to zero all fields except for $(A_a, \Phi, \chi^i_{\dot{\alpha}})$.

It remains to clarify the rôle of the Nahm boundary conditions in detail. In [75], this was done by considering a D1-brane probe in a T-dualized configuration consisting of D7- and D5-branes. This picture evidently cannot be translated into twistor space. It would be interesting to see explicitly what the boundary conditions correspond to in the topological setup. Furthermore, it could be enlightening to study the topological analogue of the Myers effect, which creates a funnel at the point where the physical D1-branes end on the physical D3-branes. Particularly the core of this "bion" might reveal interesting features in the topological theory.

§24 Summary of D-brane equivalences. We gave an interpretation of the matrix models derived from holomorphic Chern-Simons theory in terms of D-brane configurations within B-type topological string theory. During this discussion, we established connections between topological branes and physical D-branes of type IIB superstring theory, whose worldvolume theory had been reduced by an additional BPS condition due to the presence of a further physical brane. Let us summarize the correspondences in the

[20] In our presentation of the Nahm construction, we chose this distance to be $1 - (-1) = 2$.

following table:

$$
\begin{aligned}
\text{D(5|4)-branes in } \mathcal{P}^{3|4} \;&\leftrightarrow\; \text{D(3|8)-branes in } \mathbb{R}^{4|8}\\
\text{D(3|4)-branes wrapping } \mathcal{P}^{2|4} \text{ in } \mathcal{P}^{3|4} \text{ or } \hat{\mathcal{Q}}^{3|4} \;&\leftrightarrow\; \text{static D(3|8)-branes in } \mathbb{R}^{4|8}\\
\text{D(3|4)-branes wrapping } \hat{\mathcal{Q}}^{2|4} \text{ in } \hat{\hat{\mathcal{Q}}}^{3|4} \;&\leftrightarrow\; \text{static D(1|8)-branes in } \mathbb{R}^{4|8}\\
\text{D(1|4)-branes in } \mathcal{P}_{\varepsilon}^{3|4} \;&\leftrightarrow\; \text{D(-1|8)-branes in } \mathbb{R}^{4|8}\;.
\end{aligned}
$$

It should be stressed that the fermionic parts of all the branes in $\mathcal{P}_{\varepsilon}^{3|4}$ and $\hat{\mathcal{Q}}^{3|4}$ only extend into holomorphic directions. It is straightforward to add to this list the diagonal line bundle $\mathcal{D}^{2|4}$, which is obtained from $\mathcal{P}^{3|4}$ by imposing the condition[21] $z_{\pm}^1 = z_{\pm}^2$ on the local sections

$$\text{D(3|4)-branes wrapping } \mathcal{D}^{2|4} \text{ in } \mathcal{P}^{3|4} \;\leftrightarrow\; \text{D(1|8)-branes in } \mathbb{R}^{4|8}\;.$$

[21]or an appropriate modification in the Euclidean case

Chapter IX

Conclusions and Open Problems

IX.1 Summary

Let us briefly summarize the results presented in this thesis, grouped according to the papers they were first published in.

§1 Non-anticommutative deformations of superspaces. One can define a non-anticommutative deformation of $\mathcal{N} = 4$ super Yang-Mills theory by using the corresponding constraint equations. A Seiberg-Witten map can be motivated in that context.

Using Drinfeld twist techniques, one can make manifest the twisted supersymmetry on non-anticommutative superspaces. This twisted supersymmetry can take over the rôle of ordinary supersymmetry in the definition of chiral rings, supersymmetric Ward-Takahashi identities and probably even non-renormalization theorems. These constructions based on twisted supersymmetry may prove to be very useful in explicit calculations within non-anticommutative field theories, similarly to their cousins in ordinary supersymmetric field theories.

§2 Twistor string theory. The concept of marrying twistor geometry and Calabi-Yau geometry using the supertwistor space $\mathbb{C}P^{3|4}$ looks very promising. One can carry over the whole Penrose-Ward transform to the case of supertwistors and describe solutions to the $\mathcal{N} = 4$ self-dual Yang-Mills equations in terms of solutions to the holomorphic Chern-Simons equations on $\mathbb{C}P^{3|4}$.

Fattened complex manifolds (or exotic supermanifolds) arising naturally from the supertwistor space $\mathbb{C}P^{3|4}$ can be used to describe certain bosonic subsectors of $\mathcal{N} = 4$ self-dual Yang-Mills theory. Furthermore, the concept of exotic Calabi-Yau supermanifolds fits nicely into the framework of ordinary Calabi-Yau supermanifolds

The mini-twistor space $\mathcal{O}(2)$ can be supersymmetrically extended to a Calabi-Yau supermanifold. The topological B-model with this space as its target space is equivalent to the supersymmetrically extended Bogomolny equations in three dimensions.

A corresponding mini-superambitwistor space can be defined, although this space is neither the total space of a vector bundle nor a manifold. Nevertheless, this space is still a fibration and has all the necessary features for a Penrose-Ward transform between certain generalized bundles over this fibration and solutions to the $\mathcal{N} = 8$ Yang-Mills-Higgs equations in three dimensions.

Also matrix models can be consistently defined via dimensional reduction of both the topological B-model on the supertwistor space $\mathbb{C}P^{3|4}$ and its equivalent $\mathcal{N} = 4$ self-dual Yang-Mills theory in four dimensions. One can interpret these matrix models in terms of topological D-branes, (bound states of) D-branes in type IIB superstring theory and D-branes within critical $\mathcal{N} = 2$ string theory. Furthermore, one can extend the matrix models obtained from the topological B-model to be equivalent to the ADHM or even the Nahm equations. This extension allows furthermore for establishing a Penrose-Ward transform between D-brane systems in type IIB on the one side and topological B-branes on the other side.

IX.2 Directions for future research

In the derivation of the above results, several questions were raised, which can be taken as starting point for quite interesting future research.

§1 Non-anticommutative deformations of superspaces. In the definition of non-anticommutative $\mathcal{N}=4$ super Yang-Mills theory, it would be clearly interesting to see whether this definition yields compatible results with the canonical definition of non-anticommutative field theories by inserting star-products into a superfield action. For this, one could either reduce the amount of supersymmetry, or restrict the deformation tensor such that it fits with the formulation of $\mathcal{N}=4$ super Yang-Mills theory in the language of $\mathcal{N}=1$ superspace.

Furthermore, it would be very interesting to substantiate the definition of a Seiberg-Witten map in the non-anticommutative setting. In [245], some arguments in favor of such a map were given; however, these arguments are clearly not sufficient.

It would also be illuminating to explore the connection of the constraint equations and the underlying linear system of partial differential equations. This system could subsequently be used to generalize solution generating techniques (as e.g. the dressing and splitting methods) available for the corresponding linear system in the undeformed case.

Within the framework of Drinfeld twists, clearly the study of twist-deformed superconformal invariance following the discussion of twisted conformal invariance in [196], could potentially yield further interesting results.

Moreover, our results on Drinfeld twists for non-anticommutative superspaces may prove valuable for introducing a non-anticommutative deformation of supergravity. Building upon the discussion presented in [10], one could try to construct a local version of the twisted supersymmetry.

Also, Seiberg's naturalness argument should be verified or at least be motivated stronger in the non-anticommutative setting to clarify the apparent inconsistency between the non-renormalization theorem conjectured in [136] and the further results for one-loop calculations in the literature.

§2 Twistor string theory. There remain essentially two open questions concerning the general supertwistor correspondence and its application within topological string theory. First, it would be desirable to find an appropriate action functional for holomorphic Chern-Simons theory on the superambitwistor space $\mathcal{L}^{5|6}$. Up to now, there have been two attempts in this direction [199, 193, 194], but a more direct construction would be desirable. Also, one could use the Penrose-Ward transform built upon the supertwistor correspondence to establish or strengthen the long-sought relation between the $\mathcal{N}=2$ and $\mathcal{N}=4$ topological strings.

The results on the supertwistor correspondence over exotic supermanifolds and in particular the results on the extension of Yau's theorem suggest that these space are quite natural to consider as target spaces within topological string theory. One could try to establish mirror symmetry conjectures between certain exotic Calabi-Yau supermanifolds to enrich the set of examples of such a conjecture.

Partially holomorphic Chern-Simons theory leads naturally to studying twistor correspondences for further geometries, as shown also in the section discussing matrix models for the Nahm equations. Other examples are certainly the Cauchy-Riemann manifolds and the deformations of the mini-supertwistor space corresponding to turning on mass terms in the Bogomolny equations. The most interesting question is certainly whether one can use partially holomorphic geometry in the context of topological M-theory [77].

The construction of the mini-superambitwistor space $\mathcal{L}^{4|6}$ leads to a number of interesting questions. First of all, one should find out, how to construct the topological B-model, which has this space as a target space and its holomorphic Chern-Simons-type equivalent theory. Second, one should substantiate the mirror conjecture between the mini-supertwistor space $\mathcal{P}^{2|4}$ and the mini-superambitwistor space $\mathcal{L}^{5|6}$, which naturally arises due to a similar conjecture between the supertwistor space $\mathcal{P}^{3|4}$ and the superambitwistor space $\mathcal{L}^{5|6}$. As a third point, one might try to find the analogous construction of an action for holomorphic Chern-Simons theory on the mini-superambitwistor space to the one proposed by Movshev [199]. This might shed more light on the relevance and usefulness of both the constructions of $\mathcal{L}^{4|6}$ and the action in [199].

In the area of matrix models within twistor string theory, one should first examine in more detail the topological D-brane configuration yielding the Nahm equations. In particular, it is desirable to obtain more results on the Myers effect and the core of the "bion" in the topological setting. Second, one could imagine to strengthen and extend the relations between D-branes in type IIB superstring theory and the topological D-branes in the B-model. In the latter theory, the strong framework of derived categories (see e.g. [11]) might then be carried over in some form to the full ten-dimensional string theory. Eventually, it might also be interesting to look at the mirror of the presented configurations within the topological A-model.

Appendices

A. Further definitions

In this appendix, we recall the notions of some elementary mathematical objects. It turned out that the web page of Wikipedia[1] is a surprisingly useful reference for looking up further mathematical definitions.

§1 Morphisms. Given two groupoids $G_1, +_1$ and $G_2, +_2$, a map $f : G_1 \to G_2$ is called a *homomorphism* if it satisfies

$$\forall a, b \in G_1 : f(a +_1 b) \ = \ f(a) +_2 f(b) \ . \tag{A.1}$$

A bijective, injective, surjective homomorphism is called *isomorphism*, *monomorphism*, *epimorphism*. If the groupoids are identical: $(G_1, +_1) = (G_2, +_2)$ then the map is an *automorphism*.

§2 Groups and representations. A *representation* of a group $\mathscr{G}$ is a homomorphism from $\mathscr{G}$ to the space of linear transformations on a vector space. A *faithful representation* is a representation which is furthermore isomorphic to $\mathscr{G}$. The *fundamental representation* of a group is its lowest dimensional faithful representation. The *trivial representation* of $\mathscr{G}$ maps the whole group to the identity map $\mathbb{1}$ on some vector space. A representation is *irreducible* if it cannot be decomposed into block diagonal form. All reducible representations can be built of the irreducible representations.

Given an element g of a group G, the *stabilizer subgroup* of g (also called the *isotropy group* or *little group*) is the subgroup G_g of G leaving g invariant:

$$G_g \ := \ \{x \in G | x \cdot g = g\} \ . \tag{A.2}$$

§3 Short exact sequence. A sequence of groups

$$\dots A_i \ \xrightarrow{f_i} \ A_{i+1} \ \xrightarrow{f_{i+1}} \ A_{i+2} \dots \tag{A.3}$$

is called *exact* if the image of f_i is equal to the kernel of f_{i+1}. A *short exact sequence* is an exact sequence of the form

$$0 \ \longrightarrow \ A \ \xrightarrow{f} \ B \ \xrightarrow{g} \ C \ \longrightarrow \ 0 \ . \tag{A.4}$$

It follows that f is a monomorphism and g is an epimorphism. Furthermore, if A, B, C are vector bundles over a manifold M then $B = C \oplus A$.

B. Conventions

§1 Metric conventions. Our Minkowski metric follows the "east coast convention", i.e. it is mostly $+$.

[1] `http://en.wikipedia.org`

§2 Interior product. For the interior product of a vector V with a one-form A, we use the notation $V \lrcorner A := \langle V, A\rangle$. A second common notation for this product is $\mathrm{i}_V A$.

§3 Dual space. Following Grothendieck, we denote the dual of a space X, i.e. the space of linear maps $X \to \mathbb{K}$, by $X^\vee$.

§4 Commutators and parity. We use square brackets $[\cdot]$ for the commutator, curly brackets $\{\cdot\}$ for the anticommutator and a combination of both $[\![\cdot]\!\}$ for the graded or supercommutator:

$$[A,B] := A.B - B.A\ ,\quad \{A,B\} := A.B + B.A\ ,\quad [\![A,B]\!\} := A.B - (-1)^{\tilde{A}\tilde{B}} B.A\ , \tag{B.1}$$

where $\tilde{A} \in \{0,1\}$ denotes the Graßmann parity of A and . denotes a product defined between A and B.

§5 Lie Algebra and gauge field conventions. Almost all the fields in this thesis live in the adjoint representation of a gauge group: $(T^a)_{bc} = (f^a)_{bc}$. For this, we fix the trace $\mathrm{tr}\,(T^a T^b) = -\delta^{ab}$ and choose the generators of the gauge group to be anti-Hermitian: $[T^a, T^b] = f^{ab}{}_c T^c$ (note that $f^a{}_{bc} = f^{ab}{}_c = f_{abc} = \ldots$). For the field strength, we use the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and the covariant derivative of a field in the adjoint representation is $D_\mu \psi = \partial_\mu \psi + [A_\mu, \psi]$. In terms of gauge components, this reads: $F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^a{}_{bc} A^b_\mu A^c_\nu$ and $(D_\mu \lambda)^a = \partial_\mu \lambda^a + f^a{}_{bc} A^b_\mu \lambda^b$.

§6 SUSY conventions and identities. We follow essentially the conventions of Wess and Bagger [287]. Indices are raised with the epsilon tensors according to $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$. We choose $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = -1$ which implies the relation $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta^\gamma_\alpha$, and a similar one for dotted indices. Short hand notations are: $\psi\chi = \psi^\alpha \chi_\alpha = \chi^\alpha \psi_\alpha = \chi\psi$ and $\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$. Derivatives with respect to Graßmann variables are defined in the following manner: $\overrightarrow{\partial}_\alpha \theta^\beta = \delta^\beta_\alpha$ and $\theta^\beta \overleftarrow{\partial}_\alpha = -\delta^\beta_\alpha$, together with the super Leibniz rule for differential operators $\partial(fg) = (\partial f)g + (-1)^{\tilde{\partial}\tilde{f}} f(\partial g)$, where $\tilde{a}$ denotes again the Graßmann parity of a, i.e. $\tilde{a} = 0$ for bosonic a and $\tilde{a} = 1$ for fermionic a.

C. Dictionary: homogeneous ↔ inhomogeneous coordinates

This appendix is a complement to the discussion of chapter VII and provides information on how to switch between homogeneous and inhomogeneous coordinates on twistor space.

§1 The Riemann sphere $\mathbb{C}P^1$. The sphere S^2 is diffeomorphic to the complex projective space $\mathbb{C}P^1$. Let us recall from section II.1.1, §3 that this space can be parametrized globally by complex homogeneous coordinates $\lambda_{\dot{1}}$ and $\lambda_{\dot{2}}$ which are not simultaneously zero (in projective spaces, the origin is excluded). Therefore the Riemann sphere $\mathbb{C}P^1$ can be covered by two coordinate patches

$$U_+ = \{[\lambda_{\dot{1}}, \lambda_{\dot{2}}] \,|\, \lambda_{\dot{1}} \neq 0\} \quad \text{and} \quad U_- = \{[\lambda_{\dot{1}}, \lambda_{\dot{2}}] \,|\, \lambda_{\dot{2}} \neq 0\}\ , \tag{C.1}$$

with coordinates

$$\lambda_+ := \frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}} \quad \text{on } U_+ \quad \text{and} \quad \lambda_- := \frac{\lambda_{\dot{1}}}{\lambda_{\dot{2}}} \quad \text{on } U_-\ . \tag{C.2}$$

On the intersection $U_+ \cap U_-$, we have $\lambda_+ = 1/\lambda_-$.

§2 Line bundles A global section of the holomorphic line bundle $\mathcal{O}(n)$ over $\mathbb{C}P^1$ exists only for $n \geq 0$. Over $U_\pm$, it is represented by a polynomial $p_\pm^{(n)}$ of degree n in the coordinates $\lambda_\pm$ with $p_+^{(n)} = \lambda_+^n p_-^{(n)}$ on $U_+ \cap U_-$. The explicit expansion will look like

$$\begin{aligned} p_+^{(n)} &= a_0 + a_1\lambda_+ + a_2\lambda_+^2 + \ldots + a_n\lambda_+^n \ , \\ p_-^{(n)} &= a_0\lambda_-^n + \ldots + a_{n-2}\lambda_-^2 + a_{n-1}\lambda_- + a_n \ , \end{aligned} \tag{C.3}$$

and, multiplying the expansion in λ_+ by $\lambda_{\dot 1}^n$ (or the expansion in λ_- by $\lambda_{\dot 2}^n$), one obtains a homogeneous polynomial of degree n:

$$a_0\lambda_{\dot 1}^n + a_1\lambda_{\dot 1}^{n-1}\lambda_{\dot 2} + \ldots + a_{n-1}\lambda_{\dot 1}\lambda_{\dot 2}^{n-1} + a_n\lambda_{\dot 2}^n \quad =: \quad Q^{\dot\alpha_1\ldots\dot\alpha_n}\lambda_{\dot\alpha_1}\ldots\lambda_{\dot\alpha_n} \ . \tag{C.4}$$

§3 Gauge potentials. Now let us consider the expansion (VII.205a) and (VII.205b) of the super gauge potentials of hCS theory on the supertwistor space. We get the following list of objects:

$$\begin{array}{ccl} \eta_i^+ & \mathcal{O}(1) & \eta_i = \lambda_{\dot 1}\eta_i^+ \\ \gamma_+ & \mathcal{O}(-1)\otimes\bar{\mathcal{O}}(-1) & \gamma = \dfrac{1}{\lambda_{\dot 1}\bar\lambda_{\dot 1}}\,\gamma_+ \left(= \dfrac{1}{\lambda^{\dot\alpha}\hat\lambda_{\dot\alpha}}\right) \\ \hat{\mathcal{A}}_\alpha^+ & \mathcal{O}(1) & \hat{\mathcal{A}}_\alpha = \lambda_{\dot 1}\ \hat{\mathcal{A}}_\alpha^+ \\ \hat{\mathcal{A}}_{\bar\lambda_+} & \bar{\mathcal{O}}(-2) & \hat{\mathcal{A}}_3 = \dfrac{1}{\bar\lambda_{\dot 1}\bar\lambda_{\dot 1}}\ \hat{\mathcal{A}}_{\bar\lambda_+} \ . \end{array}$$

This implies the following expansions in homogeneous coordinates:

$$\begin{aligned} \hat{\mathcal{A}}_\alpha &= \lambda^{\dot\alpha} A_{\alpha\dot\alpha}(x_R) + \eta_i\chi_\alpha^i(x_R) + \gamma\,\tfrac{1}{2!}\,\eta_i\eta_j\,\hat\lambda^{\dot\alpha}\,\phi_{\alpha\dot\alpha}^{ij}(x_R) + \\ &\quad + \gamma^2\,\tfrac{1}{3!}\,\eta_i\eta_j\eta_k\,\hat\lambda^{\dot\alpha}\,\hat\lambda^{\dot\beta}\,\tilde\chi_{\alpha\dot\alpha\dot\beta}^{ijk}(x_R) + \gamma^3\,\tfrac{1}{4!}\,\eta_i\eta_j\eta_k\eta_l\,\hat\lambda^{\dot\alpha}\,\hat\lambda^{\dot\beta}\,\hat\lambda^{\dot\gamma}\,G_{\alpha\dot\alpha\dot\beta\dot\gamma}^{ijkl}(x_R) \ , \end{aligned} \tag{C.5a}$$

$$\begin{aligned} \hat{\mathcal{A}}_3 &= \gamma^2\,\tfrac{1}{2!}\,\eta_i\eta_j\,\phi^{ij}(x_R) + \gamma^3\,\tfrac{1}{3!}\,\eta_i\eta_j\eta_k\,\hat\lambda^{\dot\alpha}\,\tilde\chi_{\dot\alpha}^{ijk}(x_R) + \\ &\quad + \gamma^4\,\tfrac{1}{4!}\,\eta_i\eta_j\eta_k\eta_l\,\hat\lambda^{\dot\alpha}\,\hat\lambda^{\dot\beta} G_{\dot\alpha\dot\beta}^{ijkl}(x_R) \ . \end{aligned} \tag{C.5b}$$

§4 Equations of motion. For rewriting the equations of motion in terms of this gauge potential, we also need to rewrite the vector fields (VII.174a) and $\bar V_3^\pm = \frac{\partial}{\partial\bar\lambda_\pm}$ in homogeneous coordinates. The vector fields along the fibres are easily rewritten, analogously to the corresponding components of the gauge potential. The vector field on the sphere can be calculated by considering $\hat{\mathcal{A}}_{\bar\lambda_+}\mathrm{d}\bar\lambda_+ = \hat{\mathcal{A}}_3\bar\Theta^3$. This implies $\bar\Theta^3 = \bar\lambda_{\dot 1}\mathrm{d}\bar\lambda_{\dot 2} - \bar\lambda_{\dot 2}\mathrm{d}\bar\lambda_{\dot 1}$, which has a dual vector field $\bar V_3$ defined by $\bar V_3 \lrcorner\, \bar\Theta^3 = 1$. Altogether, we obtain the basis

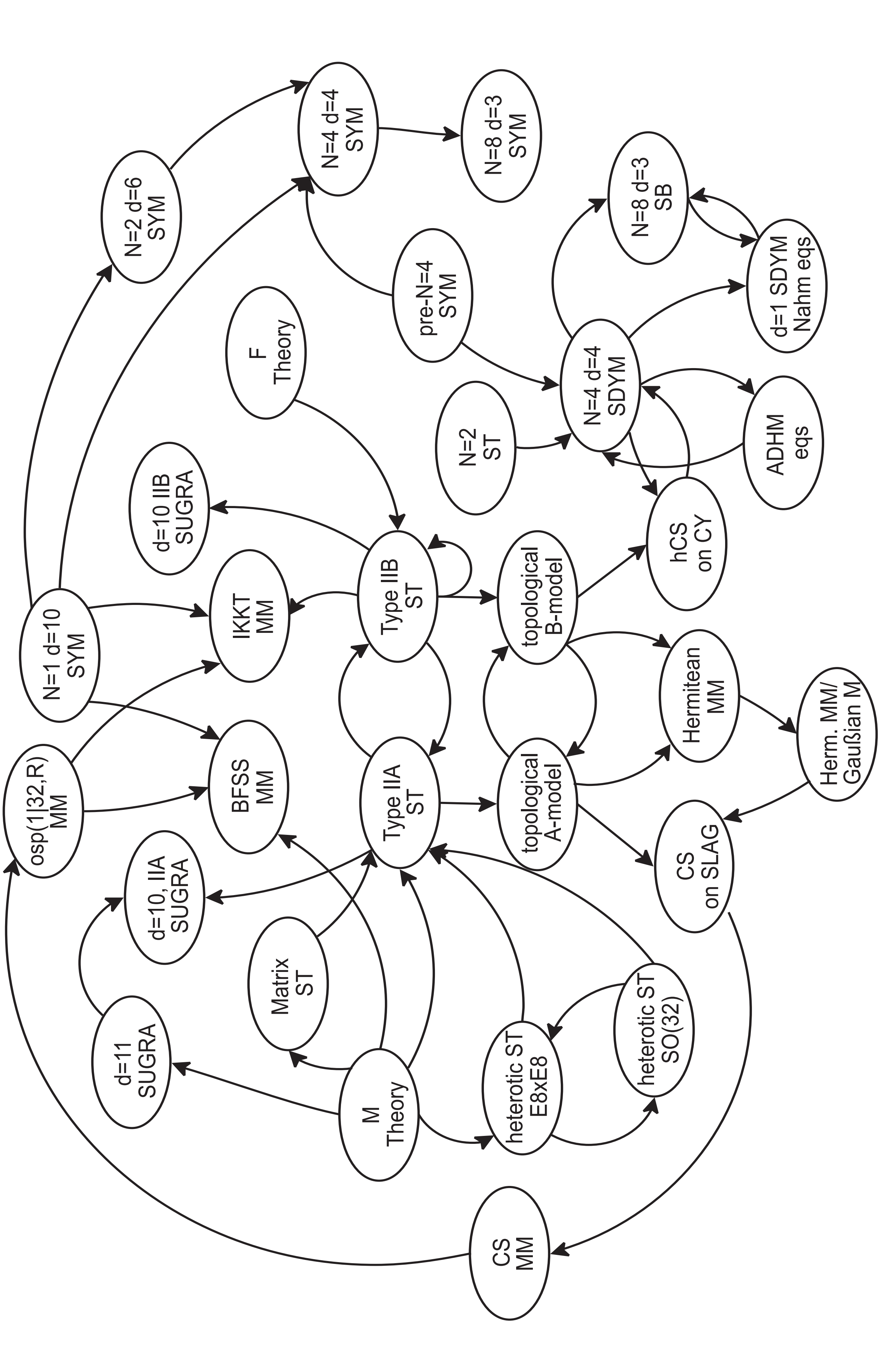
$$\bar V_\alpha \ = \ \lambda^{\dot\alpha}\frac{\partial}{\partial x_R^{\alpha\dot\alpha}} \quad \text{and} \quad \bar V_3 \ = \ -\gamma\lambda^{\dot\alpha}\frac{\partial}{\partial\hat\lambda^{\dot\alpha}} \ . \tag{C.6}$$

The field equations (VII.204a) and (VII.204b) now take the form

$$\begin{aligned} \bar V_\alpha\hat{\mathcal{A}}_\beta - \bar V_\beta\hat{\mathcal{A}}_\alpha + [\hat{\mathcal{A}}_\alpha, \hat{\mathcal{A}}_\beta] &= 0 \ , \\ \bar V_3\hat{\mathcal{A}}_\alpha - \bar V_\alpha\hat{\mathcal{A}}_3 + [\hat{\mathcal{A}}_3, \hat{\mathcal{A}}_\alpha] &= 0 \ , \end{aligned} \tag{C.7}$$

and yield the same equations (IV.62) for the physical fields.

D. Map to (a part of) "the jungle of TOE"



To put the different theories mentioned in this thesis into context, the opposite map to a small part of "the jungle of the theory of everything" might be helpful. The abbreviations used are the following:

CS	Chern-Simons theory	N	number of supersymmetries
CY	Calabi-Yau manifold	SB	super Bogomolny model
d	number of dimensions	SDYM	self-dual Yang-Mills theory
eqs	equations	SLAG	special Lagrangian manifold
Gaußian M	Gaußian measure	ST	string theory
hCS	holomorphic CS theory	SUGRA	supergravity
MM	matrix model	SYM	super Yang-Mills theory

Furthermore, let us briefly comment on some of the arrows in the map. Type IIA and type IIB superstring theory are linked by T-duality and the embedded topological models are related via mirror symmetry. The arrow from type IIB to itself is so-called S-duality. The connection between holomorphic Chern-Simons theory and self-dual Yang-Mills theory is the Penrose-Ward transform discussed extensively in this thesis. Most of the other links correspond to dimensional reductions or to taking certain weak coupling limits.

E. The quintic and the Robinson congruence

On the title page and on the back of this thesis there are two pictures representing the essential geometries encountered in this thesis: Calabi-Yau geometry and twistor geometry. The first is a cross-section of the quintic hypersurface, the second represents the Robinson congruence.

§1 The quintic. To plot a cross-section through the quintic with Mathematica, we first mod out the projective symmetry and arrive at

$$z^1 + z^2 + z^3 + z^4 + 1 \;=\; 0\;, \tag{E.1}$$

with $z^i \in \mathbb{C}$. Furthermore, we assume constant values for z^3 and z^4 and put them to zero. The remaining real four-dimensional object is then projected into three-dimensional space by adding the real and imaginary parts of $z^2 = u + \mathrm{i}v$ as a function of $z^1 = r + \mathrm{i}s$. The source code reads:

```
NSq[z_,n_]:=Abs[z]^(1/5)*Exp[I (Arg[z]+n*2*Pi)/5];

u[r_,s_,n_]:=2*Re[NSq[(1-(r+I s)^5),n]];
v[r_,s_,n_]:=2*Im[NSq[(1-(r+I s)^5),n]];

P=ParametricPlot3D[{{r,s,(u[r,s,1]+v[r,s,1])},
         {r,s,(u[r,s,2]+v[r,s,2])},{r,s,(u[r,s,3]+v[r,s,3])},
         {r,s,(u[r,s,4]+v[r,s,4])},{r,s,(u[r,s,5]+v[r,s,5])}},
         {r,-1,1},{s,-1,1}]
```

In the notebook, we first defined a function $NSq(z, n)$ for the nth of the five 5th roots of a complex number z. Then we defined $z^2(z^1)$ for the five roots and plot all of them in a single frame.

§2 The Robinson congruence. As discussed in section VII.1.1, a null twistor corresponds to a geodesic in Minkowski space. A non-null twistor Z^i, on the other hand, gives rise to a subspace of the dual twistor space $(\mathbb{C}P^3)^\vee$ via

$$S = \{W_i \in (\mathbb{C}P^3)^\vee | Z^i W_i \ = \ 0\} \ . \tag{E.2}$$

The intersection of this space with the space of null twistors $\mathbb{P}\mathbb{T}_N$ is a three-dimensional space, which parameterizes a spacetime filling family of geodesics in the compactified spacetime M. By taking a time slice and projecting the tangent vectors in M onto this time slice, we recover Penrose's picture of this Robinson congruence: A set of nested tori, with an axis in their middle as printed on the back of this thesis. The null axis corresponds to a null twistor and the time evolution is a movement of the whole configuration along this axis, while the twisted tangent vectors on the tori rotate. This observation originally gave rise to the name twistor. For more details on this picture and its relation to the Hopf fibration, see [17].

The Mathematica source code for generating the three-dimensional projection reads:

```
r[theta_,omega_,phi_]:=2(-Tan[theta]Cos[omega+phi]+Sec[theta])/
                        (1+Tan[theta]Sin[omega+phi]Sin[omega+phi])

P[theta_]:=ParametricPlot3D[{r[theta,u,phi]Cos[phi],
        r[theta,u,phi]Sin[phi],-r[theta,u,phi]Tan[theta]Sin[u+phi]},
        {u,0,2 Pi},{phi,0,2 Pi}];

Show[{P[0.2],P[0.5],P[1.2]},
        PlotRange->{{-3.5,3.5},{-1,3.5},{-1,1}},
        ViewPoint->{0,-2.6,0.9}]
```

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