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# On Topologically Nontrivial Field Configurations in Noncommutative Field Theories

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*To My Parents*

**Abstract**

In this diploma thesis we derive nonperturbative solutions to the equations of motion of various noncommutative field theoretic models. Firstly, we consider pure Yang-Mills theory defined on noncommutative Euclidean four-dimensional space and derive the noncommutative version of the self-dual $U(2)$ Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton solution using the method of dressing transformations, an iterative procedure for generating solutions from a given seed solution, and thereby we generalize Belavin's and Zakharov's work to the noncommutative setup. Furthermore, we relate the dressing approach with Ward's splitting method based on the twistor construction and rederive the solution in this context. Secondly, we study the noncommutative extension of a modified $U(n)$ sigma model in $2+1$ dimensions. Using the method of dressing transformations, we demonstrate the construction of multi-soliton and soliton-antisoliton configurations for general $n$. As examples, we investigate $U(3)$ solitons and $U(2)$ soliton-antisoliton scattering configurations. Thirdly, we consider the noncommutative version of the Abelian Seiberg-Witten monopole equations on Euclidean four-dimensional space. It has been known for quite some time that due to vanishing theorems there are no regular square integrable solutions to these equations on Euclidean spaces. For a vanishing spinor field, these equations reduce to the Abelian anti-self-dual Yang-Mills (ASDYM) equations, which have no regular solutions either. However, Nekrasov and Schwarz have shown that regular Abelian instanton solutions do exist on noncommutative Euclidean spaces. We generalize their results by constructing several regular solutions to the noncommutative Abelian Seiberg-Witten monopole equations.

**Zusammenfassung**

In der vorliegenden Diplomarbeit werden nichtperturbative Lösungen zu den Bewegungsgleichungen verschiedener Feldtheorien untersucht. Als erstes betrachten wir reine Yang-Mills Theorie, welche auf dem nichtkommutativen vierdimensionalen Euklidischen Raum definiert ist. Unter Verwendung der Methode der "dressing transformations", welche eine iterative Methode zur Generierung von Lösungen aus einer gegebenen darstellt, leiten wir die nichtkommutative Variante des selbst-dualen $U(2)$ Belavin-Polyakov-Schwarz-Tyupkin (BPST) Instantons her. Dabei verallgemeinern wir die Arbeit von Belavin und Zakharov auf den nichtkommutativen Bereich. Ferner stellen wir den Zusammenhang zu Wards "splitting method" her, welche auf der Twistor-Konstruktion basiert und leiten in diesem Rahmen das nichtkommutative BPST Instanton nochmals ab. Zum zweiten diskutieren wir die nichtkommutative Erweiterung eines modifizierten $U(n)$ Sigma-Modells in $2+1$ Dimensionen. Wiederum unter Nutzung von "dressing transformations" konstruieren wir Multi-Soliton und Soliton-Antisoliton Konfigurationen. Als veranschaulichende Beispiele diskutieren wir $U(3)$ Solitonen und betrachten $U(2)$ Soliton-Antisoliton Streuung. Zum dritten untersuchen wir die Erweiterung der Seiberg-Witten Monopol Gleichungen auf den nichtkommutativen vierdimensionalen Euklidischen Raum. Es ist schon lange bekannt, daß diese Gleichungen auf Euklidischen Räumen keine regulären quadrat-integrablen Lösungen besitzen. Für den speziellen Fall eines identisch verschwindenden Spinors reduzieren sich diese Gleichungen zu den Abelschen anti-selbst-dualen Yang-Mills (ASDYM)-Gleichungen, welche ebenfalls keine regulären Lösungen erlauben. Allerdings haben Nekrasov und Schwarz gezeigt, daß reguläre Abelsche Instantonen auf nichtkommutativen Euklidischen Räumen existieren. Wir verallgemeinern ihre Resultate und konstruieren verschiedene reguläre Lösungen zu den nichtkommutativen Abelschen Seiberg-Witten-Monopol-Gleichungen.

*"In jeder reinen Naturlehre ist nur soviel an eigentlicher Wissenschaft enthalten, als Mathematik in ihr angewandt werden kann."*

Immanuel Kant

# Introduction

In mathematics and physics, noncommutativity has a long tradition and plays an important role. For instance, the central mathematical concept in expressing the uncertainty principle in quantum mechanics is based on noncommutativity reflected in any pair of conjugate variables. The idea to introduce noncommutative spacetimes is rather old. Such kind of spaces were already discussed by Snyder [180] in the 40ies of the last century. The studies of noncommutative field theories were lately stimulated by the discovery that quantum field theories on a noncommutative spacetime naturally arise in a certain zero-slope limit ($\alpha' \to 0$) as effective field theories of open string modes coupled to a two-form background field ($B$-field) [175]. Studying various noncommutative field models and revealing their stringy origin constitutes one of the mainstream directions in modern field theory. Independently of string theory, noncommutativity of spatial coordinates is interesting in its own right because it offers a way to introduce nonlocality into (quantum) field theory without loosing control over its structure, thereby providing some hope to overcome the quandaries of quantum gravity. Yet, before attempting to quantize noncommutative field theories, it is certainly warranted to characterize the moduli space of their classical configurations, especially instantons and solitons. Such configurations play an essential role in our understanding of field and string theories beyond perturbation theory.

In this thesis we will cover three topics:

(i) The construction of the noncommutative version of the self-dual $U(2)$ Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton in pure Yang-Mills theory.

(ii) The studies of noncommutative soliton and soliton-antisoliton configurations in a modified $U(n)$ sigma model.

(iii) The extension of Abelian Seiberg-Witten theory to the noncommutative Euclidean four-dimensional space and the construction of regular finite action solutions.

**Part (i).** Nekrasov and Schwarz gave first examples of noncommutative instantons [147]. They modified the Atiyah-Drinfeld-Hitchin-Manin (ADHM) method [6] and showed that (anti-self-dual) noncommutativity resolves the singularities of the instanton moduli space. Moreover, they discovered that the noncommutative Euclidean space $\mathbb{R}^4_\theta$ permits nonsingular Abelian instanton solutions, which do not survive in the commutative limit. Since then, a lot of work has been devoted to this subject (see e.g. [26, 60, 110, 112, 184, 61, 2, 62, 149, 63, 172, 35, 85, 29, 76, 97, 113, 129, 98, 157, 114, 88, 131, 37, 190, 171, 56] and references therein). For reviews on this matter see [148, 83, 49, 117, 150, 78].

Most authors used the modified ADHM equations for the construction of instantons on $\mathbb{R}^4_\theta$. In this part of the thesis we advocate two complementing methods termed the dressing approach and the twistor approach. These proved to be successful in the commutative case: The twistor approach underlies the ADHM scheme [6] and can produce generic $N$-instantons via a sequence of Atiyah-Ward ansätze [5, 197]. In particular, the 't Hooft $N$-instanton solution can be obtained in a very explicit way. The dressing approach [214] (see also [214, 55, 196]) produces this solution equally well [54].

In the noncommutative extension of $\mathbb{R}^4$ the classical fields are best represented as (Lie-algebra valued) operators acting on some two-oscillator Fock space. The authors of [129] employed the twistor approach to construct $U(2)$ 't Hooft $N$-instantons on $\mathbb{R}^4_\theta$. However, they were led to a gauge field which violates self-duality on an $N$-dimensional subspace of the two-oscillator Fock space. This deficiency originated from the singular gauge choice pertinent to the 't Hooft solution was repaired by a suitable Murray-von Neumann transformation after a specific projection of the gauge potential. The proper noncommutative 't Hooft multi-instanton field strength was written down explicitly but its associated gauge potential could be given only implicitly. In order to get around these difficulties, Correa et al. [35] suggested to use the BPST [11] ansatz for constructing the noncommutative $U(2)$ one-instanton. In contrast to [129] they obtained an explicit expression for the self-dual gauge potential but the reality of the gauge potential and field strength was lost.

In this part of the thesis we concentrate on the one-instanton case for self-dual $\mathbb{R}^4_\theta$. Instead of trying to generalize the BPST ansatz [11] we choose the method of dressing transformations and generalize the approach due to Belavin and Zakharov [12] to the noncommutative case. This eventually results in explicit expressions for a real gauge potential with self-dual field strength for the noncommutative $U(2)$ BPST instanton. Exploiting the connection between the dressing method and Ward's splitting method, we rederive the same configuration by generalizing Crane's construction [40]. In fact, Crane's ansatz for the transition matrix substitutes the Atiyah-Ward ansatz and leads directly to the nonsingular instanton configuration without the necessity of a singular gauge transformation. Its generalization may pave the way to nonsingular noncommutative multi-instantons.

**Part (ii).** More than a decade ago it was shown by Ooguri and Vafa [155, 156] that the open $\mathcal{N}=2$ fermionic string is identical (at tree level) to self-dual Yang-Mills (SDYM) theory in $2+2$ dimensions. The noncommutative SDYM theory appears as the effective field theory describing the open critical $\mathcal{N}=2$ string in 2+2 dimensions with nonvanishing $B$-field [124, 126]. Furthermore, in [124, 126] it has also been shown that in this case the $B$-field background induces a noncommutative generalization of a modified $U(n)$ sigma model in $2+1$ dimensions on the world volume of $n$ coincident $D2$-branes, which was introduced by Ward in the commutative setup [195].

In this part of the thesis we generalize the ideas given in [128] of the construction of multi-soliton solutions of the aforementioned noncommutative modified $U(n)$ sigma model to general $n > 1$ and present the construction of soliton-antisoliton configurations using the method of dressing tranformations. As illustrative examples, we discuss $U(3)$ solitons and consider the head-on collision of a $U(2)$ soliton and an antisoliton explicitly, which will result in a $90^{\circ}$ angle scattering. Furthermore, we discuss the head-on collision of one $U(2)$ soliton with two antisolitons. This eventually results in a $60^{\circ}$ angle scattering.

**Part (iii).** The Seiberg-Witten monopole equations [206] turned out to be a powerful tool in the theory of smooth four-manifolds. They provided new differential-topological invariants known as Seiberg-Witten invariants closely related to Donaldson invariants. The latter ones are defined by using the anti-self-dual Yang-Mills (ASDYM) equations. The Seiberg-Witten equations are associated with the Abelian group $U(1)$ and moreover, they have a compact moduli space of solutions while the ASDYM equations considered in Donaldson-Witten theory [46, 47, 203] have the non-Abelian gauge group $SU(2)$ and non-compact moduli space. That is why Seiberg-Witten theory is much easier to handle than Donaldson-Witten theory. The remarkable properties of these equations have given rise to an unprecedented euphory among mathematicians working in four-dimensional topology and inspired a huge amount of papers dealing with the Seiberg-Witten equations and their applications. Several difficult and well known mathematical problems, such as the famous Thom conjecture (cf. e.g. [121]) were quickly solved with the help of Seiberg-Witten invariants constructed from the moduli space of solutions.

An important vanishing theorem of the Seiberg-Witten equations, reminiscent of the Lichnerowicz-Weitzenböck vanishing theorem [206], shows the nonexistence of nontrivial regular ($L^2$-) solutions to the Seiberg-Witten equations on manifolds with non-negative Riemannian scalar curvature, in particular on Euclidean spaces. Nevertheless, while probably not useful for Donaldson-Witten theory, one may construct non-$L^2$ solutions, as has been done by the authors of [57, 151, 1, 133]. As already mentioned, Nekrasov and Schwarz gave first examples of noncommutative instantons [147], i.e. solutions to the noncommutative ASDYM equations (or SDYM equations depending on the respective situation). It is well known that for vanishing spinor field the Seiberg-Witten equations turn into the Abelian ASDYM equations.

In this last part we formulate the noncommutatively deformed Seiberg-Witten equations in four dimensions by a little digression to eight dimensions. Namely, we consider an eight-dimensional generalization of the four-dimensional $U(2)$ SDYM equations [39]. In the commutative setup these generalized equations have been proposed for the construction of semi-topological field theories by the authors of [10]. A dimensional reduction of these equations to four dimensions yields noncommutative non-Abelian Seiberg-Witten theory. After a restriction to the $U(1)\times U(1)$ subgroup of $U(2)$ one discovers the noncommutative Abelian Seiberg-Witten theory. Using partial isometries, we then construct several solutions to the Seiberg-Witten equations. This shows that imposing noncommutativity on coordinates circumvents the vanishing theorems. It is well known that instantons are important objects in gauge theories and particularly in QCD. At the same time, the physical significance of solutions to the Seiberg-Witten equations is less clear. In this respect one should mention that the Seiberg-Witten equations on Kähler manifolds are similar to the vortex equations in two dimensions. Based on this observation, we explain that regular (vortex-like) solutions to the Seiberg-Witten equations on $\mathbb{R}^4_\theta$ may be interpreted as $D(p-4)$- and $\overline{D(p-4)}$-branes in $Dp$-$\overline{Dp}$ brane-antibrane configurations in type II superstring theory. This interpretation can also be extended to the commutative case of the Seiberg-Witten equations on Kähler manifolds.

One might wonder, whether there is some relationship between these at first sight independent looking models. Of course, there is. Starting point is an $\mathcal{N}=2$ $SU(2)$ supersymmetric Yang-Mills (SYM) theory which is defined on $\mathbb{R}^4$ and contains only the vector multiplet. A topological twist of this theory yields a theory which is independent of the coupling constant. Then in the small coupling regime of the twisted theory one discovers the $SU(2)$ Donaldson-Witten theory [203]. On the other hand, considering a spontaneous symmetry breaking of the gauge group $SU(2)$ of the $\mathcal{N}=2$ SYM theory to the $U(1)$ subgroup by the expectation value of the scalar $\phi$ contained in the $\mathcal{N}=2$ multiplet, constructing the low energy effective action which describes the physics of the remaining massless $U(1)$ multiplet, $S$-dualizing, looking at a certain singularity of the moduli space of different vacua of the scalar $\phi$ and twisting the theory, one again obtains a topological field theory which yields in the small coupling regime (or the strong coupling regime of the theory one has started with, i.e. before the $S$-dualization) the Abelian Seiberg-Witten equations [173, 174, 140]. For reviews on this subject see [22, 34, 21, 92] and references therein. Finally, Ward's model can be obtained from pure Yang-Mills theory defined on $\mathbb{R}^4$ after an analytic continuation to $\mathbb{R}^{2,2}$, choosing a particular gauge and doing a dimensional reduction to $\mathbb{R}^{2,1}$.

This thesis is organized as follows. Essentially, it can be divided into two main parts: The commutative and the noncommutative part. In the first chapter we discuss in detail the aforementioned connection between our models. It starts with a brief review on the basics of supersymmetric gauge theories by specializing to the $\mathcal{N}=2$ case. We then show explicitly how one can obtain Donaldson-Witten theory and Seiberg-Witten theory from the $\mathcal{N}=2$ SYM theory. The explicit derivation of Ward's model is postponed to the beginning of chapter IV. The intention of this thesis is also to present the general geometric picture of the discussed theories. For the sake of clarity we have therefore included two purely mathematical chapters reviewing the basic mathematical concepts subsequently used in this thesis. These chapters are denoted by "Mathematical Preliminaries - Part I/II". In part I we will cover topics like Lie groups and algebras, theory of fiber bundles, homotopy groups, spin geometry and characteristic classes. These topics are of great importance for the understanding of the commutative setup of our theories. We then discuss instantons in ordinary Yang-Mills theory and review the construction of the $SU(2)$ BPST instanton using the twistor approach and the dressing method, thereby pointing out the generalization to $SU(n)$ and multi-instantons. Chapter IV deals with Ward's model. Herein we show explicitly its connection to Yang-Mills theory. We eventually discuss the construction of soliton and soliton-antisoliton configurations using dressing transformations. In chapter V, we present the mathematical basics of Seiberg-Witten theory on Riemannian and Kähler manifolds. Moreover, we prove the non-existence of (regular) solutions to the monopole equations on $\mathbb{R}^4$. Mathematical preliminaries part II starts with an introduction on categories, functors and $C^*$-algebras. In this chapter we then demonstrate how notions and definitions familiar from (commutative) differential geometry can be made algebraic using the Gelfand-Naimark contravariant functors. Eventually, we discuss how their noncommuative generalization looks like. We then specify to noncommutative Euclidean spaces and formulate Yang-Mills theory on the latter ones. The remaining three chapters are devoted to the noncommutative generalization of our three field theoretic models in the sense of the explicit construction of noncommutative instantons and solitons and the latter's stringy interpretation. Throughout the whole thesis Einstein's summation convention for repeated indices is used. The scientific results underlying this thesis can be found in [211, 91, 163].

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Martin Wolf

**Note added:** The present version of this diploma thesis is a revised version of the original one which has been submitted to the Dresden University of Technology on July 4th 2003. Some equations have been corrected, certain paragraphs which are not needed for the understanding have been erased and several typos have been fixed.

Hannover, December 12, 2004

Martin Wolf

# Contents

# Chapter I

## Physical Setup

In the introduction we have already said a few words on the connection between the Donaldson-Witten theory and the Seiberg-Witten theory. The goal of this chapter is to explain this relationship in detail. Following the original way of treatment, one would start with an $\mathcal{N} = 2$ $SU(2)$ SYM theory on $\mathbb{R}^4$ containing only the vector multiplet. The Donaldson-Witten theory is then obtained by twisting this theory and by considering the weak coupling limit [203]. On the other hand, Abelian Seiberg-Witten theory can be obtained after a spontaneous symmetry breaking of the gauge group $SU(2)$ to $U(1)$ and by looking at the low energy effective action. The latter one is fully described in terms of an $\mathcal{N} = 2$ prepotential $\mathcal{F}$, which is holomorphic as required by supersymmetry. Moreover, one realizes that for the scalar $\phi$ the moduli space of gauge inequivalent vacua $\langle\phi\rangle = \frac{1}{2}a\sigma_3$ for $a \in \mathbb{C} \cup \{\infty\}$, which is parametrized by $u = \langle \operatorname{tr} \phi^2 \rangle \in \mathbb{C} \cup \{\infty\}$, has three singularities at $u = \infty$ and $u = \pm 1$ after a suitable rescaling. Here, $\sigma^3$ denotes the third Pauli matrix. Note that we have $u = \frac{1}{2}a^2$ in the semiclassical approximation, which is, of course, not true when quantum fluctuations become important. The singularity at $u = \infty$ corresponds to the semiclassical limit, while the singularities at $\pm 1$ correspond to cases where magnetic monopoles and dyons become massless. At the singularity where monopoles become massless the proper coordinate $a_D$ on the moduli space is obtained via a Legendre transformation of the prepotential, i.e. $a_D = \frac{\partial \mathcal{F}}{\partial a}$ , where the subscript 'D' refers to the dual picture of the theory. This transformation is known as $S$-duality transformation, since the coupling constant $\tau(a)$ gets mapped to $-1/\tau_D(a_D)$, i.e. the strong and weak coupling regime of the theory are exchanged. Moreover, it is an electromagnetic duality, since purely electrically charged states are mapped to purely magnetically charged states and vice versa. Magnetic monopoles are described by hypermultiplets of $\mathcal{N} = 2$ supersymmetry which couple locally to the dual vector multiplet. For a detailed discussion we refer the reader to [173, 174]. Twisting this Abelian $\mathcal{N} = 2$ SYM theory containing vector and hypermultiplet eventually yields the Abelian Seiberg-Witten monopole equations. In our discussion presented in the sequel, however, we shall proceed differently. We start instead with an $\mathcal{N} = 2$ $SU(n)$ SYM theory containing both the vector multiplet and the hypermultiplet. By twisting this theory we will eventually obtain non-Abelian Seiberg-Witten theory. Note that these equations can be used to prove the equivalence between the Donaldson-Witten theory and the Abelian Seiberg-Witten theory in a mathematically rigorous way, without using the functional integral approach (see e.g. [160, 27, 138, 137] and references therein). In this chapter we shall not prove this equivalence but rather physically motivate our equations. We will mainly follow the treatment given in [203, 34, 138]. The precise mathematical definitions of most of the terminology are given in the subsequent chapters.

## § I.1 Supersymmetric Yang-Mills Theory

Before talking about twisting, etc. we briefly review the basics of supersymmetry and thereby fix our notation.

### § I.1.1 Supersymmetry Algebra

**Spinor conventions.** Let us restrict to Euclidean four-dimensional space $\mathbb{R}^4$. The rotation group $SO(4)$ of $\mathbb{R}^4$ is locally given by $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$. We denote the latter decomposition by $SU_L(2) \times SU_R(2)$. Thus, the complex representation of Spin(4) is given by two copies of $SU(2)$ acting on $\mathbb{R}^4 \cong \mathbb{C}^2$. These are known as the positive and negative chirality representations. The two-dimensional complex vector spaces associated to these representations are denoted by $S^\pm$ and hence we can write the total representation space as a direct sum $S \cong S^+ \oplus S^-$. In the sequel we use $A, B, C, \ldots = 1, 2$ as indices for the fundamental representation of $SU_L(2)$ and $A', B', C', \ldots = 1, 2$ for $SU_R(2)$. We therefore write $\Psi \in S$ as

$$\Psi = \begin{pmatrix} M_A \\ N^{A'} \end{pmatrix}. \tag{I.1}$$

An important observation is that the subspaces $S^\pm$ are naturally equipped with a complex symplectic structure implemented by the second Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{I.2}$$

In what follows we use the conventions of [64]. We define

$$C_{AB} := (\sigma_2)_{AB} \qquad \text{and} \qquad C_{A'B'} := (\sigma_2)_{A'B'}. \tag{I.3a}$$

Their inverses are given by

$$C^{AB} = -(\sigma_2)_{AB} \qquad \text{and} \qquad C^{A'B'} = -(\sigma_2)_{A'B'}, \tag{I.3b}$$

such that

$$C^{AB} C_{CB} = \delta^A_C \qquad \text{and} \qquad C^{A'B'} C_{C'B'} = \delta^{A'}_{C'}. \tag{I.3c}$$

Thus, we have the isomorphisms between $S^{\pm}$ and their dual vector spaces $(S^{\pm})^*$,

$$\begin{aligned} C^+ : S^+ &\to (S^+)^* \\ M_A &\mapsto M^A = C^{AB} M_B, \end{aligned} \tag{I.4a}$$

as well as

$$\begin{aligned} C^- : S^- &\to (S^-)^* \\ N^{A'} &\mapsto N_{A'} = N^{B'} C_{A'B'}. \end{aligned} \tag{I.4b}$$

Moreover, we define

$$(\sigma^\mu)_{AA'} = (i\sigma^i, \mathbb{1}_2)_{AA'} \qquad \text{and} \qquad (\bar\sigma^\mu)^{A'A} = (-i\sigma^i, \mathbb{1}_2)^{A'A}, \tag{I.5}$$

where the $\sigma^i$'s are the Pauli matrices. Using this, we may write for any one-form $\omega = \omega_\mu \, \mathrm{d}\, x^\mu$

$$\omega_{AA'} = (\sigma^\mu)_{AA'} \omega_\mu. \tag{I.6}$$

Besides all that we also need metrics on $S^{\pm}$. To introduce them we define for an element of $S^+$, ${}^t(M_A) = (a, b)$ the conjugate element ${}^t(\bar M^A) = (a^*, b^*)$ and for an element of $S^-$, ${}^t(N^{A'}) = (a, b)$ the conjugate element ${}^t(\bar N_{A'}) = (a^*, b^*)$. Here, '$*$' denotes complex conjugation. A Hermitian metric on $S^+ \cong \mathbb{R}^4$ is given by

$$\langle M, N\rangle_+ := \bar M^A N_A \tag{I.7}$$

and similarly on $S^-$. Both together define a Hermitian metric on $S$. Furthermore, we define

$$\bar M_A = \bar M^B C_{AB} \qquad \text{and} \qquad \bar N^{A'} = \bar N_{B'} C^{B'A'}. \tag{I.8}$$

Then one may readily check that $\bar M^A M_A = \bar M_A M^A$ and $\bar N^{A'} N_{A'} = \bar N_{A'} N^{A'}$.

**Supersymmetry algebra.** Let us now specify to the $\mathcal{N} = 2$ supersymmetry case, i.e. the case of two supercharges $Q_{iA}$ and $\bar Q^{iA'}$ with $i = 1, 2$. On $\mathbb{R}^4$ the global symmetry group of $\mathcal{N} = 2$ supersymmetry is given by

$$H = SU_L(2) \times SU_R(2) \times SU_I(2) \times U_{\mathcal{R}}(1), \tag{I.9}$$

where $SU_I(2)$ and $U_{\mathcal{R}}(1)$ are internal symmetries. Note that in the quantum theory $U_{\mathcal{R}}(1)$ is broken down to $\mathbb{Z}_8$ by one-loop and instanton corrections. The $\mathcal{N} = 2$ supersymmetry algebra reads

$$\{Q_{iA}, \bar Q^j_{B'}\} = \delta_i^j P_{AB'}, \tag{I.10a}$$
$$\{Q_{iA}, Q_{jB}\} = \epsilon_{ij} C_{AB} Z, \tag{I.10b}$$
$$[Q_{iA}, P_{BB'}] = [P_{AA'}, P_{BB'}] = [\bar M_{A'B'}, Q_{iC}] = 0, \tag{I.10c}$$
$$[M_{AB}, Q_{iC}] = \tfrac{i}{2} C_{C(A} Q_{iB)}, \tag{I.10d}$$
$$[M_{AB}, P_{CC'}] = \tfrac{i}{2} C_{C(A} P_{B)C'}, \tag{I.10e}$$
$$[M_{AB}, M^{CD}] = -\tfrac{i}{2} \delta_{(A}^{\ (C} M_{B)}^{\ \ D)}, \tag{I.10f}$$
$$[M_{AB}, \bar M_{A'B'}] = 0. \tag{I.10g}$$

Let us comment our notation. First of all $\{\ ,\ \}$ denotes the anti-commutator and $[\ ,\ ]$ the commutator, respectively. The $P_{AA'}$ correspond to the generators of spacetime translations $P_\mu$ via (I.6), $\epsilon_{ij}$ is the Levi-Civita symbol with $\epsilon_{12} = 1$ and $Z$ denotes the central charge of the algebra. The generators $M_{AB} = M_A^{\ C} C_{BC}$ correspond to the rotation generators $M_{\mu\nu}$ via the following relation

$$\omega_A^{\ B} = -\tfrac{1}{2}(\bar{\sigma}^\mu)_{AA'}(\sigma^\nu)^{A'B}\omega_{\mu\nu} \tag{I.11}$$

valid for arbitrary two-forms $\omega = \frac{1}{2}\omega_{\mu\nu}\,\mathrm{d}\,x^\mu \wedge \mathrm{d}\,x^\nu$. Note that the above equation is equivalent to

$$\omega_{AB} = -\tfrac{1}{2}C^{A'B'}(\bar{\sigma}^\mu)_{AA'}(\bar{\sigma}^\nu)_{BB'}\omega_{\mu\nu}. \tag{I.12}$$

The parentheses ( ) denote symmetrization of the enclosed indices but without the usual factor of $n!$. The supercharges $Q_{iA}$ and $\bar{Q}^{iA'}$ transform under $H$ as $(\frac{1}{2}, 0, \frac{1}{2})^1$ and $(0, \frac{1}{2}, \frac{1}{2})^{-1}$, respectively. The powers 1 and $-1$ denote the charge under the internal $U_{\mathcal{R}}(1)$ symmetry.

### § I.1.2 Supermultiplets

Now we have the ingredients in order to write down the $\mathcal{N} = 2$ supermultiplets. Let us consider the massless case. One can show that for vanishing mass the central charge $Z$ has to vanish, as well ($m \geq \sqrt{2}|Z|$). Under the assumption that the helicity of the considered state is not greater than one, we have in the $\mathcal{N} = 2$ case only two different massless supermultiplets, namely the vector multiplet and the hypermultiplet. The vector multiplet is given by:

| Field | Spin | Statistics | $H$ |
|---|---|---|---|
| $\mathcal{A}_\mu$ | 1 | boson | $(\frac{1}{2}, \frac{1}{2}, 0)^0$ |
| $\lambda_{iA}$ | $\frac{1}{2}$ | fermion | $(\frac{1}{2}, 0, \frac{1}{2})^{-1}$ |
| $\bar{\lambda}_i^{A'}$ | $\frac{1}{2}$ | fermion | $(0, \frac{1}{2}, \frac{1}{2})^1$ |
| $B$ | 0 | boson | $(0,0,0)^{-2}$ |
| $B^\dagger$ | 0 | boson | $(0,0,0)^2$ |
| $D_{ij} = D_{ji}$ | 0 | boson | $(0,0,1)^0$ |

The $\mathcal{N} = 2$ hypermultiplet reads:

| Field | Spin | Statistics | $H$ |
|---|---|---|---|
| $q_i$ | 0 | boson | $(0, 0, \frac{1}{2})^0$ |
| $\psi_{iA}$ | $\frac{1}{2}$ | fermion | $(0, \frac{1}{2}, 0)^1$ |
| $\bar{\psi}_i^{A'}$ | $\frac{1}{2}$ | fermion | $(\frac{1}{2}, 0, 0)^{-1}$ |
| $F_i$ | 0 | boson | $(0, 0, \frac{1}{2})^2$ |
| $F_i^\dagger$ | 0 | boson | $(0, 0, \frac{1}{2})^{-2}$ |

It will turn out to be useful to think of the $\mathcal{N}=2$ vector multiplet as an $\mathcal{N}=1$ vector multiplet plus an $\mathcal{N}=1$ chiral multiplet. The latter one consists of a chiral fermion and a complex scalar. Furthermore, the $\mathcal{N}=2$ hypermultiplet can be described by two $\mathcal{N}=1$ chiral multiplets. The reason for this rewriting is that the $\mathcal{N}=2$ SYM action in the language of $\mathcal{N}=1$ superspace takes a very concise form.

### § I.1.3 $\mathcal{N}=2$ Supersymmetric Yang-Mills Action

**Superspace.** Consider the unextended supersymmetry case $\mathcal{N}=1$. The idea is to enlarge the ordinary space $\mathbb{R}^4$ spanned by the coordinates $\{x^\mu\}$ to $\mathbb{R}^{(4|4)}$ by adding two plus two (anticommuting) Graßmann coordinates $\theta_A$ and $\bar{\theta}_{A'}$. For $\mathcal{N}=1$ (I.10a) and (I.10b) simply reduce to

$$\{Q_A, \bar{Q}_{B'}\} = P_{AB'} \qquad \text{and} \qquad \{Q_A, Q_B\} = 0. \tag{I.13}$$

Note that in the unextended case there are no central charges at all. Remember that $P_{AB'} = (\sigma^\mu)_{AB'}\partial_\mu$. Then one can show (see e.g. [201]) that

$$Q_A = \frac{\partial}{\partial\theta^A} - \frac{\bar{\theta}^{B'}}{2}P_{AB'} \qquad \text{and} \qquad \bar{Q}_{B'} = -\frac{\partial}{\partial\bar{\theta}^{B'}} + \frac{\theta^A}{2}P_{AB'}. \tag{I.14}$$

Moreover, we introduce supercovariant derivatives by

$$D_A = \frac{\partial}{\partial\theta^A} + \frac{\bar{\theta}^{B'}}{2}P_{AB'} \qquad \text{and} \qquad \bar{D}_{B'} = -\frac{\partial}{\partial\bar{\theta}^{B'}} - \frac{\theta^A}{2}P_{AB'}, \tag{I.15}$$

which anticommute with the generators $Q_A$ and $\bar{Q}_{B'}$.

Any function on $\mathbb{R}^{(4|4)}$ can be expanded in terms of the Graßmann variables $\theta_A$ and $\bar{\theta}_{A'}$. Then the $\mathcal{N}=1$ chiral superfield $\Phi$ is defined by the constraint

$$\bar{D}_{A'}\Phi = 0. \tag{I.16}$$

By definition the $\mathcal{N}=1$ vector superpotential $V$ is subject to

$$V = V^\dagger. \tag{I.17}$$

From this one constructs the vector superfield $W_A$ via

$$W_A = -\tfrac{1}{4}\bar{D}_{A'}\bar{D}^{A'}e^{-V}D_A e^V, \tag{I.18}$$

which satisfies $\bar{D}_{A'}W_A = 0$ as one may readily check. The explicit form of $\Phi$ and $V$ can be found in [201], for instance. Here, we only want to give the component fields of the superfields $\Phi$ and $W_A$

$$\Phi,\, \Phi^\dagger \qquad : \qquad B,\, \lambda_{2A},\, D_{22},\, B^\dagger,\, \bar{\lambda}^{2A'},\, D_{11} \tag{I.19a}$$

$$W_A,\, \bar{W}_{A'} \qquad : \qquad \mathcal{A}_{AA'},\, \lambda_{1A},\, \bar{\lambda}^{1A'},\, D_{12}. \tag{I.19b}$$

Under the internal symmetry group $U_{\mathcal{R}}(1)$ the superfields $\Phi$ and $W_A$ transform as

$$\Phi \mapsto e^{-2i\phi}\Phi(e^{i\phi}\theta) \qquad \text{and} \qquad W_A \mapsto e^{-i\phi}W_A(e^{i\phi}\theta), \tag{I.20}$$

i.e. $\Phi$ carries a $U_{\mathcal{R}}(1)$ charge of $-2$ and $W_A$ of $-1$.

$\mathcal{N}=2$ **SYM action.** Let $G=SU(n)$ be the gauge group at issue. In $\mathcal{N}=1$ superspace language the $\mathcal{N}=2$ SYM action which contains only the vector multiplet takes the form

$$S_{\mathrm{SYM}}=\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\;\Phi^\dagger e^V\Phi+\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\theta\;\mathrm{tr}\,W^AW_A+\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\bar\theta\;\mathrm{tr}\,\bar W^{A'}\bar W_{A'}. \tag{I.21}$$

Note that all considered fields live in the adjoint representation of $G$. Here, 'tr' denotes the trace over the gauge algebra $\mathfrak{su}(n)$. Remember that the superfield $W_A$ satisfies also the constraint

$$D^AW_A+\bar D^{A'}\bar W_{A'}=0. \tag{I.22}$$

This implies, that the last two terms in (I.21) differ by a term which is proportional to the second Chern class.

Since we want to describe non-Abelian Seiberg-Witten theory, we also need to include to the action (I.21) the $\mathcal{N}=2$ hypermultiplet. As already mentioned, the $\mathcal{N}=2$ hypermultiplet consists of two $\mathcal{N}=1$ chiral multiplets $\mathcal{M}$ and $\tilde{\mathcal{M}}$ subject to the constraint (I.16). They are both neutral with respect to the $U_{\mathcal{R}}(1)$ symmetry, i.e.

$$\mathcal{M}\mapsto\mathcal{M}(e^{i\phi}\theta)\qquad\text{and}\qquad\tilde{\mathcal{M}}\mapsto\tilde{\mathcal{M}}(e^{i\phi}\theta). \tag{I.23}$$

The component fields of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ read

$$\mathcal{M},\,\mathcal{M}^\dagger\quad:\quad q_1,\,\psi_{qA},\,F_2,\,q^{1\dagger},\,\bar\psi_{qA'},\,F^{2\dagger}, \tag{I.24a}$$
$$\tilde{\mathcal{M}},\,\tilde{\mathcal{M}}^\dagger\quad:\quad q^{2\dagger},\,\psi_{\tilde qA},\,F^{1\dagger},\,q_2,\,\bar\psi_{\tilde qA'},\,F_1. \tag{I.24b}$$

The superfield $\mathcal{M}$ lives in the fundamental representation of the gauge group while $\tilde{\mathcal{M}}$ sits in the antifundamental representation.

Finally, in $\mathcal{N}=1$ superspace language the $\mathcal{N}=2$ hypermultiplet couples to (I.21) through

$$\begin{aligned}S_{\mathrm{HM}}=&\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,(\mathcal{M}^\dagger e^V\mathcal{M}+\tilde{\mathcal{M}}^\dagger e^{-V}\tilde{\mathcal{M}})\\&-i\sqrt2\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\theta\;\tilde{\mathcal{M}}\Phi\mathcal{M}+i\sqrt2\int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^2\bar\theta\;\tilde{\mathcal{M}}^\dagger\Phi^\dagger\mathcal{M}^\dagger,\end{aligned} \tag{I.25}$$

i.e. $S=S_{\mathrm{SYM}}+S_{\mathrm{HM}}$.

## § I.2 Cohomological Field Theory

Before we discuss explicitly the twisting of our $\mathcal{N}=2$ SYM theory let us collect a few general ideas of cohomological field theory.

### § I.2.1 Some Facts about Topological Field Theory

**BRST operator.** In the following we denote by $\Phi$ the collective field which includes gauge fields, ghosts, etc. Let $Q$ be the Becchi-Rouet-Stora-Tyutin (BRST) operator. Remember that $Q$ is nilpotent, that is $Q^2 = 0$. For the variation of any functional $\mathcal{O}$ of the fields $\Phi$ we write

$$\delta\mathcal{O} = [Q, \mathcal{O}] \qquad \text{and} \qquad \delta\mathcal{O} = \{Q, \mathcal{O}\} \tag{I.26}$$

for $\mathcal{O}$ Graßmann even and odd, respectively. Recall also that the full quantum action $S_q$ is by construction $Q$-invariant, i.e. $\delta S_q = 0$. The physical Hilbert space is defined by the condition

$$Q|\text{phys}\rangle = 0. \tag{I.27}$$

A state which is annihilated by $Q$ is said to be $Q$-closed. Two physical states $|\text{phys}\rangle$ and $|\text{phys}\rangle'$ are equivalent if they differ by a $Q$-exact state, i.e. $|\text{phys}\rangle' = |\text{phys}\rangle + Q|\chi\rangle$. Thus, this equivalence relations divides the physical Hilbert space into $Q$-cohomology classes. The BRST invariance of the vaccum implies for any functional $\mathcal{O}$ that

$$\langle 0|[Q, \mathcal{O}]|0\rangle \equiv \langle [Q, \mathcal{O}]\rangle = 0 \qquad \text{and} \qquad \langle 0|\{Q, \mathcal{O}\}|0\rangle \equiv \langle \{Q, \mathcal{O}\}\rangle = 0, \tag{I.28}$$

since $Q$ is self-adjoint.

**Topological field theory.** Suppose that our theory is defined on some Riemannian manifold $M$ with metric $g$. Consider the variation of $S_q$ with respect to the metric $g$, i.e.

$$\delta_g S_q =: \frac{1}{2}\int_M \mathrm{d^n x}\,\sqrt{g}\,\delta g^{\mu\nu} T_{\mu\nu}. \tag{I.29}$$

This defines the energy-momentum tensor $T_{\mu\nu}$. In the following we assume that the functional measure in the path integral is $Q$-invariant as well as metric independent. Then a topological field theory consists of the following ingredients [22]:

(i) a collection of fields $\Phi$ which are Graßmann graded and are defined on $(M, g)$,

(ii) a nilpotent operator $Q$ which is odd with respect to the Graßmann grading,

(iii) physical states defined to be $Q$-cohomology classes,

(iv) an energy momentum tensor which is $Q$-exact, i.e.

$$T_{\mu\nu} = \{Q, V_{\mu\nu}(\Phi, g)\}. \tag{I.30}$$

In this context the nilpotency of $Q$ is assumed to hold on the quotient space $\mathcal{E}/\mathcal{G}$ where $\mathcal{E}$ represents all field configurations and $\mathcal{G}$ is the group of gauge transformations. For instance, in our twisted $\mathcal{N} = 2$ SYM theory $Q^2$ will not be zero on $\mathcal{E}$ but will be a gauge transformation. Then the observables of the theory (operators in the $Q$-cohomology) must be gauge invariant.

Therefore, the relevant cohomology in these cases is the equivariant cohomology with respect to $\mathcal{G}$.

Now we are interested in the change of the partition function $Z$,

$$Z = \int [\mathcal{D}\Phi]\, e^{-S_q}, \tag{I.31}$$

under an infinitesimal change of the metric. We get

$$\begin{aligned} \delta_g Z &= \int [\mathcal{D}\Phi]\, e^{-S_q} \left( -\frac{1}{2} \int_M \mathrm{d^n x}\, \sqrt{g}\, \delta g^{\mu\nu} T_{\mu\nu} \right) \\ &= \int [\mathcal{D}\Phi]\, e^{-S_q} \left( -\frac{1}{2} \int_M \mathrm{d^n x}\, \sqrt{g}\, \delta g^{\mu\nu} \{Q, V_{\mu\nu}\} \right) \\ &= -\frac{1}{2} \left\langle \left\{ Q, \int_M \mathrm{d^n x}\, \sqrt{g}\, \delta g^{\mu\nu} V_{\mu\nu} \right\} \right\rangle = 0. \end{aligned}$$

Here, we essentially used the metric independence of the functional measure. To show that our assumption is really justified one needs to check for metric anomalies. Thus, we realize that for a topological field theory the partion function is independent of the metric $g$ of the underlying manifold $M$, or in other words $Z$ defines a topological invariant of $M$. Hence, a natural question arises: Are there other topological invariants? Let $\mathcal{O}$ be some observable and consider its vacuum expectation value

$$\langle \mathcal{O} \rangle = \int [\mathcal{D}\Phi]\, e^{-S_q} \mathcal{O}(\Phi). \tag{I.32}$$

The variation of $\langle \mathcal{O} \rangle$ with respect to the metric computes to

$$\delta_g \langle \mathcal{O} \rangle = \int [\mathcal{D}\Phi]\, e^{-S_q} (\delta_g \mathcal{O} - (\delta_g S_q)\mathcal{O}).$$

Under the assumption that $\delta_g \mathcal{O}$ is $Q$-exact and $\mathcal{O}$ is $Q$-invariant we find

$$\delta_g \langle \mathcal{O} \rangle = 0. \tag{I.33}$$

Thus, one may construct a larger class of topological invariants.

In general one distinguishes two kinds of topological field theories: Witten type theories, also known as cohomological field theories and Schwarz type theories. Their difference lies in the following:

$$\text{Witten type:} \quad S_q = \{Q, V\} \qquad\qquad \text{Schwarz type:} \quad S_q = S_c + \{Q, V\}, \tag{I.34}$$

where $S_c$ refers to the classical action, which is assumed to be metric independent. We shall only deal with Witten type theories. Clearly, as a consequence of $S_q = \{Q, V\}$, we have

$$T_{\mu\nu} = \left\{ Q, \tfrac{2}{\sqrt{g}} \tfrac{\delta V}{\delta g^{\mu\nu}} \right\}, \tag{I.35}$$

so that (iv) of our definition of a topological field theory is satisfied. Cohomological theories have the property of being invariant with respect to the following scaling. Consider $S_q \to \lambda S_q$ with $\lambda \in \mathbb{R}$. Then we find

$$\delta_\lambda Z \;=\; \int [\mathcal{D}\Phi]\, e^{-\lambda S_q} \delta\lambda\, S_q = \delta\lambda \int [\mathcal{D}\Phi]\, e^{-\lambda S_q} \{Q, V\} = \delta\lambda \, \langle \{Q, V\} \rangle = 0.$$

**Witten's localization theorem.** The explicit computation of the path integral (I.32) can be very difficult. However, we have a nice theorem due to Witten [204] which simplifies essentially this computation. Consider an arbitrary cohomological field theory. Let $\mathcal{E}$ be the function space over which one wishes to integrate. We denote by $F$ the $(0|1)$ dimensional supergroup generated by the BRST operator $Q$. Firstly, suppose that $F$ acts freely, i.e. there are no fixed points on $\mathcal{E}$. Then one has the fibration $\mathcal{E} \to \mathcal{E}/F$. This implies, that one can reduce the path integral over $\mathcal{E}$ to an integral over $\mathcal{E}/F$. Let $\mathcal{O}$ be an $F$-invariant observable. Then one discovers

$$\int_{\mathcal{E}} [\mathcal{D}\Phi]\, e^{-S_q} \mathcal{O} = \mathrm{Vol}(F) \int_{\mathcal{E}/F} [\mathcal{D}\Phi]\, e^{-S_q} \mathcal{O}. \tag{I.36}$$

But the volume of our $F$ is zero, since for a Graßmann variable we have $\int \mathrm{d}\theta = 0$ and hence the integral vanishes. Here, $\theta$ is the collective coordinate for the BRST symmetry. In general, however, $F$ does not act freely but has some fixed point locus $\mathcal{E}_0$. Let us denote by $\mathcal{C}$ an $F$-invariant neighborhood of $\mathcal{E}_0$ and by $\mathcal{E}'$ its complement. By the above arguments, the path integral restricted to $\mathcal{E}'$ vanishes. For that reason the entire contribution comes from the integral over $\mathcal{C}$. However, we can choose $\mathcal{C}$ as small as we wish, such that a perturbative approximation of the path integral in direction normal to $\mathcal{E}_0$ can be arbitrarily good. The integral along $\mathcal{E}_0$ must be performed exactly. One can show that the normal part can be done in a one-loop approximation. Thus, in the semi-classical limit one needs only the BRST invariant configurations to determine (I.32). One says that the path integral (I.32) is localized at the fixed points of the BRST operator.

### § I.2.2 Twisting of $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

**Twisting procedure.** Having fixed the setup we are now ready to discuss the twisting in its full glory. Consider the $SU(n)$ $\mathcal{N}=2$ SYM theory on $\mathbb{R}^4$ introduced in the previous sections. Now we consider an exotic action of the rotation group in (I.9) and write

$$H' = SU_L(2)' \times SU_R(2) \times U_{\mathcal{R}}(1), \tag{I.37}$$

where $SU_L(2)'$ is the diagonal subgroup of $SU_L(2) \times SU_I(2)$. Then the supercharges $Q_{iA}$ and $\bar{Q}^{iA'}$ transform under this new group $H'$ as

$$(\tfrac{1}{2}, \tfrac{1}{2})^{-1} \oplus (1,0)^1 \oplus (0,0)^1.$$

The twisting is achieved simply by replacing any isospin index $i$ by a spinor index $A$, i.e.

$$Q_{iA} \to Q_{BA} \qquad \text{and} \qquad \bar{Q}_{iA'} \to \bar{Q}_{BA'}. \tag{I.38}$$

The trace[1] of $Q_{BA}$,

$$Q = Q_A{}^A = Q_{\bar{1}2} - Q_{\bar{2}1}, \tag{I.39}$$

[1]Here, bar denotes the isospin index.

becomes the $(0,0)^1$ rotation invariant operator satisfying the twisted version of the $\mathcal{N}=2$ supersymmetry algebra (I.10a) and (I.10b)

$$\{Q, \bar{Q}_{AB'}\} = P_{AB'} \qquad \text{and} \qquad \{Q, Q\} = Z. \tag{I.40}$$

Thus, the momentum tensor is $Q$-exact. For vanishing central charge $Z$ the last equation clearly implies the nilpotency of $Q$.

Firstly, we discuss the twisting of the SYM action given by (I.21). The twist of this action can be understood as gauging of the $SU_I(2)$ group as the spacetime symmetry group $SU_L(2)$. The $SU_I(2)$ current of (I.21) is given by

$$j_a^\mu = \bar{\lambda}\sigma^\mu\sigma_a\lambda, \tag{I.41}$$

as one may check by going through the usual Noether procedure. Here, $\sigma^\mu$ is defined by (I.5) and the $\sigma^a$'s are the Pauli matrices ($a = 1, 2, 3$). Spinors are the only fields charged with respect to $SU_I(2)$. Hence, this gauging can be achieved after adding to the Lagrangian the term

$$-\omega_\mu^a j_a^\mu = -\bar{\lambda}^i_{A'}(\sigma^\mu)^{AA'}(\sigma_a)_i^{\ j}\omega_\mu^a\lambda_{jA}, \tag{I.42}$$

where $\omega_\mu$ is the connection on $S^+ \otimes S^+$. Note that $S^+ \otimes S^+ \cong (\Omega^0 \oplus \Omega^2_+) \otimes \mathbb{C}$; cf. subsection § II.2.4. The only change is in the fermion kinetic term of the Lagrangian. Introducing the spacetime fields

$$\eta = -\lambda_A^{\ A}, \quad \psi_\mu = i(\sigma_\mu)^{A'A}\bar{\lambda}_{AA'} \quad \text{and} \quad \lambda_{(AB)} = -\tfrac{1}{2\sqrt{2}}C^{A'B'}(\bar{\sigma}_\mu)_{AA'}(\bar{\sigma}_\nu)_{BB'}\chi^{\mu\nu}, \tag{I.43}$$

i.e. a scalar $\eta$, a one-form $\psi_\mu$ and a self-dual two-form $\chi_{\mu\nu}$, we can rewrite the fermion kinetic term as

$$-\tfrac{i}{2}\psi^\mu\nabla_\mu\eta - \sqrt{2}\psi_\mu\nabla_\nu\chi^{\nu\mu}. \tag{I.44}$$

The twisting procedure yields the following translation table:

| Old field | $H$ | New field | $H'$ | Spin | Statistics |
|---|---|---|---|---|---|
| $\mathcal{A}_\mu$ | $(\frac{1}{2},\frac{1}{2},0)^0$ | $\mathcal{A}_\mu$ | $(\frac{1}{2},\frac{1}{2})^0$ | 1 | boson |
| $\lambda_{iA}$ | $(\frac{1}{2},0,\frac{1}{2})^{-1}$ | $\eta$ | $(0,0)^{-1}$ | 0 | fermion |
| | | $\chi_{AB}$ | $(1,0)^{-1}$ | 1 | fermion |
| $\bar{\lambda}_i^{A'}$ | $(0,\frac{1}{2},\frac{1}{2})^1$ | $\psi_{AA'}$ | $(\frac{1}{2},\frac{1}{2})^1$ | 1 | fermion |
| $B$ | $(0,0,0)^{-2}$ | $i\sqrt{2}\lambda$ | $(0,0)^{-2}$ | 0 | boson |
| $B^\dagger$ | $(0,0,0)^2$ | $\frac{1}{2\sqrt{2}}\phi$ | $(0,0)^2$ | 0 | boson |
| $D_{ij} = D_{ji}$ | $(0,0,1)^0$ | $H_{AB}$ | $(1,0)^0$ | 1 | boson |

Using this table and the definitions (I.43) we can express the untwisted fields in terms of the twisted ones by

$$\lambda_{\bar{1}2} = \tfrac{i}{2}\eta + \tfrac{1}{\sqrt{2}}\chi_{12} \qquad \text{and} \qquad \lambda_{\bar{2}1} = -\tfrac{i}{2}\eta + \tfrac{1}{\sqrt{2}}\chi_{12}, \tag{I.45}$$

with $\chi_{AB} = \sqrt{2}\lambda_{(AB)}$. Note that the twisted SYM theory which contains only the vectormultiplet can be formulated on an arbitrary Riemannian four-manifold $M$, since in the twisted theory all fields are differential forms. This is not possible in the untwisted version since there one encounters a topological obstruction associated to the second Stiefel-Whitney class of $M$. In trying to formulate the theory on $M$ one usually uses the principle of minimal coupling to gravity. However, if $M$ is not a spin manifold, i.e. its second Stiefel-Whitney class is nontrivial, one cannot perform this construction consistently. The twisted version of (I.21) takes on $M$ the following form

$$\begin{aligned} S_1 = \int_M \mathrm{dVol}\ \mathrm{tr}\Big\{ & \tfrac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{i}{2\sqrt{2}}\chi^{AB}\mathcal{D}_{AA'}\psi_B^{\ A'} - \tfrac{i}{8}\chi^{AB}[\phi,\chi_{AB}] \\ & + \tfrac{i}{2}\nabla_\mu\lambda\nabla^\mu\phi + \tfrac{i}{2}\eta\nabla^\mu\psi_\mu - \tfrac{1}{2}\psi_\mu[\psi^\mu,\lambda] + \tfrac{1}{8}[\phi,\lambda]^2 + \tfrac{i}{8}\eta[\phi,\eta]\Big\}\,. \end{aligned} \tag{I.46}$$

Here, $\mathcal{D}_{AA'}$ is the Dirac operator obtained from $\nabla_\mu$ via (I.6).

Secondly, consider the hypermultiplet part (I.25). The hypermultiplet $SU_I(2)$ current contains a contribution coming from the bosonic part of the hypermultiplet, namely

$$j_a^\mu = -iq^\dagger\sigma_a D^\mu q + i(D^\mu q^\dagger)\sigma_a q, \tag{I.47}$$

where $D_\mu$ denotes the covariant derivative on scalars in the fundamental representation of $SU(n)$. Note that in the hypermultiplet only the bosons are charged with respect to $SU_I(2)$. The gauging is then again achieved by adding to the Lagrangian the term

$$-\omega_\mu^a j_\mu^a + q^\dagger\omega_\mu^a\omega^{\mu b}\sigma_a\sigma_b q, \tag{I.48}$$

such that the kinetic term for the bosons takes the form

$$((D_\mu + i\omega_\mu^a\sigma_a)q^\dagger)(D_\mu - i\omega^{\mu a}\sigma_a)q. \tag{I.49}$$

For the hypermultiplet we find the following translation table:

| Old field | $H$ | New field | $H'$ | Spin | Statistics |
|---|---|---|---|---|---|
| $q_i$ | $(0,0,\frac{1}{2})^0$ | $M_A$ | $(\frac{1}{2},0)^0$ | $\frac{1}{2}$ | boson |
| $\psi_{qA}$ | $(\frac{1}{2},0,0)^1$ | $-\frac{1}{\sqrt{2}}\mu_A$ | $(\frac{1}{2},0)^1$ | $\frac{1}{2}$ | fermion |
| $\bar\psi_{\tilde q}^{A'}$ | $(0,\frac{1}{2},0)^{-1}$ | $\frac{1}{\sqrt{2}}v^{A'}$ | $(0,\frac{1}{2})^{-1}$ | $\frac{1}{2}$ | fermion |
| $F_i$ | $(0,0,\frac{1}{2})^2$ | $K_A$ | $(\frac{1}{2},0)^2$ | $\frac{1}{2}$ | boson |
| $q^{i\dagger}$ | $(0,0,\frac{1}{2})^0$ | $\bar M^A$ | $(\frac{1}{2},0)^0$ | $\frac{1}{2}$ | boson |
| $\bar\psi_{qA'}$ | $(0,\frac{1}{2},0)^{-1}$ | $\frac{1}{\sqrt{2}}\bar v_{A'}$ | $(0,\frac{1}{2})^{-1}$ | $\frac{1}{2}$ | fermion |
| $\psi_{\tilde q}^A$ | $(\frac{1}{2},0,0)^1$ | $\frac{1}{\sqrt{2}}\bar\mu^A$ | $(\frac{1}{2},0)^1$ | $\frac{1}{2}$ | fermion |
| $F^{i\dagger}$ | $(0,0,\frac{1}{2})^{-2}$ | $\bar K^A$ | $(\frac{1}{2},0)^{-2}$ | $\frac{1}{2}$ | boson |

If now the Riemannian four-manifold $M$ is in addition a spin manifold, the twisted action corresponding to (I.25) on $M$ is given by

$$\begin{aligned} S_2 = \int_M \mathrm{dVol}\, \Big\{ &\nabla_\mu \bar{M}^A \nabla^\mu M_A + \tfrac{1}{4} R \bar{M}^A M_A - \tfrac{1}{4} \bar{M}^{(A} X^a M^{B)} \bar{M}_{(A} X^a M_{B)} \\ &- \tfrac{i}{2} (\bar{v}_{A'} \mathcal{D}^{A'A} \mu_A + \bar{\mu}^A \mathcal{D}_{AA'} v^{A'}) + \tfrac{i}{2} \bar{M}^A \{\phi, \lambda\} M_A \\ &- \tfrac{1}{\sqrt{2}} (\bar{M}_A \chi^{AB} \mu_B - \bar{\mu}_A \chi^{AB} M_B) + \tfrac{1}{2} (\bar{M}^A \psi_{AA'} v^{A'} + \bar{v}_{A'} \psi^{A'A} M_A) \\ &- \tfrac{1}{2} (\bar{\mu}^A \eta M_A + \bar{M}^A \eta \mu_A) + \tfrac{i}{4} \bar{v}^{A'} \phi v_{A'} - \bar{\mu}^A \lambda \mu_A \Big\}, \end{aligned} \tag{I.50}$$

and the twisted action $S'_{\mathrm{top}}$ is given by the sum $S'_{\mathrm{top}} = S_1 + S_2$. Here, $R$ denotes the scalar curvature of $M$ and the $X^a$'s are the skew-Hermitian generators of the gauge algebra $\mathfrak{su}(n)$.

**BRST transformations.** Finally, we are interested in the BRST transformation laws of our twisted fields in order to construct a cohomological action $S_{\mathrm{top}}$ and moreover, to say something about the localization of the path integral (I.32). The on-shell $Q = Q_{\bar{1}2} - Q_{\bar{2}1}$ transformations of the fields are given by[2]

$$[Q, \mathcal{A}_\mu] = i\psi_\mu, \quad \{Q, \psi_\mu\} = \nabla_\mu \phi, \quad [Q, \lambda] = \eta, \quad \{Q, \eta\} = i[\lambda, \phi], \quad [Q, \phi] = 0, \tag{I.51a}$$

$$\{Q, \chi^a_{AB}\} = -i\sqrt{2}(F^a_{AB} + \bar{M}_{(A} X^a M_{B)}) \tag{I.51b}$$

as well as

$$[Q, M_A] = \mu_A, \quad [Q, \bar{M}_A] = \bar{\mu}_A, \quad \{Q, \mu_A\} = -i\phi M_A, \quad \{Q, \bar{\mu}_A\} = i\bar{M}_A \phi, \tag{I.51c}$$

$$\{Q, v^{A'}\} = -2i\mathcal{D}^{A'A} M_A, \quad \{Q, \bar{v}_{A'}\} = -2i\mathcal{D}_{AA'} \bar{M}^A. \tag{I.51d}$$

Note that one can show that the above $Q$-transformations close on-shell up to gauge transformations with the scalar field $\phi$ as gauge parameter. Finally, after adding the topological term corresponding to the second Chern class,

$$S_3 = \frac{1}{4} \int_M \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}), \tag{I.52}$$

to the action $S'_{\mathrm{top}} = S_1 + S_2$, we can write the resulting action $S_{\mathrm{top}} = S_1 + S_2 + S_3$ in the promised $Q$-exact form,

$$S_{\mathrm{top}} = \{Q, V\}, \tag{I.53}$$

with

$$\begin{aligned} V = \int_M \mathrm{dVol}\, \Big[ &-\tfrac{i\sqrt{2}}{8} \chi^{aBA} (\mathcal{F}^a_{AB} + \bar{M}_{(A} X^a M_{B)}) + \tfrac{i}{4} (\bar{v}_{A'} \mathcal{D}^{A'A} M_A + (\mathcal{D}_{AA'} \bar{M}^A) v^{A'}) \\ &- \tfrac{1}{2} (\bar{\mu}^A \lambda M_A - \bar{M}^A \lambda \mu_A) + \mathrm{tr}\left\{ \tfrac{i}{2} \lambda \nabla^\mu \psi_\mu - \tfrac{i}{8} \eta [\phi, \lambda] \right\} \Big], \end{aligned} \tag{I.54}$$

[2]We use the convention $\mathcal{F}_{\mu\nu} = F^a_{\mu\nu} X_a$ and hence $\mathcal{F}_{AB} = F^{a+}_{AB} X_a$. Here, '+' denotes the self-dual part. Remember that $\mathcal{F}_{AB}$ corresponds to the self-dual part while $\mathcal{F}_{A'B'}$ corresponds to the anti-self-dual part of $\mathcal{F}_{\mu\nu}$; cf. [197].

as one may check using the transformation laws (I.51).

Witten's localization theorem, which we have discussed in the previous subsection, implies that the path integral (I.32) is localized on $Q$-invariant configurations. Employing this theorem the $Q$-transformation laws for $\chi_{AB}$ and $v^A$ given by (I.51) yield the equations

$$\mathcal{F}_{AB} = -\bar{M}_{(A}X^aM_{B)}X_a \qquad \text{and} \qquad \mathcal{D}^{A'A}M_A = 0. \tag{I.55}$$

These are known as a non-Abelian generalization of the Abelian Seiberg-Witten monopole equations (in spinor notation). The Abelian equations are contained as a special case of the above equations, namely

$$\mathcal{F}_{AB} = \bar{M}_{(A}M_{B)} \qquad \text{and} \qquad \mathcal{D}^{A'A}M_A = 0. \tag{I.56}$$

On the other hand choosing the matter field $M^A$ to be identically zero one recovers from (I.55) the $SU(n)$ ASDYM equations

$$\mathcal{F}_{AB} = 0 \tag{I.57}$$

which are used in the Donaldson-Witten theory.

Summarizing the above discussion, we have obtained a non-Abelian generalization of the Abelian Seiberg-Witten monopole equations after twisting an $\mathcal{N}=2$ SYM theory containing a vector multiplet and a hypermultiplet. The non-Abelian Seiberg-Witten equations reduce in the case of vanishing spinor field to the ASDYM equations. This is the promised bridge between the Seiberg-Witten theory and the Donaldson-Witten theory.

**Remark.** In our discussion, we have never used the auxiliary fields $K_A$ and $H_{AB}$. These were not needed for the construction of a twisted theory whose action functional was $Q$-exact on-shell. However, auxiliary fields permit to close the supersymmetry off-shell. Thus, our model can be extended to a theory which is $Q$-exact off-shell. We refer the interested reader to [138], where this is extensively discussed.

# Chapter II

## Mathematical Preliminaries - Part I

In this chapter we briefly review the mathematical background material needed for the first part of this thesis. Starting point is a rather basic discussion of Lie groups and their associated algebras. The most important mathematical objects for our purposes are fiber bundles. There are two main classes of fiber bundles: vector bundles and principal bundles. We study their geometry by introducing notions like connection, curvature and parallel transport and by analyzing their topological structure using homotopy theory. In addition we introduce several characteristic classes subsequently used in this thesis. Not only the studies of Seiberg-Witten theory require some amount of spin geometry, but also the discussion of instantonic solutions in pure Yang-Mills theory. For these reasons we discuss spin structures, spin bundles, etc., as well. This eventually results in the Atiyah-Singer index theorem for the spin complex. However, we cannot give here a detailed discussion of these subjects, especially of the underlying proofs, since this would carry us too far afield from the main thread of development. We therefore refer the reader to [87, 170, 45, 146, 51, 86, 182, 122, 197, 179, 68]. The ideas presented in this chapter are mainly taken from these references.

## § II.1 Lie Groups and Lie Algebras

**Lie groups and Lie algebras.** In general we are going to consider a gauge theory where the Lagrangian density belonging to this theory is invariant under the action of some Lie group $G$ via so-called local gauge transformations. A Lie group $G$ is a smooth manifold with an abstract group structure consisting of countably many connected components, where the multiplication $\mu : G \times G \to G$ defined by $(g, h) \mapsto gh$ and the inversion $i : G \to G$ given by $g \mapsto g^{-1}$ are smooth maps.

For all $g \in G$ we define the left and right translation $L_g$, $R_g : G \to G$ by $L_g(h) := gh$ and $R_g(h) := hg$, respectively. These maps are smooth and even diffeomorphisms. The inverse maps are just $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. Let $T_eG$ be the tangent space at the identity and denoted by $\mathfrak{g}$. Obviously, $\mathfrak{g} = T_eG$ and $T_gG$ are isomorphically related via the induced linear mappings $d(L_g)_e$ or $d(R_g)_e$. Sometimes, $d(L_g)$ and $d(R_g)$ are also denoted by $(L_g)_*$ and $(R_g)_*$.

Given $V \in \mathfrak{g}$ there exists a vector field $X_V$ defined by $X_V(g) := d(L_g)_e(V)$ which is left invariant , i.e. $d(L_g)(X) = X$. Thus, $\mathfrak{g}$ is isomorphic to the $\mathcal{F}(G)$-module of all left invariant vector fields on $G$ denoted by $\mathrm{Vect}_l(G) \subseteq \mathcal{X}(G)$, where $\mathcal{X}(G)$ is the $\mathcal{F}(G)$-module of all tangent vector fields on $G$. Here, $\mathcal{F}(G)$ denotes the ring of all smooth functions from $G$ to some field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$). Remember that we have for two vector fields $X$ and $Y$ another vector field $[X, Y]$ defined by $[X, Y]_g(f) := X_g(Y(f)) - Y_g(X(f))$, known as the Lie bracket, where $f \in \mathcal{F}(G)$. If $X$ and $Y$ are left invariant, so is $[X, Y]$. Thus, we have a map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$X_{[V,W]} := [X_V, X_W] \qquad \text{with} \qquad V, W \in \mathfrak{g}. \tag{II.1}$$

Examples of Lie groups are the matrix groups $GL(n, \mathbb{F})$ with their subgroups.

For any vector field $X$ on $G$ there exists a local flow $\phi(t, g)$ with $\phi(0, g) = g$ and $\partial_t\phi(t, g) = X_{\phi(t,g)}$ where $g \in G$ and $X \in \mathcal{X}(G)$. If $X \in \mathrm{Vect}_l(G)$ we have $\phi(t, g) = g\phi(t, e)$. Let $\exp : \mathfrak{g} \to G$ be defined by

$$\exp(V) := \phi(1, e) \qquad \text{with} \qquad V \in \mathfrak{g}, \tag{II.2}$$

called the exponential map, where $\phi$ is the local flow of $X$. Moreover, we know that $\exp : O \to U$ is a local diffeomorphism with $O \subseteq \mathfrak{g}$ an open neighborhood of 0 and $U \subseteq G$ an open neighborhood of $e$. Consider the following diagram

$$\begin{tikzcd} O \times O \arrow[r] \arrow[d, "\exp\times\exp"'] & \mathfrak{g} \arrow[d, "\exp"'] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

where $O \times O$ is an open neighborhood of $(0, 0)$ in $\mathfrak{g} \times \mathfrak{g}$. In order to obtain a commuting diagram one can show that

$$\exp(tV)\exp(tW) = \exp\Big(t(V + W) + \tfrac{t^2}{2}[V, W] + \mathcal{O}(t^3)\Big), \tag{II.4}$$

which is the famous Campbell-Baker-Hausdorff formula.

Note that for $G = GL(n, \mathbb{F})$[1] the exponential map $\exp(V)$ for $V \in \mathfrak{gl}(n, \mathbb{F})$ is really the exponential $e^V$ defined via the Taylor series expansion

$$e^V := \sum_{n\in\mathbb{N}_0} \tfrac{1}{n!}V^n, \tag{II.5}$$

since all left invariant vector fields on $GL(n, \mathbb{F})$ are of the form $X_g = gV$ for $g \in GL(n, \mathbb{F})$ and $V \in \mathfrak{gl}(n, \mathbb{F})$. Hence, $\phi(t, e) = \exp(tV)$[2] satisfies the differential equation $\partial_t\phi(t, e) = \phi(t, e)V$ with the initial condition $\phi(0, e) = e$. Since this equation is also satisfied by the matrix exponential function, we conclude by the uniqueness theorem for differential equations that $\exp(V) = e^V$.

[1] That is also true for all subgroups of $G = GL(n, \mathbb{F})$.

[2] Do not confuse $e$ with the Euler number in this context.

Moreover, we have the remarkable property that for compact and connected Lie groups $G$ the exponential map is surjective. This follows from the fact that for $G$ being compact and connected, the curves $\gamma(t) := \exp(tV)$ for $V \in \mathfrak{g}$ are the geodesics of $G$. Thus, $G$ is geodesically complete, therefore by the Hopf-Rinow theorem complete as a metric space, and consequently any two points in $G$ can be connected by a geodesic.

Given a Lie group $G$ we endow $\mathfrak{g} = T_eG$ (a vectorspace over $\mathbb{F}$) with $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $(\mathfrak{g}, [\ ,\ ])$ forms a Lie algebra over $\mathbb{F}$. The bracket is bilinear, skew-symmetric and satisfies the Jacobi identity

$$[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0 \quad \text{with} \quad U,V,W \in \mathfrak{g}. \tag{II.6}$$

Let $\{X_a\}$ be a basis of $\mathfrak{g}$. Then we write

$$[X_a, X_b] = C_{ab}{}^c X_c. \tag{II.7}$$

The $C_{ab}{}^c$'s are called the structure constants and the $X_a$'s the generators.

**Representations.** Let $GL(V)$ be the automorphism group of a vector space $V$ over $\mathbb{F}$. A representation $\mathfrak{R}$ of a group $G$ on $V$ is a homomorphism $\mathfrak{R} : G \to GL(V)$, i.e. for all $g \in G$ we have a linear bijection $\mathfrak{R}(g) : V \to V$ and $\mathfrak{R}(gh) = \mathfrak{R}(g)\mathfrak{R}(h)$. We call a representation irreducible if there is no nontrivial invariant subspace $W \subseteq V$, i.e. $\mathfrak{R}(G)W \subseteq W$. Similarly, we define representations of a Lie algebra $\mathfrak{g}$ via $\mathfrak{R} : \mathfrak{g} \to \mathfrak{gl}(V)$. The map $\mathrm{Int}_g : G \to G$ given by $h \mapsto ghg^{-1}$ defines a diffeomorphism which is called the adjoint map. Since $\mathrm{Int}_{gh} = \mathrm{Int}_g \mathrm{Int}_h$, we call it the adjoint representation of $G$ on itself. The linearization of $\mathrm{Int}_g$ at the identity is given by $d(\mathrm{Int}_g)_e =: \mathrm{Ad}_g$. Furthermore, we have $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h$ and therefore $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defines a representation called the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$. We have the following commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"'] \\ G \arrow[r, "\mathrm{Int}_g"'] & G \end{tikzcd}$$

or equally $g(\exp(V))g^{-1} = \exp(\mathrm{Ad}_g V)$. On the other hand, we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d(\mathrm{Ad})_e"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"'] \\ G \arrow[r, "\mathrm{Ad}"'] & GL(\mathfrak{g}) \end{tikzcd}$$

since Ad is a Lie homomorphism and we denote $d(\mathrm{Ad})_e$ by ad. Hence, $\mathrm{Ad}_{\exp(V)} = e^{\mathrm{ad}(V)}$. A short calculation shows that $\mathrm{ad}(V)(W) = [V,W]$. Since ad respects the bracket, $\mathrm{ad}([V,W]) = [\mathrm{ad}(V), \mathrm{ad}(W)]$, it forms a representation of the Lie algebra $\mathfrak{g}$ on itself and is called the adjoint representation and therefore given by the structure constants. A Lie algebra $\mathfrak{g}$ is said to be simple if it is non-Abelian and has no ideals, i.e. $\mathfrak{h} \subseteq \mathfrak{g}$ with $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, except for $\{0\}$ and $\mathfrak{g}$. If a Lie algebra $\mathfrak{g}$ does not contain Abelian ideals it is called semisimple. A Lie group $G$ is called simple (semisimple) if its Lie algebra $\mathfrak{g}$ is simple (semisimple). Thus, simple Lie algebras are always semisimple, while the converse may not be true. One can prove that for compact and semisimple Lie groups the structure constants $C_{abc}$ may be chosen to be totally antisymmetric, where $C_{abc} = g_{cd}\, C_{ab}{}^{d}$ and $g_{ab}$ is the Cartan-Killing form $g(V, W) = \mathrm{tr}(\mathrm{ad}(V)\, \mathrm{ad}(W))$ in the basis $\{X_a\}$.

**Examples.** The most interesting Lie groups for our purposes will be the unitary goups $U(n) = \{g \in GL(n, \mathbb{C}) \,|\, g^\dagger = g^{-1}\}$ and $SU(n) = \{g \in U(n) \,|\, \det(g) = 1\}$. The $SU(n)$ group is an example of a compact, connected and simple Lie group. A complex $n \times n$ matrix has $2n^2$ real parameters. However, we have $n^2 + 1$ constraints due to the unitary and determinant (equal to one) condition. Hence, $U(n)$ is $n^2$ and $SU(n)$ is $n^2 - 1$ dimensional. Let us consider as simplest examples the $SU(2)$ and $U(2)$ groups. For $SU(2)$ we have three free parameters. The Lie algebra $\mathfrak{su}(2)$ is given by $\mathfrak{su}(2) = \{C \in \mathfrak{gl}(2, \mathbb{C}) \,|\, C^\dagger = -C, \mathrm{tr}\, C = 0\}$. Constructing those matrices yields the following commutation relations

$$[X_a, X_b] = \epsilon_{abc} X_c, \qquad \text{with} \qquad a, b, c = 1, 2, 3, \tag{II.10}$$

where $\epsilon_{abc}$ is the Levi-Civita symbol with $\epsilon_{123} = 1$. In the fundamental representation the generators are given by

$$X_a^{(f)} := \tfrac{1}{2i} \sigma_a, \tag{II.11}$$

with the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{II.12}$$

In this representation the generators satisfy

$$\mathrm{tr}(X_a^{(f)} X_b^{(f)}) = -\tfrac{1}{2} \delta_{ab}. \tag{II.13}$$

Using our relation for the exponential map and remembering that for matrix groups it coincides with the ordinary exponential we obtain all group elements $g \in SU(2)$ via

$$g(x) = e^{\alpha^a(x) X_a^{(f)}}, \tag{II.14}$$

$\alpha^a(x)$ being smooth (real valued) functions of some parameter $x$.

Now let $\phi : U(1) \times SU(2) \to U(2)$ be given by $(\lambda, g) \mapsto \lambda g$. Note that this defines a surjective homomorphism of Lie groups, since one can rewrite $q \in U(2)$ as follows

$$q = \underbrace{(\det(q))^{1/2}}_{\in U(1)} \underbrace{((\det(q))^{-1/2} q)}_{\in SU(2)}.$$

The kernel of $\phi$ is given by $\ker(\phi) = \{(\pm 1, \pm e)\}$ and therefore by the first isomorphism theorem of groups

$$U(2) \cong (U(1) \times SU(2))/\{\pm(1,e)\} \cong (U(1) \times SU(2))/\mathbb{Z}_2, \tag{II.16}$$

where $\mathbb{Z}_2$ is the group of two elements. Topologically, $U(2)$ is

$$U(2) \simeq U(1) \times SU(2) \cong S^1 \times S^3, \tag{II.17}$$

since we have a diffeomorphism

$$q \mapsto \left(\det(q), \begin{pmatrix} (\det(q))^{-1} & 0 \\ 0 & 1 \end{pmatrix} q\right)$$

with $q \in U(2)$, which of course is not a group homomorphism. $S^n$ denotes the $n$-sphere.

For the Lie algebra $\mathfrak{u}(2) = \{C \in \mathfrak{gl}(2, \mathbb{C}) \,|\, C^\dagger = -C\}$ we find

$$\mathfrak{u}(2) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2) \cong i\mathbb{R} \oplus \mathfrak{su}(2), \tag{II.19}$$

since the tangent bundle of a product manifold is the sum of the tangent bundles of each manifold.

Generally speaking, we have the isomorphism

$$U(n) \cong (U(1) \times SU(n))/\mathbb{Z}_n \tag{II.20}$$

and hence for the Lie algebra,

$$\mathfrak{u}(n) \cong i\mathbb{R} \oplus \mathfrak{su}(n), \tag{II.21}$$

what shows that $U(n)$ is neither simple nor semisimple.

# § II.2 Geometry of Fiber Bundles

## § II.2.1 Fiber Bundles

**Vector and principal bundles.** Given a smooth manifold $M$ which we call the base manifold and another smooth manifold $F$ which we call the fiber, we define a smooth manifold $E$, the fiber bundle over $M$, together with a projection $\pi : E \to M$ which satisfies the local triviality condition: For each point $x \in M$ there exists an open neighborhood $U_i$ of $x$, where the set $\{U_i\}$ covers $M$, and a diffeomorphism[3] $\phi_i : U_i \times F \to \pi^{-1}(U_i)$. Let $(x, v)$ denote a point of $U_i \times F$. Then we require that $\pi(\phi_i(x, v)) = x$ as a consistency condition. We call $\pi^{-1}(x) = F_x$ the fiber over $x$; $E = \bigcup_{x \in M} F_x$. $E$ is also called the total space and sometimes denoted by $E(M, F)$, if it is not clear from the context. So far, this definition has no information about the global topology of $E$. We specify further by introducing the transition functions which describe the

[3]Later on, we will also deal with holomorphic vector bundles. Then we require the $\phi_i$'s to be biholomorphisms.

"gluing" of the fibers: Let $x$ be in the intersection of $U_i$ and $U_j$, i.e. $x \in U_i \cap U_j$. Then for all bundle points $b \in E$ with $\pi(b) = x$ we assign two local coordinates $(x, v_i)$ and $(x, v_j)$ and require that the change of local coordinates is given by $v_i = \phi_{ij} v_j$ with $\phi_{ij} = \phi_i^{-1} \circ \phi_j$ which are called the transition functions and belong to a group $G$ of transformations of the fiber space $F$. We call $G$ the structure group of the fiber bundle. The transition functions satisfy the cocycle condition, i.e. $\phi_{ii} = \mathrm{id}$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for $x \in U_i \cap U_j \cap U_k$. Hence, the global topology of $E$ might be quite complicated due to the relative twisting of neighboring fibers. Note that in the case when all transition functions can be taken to be the identity, the global topology of $E$ is that of a direct product, i.e. $E = M \times F$.

A map $s : M \to E$ with $\pi \circ s = \mathrm{id}$ is called a section of the fiber bundle, i.e. $s_x \in F_x$ for all $x \in M$. The $\mathcal{F}(M)$-module of all smooth sections of $E$ is denoted by $\Gamma(M, E)$, where $\mathcal{F}(M) := \{f : M \to \mathbb{F} \,|\, f \text{ smooth }\}$. Sometimes we also use the notation $C^\infty(M, E)$.

In general one distinguishes two main classes of fiber bundles: vector bundles and principal bundles. Principal bundles with their associated vector bundles play the essential role in gauge field theories.

A vector bundle $E = E(M, V)$ with the fiber $V$, $V$ being a vector space over $\mathbb{F}$, is a surjection $\pi : E \to M$ between the bundle manifold $E$ and the base manifold $M$ which satisfies the local triviality condition. Moreover, the transition functions belong to $GL(V)$. Examples of vector bundles are the tangent bundle $TM$ and its dual bundle $T^*M$, where the dimensions of the fibers are the same as the dimension of the base manifold.

A principal bundle $P = P(M, G)$ with the fiber $G$, a Lie group, is a surjection $\pi : P \to M$ between the bundle manifold $P$ and the base manifold $M$ which satisfies the local triviality condition. Furthermore, the structure group is $G$, as well. The transition functions of $P$ act on $G$ by left multiplication, i.e. $g_i = \phi_{ij} g_j$, where $x \in U_i \cap U_j$ and $(x, g_i)$ and $(x, g_j)$ are the two local coordinates. We can define a right action of the structure group $G$ on $P$ in a natural manner since left and right multiplication commute. This gives a map from $P \times G \to P$ which acts fiberwise, i.e. $\pi(bg) = \pi(b)$ for all $g \in G$ and $b \in P$ or explicitly in local coordinates $\pi(x, g_j g) = \pi(x, g_j) = x$.

Let $\pi : P \to M$ be a principal bundle with structure group $G$ and $\mathfrak{R} : G \to GL(V)$ be a representation of $G$ on $V$. The vector bundle $\pi_E : E \to M$ with fiber $V$ is said to be associated to the principal bundle $\pi : P \to M$ if $\phi_{ij}^E = \mathfrak{R}(\phi_{ij}^P)$. If all $\phi_{ij}^E$ lie in a subgroup $H$ of $GL(V)$, then $H$ is the structure group of $E$.

## § II.2.2 Connection and Curvature

**Connection.** So far, we have only discussed fiber bundles in a global geometric context. However, to study local properties of fiber bundles we need the notion of a connection in order to construct a covariant derivative, for instance. Let us start with principal bundles. There are several approaches to introduce a connection. Let $P$ be some principal bundle and $b \in P$ a bundle point with $G_x$ a fiber over $x$; $x = \pi(b)$. The vertical subspace $V_bP \subseteq T_bP$ is the subspace which is tangent to $G_x$. Choose an arbitrary $V \in \mathfrak{g}$. Then $R_{\exp(tV)}b = b\exp(tV)$ defines a curve through $b$ in $P$. Since $\pi(b) = \pi(b\exp(tV)) = x$ this curve lies within $G_x$. Now we define a vector field $V^\# \in T_bP$ by

$$V^\#(f)(b) := \frac{\mathrm{d}}{\mathrm{d}\,t} f(b\exp(tV))\bigg|_{t=0} \qquad \text{with} \qquad f \in \mathcal{F}(P). \tag{II.22}$$

The vector $V^\#$ is tangent to $G_x$ and hence $V^\# \in V_bP$. Thus, we have constructed a vector field at each point of $P$ and call it the fundamental vector field generated by $V$ via the isomorphism $\# : \mathfrak{g} \to V_bP$ which sends $V$ to $V^\#$. The horizontal subspace $H_bP$ is a complement of $V_bP$ in $T_bP$. Note that $\#$ preserves the bracket, that is $[V, W]^\# = [V^\#, W^\#]$.

A connection on $P$ is then a unique seperation of the tangent space $T_bP$ into the two subspaces $V_bP$ and $H_bP$ such that the following holds:

(i) $T_bP = V_bP \oplus H_bP$.

(ii) $H_{bg}P = d(R_g)H_bP$ with $b \in P$ and $g \in G$, i.e. $H_bP$ and $H_{bg}P$ on the same fiber are related by the linear map $d(R_g)$ and therefore $H_bP$ at $b$ generates all the horizontal subspaces on the same fiber.

Using a Lie algebra valued one-form $\mathcal{A} \in \mathfrak{g} \otimes T^*P$ called the connection one-form $\mathcal{A}$, we can seperate $T_bP$ into $V_bP$ and $H_bP$ in a systematic way. This form is defined as

$$\mathcal{A}(V^\#) := V \quad \text{and} \quad R_g^*\mathcal{A} := \mathrm{Ad}_{g^{-1}}\mathcal{A} \quad \text{with} \quad V \in \mathfrak{g} \quad \text{and} \quad g \in G. \tag{II.23}$$

So, $\mathcal{A}$ is a projection of $T_bP$ onto the vertical component $V_bP \cong \mathfrak{g}$. The second condition of (II.23) reads

$$R_g^*\mathcal{A}_{bg}(X) = \mathcal{A}_{bg}(d(R_g)X) = g^{-1}\mathcal{A}_b(X)g \qquad \text{with} \qquad X \in T_bP.$$

The horizontal subspace $H_bP$ is then defined as the kernel of $\mathcal{A}$,

$$H_bP = \{X \in T_bP \,|\, \mathcal{A}(X) = 0\}. \tag{II.25}$$

Note that (ii) is satisfied. Indeed, let $X \in H_bP$ and $d(R_g)X \in T_{bg}P$. Then we have

$$\mathcal{A}(d(R_g)(X)) = R_g^*\mathcal{A}(X) = g^{-1}\mathcal{A}(X)g = 0,$$

that is $d(R_g)(X) \in H_{bg}P$. Since $d(R_g)$ is an invertible linear map, any $Y \in H_{bg}$ can be expressed via some $X \in H_bP$ using the right translation.

Let $\{U_i\}$ be an (open) covering of $M$ and $\sigma_i$ be a local section defined on each $U_i$. We define a Lie algebra valued one form $\mathcal{A}_i := \sigma_i^*\mathcal{A} \in \mathfrak{g} \otimes \Omega^1(U_i)$, where $\Omega^p(U_i)$ is the space of all

$p$-forms on $U_i$. Note that $\Omega^p(U_i)$ is also denoted by $\Lambda^p(T^*U_i)$. We shall use both symbols in this thesis. Remember that for a given Lie algebra valued one-form $\mathcal{A}_i$ on $U_i$ and a local section $\sigma_i : U_i \to \pi^{-1}(U_i)$ there exists a connection one-form $\mathcal{A}$ with $\mathcal{A}_i := \sigma_i^* \mathcal{A}$ on $U_i$. This follows from the fact that the Lie algebra valued one-form

$$\mathcal{A}_i = g_i^{-1} \pi^* \mathcal{A}_i g_i + g_i^{-1} \, \mathrm{d}\, g_i, \tag{II.27}$$

where 'd' stands for the exterior derivative on $P$ and $g_i$ is the canonical local trivialization defined by $\phi_i^{-1}(b) = (x, g_i)$ for $b = \sigma_i(x) g_i$, satisfies the properties of a connection one-form and $\mathcal{A}_i = \sigma_i^* \mathcal{A}_i$.

However, for $\mathcal{A}$ defined uniquely on $P$ we need $\mathcal{A}_i = \mathcal{A}_j$ on $U_i \cap U_j$ and $\mathcal{A}$ is then given by $\mathcal{A}|_{U_i} = \mathcal{A}_i$. In order to have this property, the pullbacks $\mathcal{A}_i$ must be related in a particular way on intersection of charts, namely

$$\mathcal{A}_j = \phi_{ij}^{-1} \mathcal{A}_i \phi_{ij} + \phi_{ij}^{-1} \, \mathrm{d}\, \phi_{ij}. \tag{II.28}$$

Note that a nontrivial principal bundle does not admit a global section; therefore the pullbacks $\mathcal{A}_i$ exist locally but not necessarily globally. Physically, $\mathcal{A}_i$ is identified with the gauge potential on $U_i$.

Let $P$ be a principal bundle over $M$ and $\sigma$ $(\sigma^g)$ two local sections over $U \subseteq M$ with $\sigma^g(x) = \sigma(x) g(x)$. Then we have

$$\mathcal{A}^g = g^{-1} \mathcal{A} g + g^{-1} \, \mathrm{d}\, g, \tag{II.29}$$

what we call a gauge transformation.

**Parallel transport.** Similarly to Riemannian geometry the notion of parallel transport in a principal bundle is of crucial importance. It is provided by the horizontal lift of a curve lying in the base manifold $M$. Let $P$ be a principal bundle and $c : [0,1] \to M$ a smooth curve. A curve $\tilde{c} : [0,1] \to P$ is called the horizontal lift of $c$ if $\pi \circ \tilde{c} = c$ and the tangent vector of $\tilde{c}(t)$ always belongs to the horizontal subspace $H_{\tilde{c}(t)} P$. The tangent vector $\tilde{X}$ to such a $\tilde{c}$ satisfies $\mathcal{A}(\tilde{X}) = 0$. Now let $b_0 \in \pi^{-1}(c(0))$. Then there exists a unique horizontal lift $\tilde{c}(t)$ in $P$ such that $\tilde{c}(0) = b_0$. The question is, how we can construct such a curve $\tilde{c}$. To answer this question we write the differential equation $\mathcal{A}(\tilde{X}) = 0$ in a local form. Let $U_i$ be a chart and $\sigma_i$ a section over $U_i$. The fundamental theorem of ordinary differential equations guarantees the local existence and uniqueness of horizontal lifts. Hence, we may express $\tilde{c}$ as $\tilde{c}(t) = \sigma_i(t) g_i(t)$, where $\sigma_i$ and $g_i$ are taken at $c(t)$. Furthermore, we may assume that $\sigma_i(0) = \tilde{c}(0)$, i.e. $g_i(0) = e$. If $X$ is the tangent vector to $c$, then $\tilde{X} = d\tilde{c}(X)$ is the tangent vector to $\tilde{c}$. Thus, we have

$$\begin{aligned}
\tilde{X} = d\tilde{c}(X) &= \frac{\mathrm{d}}{\mathrm{d}\,t} \tilde{c}(c(t)) \\
&= \Big( \frac{\mathrm{d}}{\mathrm{d}\,t} \sigma_i(t) \Big) g_i(t) + \sigma_i(t) \frac{\mathrm{d}}{\mathrm{d}\,t} g_i(t) \\
&= d(R_{g_i(t)}) d\sigma_i(X) + \tilde{c}(t) g_i^{-1}(t) \frac{\mathrm{d}}{\mathrm{d}\,t} g_i(t).
\end{aligned}$$

The second term of the last line represents the vector field $(g_i^{-1}\,\mathrm{d}\,g_i(X))^{\#}$ taken at $\tilde{c}(t)$. The application of the connection one-form yields

$$\mathcal{A}(\tilde{X}) = 0 = g_i^{-1}(t)\mathcal{A}(d\sigma_i(X))g_i(t) + g_i^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}\,t}g_i(t).$$

Remembering that $\mathcal{A}(d\sigma_i(X)) = \sigma_i^*\mathcal{A}(X) = \mathcal{A}_i(X)$, we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}\,t}g_i(t) + \mathcal{A}_i(X)g_i(t) = 0 \qquad \text{with} \qquad g_i(0) = e. \tag{II.31}$$

Using the path-ordering symbol, we can formally write down a solution to this differential equation

$$g_i(c(t)) = \mathrm{P}\exp\left(-\int_0^t \mathcal{A}_{i\mu}\frac{\mathrm{d}\,x^\mu}{\mathrm{d}\,t}\,\mathrm{dt}\right) = \mathrm{P}\exp\left(-\int_{c(0)}^{c(t)} \mathcal{A}_{i\mu}\,\mathrm{dx}^\mu\right) \tag{II.32}$$

with $\tilde{c}(t) = \sigma_i(c(t))g_i(c(t))$. The point $\tilde{c}(t)$ is then called the parallel transport of $b_0$ along the curve $\tilde{c}$. In physics the object (II.32) is called a Wilson line.

**Curvature.** Having introduced the notion of connection it is natural to ask what the curvature for a principal bundle should be. The curvature form is defined as the covariant derivative of the connection one-form

$$\mathcal{F} := \mathrm{D}\,\mathcal{A} \in \mathfrak{g} \otimes \Omega^2(P), \tag{II.33}$$

where we have used the definition of the covariant derivative for Lie algebra valued differential forms

$$\mathrm{D}\,\omega(X_1,\ldots,X_{p+1}) := \mathrm{d}\,\omega(X_1^H,\ldots,X_{p+1}^H) \quad \text{with} \quad \omega \in \mathfrak{g}\otimes\Omega^p(P) \quad \text{and} \quad X_i \in T_bP. \tag{II.34}$$

Here, $X_a^H$ denotes the horizontal part of $X_a$. The connection one-form and the curvature two-form satisfy the Cartan's structure equation

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}] = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}. \tag{II.35}$$

Here, we have used the definition $[\omega,\rho] := [X_a,X_b]\otimes(\omega^a\wedge\rho^b)$ for Lie algebra valued forms $\omega = X_a\otimes\omega^a$ and $\rho = X_a\otimes\rho^a$ where $\omega^a$ and $\rho^a$ are ordinary differential forms while $\{X_a\}$ forms a basis of $\mathfrak{g}$. Remark: In the following we shall omit '$\otimes$' and simply write $\omega = X_a\omega^a$.

Let again $\{U_i\}$ be an (open) covering of $M$ and $\sigma_i$ be a local section defined on each $U_i$. Then we define a Lie algebra valued two-form as the pullback $\mathcal{F}_i := \sigma_i^*\mathcal{F} \in \mathfrak{g}\otimes\Omega^2(U_i)$, which is expressed in terms of the gauge potential $\mathcal{A}_i$ as

$$\mathcal{F}_i = \mathrm{d}\mathcal{A}_i + \mathcal{A}_i\wedge\mathcal{A}_i \tag{II.36}$$

and identified with the field strength on $U_i$. The component expression of $\mathcal{F}_i$ on the chart $U_i$ is readily computed using $\mathcal{A}_i = \mathcal{A}_{i\mu}\,\mathrm{d}\,x^\mu$ with $\mathcal{A}_{i\mu}\in\mathfrak{g}$

$$\mathcal{F}_i = \tfrac{1}{2}\mathcal{F}_{i\mu\nu}\,\mathrm{d}\,x^\mu\wedge\mathrm{d}\,x^\nu \quad \text{and} \quad \mathcal{F}_{i\mu\nu} = \partial_\mu\mathcal{A}_{i\nu} - \partial_\nu\mathcal{A}_{i\mu} + [\mathcal{A}_{i\mu},\mathcal{A}_{i\nu}]. \tag{II.37}$$

On the overlapping region $U_i \cap U_j$ the field strengths $\mathcal{F}_i$ and $\mathcal{F}_j$ satisfy

$$\mathcal{F}_j = \mathrm{Ad}_{\phi_{ij}^{-1}} \mathcal{F}_i = \phi_{ij}^{-1} \mathcal{F}_i \phi_{ij}. \tag{II.38}$$

Similarly, under a gauge transformation (II.29) $\mathcal{F}$ transforms as

$$\mathcal{F}^g = \mathrm{Ad}_{g^{-1}} \mathcal{F} = g^{-1} \mathcal{F} g. \tag{II.39}$$

Furthermore, $\mathfrak{F}_i$ satisfies a Bianchi identity

$$\mathrm{D}\, \mathfrak{F}_i = 0 \tag{II.40}$$

or in the pulled back version

$$\mathcal{D}\mathcal{F}_i = \mathrm{d}\mathcal{F}_i + \mathcal{A}_i \wedge \mathcal{F}_i - \mathcal{F}_i \wedge \mathcal{A}_i = 0 \tag{II.41}$$

with the definition

$$\mathcal{D}\omega := \mathrm{d}\omega + [\mathcal{A}, \omega] = \mathrm{d}\omega + \mathcal{A} \wedge \omega - (-1)^p \omega \wedge \mathcal{A} \qquad \text{for} \qquad \omega \in \mathfrak{g} \otimes \Omega^p(M). \tag{II.42}$$

**Covariant derivative.** In fact, above we have introduced covariant differentiation on principal bundles. However, we know that physical (matter) fields are sections of associated vector bundles.[4] Therefore we need also there the notion of covariant differentiation. The connection one-form $\mathcal{A}$ provides us in a natural way the covariant derivative in associated vector bundles.

Let $P$ be a principal bundle and $E = P \times_G V$ the associated vector bundle with $\mathfrak{R} : G \to GL(V)$ a representation of $G$ on $V$. Note that $E$ is defined as the set of equivalence classes $[(b, v)] = \{(bg, (\mathfrak{R}(g))^{-1} v) \,|\, b \in P, v \in V, g \in G\}$. In the following we omit the letter $\mathfrak{R}$ for notational simplicity. Let $s : M \to E$ be a section on $M$ given by $s(x) := [(\sigma_i(x), v(x))]$. A vector is said to be parallel transported if it is constant with respect to a horizontal lift $\tilde{c}$ of $c$ in $P$, i.e. the section $s(c(t)) := [(\tilde{c}(t), v(c(t)))]$ is parallel transported if $v$ is constant along $c(t)$. The covariant derivative $\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$ of $s$ along $c$ at $c(0)$ is defined by

$$\nabla_X s := \left[\left(\tilde{c}(0), \left.\frac{\mathrm{d}}{\mathrm{d}\,t} v(c(t))\right|_{t=0}\right)\right], \tag{II.43}$$

where $X$ is the tangent vector to $c$ at $c(0)$. It has the following properties

$$\begin{aligned} \nabla_X(s_1 + s_2) &= \nabla_X s_1 + \nabla_X s_2, && \text{(II.44a)} \\ \nabla_{f X + g Y} s &= f \nabla_X s + g \nabla_Y s, && \text{(II.44b)} \\ \nabla_X (f s) &= f \nabla_X s + X(f) s && \text{(II.44c)} \end{aligned}$$

[4] To be more precise matter fields can additionally live in other bundles. For instance, in spinor bundles when dealing with spinorial fields.

with $s, s_1, s_2 \in \Gamma(M,E)$, $X, Y \in \mathcal{X}(M)$ and $f, g \in \mathcal{F}(M)$. Again, we write $\tilde{c}$ as $\tilde{c}(t) = \sigma_i(t) g_i(t)$ and choose a particular section of $E$ given by $e_a(x) := [(\sigma_i(x), e_a^0)]$. Here $e_a^0$ is the $a^{\text{th}}$ basis vector of $V$ with $(e_a^0)^b = \delta_a^{\ b}$. Then we can write

$$e_a(t) = [(\tilde{c}(t) g_i^{-1}(t), e_a^0)] = [(\tilde{c}(t), g_i^{-1}(t) e_a^0)].$$

Equation (II.31) then yields

$$\begin{aligned} \nabla_X e_a &= [(\sigma_i(0), \mathcal{A}_i(X) e_a^0)] \\ &= \left[\left(\sigma_i(0), \frac{\mathrm{d}\, x^\mu}{\mathrm{d}\, t} \mathcal{A}_{i\mu}{}^b{}_a e_b^0\right)\right] \\ &= \frac{\mathrm{d}\, x^\mu}{\mathrm{d}\, t} \mathcal{A}_{i\mu}{}^b{}_a e_b. \end{aligned} \tag{II.46}$$

Finally, for a section $s_i(x) = [(\sigma_i(x), v_i(x))] = v_i^a e_a$ we obtain the local form of the covariant derivative

$$\nabla_X s_i = \left[\left(\sigma_i(0), \frac{\mathrm{d}\, v_i}{\mathrm{d}\, t} + \mathcal{A}_i(X) v_i|_{t=0}\right)\right] = \frac{\mathrm{d}\, x^\mu}{\mathrm{dt}} \left\{\frac{\partial v_i^a}{\partial x^\mu} + \mathcal{A}_{i\mu}{}^a{}_b v_i^b\right\} e_a = \left\{\frac{\mathrm{D}}{\mathrm{d}\, t} v_i\right\}^a e_a \tag{II.47}$$

with

$$D := \mathrm{d} + \mathcal{A} = \mathrm{d}\, x^\mu \left\{\frac{\partial}{\partial x^\mu} + \mathcal{A}_\mu\right\} = \mathrm{d}\, x^\mu \nabla_{\partial_\mu}. \tag{II.48}$$

Furthermore, we have with $\nabla_\mu := \nabla_{\partial_\mu}$

$$[\nabla_\mu, \nabla_\nu] s_i = \mathcal{F}_{i\mu\nu} s_i. \tag{II.49}$$

Thus, the curvature measures the extent to which covariant differentiation fails to commute. Note that under a gauge transformation $s_i \mapsto s_i^g$ given by $s_i^g = [(\sigma_i^g, g^{-1} v_i)]$ for $g \in G$ the covariant derivative transforms by construction in the same way as $s$, that is

$$(\nabla_X s_i)^g = \left[\left(\sigma_i^g(0), g^{-1}\left\{\frac{\mathrm{d}\, v_i}{\mathrm{d}\, t} + \mathcal{A}_i(X) v_i|_{t=0}\right\}\right)\right]. \tag{II.50}$$

### § II.2.3 Homotopy Groups

This subsection is a brief résumé of homotopy theory. Later on, we are dealing with field configurations having finite action or energy. Nontrivial topological properties of such configurations can be described via homotopy groups.

**The fundamental group.** Let $X$ and $Y$ denote two topological spaces. In most of our examples $X$ will be the physical coordinate space and $Y$ the target space. If we can deform a field configuration $f : X \to Y$ continuously into another field configuration $g : X \to Y$ without passing through forbidden configurations where the considered functional becomes infinite, we shall call these two configurations topologically equivalent and write $f \simeq g$. One also says that $f$ is homotopic to $g$ via the continuous map $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x) \quad \forall\, x \in X.$$

$F$ is called a homotopy. Homotopy is an equivalence relation on the set of all continuous maps from $X \to Y$. The equivalence classes under this equivalence relation are denoted by $[f]$ and are called homotopy classes.

Firstly, we restrict our attention to the case where $X = [0,1]$. Then the continuous map $f : [0,1] \to Y$ is called a path in $Y$ and if $f(0) = y_0$ and $f(1) = y_1$, one says that $f$ is a path from $y_0$ to $y_1$. The space $Y$ is said to be path connected if, for every pair $y_0, y_1 \in Y$, there exists a path in $Y$ from $y_0$ to $y_1$. Given two paths $f, g : [0,1] \to Y$ such that $f(1) = g(0)$, there is a product path $f \bullet g$ that traverses first $f$ and then $g$, given by

$$(f \bullet g)(t) := \begin{cases} f(2t) & \text{for} \quad 0 \le t \le 1/2, \\ g(2t-1) & \text{for} \quad 1/2 \le t \le 1. \end{cases} \tag{II.52}$$

This product operation respects the homotopy classes, i.e. if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $f_0 \bullet g_0 \simeq f_1 \bullet g_1$.

Now let us specify further and concentrate on closed paths (also referred to as loops), i.e. $f(0) = f(1) = y_0$, where $y_0$ is any fixed point of $Y$ known as the base point. The set of all homotopy classes $[f]$ of loops $f : [0,1] \to Y$ with basepoint $y_0$ is denoted by $\pi_1(Y, y_0)$. One can show that $\pi_1(Y, y_0)$ with the multiplication $[f][g] := [f \bullet g]$ forms a group, the fundamental group of $Y$ at the base point $y_0$. If $Y$ is path connected then the groups $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$ are isomorphic. In that sense the group $\pi_1(Y, y_0)$ is independent of the choice of the basepoint. Then we simply write $\pi_1(Y)$.

The standard example of a manifold $M$ with a nontrivial homotopy group is the circle, i.e. $M = S^1$. Let us view $S^1$ as the set of all complex numbers $z$ with $|z| = 1$. Obviously, the map $z \mapsto z^1$, which wraps $[0,1]$ around $S^1$ once, is not homotopic to the constant map $z \mapsto z^0 = 1$. Thus, we need a way to distinguish these two maps. From elementary complex calculus we know that these functions can be distinguished by a particular line integral known as the winding number

$$N(f) := \frac{1}{2\pi i} \oint_f \frac{\mathrm{dz}}{z}, \tag{II.53}$$

which can readily be evaluated by writing $f(t) = e^{2\pi i h(t)}$

$$N(f) = \frac{1}{2\pi i} \oint_f \frac{\mathrm{dz}}{z} = \int_0^1 \mathrm{d}t \, h'(t) = h(1) - h(0).$$

With $h(t) = nt + k$, $n \in \mathbb{Z}$ and $k \in \mathbb{R}$ we find $N(f) = n$. These statements do not provide a proof, rather they are supposed to give the intuition for the result

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{II.55}$$

A topological space $Y$ is said to be simply connected if any closed path in $Y$ is nullhomotopic. For example, all spheres $S^n$ with $n > 1$ are simply connected resulting in a trivial fundamental group, i.e.

$$\pi_1(S^n) \cong \{e\} \qquad \text{with} \qquad n > 1. \tag{II.56}$$

**Higher homotopy groups.** The fundamental group $\pi_1(Y, y_0)$ is the first in a sequence of groups $\pi_n(Y, y_0)$ called the homotopy groups. These groups are defined in a similar manner. Instead of considering maps from the interval $[0,1] = I$ into a topological space $Y$, we consider maps from the unit $n$-cube $I^n = [0,1] \times \cdots \times [0,1]$ into $Y$. The multiplication of two maps $f, g : I^n \to Y$ such that $f(1, t_2, \ldots, t_n) = g(0, t_2, \ldots, t_n)$ is defined by

$$(f \bullet g)(t_1, t_2, \ldots, t_n) := \begin{cases} f(2t_1, t_2, \ldots, t_n) & \text{for} \quad 0 \leq t_1 \leq 1/2, \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{for} \quad 1/2 \leq t_1 \leq 1 \end{cases} \tag{II.57}$$

and respects the homotopy classes.

Again, the map $f : I^n \to Y$ is chosen to be a "loop" in the sense of taking the same value on the boundary $\partial I^n$ of the unit $n$-cube. Then $\pi_n(Y, y_0)$ is the group of all homotopy classes of "loops" with the product being $[f][g] := [f \bullet g]$. If $Y$ is path connected the $n$-th homotopy group is independent of the choice of the base point. Then we write $\pi_n(Y)$. One property of the higher homotopy groups is that they are always Abelian. Choosing $Y$ to be the $n$-sphere $S^n$, we can again introduce winding numbers which measure how often the unit $n$-cube $I^n$ is wrapped around $S^n$. This finally results in

$$\pi_n(S^n) \cong \mathbb{Z} \qquad \text{with} \qquad n \geq 1. \tag{II.58}$$

Demanding that $f$ maps the boundary $\partial I^n$ of $I^n$ to a single point, we have effectively obtained the $n$-sphere $S^n$ from $I^n$, that means we have identified the boundary $\partial I^n$ of $I^n$ to a single point $S^n \cong I^n / \partial I^n$. For instance, considering the interval $[0,1]$ and identifying its endpoints yields the circle $S^1$, or considering the unit square $[0,1] \times [0,1]$ and identifying the edges yields the two-sphere $S^2$, etc. In this sense $f$ becomes a map from $S^n$ into $Y$.

There is a remarkable property between the higher homotopy groups, i.e. $n \geq 2$, of some topological space and its universal covering space. Remember the definition of a covering space. Let $X$ and $\tilde{X}$ be two connected topological spaces. Then we call the pair $(\tilde{X}, p)$ the covering space of $X$ if the following properties hold: There exists an open cover $\{U_i\}$ of $X$ such that for each $i$, $p^{-1}(U_i)$ is the disjoint union of open sets in $\tilde{X}$, each of which is mapped by $p$ homeomorphically onto $U_i$. When dealing with manifolds we require the mapping to be diffeomorphic (holomorphic). In particular, if $\tilde{X}$ is simply connected we call it the universal cover. Then the covering projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces an isomorphism[5] $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ for all $n \geq 2$. As an example, we consider the real projective space $\mathbb{RP}^n$. It is obtained from the $n$-sphere $S^n$ by identifying the pair of anti-podal points $(x, -x)$. For $n \geq 2$ the $n$-sphere $S^n$ is the universal covering space of $\mathbb{RP}^n$ and hence we have

$$\pi_n(\mathbb{RP}^m) \cong \pi_n(S^m) \qquad \forall\, n, m \geq 2. \tag{II.59}$$

[5]If $\phi : X \to Y$ is a map taking the base point $x_0 \in X$ to the base point $y_0 \in Y$, then $\phi$ induces a homomorphism between the corresponding homotopy groups by $\phi_*[f] = [\phi \circ f]$.

Furthermore, we are dealing with fiber bundles. There is a nice way of characterizing the homotopy groups of fiber bundles in terms of a long exact sequence. But to explain this we need the notion of relative homotopy groups $\pi_n(X, A, x_0)$ of a pair $(X, A)$ with a base point $x_0 \in A$. For this we regard $I^{n-1}$ as the face of $I^n$ with the last coordinate $t_n = 0$. Moreover, let $J^{n-1}$ be the closure of $\partial I^n \setminus I^{n-1}$. Then $\pi_n(X, A, x_0)$ for $n \geq 1$ is defined to be the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ with homotopies through maps of the same form. Of course, the absolute homotopy groups $\pi_n(X, x_0)$ are contained in the relative ones by setting $A$ to be the base point $x_0$. Now let $\pi : E \to X$ be a fiber bundle over $X$ with the fiber $F_0 = \pi^{-1}(x_0)$ over $x_0 \in X$. Then the bundle projection $\pi$ induces an isomorphism $\pi_* : \pi_n(E, F_0, f_0) \to \pi_n(X, x_0)$ with $\pi(f_0) = x_0$. Let $i : F_0 \to E$ be the inclusion map and $\partial$ be the boundary homomorphism which comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (E, F_0, x_0)$ to $I^{n-1}$. Then the homotopy sequence

$$\cdots \longrightarrow \pi_n(F_0) \xrightarrow{i_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(X) \xrightarrow{\Delta} \pi_{n-1}(F_0) \longrightarrow \cdots \longrightarrow \pi_1(X) \tag{II.60}$$

with $\Delta := \partial \circ \pi_*^{-1}$ is a long exact sequence, that is $\ker \pi_* = \operatorname{im} i_*$, $\ker \Delta = \operatorname{im} \pi_*$, etc. A nice example is the Hopf fibration. Due to Hopf we know that $S^3$ is a $S^1$-bundle over $S^2$. Then the long exact sequence of homotopy groups for fiber bundles teaches us that

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1).$$

But the higher homotopy groups of $S^1$ are trivial which follows from the fact that $\mathbb{R}$ is the universal covering space of $S^1$ which is contractible. Therefore $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Let us list some important homotopy groups of various spaces:

| | |
|---|---|
| $\pi_n(S^m) \cong \{e\}$ | for $n < m$ |
| $\pi_n(S^n) \cong \mathbb{Z}$ | |
| $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$ | except for $\pi_2(S^1) \cong \{e\}$ and $\pi_3(S^2) \cong \mathbb{Z}$ |
| $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ | except for $\pi_3(S^1) \cong \{e\}$ |
| $\pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$ | except for $\pi_4(S^1) \cong \{e\}$, $\pi_5(S^2) \cong \mathbb{Z}_2$, $\pi_6(S^3) \cong \mathbb{Z}_{12}$ and $\pi_7(S^4) \cong \mathbb{Z} \times \mathbb{Z}_{12}$ |
| $\pi_n(S^1) \cong \{e\}$ | for $n > 1$ |
| $\pi_1(G) \cong \mathbb{Z}$ | $G = U(1)$ |
| $\pi_1(G) \cong \mathbb{Z}_2$ | $G = SO(n)$ for $n \geq 3$ |
| $\pi_1(G) \cong \{e\}$ | for all other simple, compact and connected Lie groups |
| $\pi_2(G) \cong \{e\}$ | for any compact and connected Lie group |
| $\pi_3(G) \cong \mathbb{Z}$ | for any simple, compact and connected Lie group |

Furthermore, we have the following useful statements

(i) Let $X$ and $Y$ be two topological spaces and $x_0 \in X$ and $y_0 \in Y$. Then

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0). \tag{II.62}$$

(ii) For any Lie group $G$ and any Lie subgroup $H \subseteq G$ we have

$$\pi_2(G/H) \cong \ker\{\pi_1(H) \mapsto \pi_1(G)\}. \tag{II.63}$$

Choosing $\pi_1(G) \cong \{e\}$ we obtain $\pi_2(G/H) \cong \pi_1(H)$.

### § II.2.4 Clifford Algebras and Spin Geometry

**Clifford algebras.** Throughout this section let $V$ be a real vector space of dimension $2n$ endowed with an inner product $(\ ,\ )$ defined by a positive definite quadratic form $Q$. Choose an orthonormal basis $\{e_i\}$ of $V$. Let $\mathcal{T}(V)$ be the tensor algebra of $V$, i.e.

$$\mathcal{T}(V) := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots \tag{II.64}$$

and consider the two-sided ideal $\mathcal{I}$ in $\mathcal{T}(V)$ generated by the relation $v \otimes v + Q(v)$ for $v \in V$. Then the Clifford algebra $\mathrm{Cl}(V)$ is the quotient algebra

$$\mathrm{Cl}(V) := \mathcal{T}(V)/\mathcal{I}. \tag{II.65}$$

The product of $x, y \in \mathrm{Cl}(V)$ induced by the tensor product in $\mathcal{T}(V)$ is known as Clifford multiplication and is denoted by $x \cdot y \equiv xy$. As a real vector space, $\mathrm{Cl}(V)$ has dimension $2^{2n}$ if the dimension of $V$ is $2n$. The basis of $\mathrm{Cl}(V)$ consists of $1$, $e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$ where $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, 2n\}$ such that $i_1 < i_2 < \cdots i_k$ and $k = |I|$. We have

$$e_i^2 = -1 \quad \text{for} \quad i = 1, \ldots, 2n \quad \text{and} \quad e_i e_j + e_j e_i = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad i, j = 1, \ldots, 2n. \tag{II.66}$$

This can be seen as follows. The quadratic form $Q$ defines the inner product $(\ ,\ )$ on $V$ via

$$(u, v) := \tfrac{1}{4}(Q(u+v) - Q(u-v)) \qquad \text{for} \qquad u, v \in V. \tag{II.67}$$

Hence, we have

$$\begin{aligned} e_i e_j + e_j e_i &= \tfrac{1}{2}((e_i + e_j)(e_i + e_j) - (e_i - e_j)(e_i - e_j)) \\ &= -\tfrac{1}{2}(Q(e_i + e_j) - Q(e_i - e_j)) = -2(e_i, e_j), \end{aligned} \tag{II.68}$$

which is the desired result, since the set $\{e_i\}$ forms an orthonormal basis.

On $\mathcal{T}(V)$ we define the transpose by

$$^t(v_1 \otimes \cdots \otimes v_k) := v_k \otimes \cdots \otimes v_1. \tag{II.69}$$

Since ${}^t\mathcal{I} = \mathcal{I}$ the transpose extends to $\mathrm{Cl}(V)$. We can decompose

$$\mathcal{T}(V) \cong \mathcal{T}(V)_e \oplus \mathcal{T}(V)_o \tag{II.70}$$

into tensors of even and odd degrees to give $\mathcal{T}(V)$ a $\mathbb{Z}_2$ grading. Since $\mathcal{I}$ respects this grading we have with

$$\mathrm{Cl}(V)_e := \bigoplus_{k\,:\,\mathrm{even}} \mathrm{Cl}(V)_k \qquad \text{and} \qquad \mathrm{Cl}(V)_o := \bigoplus_{k\,:\,\mathrm{odd}} \mathrm{Cl}(V)_k, \tag{II.71}$$

where $\mathrm{Cl}(V)_k$ denotes the subset of the order $k$ elements, i.e. $x = \sum_{|I|=k} x_I e_I$ for $x \in \mathrm{Cl}(V)_k$,

$$\mathrm{Cl}(V) \cong \mathrm{Cl}(V)_e \oplus \mathrm{Cl}(V)_o. \tag{II.72}$$

Examples are $\mathrm{Cl}(\mathbb{R}^2) \cong \mathbb{H}$ and $\mathrm{Cl}(\mathbb{R}^4) \cong \mathrm{Mat}(2, \mathbb{H})$. Here, $\mathrm{Mat}(n, \mathbb{F})$ denotes the $n \times n$ matrices with entries in $\mathbb{F}$.

**Multiplicative group.** Let $\mathrm{Cl}^*(V)$ be the group of invertible elements of $\mathrm{Cl}(V)$. Then $V - \{0\}$ is contained in $\mathrm{Cl}^*(V)$, since for $v \in V - \{0\}$ we have $v^{-1} := -v/\|v\|^2$. The group $\mathrm{Cl}^*(V)$ acts on $\mathrm{Cl}(V)$ by the adjoint representation

$$g \mapsto \mathrm{Ad}_g(x) = gxg^{-1}, \tag{II.73}$$

where $g \in \mathrm{Cl}^*(V)$. Note that for any $u \in V - \{0\}$ and $v \in V$, we have

$$-\mathrm{Ad}_u(v) = v - \frac{2(u, v)}{\|u\|^2} u, \tag{II.74}$$

i.e. it is a reflection with respect to the $u^\perp$ hyperplane. In order to avoid the minus sign we introduce the so-called twisted adjoint representation

$$g \mapsto \pi_g(x) := \alpha(g) x g^{-1}, \tag{II.75}$$

where $g \in \mathrm{Cl}^*(V)$, $x \in \mathrm{Cl}(V)$ and $\alpha(g) := (-1)^{\deg g} g$ is the grading map. Then for $u \in V - \{0\}$ the map $\pi_u : V \to V$ is a reflection with respect to $u^\perp$.

**Clifford, pin and spin groups.** The Clifford group $\mathrm{Cliff}(V)$ is defined as

$$\mathrm{Cliff}(V) := \{g \in \mathrm{Cl}^*(V) \,|\, \pi_g(V) = V\}. \tag{II.76}$$

Then we have the short exact sequence

$$\{e\} \longrightarrow \mathbb{R}^* \longrightarrow \mathrm{Cliff}(V) \xrightarrow{\pi} O(V) \longrightarrow \{e\}, \tag{II.77}$$

where $O(V)$ are the automorphisms of $(V, Q)$, i.e. orthogonal transformations of $V$ with respect to the quadratic form $Q$. Note that any orthogonal transformation is a composition of reflections.

On $\mathrm{Cl}(V)$ we introduce an involution $x \mapsto \bar{x}$ by $\bar{x} = \alpha({}^t x)$ for $x \in \mathrm{Cl}(V)$. The mapping $x \mapsto \| x \|^2 := \bar{x}x$ is called the square-norm mapping and it gives a group homomorphism

$$\mathrm{Cl}^*(V) \xrightarrow{\|x\|^2} \mathbb{R}^*. \tag{II.78}$$

Then we define the pin group by $\mathrm{Pin}(V) := \{x \in \mathrm{Cliff}(V) \mid \| x \|^2 = 1\}$ yielding the short exact sequence

$$\{e\} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \xrightarrow{\pi} O(V) \longrightarrow \{e\}, \tag{II.79}$$

If we let $\mathrm{Spin}(V) := \{x \in \mathrm{Pin}(V) \mid \det \pi_x = 1\}$ we can specialize further to

$$\{e\} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\pi} SO(V) \longrightarrow \{e\}, \tag{II.80}$$

where $\mathrm{Spin}(V)$ is called the spin group. Examples of spin groups are $\mathrm{Spin}(\mathbb{R}^2) \cong U(1)$ and $\mathrm{Spin}(\mathbb{R}^4) \cong SU(2) \times SU(2)$.

The Lie algebra $\mathfrak{spin}(V)$ of $\mathrm{Spin}(V)$ is obviously $\mathfrak{so}(V)$. Moreover, it can be shown that it coincides with the Lie algebra $\mathfrak{cl}_2(V)$, which is $\mathrm{Cl}(V)_2$ equipped with the Lie bracket.

Letting $G_0$ be the identity component of a Lie group $G$, we can summarize the things discussed above in the following commutative diagram:

$$\begin{array}{ccccccccc}
\{e\} & \longrightarrow & \mathbb{R}^* & \longrightarrow & \mathrm{Cliff}(V) & \longrightarrow & O(V) & \longrightarrow & \{e\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{e\} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Pin}(V) & \longrightarrow & O(V) & \longrightarrow & \{e\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{e\} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(V) & \longrightarrow & SO(V) & \longrightarrow & \{e\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{e\} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(V)_0 & \longrightarrow & SO(V)_0 & \longrightarrow & \{e\}
\end{array}$$

where the vertical arrows are the standard embeddings.

Now we want to introduce the spin$^c$ groups. Let $\mathrm{Cl}^c(V) := \mathrm{Cl}(V) \otimes \mathbb{C}$ be the complexified Clifford algebra endowed with a Hermitian inner product and a conjugation coming from those given on $\mathrm{Cl}(V)$. The generalized Clifford group is given by

$$\mathrm{Cliff}^c(V) := \{g \in (\mathrm{Cl}^c(V))^* \mid \pi_g(V) = V\}. \tag{II.81}$$

Note that we are not complexifying $V$ here and hence we obtain a mapping $\mathrm{Cliff}^c(V) \to O(V)$. Defining $\mathrm{Pin}^c(V) := \{x \in \mathrm{Cliff}^c(V) \mid \| x \|^2 = 1\}$, we have the following exact sequences

$$\{e\} \longrightarrow U(1) \longrightarrow \mathrm{Pin}^c(V) \xrightarrow{\pi} O(V) \longrightarrow \{e\} \tag{II.82}$$

and

$$\{e\} \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\pi} SO(V) \longrightarrow \{e\}, \tag{II.83}$$

where $\mathrm{Spin}^c(V) := \{x \in \mathrm{Pin}^c(V) \,|\, \det \pi_x = 1\}$. Note that $\mathrm{Spin}^c(V)$ is the circle extension of $\mathrm{Spin}(V)$, i.e.

$$\mathrm{Spin}^c(V) := \{xe^{i\theta} \,|\, x \in \mathrm{Spin}(V),\ \theta \in \mathbb{R}\}, \tag{II.84}$$

so that we have another exact sequence

$$\{e\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\delta} U(1) \longrightarrow \{e\}, \tag{II.85}$$

where the homomorphism $\delta$ is given by $\delta : xe^{i\theta} \mapsto e^{2i\theta}$. We need the '2' in the exponent, since otherwise this map would be ill defined. Thus,

$$\mathrm{Spin}^c(V) \cong (\mathrm{Spin}(V) \times U(1))/\mathbb{Z}_2. \tag{II.86}$$

The Lie algebra of $\mathrm{Spin}^c(V)$ is simply $\mathfrak{spin}^c(V) \cong \mathfrak{so}(V) \oplus i\mathbb{R}$.

**Spin representation.** A spin representation is defined as a linear map

$$\Gamma : V \to \mathrm{End}(W), \tag{II.87}$$

where $W$ is a $2^n$-dimensional Hermitian complex vector space and $\mathrm{End}(W)$ are the endomorphisms of $W$, having the properties

$$\Gamma^\dagger(v) + \Gamma(v) = 0 \qquad \text{and} \qquad \Gamma^\dagger(v)\Gamma(v) = \|v\|^2 \,\mathrm{id}\,. \tag{II.88}$$

The map $\Gamma$ extends to an algebra isomorphism $\Gamma : \mathrm{Cl}^c(V) \to \mathrm{End}(W)$. Elements of $W$ are called spinors. Defining a Clifford volume element

$$\omega := -e_1 e_2 \cdots e_{2n} \in \mathrm{Cl}(V)_{2n} \tag{II.89}$$

with the properties

$$\omega^2 = (-1)^n \qquad \text{and} \qquad \omega v + v\omega = 0 \quad \forall\, v \in V, \tag{II.90}$$

we can introduce the chiral spinor spaces

$$W^\pm := \{w \in W \,|\, \Gamma(\omega)w = \pm i^n w\}, \tag{II.91}$$

where the elements of $W^\pm$ are called spinors of positive or negative chirality. Then we have the decomposition $W \cong W^+ \oplus W^-$ and $\Gamma(v) : W^\pm \to W^\mp$ for all $v \in V$, since $\Gamma(\omega)\Gamma(v)w_\pm = \Gamma(\omega v)w_\pm = -\Gamma(v\omega)w_\pm = \mp i^n \Gamma(v)w_\pm$ for $w_\pm \in W^\pm$.

Let us discuss some examples of spinor representations. For $n = 1$ the map $\Gamma : \mathrm{Cl}^c(\mathbb{R}^2) \cong \mathbb{H} \to \mathrm{Mat}(2, \mathbb{C})$ is the complexified Pauli map $\sigma^c$, where

$$\begin{aligned} \sigma : \mathbb{H} &\to \mathrm{Mat}(2,\mathbb{C}) \\ x = (x^1, x^2, x^3, x^4) &\mapsto \sigma(x) = \begin{pmatrix} x^4 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 \end{pmatrix}. \end{aligned} \tag{II.92}$$

For $n = 2$ we have $\Gamma : \mathrm{Cl}^c(\mathbb{R}^4) \cong \mathrm{Cl}^c(\mathbb{H}) \to \mathrm{Mat}(4, \mathbb{C})$, where $\Gamma$ is now generated by the complexified Dirac map $\gamma^c$ with

$$\begin{aligned} \gamma \,:\, \mathbb{H} &\to \mathrm{Mat}(4, \mathbb{C}) \\ x = (x^1, x^2, x^3, x^4) &\mapsto \gamma(x) = \begin{pmatrix} 0 & \sigma(x) \\ -\sigma^\dagger(x) & 0 \end{pmatrix}. \end{aligned} \tag{II.93}$$

Note that $\det(\sigma(x)) = \|x\|^2_{\mathbb{R}^4}$.

**Exterior algebra.** Let $\Lambda^*(V)$ be the exterior algebra of $V$, i.e.

$$\Lambda^*(V) = \bigwedge_{p=0}^{2n} \Lambda^p(V). \tag{II.94}$$

It can be canonically identified with the Clifford algebra $\mathrm{Cl}(V)$ using the alternation map defined by

$$\begin{aligned} \mathrm{Alt}_k \,:\, V \times \cdots \times V &\to \mathrm{Cl}(V)_k \\ (v_1, \ldots, v_2) &\mapsto \frac{1}{k!} \sum_\sigma \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \cdots v_{\sigma(2)}. \end{aligned} \tag{II.95}$$

Then we have the vector space isomorphism $\mathrm{Alt} : \Lambda^*(V) \to \mathrm{Cl}(V)$ and by duality

$$\mathrm{Alt}^* \,:\, \Lambda^*(V^*) \to \mathrm{Cl}(V^*) \cong \mathrm{Cl}(V). \tag{II.96}$$

Using the spin representation $\Gamma : \mathrm{Cl}(V) \to \mathrm{End}(W)$, we define

$$\rho := \Gamma \circ \mathrm{Alt}^* \,:\, \Lambda^*(V^*) \to \mathrm{End}(W). \tag{II.97}$$

Therefore the mapping $\rho$ defines the Clifford multiplication on $W$ by forms from $\Lambda^*(V^*)$. In particular, the Clifford multiplication by a two-form leaves the chiral spinor spaces $W^\pm$ invariant, as one may readily check and furthermore, $\rho$ maps real-valued two-forms to skew-Hermitian traceless endomorphisms of $W$ and imaginary two-forms to Hermitian traceless endomorphisms of $W$.

If $V$ is four-dimensional $\Lambda^2(V^*)$ can be decomposed into vector spaces of self-dual and anti-self-dual two-forms with respect to the Hodge operator, i.e. $\Lambda^2(V^*) \cong \Lambda^2_+(V^*) \oplus \Lambda^2_-(V^*)$ and $\rho^\pm := \rho|_{W^\pm}$ induces the isomorphisms

$$\Lambda^2_\pm(V^*) \cong \mathfrak{su}(W^\pm) \qquad \text{and} \qquad \Lambda^2_\pm(V^*) \otimes i\mathbb{R} \cong \mathrm{Herm}_0(W^\pm), \tag{II.98}$$

where $\mathrm{Herm}_0(W^\pm)$ are the traceless Hermitian endomorphisms of $W^\pm$. Explicitly, we may take

$$e^1 \wedge e^2 + e^3 \wedge e^4, \qquad e^1 \wedge e^3 + e^4 \wedge e^2 \qquad \text{and} \qquad e^1 \wedge e^4 + e^2 \wedge e^3 \tag{II.99}$$

as a basis of $\Lambda^2_+(V^*)$. Using (II.92) and (II.93) we find

$$\gamma(e_1)\gamma(e_2)+\gamma(e_3)\gamma(e_4)=\begin{pmatrix}-2i&0&0&0\\0&2i&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \gamma(e_1)\gamma(e_3)+\gamma(e_4)\gamma(e_2)=\begin{pmatrix}0&2&0&0\\-2&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$
$$\gamma(e_1)\gamma(e_4)+\gamma(e_2)\gamma(e_3)=\begin{pmatrix}0&-2i&0&0\\-2i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

and hence we see that the real self-dual two forms can be identified with the skew-Hermitian traceless endomorphisms of $W^+$. A similar calculation goes through for the anti-self-dual two forms. More generally, we have the isomorphisms $\Lambda^2_\pm(V^*)\otimes\mathbb{C}\cong\mathrm{End}_0(W^\pm)$.

**Spin$^c$ structures.** Let $M$ be an orientable $2n$-dimensional Riemannian manifold and $P(M,SO(2n))$ the principal $SO(2n)$-bundle of orthonormal frames on $M$. For brevity we simply write $SO(V)\equiv SO(n)$, etc., if $\dim_{\mathbb{R}}V=n$. A spin$^c$ structure on $P(M,SO(2n))$ is defined as its extension to a principal Spin$^c(2n)$-bundle $P(M,\mathrm{Spin}^c(2n))$ together with a spin$^c$ invariant bundle epimorphism

$$\begin{array}{ccc}P(M,\mathrm{Spin}^c(2n)) & \longrightarrow & P(M,SO(2n))\\ \downarrow & & \downarrow\\ M & \longrightarrow & M\end{array}$$

where the action of Spin$^c(2n)$ on $P(M,SO(2n))$ is given by $\pi:\mathrm{Spin}^c(2n)\to SO(2n)$. Moreover, we can define an associated principal $U(1)$-bundle $P(M,U(1))$ such that

$$\begin{array}{ccc}P(M,\mathrm{Spin}^c(2n)) & \stackrel{\delta}{\longrightarrow} & P(M,U(1))\\ \downarrow & & \downarrow\\ M & \longrightarrow & M\end{array}$$

where Spin$^c(2n)$ acts on $P(M,U(1))$ by $\delta:\mathrm{Spin}^c(2n)\to U(1)$. We denote by $L\to M$ the complex line bundle associated with $P(M,U(1))$, i.e. $L:=P(M,U(1))\times_{U(1)}\mathbb{C}$. This bundle is called the characteristic line bundle of the spin$^c$ structure and its first Chern class $c_1(L)$ the characteristic class of the spin$^c$ structure; cf. section § II.2.5.

Analogously, we can define a spin$^c$ structure on an orientable Riemannian vector bundle $E\to M$ of rank $2n$ associated with $P(M,SO(2n))$, i.e. $E\cong P(M,SO(2n))\times_{SO(2n)}\mathbb{R}^{2n}$. A spin$^c$ structure on $E\to M$ is again an extension of the structure group to Spin$^c(2n)$. In other words, a spin$^c$ structure exists if there exists a principal Spin$^c(2n)$-bundle $P(M,\mathrm{Spin}^c(2n))$, such that $E\cong P(M,\mathrm{Spin}^c(2n))\times_{\mathrm{Spin}^c(2n)}\mathbb{R}^{2n}$, where Spin$^c(2n)$ acts on $\mathbb{R}^{2n}$ by $\pi:\mathrm{Spin}^c(2n)\to SO(2n)$. Note that in the case of $E=TM$ we say that $M$ has a spin$^c$ structure.

Equivalently, we can give a definition of a spin$^c$ structure on $E$ in terms of the spin representation. Let $W$ be a complex Hermitian vector bundle of rank $2^n$ associated with the principal Spin$^c(2n)$-bundle $P(M,\mathrm{Spin}^c(2n))$, i.e.

$$W:=P(M,\mathrm{Spin}^c(2n))\times_{\mathrm{Spin}^c(2n)}\mathbb{C}^{2^n}, \tag{II.102}$$

where the action of $\mathrm{Spin}^c(2n)$ on $\mathbb{C}^{2^n}$ is given by the spin representation $\Gamma : \mathrm{Cl}^c(2n) \to \mathrm{End}(\mathbb{C}^{2^n})$ yielding a bundle homomorphism $\Gamma : E \to \mathrm{End}(W)$ which satisfies the usual spin representation properties. Then a spin$^c$ structure is defined as the pair $(W, \Gamma)$. The characteristic line bundle of the spin$^c$ structure $(W, \Gamma)$ is

$$L_\Gamma := P(M, \mathrm{Spin}^c(2n)) \times_{\mathrm{Spin}^c(2n)} \mathbb{C}, \tag{II.103}$$

where $\mathrm{Spin}^c(2n)$ acts on $\mathbb{C}$ by $\delta : \mathrm{Spin}^c(2n) \to U(1)$, as previously.

Now the question is, when do spin$^c$ structures exist. It can be shown that $P(M, SO(2n))$ admits a spin$^c$ structure if and only if there is some $c \in H^2(M, \mathbb{Z})$ such that

$$w_2(P(M, SO(2n)) \equiv c \mod 2, \tag{II.104}$$

where $w_2$ is the second Stiefel-Whitney class; cf. section § II.2.5. Moreover, iff $w_2(P(M, SO(2n)))$ $= \{e\}$ the bundle $P(M, SO(2n))$ admits a spin structure. In that case we call $M$ a spin manifold.

Later on, we shall also deal with almost complex manifolds. Suppose that $M$ is an orientable Kähler manifold of (complex) dimension $n$. Remember that a Hermitian manifold $(M, g, J, \omega)$ is said to be Kähler if the associated two-form $\omega$ is closed: $\mathrm{d}\omega = 0$, where $\omega(X, Y)(p) = g(JX, Y)(p)$ with $X, Y \in T_pM$, $J$ a chosen almost complex structure and $g$ a Hermitian metric $g(JX, JY)(p) = g(X, Y)(p)$. Recall that $J$ is an endomorphism $TM \to TM$ which squares to minus the identity, i.e. $J^2 = -\,\mathrm{id}$. It is well known that in this case the $k$-forms on $M$ split with respect to the dual almost complex structure $J^*$ as

$$\Lambda^k(T^*M) \cong \bigoplus_{p+q=k} \Lambda^{(p,q)}(T^*M). \tag{II.105}$$

Then the canonical spin$^c$ structure $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ is defined by

$$W_{\mathrm{can}} := \Lambda^{(0,*)}(T^*M) = \bigoplus_{q=0}^{n} \Lambda^{(0,q)}(T^*M) \tag{II.106}$$

with the canonical spin representation $\Gamma_{\mathrm{can}} : TM \to \mathrm{End}(W_{\mathrm{can}})$

$$\Gamma_{\mathrm{can}}(v)\omega^{0,q} := \sqrt{2}(v^{0,1} \wedge \omega^{0,q} - v^{1,0} \lrcorner \omega^{0,q}), \tag{II.107}$$

where $v^* = v^{1,0} + v^{0,1} \in T^*M$ is dual to $v \in TM$ and '$\lrcorner$' denotes the interior product. Using $v^* \lrcorner (\omega \wedge \rho) = (v^* \lrcorner \omega) \wedge \rho + (-1)^k \omega \wedge (v^* \lrcorner \rho)$ for $\omega \in \Lambda^k(T^*M)$ and $v^* \lrcorner (v^* \lrcorner) = 0$ one readily verifies that the spin representation properties (II.88) are satisfied. Moreover, we have $\Gamma(v) : \Lambda^{(0,\mathrm{o/e})}(T^*M) \to \Lambda^{(0,\mathrm{e/o})}(T^*M)$, where '$e$' refers to even and '$o$' to odd, respectively. The spin bundle $W$ splits into the two chiral subbundles

$$W^+_{\mathrm{can}} = \Lambda^{(0,\mathrm{e})}(T^*M) \qquad \text{and} \qquad W^-_{\mathrm{can}} = \Lambda^{(0,\mathrm{o})}(T^*M). \tag{II.108}$$

The characteristic line bundle $L_{\mathrm{can}}$ coincides with the anti-canonical bundle $K^*$

$$K^* \cong \Lambda^{(0,n)}(T^*M). \tag{II.109}$$

Any other spin$^c$ structure on $M$ can be obtained from the canonical one via tensoring with some line bundle $E \to M$, i.e.

$$W_E \cong W_{\text{can}} \otimes E \qquad \text{and} \qquad \Gamma_E = \Gamma_{\text{can}} \otimes \text{id}\,. \tag{II.110}$$

The characteristic line bundle is given by

$$L_{\Gamma_E} \cong L_{\text{can}} \otimes E^2. \tag{II.111}$$

**Spin$^c$ connection and Dirac operator.** Let $M$ be an orientable Riemannian manifold of dimension $2n$ endowed with a spin$^c$ structure $(W, \Gamma)$. We denote by $\nabla$ the Levi-Civita connection on $M$. Then a spin$^c$ connection is an extension of $\nabla$ to $W$, i.e. it is a connection $\nabla$ on $W$ satisfying

$$\nabla_u(\Gamma(v)\Phi) = \Gamma(v)\nabla_u\Phi + \Gamma(\nabla_u v)\Phi, \tag{II.112}$$

where $u, v \in \mathcal{X}(M)$ and $\Phi \in \Gamma(M, W)$. One can show that a spin$^c$ connection preserves $W^{\pm}$ and that any two of such connections differ by a $\mathfrak{u}(1)$-valued one-form. This can be seen as follows. Let $P_\Gamma := P(\text{Spin}^c(2n), M)$ be a principal $\text{Spin}^c(2n)$-bundle given by the spin$^c$ structure $(W, \Gamma)$. Then we have

$$W \cong P(\text{Spin}^c(2n), M) \times_{\text{Spin}^c(2n)} \mathbb{C}^{2^n}, \tag{II.113}$$

where $\text{Spin}^c(2n)$ acts on $\mathbb{C}^{2^n}$ by the standard spin representation $\Gamma$. Furthermore, we have

$$TM \cong P(\text{Spin}^c(2n), M) \times_{\text{Spin}^c(2n)} \mathbb{R}^{2n}, \tag{II.114}$$

where $\text{Spin}^c(2n)$ acts on $\mathbb{R}^{2n}$ by the homomorphism $\pi : \text{Spin}^c(2n) \to SO(2n)$. Consider the standard spin representation $\Gamma : \text{Spin}^c(2n) \to \text{End}(\mathbb{C}^{2^n})$ and let $G := \Gamma(\text{Spin}^c(2n))$. Then the Lie algebra $\mathfrak{g}$ of $G$ is given by

$$\mathfrak{g} \cong \Gamma(\mathfrak{so}(2n)) \oplus i\mathbb{R}) \cong \Gamma(\mathfrak{so}(2n)) \oplus i\mathbb{R} =: \mathfrak{g}_0 \oplus i\mathbb{R}, \tag{II.115}$$

with $\mathfrak{g}_0 := \Gamma(\mathfrak{so}(2n))$. Note that $\mathfrak{g}_0 \cong \mathfrak{so}(2n)$. Then a spin$^c$ connection on $W$ is generated by a connection one-form $A \in \mathfrak{g} \otimes \Omega^1(P_\Gamma)$ and we can decompose it as $A = \mathcal{A}_0 \oplus \mathcal{A}$, where $\mathcal{A}_0 \in \mathfrak{g}_0 \otimes \Omega^1(P_\Gamma)$ is the traceless part of $A$ and $\mathcal{A} \in \mathfrak{u}(1) \otimes \Omega^1(P_\Gamma)$ is the trace part. Note that the traceless part generates a connection on the tangent bundle $TM$ and hence by the definition of the spin$^c$ connection, it coincides with the Levi-Civita connection on $M$. Therefore, $A$ is completely determined by its trace part. Since $\delta : 1 \cdot e^{i\theta} \mapsto e^{2i\theta}$, the trace part $\mathcal{A}$ generates the connection $2\mathcal{A}$ on the characteristic line bundle $L_\Gamma$. In the case of spin manifolds $L_\Gamma$ has a square root and therefore $\mathcal{A}$ also generates a connection on $L_\Gamma^{1/2}$. In general, however, we call $\mathcal{A}$ a virtual connection on the virtual line bundle $L_\Gamma^{1/2}$. We denote the space of such virtual connections $\mathcal{A}$ on the virtual line bundle $L_\Gamma^{1/2}$ by $\mathcal{A}(\Gamma)$.

Let $\nabla_\mathcal{A}$ be the covariant derivative on sections of $W$ generated by the connection one-form $A = \mathcal{A}_0 \oplus \mathcal{A}$. Then the spin$^c$ Dirac operator

$$\mathcal{D}_\mathcal{A} : \Gamma(M, W^+) \to \Gamma(M, W^-) \tag{II.116}$$

associated with the virtual connection $\mathcal{A}$ is defined by

$$\mathcal{D}_{\mathcal{A}}\Phi := \sum_{i=1}^{2n} \Gamma(e_i)\nabla_{\mathcal{A},e_i}\Phi, \tag{II.117}$$

where $\Phi \in \Gamma(M, W^+)$ and $\{e_i\}$ forms a local orthonormal basis of $TM$. Note that $\mathcal{D}_{\mathcal{A}}$ is elliptic. The adjoint spin$^c$ Dirac operator is given by

$$\mathcal{D}^\dagger_{\mathcal{A}} \,:\, \Gamma(M, W^-) \to \Gamma(M, W^+). \tag{II.118}$$

As already mentioned, we need also the notion of a spin$^c$ connection and a spin$^c$ Dirac operator on Kähler manifolds. Let $M$ be an orientable Kähler manifold of (complex) dimension $n$. Then the trace part $\mathcal{A}$ of any spin$^c$ connection $A$ on $W_E \cong W_{\mathrm{can}} \otimes E$ can be written as

$$\mathcal{A} = \mathcal{A}_{\mathrm{can}} \otimes \mathrm{id} \oplus \mathrm{id} \otimes \mathcal{B}, \tag{II.119}$$

where $\mathcal{B}$ is some fixed connection on the line bundle $E \to M$. Then the spin$^c$ Dirac operator

$$\mathcal{D}_{\mathcal{A}} \,:\, \Gamma(M, W_E^+) \to \Gamma(M, W_E^-) \tag{II.120}$$

with $W_E^\pm$ given by (II.108) coincides with

$$\mathcal{D}_{\mathcal{A}} = \sqrt{2}(\bar\partial_{\bar{\mathcal{B}}} + \bar\partial^\dagger_{\bar{\mathcal{B}}}), \tag{II.121}$$

where $\bar\partial_{\bar{\mathcal{B}}} = \bar\partial + \bar B$. To prove this formula it suffices to consider the canonical part of the Dirac operator. On Kähler manifolds one can choose local coordinates $z^a = x^{2a-1} + ix^{2a}$ for $a = 1, \ldots, n$ which are normal at $p \in M$, i.e. $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$ with $e_i(p) = \partial_i(p)$. This implies that covariant derivatives $\partial + A_{\mathrm{can}}$ can be replaced by ordinary derivatives. Using the complexified Dirac map (II.93) one arrives at the desired result.

**Weitzenböck's formula.** Let $M$ be an orientable Riemannian manifold of dimension $2n$ and furthermore, let $\nabla^\dagger_{\mathcal{A}} \,:\, \Gamma(M, W) \to \Gamma(M, W)$ be the $L^2$-adjoint operator of $\nabla_{\mathcal{A}}$. Then the Weitzenböck formula reads

$$\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi = \nabla^\dagger_{\mathcal{A}}\nabla_{\mathcal{A}}\Phi + \tfrac{1}{4}R\,\Phi + \rho^+(\mathcal{F}_{\mathcal{A}})\Phi, \tag{II.122a}$$

$$\mathcal{D}_{\mathcal{A}}\mathcal{D}^\dagger_{\mathcal{A}}\Psi = \nabla_{\mathcal{A}}\nabla^\dagger_{\mathcal{A}}\Psi + \tfrac{1}{4}R\,\Psi + \rho^-(\mathcal{F}_{\mathcal{A}})\Psi, \tag{II.122b}$$

where $\Phi \in \Gamma(M, W^+)$, $\Psi \in \Gamma(M, W^-)$, $R$ is the scalar curvature of $M$ and $\mathcal{F}_{\mathcal{A}} = \mathrm{d}\,\mathcal{A}$. Here, $\rho^\pm$ are given as previously.

**Remark.** Note that spinor fields may often live additionally in some associated vector bundle. Then the spinor is an element of $\Gamma(M, W \otimes E)$, where $E = P \times_G \mathbb{C}^r$ is the vector bundle associated to the principal bundle $P(M,G)$. The $G$-action on $E$ is given by some representation $\mathfrak{R}$. Let $\mathfrak{g}$ be the Lie algebra of $G$. The map $\rho$ given by expression (II.97) extends to

$$\rho = (\Gamma \otimes \mathfrak{R}) \circ (\mathrm{Alt}^* \otimes \mathrm{id}) \,:\, \Lambda^*(T^*M) \otimes \mathfrak{g} \to \mathrm{End}(W \otimes E). \tag{II.123}$$

The spin$^c$ Dirac operator (II.116) extends to the twisted spin$^c$ Dirac operator

$$\mathcal{D}_{\mathcal{A},\mathcal{B}} \,:\, \Gamma(M, W^+ \otimes E) \to \Gamma(M, W^- \otimes E), \tag{II.124}$$

where $\mathcal{B}$ is a fixed connection one-form on $E$. Weitzenböck's formulas generalize to

$$\begin{aligned} \mathcal{D}^\dagger_{\mathcal{A},\mathcal{B}}\mathcal{D}_{\mathcal{A},\mathcal{B}}\Phi &= \nabla^\dagger_{\mathcal{A},\mathcal{B}}\nabla_{\mathcal{A},\mathcal{B}}\Phi + \tfrac{1}{4}R\,\Phi + \rho^+(\mathcal{F}_{\mathcal{A},\mathcal{B}})\Phi, && \text{(II.125a)} \\ \mathcal{D}_{\mathcal{A},\mathcal{B}}\mathcal{D}^\dagger_{\mathcal{A},\mathcal{B}}\Psi &= \nabla_{\mathcal{A},\mathcal{B}}\nabla^\dagger_{\mathcal{A},\mathcal{B}}\Psi + \tfrac{1}{4}R\,\Psi + \rho^-(\mathcal{F}_{\mathcal{A},\mathcal{B}})\Psi, && \text{(II.125b)} \end{aligned}$$

where $\Phi \in \Gamma(M, W^+ \otimes E)$ and $\Psi \in \Gamma(M, W^- \otimes E)$. The curvature $\mathcal{F}_{\mathcal{A},\mathcal{B}}$ is computed from both connection one-forms and $\rho^\pm$ are the restrictions of (II.123) to the chiral spinor spaces $W^\pm$.

## § II.2.5 Characteristic Classes and the Atiyah-Singer Index Theorem

**Chern classes and characters.** Characteristic classes are important for the discussion of the topology of fiber bundles. In general, they are subsets of cohomology classes of the base manifold. They measure the nontriviality or the twisting of a bundle. In this subsection we briefly review the most important characteristic classes and eventually write down the Atiyah-Singer index theorem for the spin complex discussed in the previous subsection.

Let $E \to M$ be a complex vector bundle of rank $r$ and structure group $G$ which is a subgroup of $GL(r,\mathbb{C})$. The total Chern class is defined as

$$c(\mathcal{F}) := \det\left(1 + \frac{i\mathcal{F}}{2\pi}\right) =: 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots\,, \tag{II.126}$$

where $\mathcal{F}$ is the pulled back curvature (II.36) and $c_j(\mathcal{F}) \in \mathfrak{g} \otimes \Omega^{2j}(M)$ is called the $j^{\text{th}}$ Chern class, which are

$$\begin{aligned} c_0(\mathcal{F}) &= 1, && \text{(II.127a)} \\ c_1(\mathcal{F}) &= \frac{i}{2\pi}\,\mathrm{tr}\,\mathcal{F}, && \text{(II.127b)} \\ c_2(\mathcal{F}) &= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \{\mathrm{tr}\,\mathcal{F} \wedge \mathrm{tr}\,\mathcal{F} - \mathrm{tr}(\mathcal{F} \wedge \mathcal{F})\}, && \text{(II.127c)} \\ &\vdots \\ c_r(\mathcal{F}) &= \left(\frac{i}{2\pi}\right)^r \det\mathcal{F}. && \text{(II.127d)} \end{aligned}$$

For a Whitney sum bundle $E \oplus F$ we have $c(E \oplus F) = c(E) \wedge c(F)$, which is an important property of the total Chern class.

The total Chern character is defined by

$$\mathrm{ch}(\mathcal{F}) := \mathrm{tr}\exp\left(\frac{i\mathcal{F}}{2\pi}\right) = \sum_j \frac{1}{j!}\,\mathrm{tr}\left(\frac{i\mathcal{F}}{2\pi}\right)^j =: \sum_j \mathrm{ch}_j(\mathcal{F}), \tag{II.128}$$

where the Chern characters $\mathrm{ch}_j(\mathcal{F}) \in \mathfrak{g} \otimes \Omega^{2j}(M)$ are

$$\begin{aligned}
\mathrm{ch}_0(\mathcal{F}) &= r, && \text{(II.129a)}\\
\mathrm{ch}_1(\mathcal{F}) &= c_1(\mathcal{F}), && \text{(II.129b)}\\
\mathrm{ch}_2(\mathcal{F}) &= \frac{1}{2}\{c_1(\mathcal{F}) \wedge c_1(\mathcal{F}) - 2c_2(\mathcal{F})\}, && \text{(II.129c)}\\
&\ \vdots
\end{aligned}$$

We have $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F)$ and $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) \oplus \mathrm{ch}(F)$.

**Pontrjagin classes.** The Pontrjagin classes are characteristic classes associated with real vector bundles. Let $E \to M$ be a real vector bundle of rank $r$ endowed with the fiber metric. Thus, its structure group can be reduced to $O(r)$. The total Pontrjagin class is defined by

$$p(\mathcal{F}) := \det\left(1 + \frac{\mathcal{F}}{2\pi}\right) =: 1 + p_1(\mathcal{F}) + p_2(\mathcal{F}) + \cdots . \tag{II.130}$$

Since $\mathcal{F}$ is skew-symmetric, $p(\mathcal{F})$ is an even function of $\mathcal{F}$. The expansion is given by

$$\begin{aligned}
p_0(\mathcal{F}) &= 1, && \text{(II.131a)}\\
p_1(\mathcal{F}) &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{tr}\,\mathcal{F} \wedge \mathcal{F}, && \text{(II.131b)}\\
p_2(\mathcal{F}) &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^2 \{\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) - 2\,\mathrm{tr}(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F})\}, && \text{(II.131c)}\\
&\ \vdots\\
p_{r/2}(\mathcal{F}) &= \left(\frac{1}{2\pi}\right)^r \det \mathcal{F}. && \text{(II.131d)}
\end{aligned}$$

One can show that $p(E \oplus F) = p(E) \wedge p(F)$. Furthermore,

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}), \tag{II.132}$$

where $E^{\mathbb{C}}$ is the complexification of $E$.

**Stiefel-Whitney classes.** In contrast to the previously considered characteristic classes the Stiefel-Whitney class cannot be expressed in terms of the curvature of the bundle.

Let $\{U_i\}$ be an open cover of a Riemannian manifold $M$ of dimension $n$. A Čech $p$-cochain is a function $f(i_0, i_1, \ldots, i_p) \in \mathbb{Z}_2$ defined on $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \neq \emptyset$, which is totally symmetric under an arbitrary permutation of the $i_j$'s. Let $C^p(M,\mathbb{Z}_2)$ be the multiplicative group of such Čech $p$-cochains. We introduce the coboundary operator by

$$\begin{aligned}\delta\,:\,C^p(M,\mathbb{Z}_2)&\to C^{p+1}(M,\mathbb{Z}_2)\\ (\delta f)(i_0,\ldots,i_{p+1})&:=\prod_{j=0}^{p+1} f(i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}),\end{aligned}\tag{II.133}$$

where $\hat{i}_j$ means omitting the variable $i_j$. We denote the trivial element of $C^p(M,\mathbb{Z}_2)$ by 1. Note that $\delta$ is nilpotent, i.e. $\delta^2 f = 1$. The cocycle group $Z^p(M,\mathbb{Z}_2)$ and the coboundary group $B^p(M,\mathbb{Z}_2)$ are given by

$$Z^p(M,\mathbb{Z}_2) \;:=\; \{f\in C^p(M,\mathbb{Z}_2)\,|\,\delta f = 1\} \tag{II.134a}$$

$$B^p(M,\mathbb{Z}_2) \;:=\; \{f\in C^p(M,\mathbb{Z}_2)\,|\,f=\delta f',\;f'\in C^{p-1}(M,\mathbb{Z}_2)\}. \tag{II.134b}$$

The $p^{\text{th}}$ Čech cohomology group $H^p(M,\mathbb{Z}_2)$ is defined as the quotient group

$$H^p(M,\mathbb{Z}_2) := \ker\delta_p/\mathrm{im}\,\delta_{p-1} = Z^p(M,\mathbb{Z}_2)/B^p(M,\mathbb{Z}_2). \tag{II.135}$$

The Stiefel-Whitney classes $w_p$ are characteristic classes taking values in $H^p(M,\mathbb{Z}_2)$. Let now the covering $\{U_i\}$ of $M$ be simple, that is the intersection of any number of $U_i$'s is either empty or contractible. Denote by $P(M,O(n))$ the principal bundle of orthonormal frames. We we define the Čech one-cochain $f(i,j)$ by

$$f(i,j) := \det(\phi_{ij}) = \pm 1, \tag{II.136}$$

where the $\phi_{ij}\,:\,U_i\cap U_j\to O(n)$ are the transition functions as defined in section § II.2.1. The function $f$ is an element of $C^1(M,\mathbb{Z}_2)$, since $f(i,j)=f(j,i)$ for all $i$ and $j$. From the cocycle condition $\phi_{ij}\phi_{jk}\phi_{ki}=1$ we discover $\delta f(i,j,k)=\det(\phi_{ij})\det(\phi_{jk})\det(\phi_{ki})=\det(\phi_{ij}\phi_{jk}\phi_{ki})=1$, what shows that $f\in Z^1(M,\mathbb{Z}_2)$. Moreover, it defines an element $[f]$ of $H^1(M,\mathbb{Z}_2)$. It can readily be shown that this element is independent of the chosen local frame. Then the first Stiefel-Whitney class $w_1(P(M,O(n)))$ of $M$ is defined as $w_1(P(M,O(n))) := [f]\in H^1(M,\mathbb{Z}_2)$. One can prove that $M$ is orientable iff $w_1$ is trivial.

Let $M$ be an orientable $n$-dimensional Riemannian manifold. Then the structure group of $TM$ is $SO(n)$. Consider the lifting of $\phi_{ij}\in SO(n)$ to $\tilde\phi_{ij}\in\mathrm{Spin}(n)$, such that $\pi\,:\,\tilde\phi_{ij}\mapsto\phi_{ij}$, where $\pi$ is the twisted adjoint representation (II.75). Instead of $\tilde\phi_{ij}$ we could also take $-\tilde\phi_{ij}$. The Čech two-cochain is then defined by

$$f(i,j,k) := \tilde\phi_{ij}\tilde\phi_{jk}\tilde\phi_{ki}. \tag{II.137}$$

Note that $\pi\,:\,\tilde\phi_{ij}\tilde\phi_{jk}\tilde\phi_{ki}\mapsto\phi_{ij}\phi_{jk}\phi_{ki}=1$. One can easily verify that $f$ is symmetric and closed and hence $f$ defines the second Stiefel-Whitney class to be $w_2(P(M,SO(n))) := [f]\in H^2(M,\mathbb{Z}_2)$. Then one can prove that $M$ is a spin manifold iff $w_2$ is trivial or is an spin$^c$ manifold iff $w_2\equiv c \bmod 2$ for some $c\in H^2(M,\mathbb{Z})$. For instance, we have $w_1(S^n)\cong w_2(S^n)\cong\{e\}$.

**Atiyah-Singer index theorem.** This theorem is one of the monuments of twentieth century mathematics. Let us state this theorem not in its full generality, but for the spin$^c$ complex defined in section § II.2.4, i.e. for

$$\Gamma(M, W^+) \underset{\mathcal{D}^\dagger_\mathcal{A}}{\overset{\mathcal{D}_\mathcal{A}}{\rightleftarrows}} \Gamma(M, W^-). \tag{II.138}$$

The analytical index of this complex is defined as

$$\operatorname{ind} \mathcal{D}_\mathcal{A} := \dim \ker \mathcal{D}_\mathcal{A} - \dim \ker \mathcal{D}^\dagger_\mathcal{A}. \tag{II.139}$$

Let $M$ be an orientable, compact Riemannian manifold of dimension $n$. Then formally the Atiyah-Singer index theorem for the complex (II.138) reads

$$\operatorname{ind} \mathcal{D}_\mathcal{A} = \int_M \hat{A}(TM) \wedge \operatorname{ch}(L_\Gamma^{1/2}), \tag{II.140}$$

where $\hat{A}(TM)$ is Dirac genus. It is given in terms of the Pontrjagin classes (II.131) by

$$\hat{A}(TM) := 1 - \frac{1}{24} p_1(M) + \frac{1}{5760}\{7p_1(M) \wedge p_1(M) - 4p_2(M)\} + \cdots . \tag{II.141}$$

Let the spinor be a section of $W \otimes E$, where $E = P \times_G V$ is the vector bundle associated to the principal bundle $P(M, G)$ and the $G$-action on $E$ is given by some representation $\mathfrak{R}$. The spin$^c$ complex (II.138) extends to the so-called twisted spin$^c$ complex

$$\Gamma(M, W^+ \otimes E) \underset{\mathcal{D}^\dagger_{\mathcal{A},\mathcal{B}}}{\overset{\mathcal{D}_{\mathcal{A},\mathcal{B}}}{\rightleftarrows}} \Gamma(M, W^- \otimes E), \tag{II.142}$$

where $\mathcal{A} \in \mathcal{A}(\Gamma)$ and $\mathcal{B}$ is a fixed connection one-form on $E$. Then the Atiyah-Singer index theorem for the twisted spin$^c$ complex is

$$\operatorname{ind} \mathcal{D}_{\mathcal{A},\mathcal{B}} = \int_M \hat{A}(TM) \wedge \operatorname{ch}(L_\Gamma^{1/2} \otimes E) = \int_M \hat{A}(TM) \wedge \operatorname{ch}(L_\Gamma^{1/2}) \wedge \operatorname{ch}(E). \tag{II.143}$$

Later on, we will specify this theorem to the case of four-dimensional manifolds.

## § II.3 Yang-Mills Field Theory

### § II.3.1 Action and Equations of Motion

**Electrodynamics.** The simplest example of a gauge theory is electrodynamics, where the base manifold is chosen to be some pseudo-Riemannian manifold $M$ with signature $(p, q)$[6] and the fiber is the $U(1)$ group. With $g(x) = e^{-i\alpha(x)}$ the gauge transformation (II.29) reduces to

$$\mathcal{A}'_\mu = \mathcal{A}_\mu - i\partial_\mu \alpha \tag{II.144}$$

[6] We are using the mostly plus convention for the metric, e.g. $(1, 1, 1, -1)$ for the Minkowski space.

and therefore $\mathcal{F}'_{\mu\nu} = \mathcal{F}_{\mu\nu}$. Since $\mathcal{A}$ and $\mathcal{F}$ are purely imaginary, we define $\mathcal{A} := iA$ and $\mathcal{F} := iF$. The gauge invariant action functional $S : \mathfrak{u}(1) \otimes \Omega^1(M) \to \mathbb{R} \cup \{\infty\}$ is by definition

$$S(\mathcal{A}) = \frac{1}{2}\int_M \mathcal{F} \wedge *\mathcal{F} = -\frac{1}{4}\int_M \mathrm{d^n x}\ \sqrt{|g|}\, F_{\mu\nu}F^{\mu\nu}, \tag{II.145}$$

where '$*$' denotes the Hodge star and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By varying the action we obtain Maxwell's equations. Consider an infinitesimal transformation of the form $\mathcal{A} \mapsto \mathcal{A} + \delta\mathcal{A}$. Then we find

$$\begin{aligned} \delta S(\mathcal{A}) &= \frac{1}{2}\int_M (\delta\mathcal{F} \wedge *\mathcal{F} + \mathcal{F} \wedge *\delta\mathcal{F}) = \int_M \delta\mathcal{F} \wedge *\mathcal{F} = \int_M \mathrm{d}(\delta\mathcal{A}) \wedge *\mathcal{F} \\ &= \int_M \delta\mathcal{A} \wedge \mathrm{d}*\mathcal{F} \stackrel{!}{=} 0 \end{aligned}$$

and hence $\mathrm{d}*\mathcal{F} = 0 = \mathrm{d}*F$. In the second step we have used the relation $\omega \wedge *\rho = \rho \wedge *\omega$ for $\omega, \rho \in \Omega^p(M)$ and in the fourth Stokes' theorem. The Bianchi identities (II.41) are $\mathrm{d}\mathcal{F} = 0$.

**Yang-Mills action and equations of motion.** After this short motivation we generalize these ideas to general Lie groups $G$. Let $\{X_a\}$ be a basis for the Lie algebra $\mathfrak{g}$. Introducing $\mathcal{A}_\mu := g_{\mathrm{YM}} A^a_\mu X_a$ and $\mathcal{F}_{\mu\nu} := g_{\mathrm{YM}} F^a_{\mu\nu} X_a$ the curvature (II.37) becomes

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_{\mathrm{YM}}\, C_{bc}{}^a A^b_\mu A^c_\nu, \tag{II.146}$$

where $g_{\mathrm{YM}}$ is referred to as the coupling constant.

The trace of the wedge product $\mathcal{F} \wedge *\mathcal{F}$ is invariant under the gauge transformation (II.39). Therefore, the natural generalization of (II.145) is $S_{\mathrm{YM}} : \mathfrak{g} \otimes \Omega^1(M) \to \mathbb{R} \cup \{\infty\}$ with

$$S_{\mathrm{YM}}(\mathcal{A}) = \frac{1}{g^2_{\mathrm{YM}}}\int_M \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}), \tag{II.147}$$

and is referred to as the Yang-Mills action. In a similar manner one finds by varying the action the equations of motion. We have

$$\delta S_{\mathrm{YM}}(\mathcal{A}) = \frac{1}{g^2_{\mathrm{YM}}}\int_M \mathrm{tr}(\delta\mathcal{F} \wedge *\mathcal{F} + \mathcal{F} \wedge *\delta\mathcal{F}) = \frac{2}{g^2_{\mathrm{YM}}}\int_M \mathrm{tr}(\delta\mathcal{F} \wedge *\mathcal{F}).$$

In the last step we remembered that $\mathrm{tr}(\omega \wedge *\rho) = \mathrm{tr}(\rho \wedge *\omega)$ for $\omega, \rho \in \mathfrak{g} \otimes \Omega^p(M)$. Moreover, we have $\mathrm{d}(\delta\mathcal{A} \wedge *\mathcal{F}) = \mathrm{d}(\delta\mathcal{A}) \wedge *\mathcal{F} - \delta\mathcal{A} \wedge \mathrm{d}*\mathcal{F}$ and therefore

$$\delta\mathcal{F} \wedge *\mathcal{F} = \mathrm{d}(\delta\mathcal{A} \wedge *\mathcal{F}) + \delta\mathcal{A} \wedge (\mathrm{d}*\mathcal{F} + \mathcal{A} \wedge *\mathcal{F}) + \mathcal{A} \wedge \delta\mathcal{A} \wedge *\mathcal{F}.$$

Note that $\mathrm{tr}(\mathcal{A} \wedge \delta\mathcal{A} \wedge *\mathcal{F}) = -(-1)^{n-2}\,\mathrm{tr}(\delta\mathcal{A} \wedge *\mathcal{F} \wedge \mathcal{A})$, what immediatelly follows from the cyclicity property of the trace. Using Stokes' theorem, we obtain

$$\begin{aligned} \delta S_{\mathrm{YM}}(\mathcal{A}) &= \frac{2}{g^2_{\mathrm{YM}}}\int_M \mathrm{tr}\{\delta\mathcal{A} \wedge (\mathrm{d}*\mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - (-1)^{n-2}*\mathcal{F} \wedge \mathcal{A})\} \\ &= \frac{2}{g^2_{\mathrm{YM}}}\int_M \mathrm{tr}(\delta\mathcal{A} \wedge \mathcal{D}*\mathcal{F}) \stackrel{!}{=} 0 \end{aligned}$$

and hence

$$\mathcal{D}{*}\mathcal{F} = 0. \tag{II.150}$$

If the trace $\mathrm{tr}(X_a X_b)$ can be chosen to be $-\frac{1}{2}\delta_{ab}$, what is always possible for compact and semisimple Lie groups, the action (II.147) can be rewritten as

$$S_{\mathrm{YM}}(\mathcal{A}) = -\frac{1}{4}\int_M \mathrm{d^n x}\ \sqrt{|g|}\ F^a_{\mu\nu}F^{a\mu\nu}. \tag{II.151}$$

In contrast to electrodynamics we do obtain nonlinear coupled partial differential equations for the gauge potential $\mathcal{A}$. So far, no general method for solving such equations exists and hence it is most unlikely to find explicit solutions. Even if one could construct solutions, one still needs to show that they minimize the action.

However, under certain assumptions it is possible to find solutions to (II.150). We are going to see that for given boundary conditions, which will be specified by a topological charge (or winding number), one can construct spacetime-dependent field configurations, known as instantons, which minimize the Euclidean version of (II.147) for $n = 4$, where the latter can either be thought of as the Euclidean action of a $n = 4$ theory or as the energy for a theory in $4 + 1$ dimensions choosing temporal gauge for the gauge fields, i.e. $\mathcal{A}_5 = 0$. Furthermore, in lower dimensions one can find solutions, which are again characterized by a topological charge. Such configurations are referred to as monopoles, vortices, etc. Having made these statements we have implicitly assumed that the action (or the energy) is finite. It is convenient to summarize monopoles, vortices, etc. under the name soliton. The notion soliton was introduced by mathematicians for localized solutions which are stable against pertubations and do not change either velocity or shape when colliding with each other. However, we are going to follow a looser definition and also allow a change of velocity and shape.

## § II.3.2 Symmetries of the Yang-Mills Action

**Conformal group.** Let us investigate the symmetries of the action (II.147) in the case of $M = \mathbb{R}^{p,q}$. Besides the intrinsic gauge symmetry, the Yang-Mills action (II.147) is invariant with respect to the group of conformal transformations of the considered Lorentz space $\mathbb{R}^{p,q}$. This group is locally isomorphic to the group $SO(p{+}1, q{+}1)$. Infinitesimal conformal transformations are

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon^\mu, \tag{II.152a}$$

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \omega^\mu_{\ \nu} x^\nu, \tag{II.152b}$$

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \alpha x^\mu, \tag{II.152c}$$

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + b^\mu x^2 - 2x^\mu b_\nu x^\nu. \tag{II.152d}$$

where $\omega = (\omega^\mu_{\ \nu}) \in \mathfrak{so}(p,q)$, $\epsilon = (\epsilon^\mu)$ and $b = (b^\mu) \in \mathbb{R}^{p,q}$, $x^2 = g_{\mu\nu}x^\mu x^\nu$ and $\alpha \in \mathbb{R}$. These transformations are referred to as translations, (Lorentz) rotations, dilatations and special conformal transformations, respectively. It is fairly straightforward to write down their global versions

$$x^\mu \;\mapsto\; \tilde{x}^\mu = x^\mu + a^\mu, \tag{II.153a}$$
$$x^\mu \;\mapsto\; \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{II.153b}$$
$$x^\mu \;\mapsto\; \tilde{x}^\mu = e^\alpha x^\mu, \tag{II.153c}$$
$$x^\mu \;\mapsto\; \tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b_\nu x^\nu + b^2 x^2}, \tag{II.153d}$$

where now $\Lambda = (\Lambda^\mu{}_\nu) \in SO(p,q)$, $a = (a^\mu)$ and $b = (b^\mu) \in \mathbb{R}^{p,q}$, $x^2 = g_{\mu\nu}x^\mu x^\nu$ and $\alpha \in \mathbb{R}$.

The action of infinitesimal conformal transformations on smooth functions $f \in \mathcal{F}(\mathbb{R}^{p,q})$,

$$f \mapsto f + \delta_N f = f + \epsilon \mathcal{L}_N f,$$

is induced by the vector fields,

$$P_\mu = \partial_\mu, \tag{II.154a}$$
$$M_{\mu\nu} = \tfrac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu), \tag{II.154b}$$
$$D = x^\mu \partial_\mu, \tag{II.154c}$$
$$K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu. \tag{II.154d}$$

Here, $\mathcal{L}_N$ denotes the Lie derivative along the vector field $N$ where $N = N^\mu \partial_\mu$ represents any vector field of (II.154), i.e. $\mathcal{L}_N f = Nf$, and $\epsilon$ stands symbolically for the infinitesimal parameters of the transformation.

Infinitesimal transformations $\mathcal{A}_\mu \mapsto \mathcal{A}_\mu + \delta_N \mathcal{A}_\mu$ of the components of the gauge field are given by

$$\mathcal{L}_N \mathcal{A}_\mu = N^\nu \partial_\nu \mathcal{A}_\mu + \mathcal{A}_\nu \partial_\mu N^\nu. \tag{II.155}$$

Then one can show by a direct calculation that the action

$$S_{\rm YM}(\mathcal{A}) = -\frac{1}{4}\int_{\mathbb{R}^{p,q}} \mathrm{d}^{\rm n}\mathrm{x}\;\, F^a_{\mu\nu} F^{a\mu\nu}. \tag{II.156}$$

is invariant under (II.155).

Moreover, it is well known that for a semisimple structure group $G$ there are no local symmetries differing from the gauge and conformal symmetries.

# Chapter III

## Yang-Mills Instantons

We start our studies with a general discussion of instantons and then describe their construction using different methods. The term instanton refers to localized finite-action solutions to the classical Euclidean field equations of a theory. In that context "localized" means a localization in space and time. In that sense, instantons are rather events in spacetime than particles. The field strength $F^a_{\mu\nu}$ in (II.151) contains the coupling $g_{\rm YM}$. However, we can rescale the gauge potential $A'^a_\mu = A^a_\mu / g_{\rm YM}$ so that the action (II.151) takes the form

$$S_{\rm YM}(\mathcal{A}) = -\frac{1}{4g^2_{\rm YM}} \int_M \mathrm{d^n x}\ \sqrt{|g|} F'^a_{\mu\nu} F'^{a\mu\nu}, \tag{III.1}$$

where $F'^a_{\mu\nu}$ is now independent of the coupling. Therefore instantons become important in the strong coupling regime of a theory, because looking at the Euclidean path integral for a theory we have $e^{-S^{\rm E}}$ and $S^{\rm E}$ is positive-semidefinite (see below). Thus, instanton configurations will dominate the path integral.

## § III.1 Instantons - A General Discussion

**Self-dual and anti-self-dual Yang-Mills equations.** The Euclidean version of (II.147) is ($M = \mathbb{R}^n$)

$$S^{\rm E}_{\rm YM}(\mathcal{A}) = -\frac{1}{g^2_{\rm YM}} \int_{\mathbb{R}^n} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) = \frac{1}{4} \int_{\mathbb{R}^n} \mathrm{d^n x}\ F^a_{\mu\nu} F^{a\mu\nu} \geq 0, \tag{III.2}$$

where in the second step we have assumed that the generators satisfy $\mathrm{tr}(X_a X_b) = -\frac{1}{2}\delta_{ab}$. In order to have $S^{\rm E}_{\rm YM}$ finite, $\mathcal{F}_{\mu\nu}$ must vanish sufficiently rapidly as $\|x\| \to \infty$. This can be achieved if the gauge field $\mathcal{A}_\mu$ approaches a pure gauge, that is

$$\mathcal{A}_\mu(x) \to g^{-1}(\hat{x}) \partial_\mu g(\hat{x}) \tag{III.3}$$

with $\|\hat{x}\| = 1$ and $g(\hat{x}) \in G$. Obviously, this expression is invariant under the left action of a constant group element $g_0 \in G$. Hence, (III.3) defines a map from the unit sphere $S^{n-1}$ to the group manifold $G$, i.e. $g : S^{n-1} \to G$. Having in mind subsection § II.2.3, we know that such topologically distinct mappings can be characterized by the $(n-1)$-th homotopy group,

$\pi_{n-1}(G)$, since we may choose these maps so that one point of $S^{n-1}$ is mapped into a fixed element of $G$.

Following a scaling argument (cf. e.g. [200]) we can put restrictions on the values of the dimensionality $n$. Define $\mathcal{A}^\lambda_\mu(x) := \mathcal{A}_\mu(x/\lambda)/\lambda$ with $\lambda \in \mathbb{R}$ and $\lambda > 0$. Then the action (III.2) scales as

$$S^{\mathrm{E}}_{\mathrm{YM}}(\mathcal{A}^\lambda) = \lambda^{n-4} S^{\mathrm{E}}_{\mathrm{YM}}(\mathcal{A}), \tag{III.4}$$

i.e. for values $n \neq 4$ we cannot find topological nontrivial configurations minimizing the action unless the action is identically zero. This, however, essentially means that the curvature has to vanish everywhere and hence by a gauge transformation we can bring the gauge potential to zero everywhere, as well. Thus, in the following we restrict ourselves to the Euclidean four-dimensional space $\mathbb{R}^4$.

In order to find the minimum action configurations we start with the trivial inequality

$$\mp \int_{\mathbb{R}^4} \mathrm{tr}\{(\mathcal{F} \pm *\mathcal{F}) \wedge (\mathcal{F} \pm *\mathcal{F})\} \geq 0. \tag{III.5}$$

Then we find after a short calculation from (III.5)

$$-\int_{\mathbb{R}^4} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) \geq \pm \int_{\mathbb{R}^4} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F})$$

and therefore

$$S^{\mathrm{E}}_{\mathrm{YM}} \geq \pm \frac{1}{g^2_{\mathrm{YM}}} \int_{\mathbb{R}^4} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) \qquad \Longrightarrow \qquad S^{\mathrm{E}}_{\mathrm{YM}} \geq \frac{8\pi^2}{g^2_{\mathrm{YM}}} |Q|, \tag{III.7}$$

where $Q \in \mathbb{Z}$ is called the instanton number or the topological charge of the instanton,

$$Q = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}). \tag{III.8}$$

For semisimple Lie groups $G$ it coincides with the negative second Chern number of the principal $G$-bundle over $\mathbb{R}^4$.

It is well known that the integral in (III.8) reduces to an integral over a three-sphere at infinity,

$$Q = \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) = \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(g^{-1}\,\mathrm{d}\,g \wedge g^{-1}\,\mathrm{d}\,g \wedge g^{-1}\,\mathrm{d}\,g), \tag{III.9}$$

where we have used the fact that the gauge potential approaches a pure gauge configuration at infinity (III.3). To understand this, we want to remember that there is another way of satisfying the finite action requirement by formulating the theory on a compactified $\mathbb{R}^4$ what means that we have the topology of a four-sphere, since $\mathbb{R}^4 \cup \{\infty\} \cong S^4$. The sphere $S^4$ can be covered by two coordinate patches

$$S^4 = U_1 \cup U_2 \qquad \text{with} \qquad U_1 = S^4_+ \qquad \text{and} \qquad U_2 = S^4_-, \tag{III.10}$$

where $S^4_+$ denotes the northern and $S^4_-$ the southern hemisphere of $S^4$, respectively. Furthermore, the stereographic projection is a conformal map. The Yang-Mills action is conformally invariant, as we have seen previously[1], what implies the form invariance of the equations of motion. Since $S^4$ is a compact manifold, the finiteness requirement is reflected in the statement of a smooth extension of the gauge potential from $\mathbb{R}^4$ to $S^4$. Now we want to assume that the gauge potential vanishes identically on $U_2$, i.e.

$$\mathcal{A}_2(x) = 0 \qquad \text{for} \qquad x \in U_2. \tag{III.11}$$

On the overlapping region $U_1 \cap U_2 \cong S^3$ we find

$$\mathcal{A}_1 = \phi_{12}^{-1} \mathcal{A}_2 \phi_{12} + \phi_{12}^{-1} \,\mathrm{d}\, \phi_{12} = \phi_{12}^{-1} \,\mathrm{d}\, \phi_{12}. \tag{III.12}$$

Therefore $g(x)$ in (III.3) is identified with the transition function $\phi_{12}$. Next we consider the four-form $\mathcal{F} \wedge \mathcal{F}$ and note that $\mathrm{tr}(\mathcal{F} \wedge \mathcal{F})$ is closed, that is $\mathrm{d}\,\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) = 0$. This assertion can readily be shown using the Bianchi identity (II.41),

$$\begin{aligned} \mathrm{d}\,\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) &= \mathrm{tr}\{\mathrm{d}\,\mathcal{F} \wedge \mathcal{F} + \mathcal{F} \wedge \mathrm{d}\,\mathcal{F}\} \\ &= \mathrm{tr}\{-(\mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A}) \wedge \mathcal{F} - \mathcal{F} \wedge (\mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A})\} = 0. \end{aligned}$$

By Poincare's lemma, the closed form $\mathrm{tr}(\mathcal{F} \wedge \mathcal{F})$ is locally exact,

$$\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) = \mathrm{d}\,\omega, \tag{III.13}$$

where $\omega$ is a local three-form. The claim is that $\omega$ is given by

$$\omega = \mathrm{tr}\left\{\mathcal{A} \wedge \mathrm{d}\,\mathcal{A} + \tfrac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right\}. \tag{III.14}$$

This can be seen using the Bianchi identity (II.41) and the relations $\mathrm{tr}(\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) = 0$ and $\mathrm{tr}(\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{A}) = -\,\mathrm{tr}(\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F}) = -\,\mathrm{tr}(\mathcal{F} \wedge \mathcal{A} \wedge \mathcal{A})$. Exploiting Stokes' theorem, we find

$$\int_{U_1} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) = \int_{U_1} \mathrm{d}\,\omega = \int_{S^3} \omega.$$

Since $\mathcal{F} = 0$ on $S^3$, we have

$$\begin{aligned} \omega|_{S^3} &= \mathrm{tr}\left\{\mathcal{A} \wedge \mathrm{d}\,\mathcal{A} + \tfrac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right\} \\ &= \mathrm{tr}\left\{\mathcal{A} \wedge (-\mathcal{A} \wedge \mathcal{A}) + \tfrac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right\} = -\tfrac{1}{3}\,\mathrm{tr}(\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} Q &= -\frac{1}{8\pi^2} \int_{S^4} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) \\ &= -\frac{1}{8\pi^2} \int_{U_1} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) = \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \\ &= \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(g^{-1}\,\mathrm{d}\,g \wedge g^{-1}\,\mathrm{d}\,g \wedge g^{-1}\,\mathrm{d}\,g), \end{aligned} \tag{III.16}$$

[1] To be precise, we have written down the conformal properties for flat spaces only. However, for $M$ a Riemannian four-manifold, we still have a conformal symmetry, since the Yang-Mills action is invariant under a rescaling of the metric of the form $g \mapsto \lambda^2 g$ with $\lambda \in \mathcal{F}(M)$ and $\lambda \neq 0$.

since $\mathcal{F}$ vanishes identically on $U_2$.

The integer $Q$ corresponds to the third homotopy group, e.g. choosing $G$ compact, connected and simple we have from the table given in § II.2.3 that $\pi_3(G) \cong \mathbb{Z}$. For instance, if we take $G = SU(2)$ what is topologically equivalent to $S^3$, then $g$ is a map from $S^3$ to $S^3$ and $Q$ tells us how often $S^3$ is wrapped around $SU(2) \cong S^3$.

The lower bound of (III.7) is saturated if

$$\mathcal{F} = \pm{*}\mathcal{F} \qquad \Longleftrightarrow \qquad \mathcal{F}_{\mu\nu} = \pm\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{III.17}$$

The equations (III.17) are known as the SDYM and ASDYM equations. By virtue of the Bianchi identity (II.41) solutions to (III.17) automatically satisfy the Yang-Mills equations $\mathcal{D}{*}\mathcal{F} = 0$.

**Remark on Abelian instantons.** In the case of an Abelian gauge group the curvature is given by $\mathcal{F} = \mathrm{d}\,\mathcal{A}$ and hence we obtain for the topological charge (III.8)

$$-\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathcal{F}\wedge\mathcal{F} = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{d}(\mathcal{A}\wedge\mathcal{F}) = -\frac{1}{8\pi^2}\int_{S^3}\mathcal{A}\wedge\mathcal{F} = 0.$$

This means that the curvature has to vanish everywhere and therefore no Abelian instantons exist.

## § III.2 Some Twistor Geometry

Now, the task is to find solutions to the equations (III.17). Still, they are nonlinear coupled partial differential equations for the gauge potential. But in contrast to the equations of motion they are of first order. One might still expect that it is difficult to find explicit solutions. However, there is an important observation that these equations can be formulated as the compatibility conditions for a linear system of differential equations on some matrix-valued function $\psi$ which lives on the so-called twistor space.

Twistors were introduced by Penrose in order to translate massless free field equations in spacetime into holomorphic structures on a related complex manifold known as the twistor space. The twistor theory is based on an integro-geometric transformation which transforms complex analytic data on the twistor space to solutions of massless field equations. Originally suggested for the description of linear conformally invariant equations, the twistor approach turned out to be a powerful tool for the construction of solutions to the (nonlinear) equations of motion of General Relativity and Yang-Mills theories [158, 193].

In the sequel we shall describe the construction of the aforementioned linear system for the self-dual case of (III.17) eventually resulting in the Penrose-Ward correspondence. But to do this and to understand the geometrical meaning behind it we first need to introduce complex structures on $\mathbb{R}^4$ and $S^2$. The reason for $S^2$ becomes clear momentarily. Equivalently, we can take $\mathbb{CP}^1$ instead of $S^2$ since they are diffeomorphic to each other. The following discussion is mainly taken from [162, 197] and [103].

### § III.2.1 Complex Structures on $\mathbb{R}^4$ and $S^2$

**Complex structure on $\mathbb{R}^4$.** Let $M$ be an orientable Riemannian four-manifold. Suppose, that $M$ is endowed with an almost complex structure $J = (J^\mu_{\ \nu}) : T_xM \to T_xM$. It is well known that all constant almost complex structures on $M$ are parametrized by the Hermitian symmetric space $SO(4)/U(2) \cong S^2$. This follows from the fact that having chosen an oriented frame on $T_xM$ we can only perform $SO(4)$ rotations to obtain a new frame with the same orientation. Now the question is which elements of $SO(4)$ leave the almost complex structure $J$ invariant or in other words which $\Lambda \in SO(4)$ do commute with $J$. It turns out that this subgroup of $SO(4)$ is isomorphic to $U(2)$ and hence the different almost complex structures $J$ are parametrized by the quotient $SO(4)/U(2)$. Remembering equation (II.20) from the second chapter and the fact that $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$, one realizes that $SO(4)/U(2) \cong S^2$. Let now $M = \mathbb{R}^4$. We can write $J$ as follows

$$J^\mu_{\ \nu} = \xi_a \bar\eta^a_{\nu\sigma}\delta^{\sigma\mu}, \tag{III.18}$$

where the real numbers $\xi^a$ parametrize $S^2$: $\xi_a\xi^a = 1$. Here we have used the 't Hooft symbols which are defined as

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon^a_{bc} & \text{for} \quad \mu = b, \nu = c \\ \delta^a_\mu & \text{for} \quad \nu = 4 \\ -\delta^a_\nu & \text{for} \quad \mu = 4 \end{cases} \qquad \text{and} \qquad \bar\eta^a_{\mu\nu} = \begin{cases} \epsilon^a_{bc} & \text{for} \quad \mu = b, \nu = c \\ -\delta^a_\mu & \text{for} \quad \nu = 4 \\ \delta^a_\nu & \text{for} \quad \mu = 4 \end{cases} \tag{III.19}$$

Using the identities

$$\bar\eta^a_{\mu\sigma}\bar\eta^b_{\nu\sigma} = \delta^{ab}\delta_{\mu\nu} + \epsilon^{abc}\bar\eta^c_{\mu\nu}, \tag{III.20}$$

it can be shown that indeed $J^2 = -1$. Of course, the choice $\tilde J^\mu_{\ \nu} = \xi_a \eta^a_{\nu\sigma}\delta^{\sigma\mu}$ is also possible. This corresponds to the opposite orientation on $\mathbb{R}^4$ and to the transition from the SDYM to the ASDYM equations. The eigenvalues of $J$ are simply $\pm i$ and therefore we can decompose the complexified tangent space $\mathbb{C}^4 \cong \mathbb{R}^4 \otimes \mathbb{C}$ of $\mathbb{R}^4$ into two subspaces, i.e. $\mathbb{C}^4 \cong \mathcal{V} \oplus \bar{\mathcal{V}}$ with

$$\mathcal{V} = \{V \in \mathbb{C}^4 \,|\, J^\mu_{\ \nu}V^\nu = iV^\mu\} \qquad \text{and} \qquad \bar{\mathcal{V}} = \{V \in \mathbb{C}^4 \,|\, J^\mu_{\ \nu}V^\nu = -iV^\mu\}. \tag{III.21}$$

As a basis of $\mathcal{V}$ and $\bar{\mathcal{V}}$ we may take

$$(V_1^{(1)\mu}) = \left(\tfrac{1}{2}, -\tfrac{i}{2}, -\tfrac{1}{2}\bar\lambda, \tfrac{i}{2}\bar\lambda\right) \qquad \text{and} \qquad (V_2^{(1)\mu}) = \left(\tfrac{1}{2}\bar\lambda, \tfrac{i}{2}\bar\lambda, \tfrac{1}{2}, \tfrac{i}{2}\right), \tag{III.22a}$$

$$(\bar V_{\bar 1}^{(1)\mu}) = \left(\tfrac{1}{2}, \tfrac{i}{2}, -\tfrac{1}{2}\lambda, -\tfrac{i}{2}\lambda\right) \qquad \text{and} \qquad (\bar V_{\bar 2}^{(1)\mu}) = \left(\tfrac{1}{2}\lambda, -\tfrac{i}{2}\lambda, \tfrac{1}{2}, -\tfrac{i}{2}\right), \tag{III.22b}$$

where $\lambda$ and $\bar\lambda$ are the local holomorphic and anti-holomorphic coordinates on the two-sphere $SO(4)/U(2)$, i.e. $\lambda = (\xi^1 + i\xi^2)/(1+\xi^3)$. Using $J$, we can introduce vector fields $V_a^{(1)} = V_a^{(1)\mu}\partial_\mu$ of type $(1,0)$ and vector fields $\bar V_{\bar a}^{(1)} = \bar V_{\bar a}^{(1)\mu}\partial_\mu$ of type $(0,1)$ with respect to $J$ (for $a = 1, 2$). For instance, we have

$$\bar V_{\bar 1}^{(1)} = \partial_{\bar z^{\bar 1}} - \lambda\partial_{z^2} \qquad \text{and} \qquad \bar V_{\bar 2}^{(1)} = \partial_{\bar z^{\bar 2}} + \lambda\partial_{z^1}, \tag{III.23}$$

where

$$z^1 = x^1 + ix^2, \quad \bar{z}^{\bar{1}} = x^1 - ix^2, \quad z^2 = x^3 - ix^4, \quad \bar{z}^{\bar{2}} = x^3 + ix^4 \tag{III.24}$$

are the canonical complex coordinates on $\mathbb{R}^4 \cong \mathbb{C}^2$.

**Complex structure on $S^2$.** From elementary differential geometry we know that $S^2$ can be covered by two coordinate patches $U_{1,2} \cong \mathbb{R}^2 \cong \mathbb{C}$ using the stereographic projection defined by

$$\omega_1^1 = \frac{\xi^1}{1+\xi^3} \quad \text{and} \quad \omega_1^2 = \frac{\xi^2}{1+\xi^3} \quad \text{on} \quad U_1, \tag{III.25a}$$

$$\omega_2^1 = \frac{\xi^1}{1-\xi^3} \quad \text{and} \quad \omega_2^2 = \frac{\xi^2}{1-\xi^3} \quad \text{on} \quad U_2. \tag{III.25b}$$

Taking the standard complex structure $\mathfrak{J}$ on $S^2$ given by

$$\mathfrak{J} = (\mathfrak{J}^a{}_b), \qquad \mathfrak{J}^a{}_c \mathfrak{J}^c{}_b = -\delta^a_b \quad \text{and} \quad \mathfrak{J}^1{}_2 = -\mathfrak{J}^2{}_1 = -1 \tag{III.26}$$

in the coordinates $\{\omega_1^a\}$, we can introduce holomorphic and anti-holomorphic vector fields on $U_1$ by

$$V_3^{(1)} = \tfrac{1}{2}(\partial_{\omega_1^1} - i\partial_{\omega_1^2}) = \partial_\lambda \quad \text{and} \quad \mathfrak{J}^a{}_b V_3^{(1)b} = iV_3^{(1)a}, \tag{III.27a}$$

$$\bar{V}_{\bar{3}}^{(1)} = \tfrac{1}{2}(\partial_{\omega_1^1} + i\partial_{\omega_1^2}) = \partial_{\bar{\lambda}} \quad \text{and} \quad \mathfrak{J}^a{}_b \bar{V}_{\bar{3}}^{(1)b} = -i\bar{V}_{\bar{3}}^{(1)a}, \tag{III.27b}$$

where $\lambda = \omega_1^1 + i\omega_1^2$ is the complex coordinate on $U_1$. Similarly, we choose $\zeta = \omega_2^1 - i\omega_2^2$ as the complex coordinate on $U_2$ and put the vector fields $V_3^{(2)} = \partial_\zeta$ and $\bar{V}_{\bar{3}}^{(2)} = \partial_{\bar{\zeta}}$ on it. That means we cover $S^2$ by two coordinate patches $U_{1,2}$, with $U_1$ a neighborhood of $\lambda = 0$ and $U_2$ a neighborhood of $\lambda = \infty$, i.e.

$$U_1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| < \infty\} \quad \text{and} \quad U_2 = \{\lambda \in \mathbb{C} \cup \{\infty\} \,|\, |\lambda| > 0\}. \tag{III.28}$$

Thus, the sphere $S^2$ considered as the complex projective line $\mathbb{CP}^1$ is the complex manifold obtained by patching together $U_1$ and $U_2$. On the intersection $U_1 \cap U_2 \cong \mathbb{C}^*$ the coordinates $\lambda$ and $\zeta$ are related via $\lambda\zeta = 1$.

### § III.2.2 Twistor Space of $\mathbb{R}^4$

**Definition of the twistor space.** Let again $M$ be an orientable Riemannian four-manifold. Then we define another manifold by

$$\mathcal{Z}(M) := P(M, SO(4)) \times_{SO(4)} (SO(4)/U(2)), \tag{III.29}$$

where $P(M, SO(4))$ is the principal $SO(4)$-bundle of oriented orthonormal frames on $M$. The manifold $\mathcal{Z}(M)$ is called the twistor space of $M$. It describes the different complex structures on $M$. A Riemannian metric is said to be self-dual if the anti-self-dual part of the Weyl tensor vanishes. We call a manifold self-dual when it is endowed with a self-dual metric. One can show that the twistor space $\mathcal{Z}(M)$ for self-dual $M$ is a complex three-dimensional manifold. This we are going to see for $M = \mathbb{R}^4$ in the following. Note that $\mathbb{R}^4$ is trivially self-dual. Then the definition of the twistor space (III.29) takes the form

$$\mathcal{Z}(\mathbb{R}^4) \equiv \mathcal{Z} \cong \mathbb{R}^4 \times \mathbb{CP}^1. \tag{III.30}$$

This space can be covered by two coordinate patches

$$\mathcal{Z} = \mathcal{U}_1 \cup \mathcal{U}_2 \qquad \text{with} \qquad \mathcal{U}_1 = \mathbb{R}^4 \times U_1 \qquad \text{and} \qquad \mathcal{U}_2 = \mathbb{R}^4 \times U_2 \tag{III.31}$$

with the coordinates $\{x^\mu, \lambda, \bar\lambda\}$ and $\{x^\mu, \zeta, \bar\zeta\}$ on $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. On the intersection $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$ one may use any of them, and we will choose $\{x^\mu, \lambda, \bar\lambda\}$. Therefore, the twistor space $\mathcal{Z}$ can be considered as a trivial bundle $\pi : \mathcal{Z} \to \mathbb{R}^4$ over $\mathbb{R}^4$ with the fiber $\mathbb{CP}^1$ and the canonical projection $\pi : \{x^\mu, \lambda, \bar\lambda\} \mapsto \{x^\mu\}$. Later on, we will consider a sufficiently small open ball $U \subset \mathbb{R}^4$, such that $\mathcal{P} := \mathcal{Z}|_U \cong U \times \mathbb{CP}^1$, where we denote the cover of $\mathcal{P}$ by the same letters $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ with $\mathcal{U}_1 = U \times U_1$ and $\mathcal{U}_2 = U \times U_2$.

**Complex structure on $\mathcal{Z}$.** Having introduced complex structures on $\mathbb{R}^4$ and $S^2$, we are able to define a complex structure $\mathcal{J} = (J, \mathfrak{J})$ on $\mathcal{Z}$. Note that in our construction each point of the fiber $\mathbb{CP}^1$ determines the respective complex structure of $\mathbb{R}^4$. The vector fields (III.23) and (III.27b) are vector fields of the type $(0,1)$ with respect to the complex structure $\mathcal{J}$. Vector fields $\{\bar V^{(2)}_{\bar a}\}$ of type $(0,1)$ on $\mathcal{U}_2$ are of the form

$$\bar V^{(2)}_{\bar 1} = \zeta\partial_{\bar z^{\bar 1}} - \partial_{z^2}, \qquad \bar V^{(2)}_{\bar 2} = \zeta\partial_{\bar z^{\bar 2}} + \partial_{z^1} \qquad \text{and} \qquad \bar V^{(2)}_{\bar 3} = \partial_{\bar\zeta}. \tag{III.32}$$

On the intersection $\mathcal{U}_{12}$ we have

$$\bar V^{(1)}_{\bar 1} = \lambda \bar V^{(2)}_{\bar 1}, \qquad \bar V^{(1)}_{\bar 2} = \lambda \bar V^{(2)}_{\bar 2} \qquad \text{and} \qquad \bar V^{(1)}_{\bar 3} = -\bar\lambda^2 \bar V^{(2)}_{\bar 3}. \tag{III.33}$$

The canonical coordinates on the twistor space $\mathcal{Z}$ are then

$$w_1^1 = z^1 - \lambda\bar z^{\bar 2}, \qquad w_1^2 = z^2 + \lambda \bar z^{\bar 1} \qquad \text{and} \qquad w_1^3 = \lambda \qquad \text{on} \qquad \mathcal{U}_1, \tag{III.34a}$$
$$w_2^1 = \zeta z^1 - \bar z^{\bar 2}, \qquad w_2^2 = \zeta z^2 + \bar z^{\bar 1} \qquad \text{and} \qquad w_2^3 = \zeta \qquad \text{on} \qquad \mathcal{U}_2. \tag{III.34b}$$

Note that on the intersection $\mathcal{U}_{12}$ these coordinates are related by the holomorphic transition function $t_{12}$

$$w_1^a = t_{12}^a(w_2^1, w_2^2, w_2^3) \quad \Longleftrightarrow \quad w_1^1 = \frac{w_2^1}{w_2^3}, \quad w_1^2 = \frac{w_2^2}{w_2^3} \quad \text{and} \quad w_1^3 = \frac{1}{w_2^3}. \tag{III.34c}$$

Using (III.34) we obtain after few lines of algebra

$$\partial_{\bar w_1^{\bar 1}} = \gamma_1 \bar V^{(1)}_{\bar 1}, \qquad \partial_{\bar w_1^{\bar 2}} = \gamma_1 \bar V^{(1)}_{\bar 2} \qquad \text{and} \qquad \partial_{\bar w_1^{\bar 3}} = \bar V^{(1)}_{\bar 3} + z^2\gamma_1 \bar V^{(1)}_{\bar 1} - z^1 \gamma_1 \bar V^{(1)}_{\bar 2} \tag{III.35a}$$

on $\mathcal{U}_1$ with $\gamma_1 = (1+\lambda\bar\lambda)^{-1}$. Similarly, we find on $\mathcal{U}_2$

$$\partial_{\bar w_2^{\bar 1}} = \gamma_2 \bar V_{\bar 1}^{(2)}, \qquad \partial_{\bar w_2^{\bar 2}} = \gamma_2 \bar V_{\bar 2}^{(2)} \qquad \text{and} \qquad \partial_{\bar w_2^{\bar 3}} = \bar V_{\bar 3}^{(2)} - \bar z^{\bar 1}\gamma_2 \bar V_{\bar 1}^{(2)} - \bar z^{\bar 2}\gamma_2 \bar V_{\bar 2}^{(2)}, \tag{III.35b}$$

where $\gamma_2 = (1+\zeta\bar\zeta)^{-1}$.

Moreover, we can construct a local basis of $(0,1)$-forms $\bar\theta^{\bar a}_{(1,2)}(\bar V_{\bar b}^{(1,2)}) = \delta^{\bar a}_{\bar b}$ with respect to $\mathcal{J}$ given by

$$\begin{aligned} \bar\theta^{\bar 1}_{(1)} &= \gamma_1(\mathrm{d}\,\bar z^{\bar 1} - \bar\lambda\,\mathrm{d}\,z^2), \quad \bar\theta^{\bar 2}_{(1)} = \gamma_1(\mathrm{d}\,\bar z^{\bar 2} + \bar\lambda\,\mathrm{d}\,z^1) \quad \text{and} \quad \bar\theta^{\bar 3}_{(1)} = \mathrm{d}\,\bar\lambda, & \text{(III.36a)}\\ \bar\theta^{\bar 1}_{(2)} &= \gamma_2(\bar\zeta\,\mathrm{d}\,\bar z^{\bar 1} - \mathrm{d}\,z^2), \quad \bar\theta^{\bar 2}_{(2)} = \gamma_2(\bar\zeta\,\mathrm{d}\,\bar z^{\bar 2} + \mathrm{d}\,z^1) \quad \text{and} \quad \bar\theta^{\bar 3}_{(2)} = \mathrm{d}\,\bar\zeta & \text{(III.36b)} \end{aligned}$$

on $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively.

What we find is that the exterior derivative d on $\mathcal{Z}$ splits into two parts: $\mathrm{d} = \partial + \bar\partial$ with

$$\begin{aligned} \bar\partial &= \mathrm{d}\,\bar w_1^{\bar a}\,\partial_{\bar w_1^{\bar a}} = \bar\theta^{\bar a}_{(1)}\bar V_{\bar a}^{(1)} \quad \text{on} \quad \mathcal{U}_1, & \text{(III.37a)}\\ \bar\partial &= \mathrm{d}\,\bar w_2^{\bar a}\,\partial_{\bar w_2^{\bar a}} = \bar\theta^{\bar a}_{(2)}\bar V_{\bar a}^{(2)} \quad \text{on} \quad \mathcal{U}_2. & \text{(III.37b)} \end{aligned}$$

The operators $\partial$ and $\bar\partial$ are related via complex conjugation.

In summary, we have learned that the twistor space $\mathcal{Z}$ when considered as a complex manifold is not a direct product manifold like $\mathbb{C}^2 \times \mathbb{CP}^1$. It is a nontrivial holomorphic vector bundle $\mathrm{pr} : \mathcal{Z} \to \mathbb{CP}^1$. Moreover, it follows that $\mathcal{Z}$ can be identified with the total space of the rank two holomorphic vector bundle $\mathcal{O}(1)\otimes\mathbb{C}^2 \cong \mathcal{O}(1)\oplus\mathcal{O}(1)$ over $\mathbb{CP}^1$, i.e.

$$\mathrm{pr} : \mathcal{Z} \cong \mathcal{O}(1)\oplus\mathcal{O}(1) \to \mathbb{CP}^1, \tag{III.38}$$

where $\mathcal{O}(1)$ is the holomorphic line bundle over $\mathbb{CP}^1$ with the transition function $\lambda^1$ and the first Chern class $c_1$ equals to 1. To see this let us first recall the definition of $\mathcal{O}(n)$ for $n \geq 0$: The bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ is a holomorphic line bundle with the transition function $\lambda^n$ and the first Chern class $c_1 = n$. The space $\mathcal{O}(n)$ is a two-dimensional complex manifold and can be covered by two coordinate patches $\mathcal{O}(n) = \Omega_1 \cup \Omega_2$ with the coordinates $(w_1,\lambda)$ on $\Omega_1$ and $(w_2,\zeta)$ on $\Omega_2$. The canonical projection $\mathcal{O}(n) \to \mathbb{CP}^1$ is then given by $(w_1,\lambda) \mapsto \lambda$ and $(w_2,\zeta) \mapsto \zeta$. On the overlap region $\Omega_1\cap\Omega_2$ these coordinates are related via $(w_2,\zeta) = (\lambda^{-n}w_1,\lambda^{-1})$. The module $\Gamma(\mathbb{CP}^1,\mathcal{O}(n))$ of global holomorphic sections of the bundle $\mathcal{O}(n)$ coincides with the $\mathbb{C}$-module of polynomials of maximal degree $n$ in $\lambda$ (or $\zeta$) with complex coefficients, i.e. holomorphic sections are projective lines

$$\mathbb{CP}^1_z = \begin{cases} w_1 = \sum_{i=0}^n z_i\lambda^i & \text{for} \quad \lambda \in U_1,\\[1ex] w_2 = \sum_{i=0}^n z_i\zeta^{n-i} & \text{for} \quad \zeta \in U_2, \end{cases}$$

which are parametrized by points $(z_0,\ldots,z_n) \in \mathbb{C}^{n+1}$. Using this definition of $\mathcal{O}(n)$ for $n=1$ and considering the tensor product of $\mathcal{O}(1)$ with $\mathbb{C}^2$, we realize by virtue of equations (III.34) that it coincides with $\mathcal{Z}$.

Holomorphic sections of the bundle (III.38) are

$$\mathbb{CP}^1_z = \begin{cases} w_1^1 = z^1 + \lambda \tilde{z}^2,\ w_1^2 = z^2 + \lambda \tilde{z}^1 & \text{for} \quad \lambda \in U_1, \\ w_2^1 = \zeta z^1 + \tilde{z}^2,\ w_2^2 = \zeta z^2 + \tilde{z}^1 & \text{for} \quad \zeta \in U_2 \end{cases} \tag{III.40}$$

with $z = (z^1, z^2, \tilde{z}^1, \tilde{z}^2) \in \mathbb{C}^4$.

**Real structure on $\mathcal{Z}$.** Furthermore, we need the notion of a real structure on the complex twistor space $\mathcal{Z}$. It is defined as an anti-holomorphic involution $\tau : \mathcal{Z} \to \mathcal{Z}$, such that

$$\tau : (x^\mu, \lambda) \mapsto (x^\mu, -\bar{\lambda}^{-1}) \qquad \text{with} \qquad \tau^2 = 1. \tag{III.41}$$

This involution sends the complex structure $\mathcal{J}$ on $\mathcal{Z}$ to its conjugate, i.e. it is anti-holomorphic. Obviously, $\tau$ has no fixed points on $\mathcal{Z}$ but leaves the fibers $\mathbb{CP}^1_x$ with $x \in \mathbb{R}^4$ of the bundle $\pi : \mathcal{Z} \to \mathbb{R}^4$ invariant. Note that fibers $\mathbb{CP}^1_x$ of the bundle $\pi : \mathcal{Z} \to \mathbb{R}^4$ are real holomorphic sections of the bundle (III.38) with $\tilde{z}^1 = \bar{z}^{\bar{1}}$ and $\tilde{z}^2 = -\bar{z}^{\bar{2}}$. The extension of the involution $\tau$ to complex-valued functions is given by

$$\tau : f(x^\mu, \lambda) \mapsto \overline{f(x^\mu, -\bar{\lambda}^{-1})}. \tag{III.42}$$

Note that all holomorphic data are compatible with $\tau$: The involution $\tau$ acts on the coordinates $\{w^a_{1,2}\}$ as $\tau(w_1^1) = w_2^2$, $\tau(w_1^2) = -w_2^1$ and $\tau(w_1^3) = -w_2^3$, i.e. $\tau(w_1^a) = B^a_b w_2^b$ with $B^1_2 = 1$, $B^2_1 = -1$, $B^3_3 = -1$ and all other components equal to zero. For the transition functions $t^a_{12}$ of (III.34c) we find $\tau(t^a_{12}) = B^a_b t^b_{21}$ with $t_{21} = (t_{12})^{-1}$.

### § III.2.3 Penrose-Ward Correspondence

**Pulled back bundle over $\mathcal{P}$.** Let $P = U \times G$ be a principal $G$-bundle over $U \subset \mathbb{R}^4$ and $\mathfrak{R} : G \to GL(n, \mathbb{C})$ a representation of $G$. Then as usual we associate to $P$ the complex vector bundle $E = P \times_G \mathbb{C}^n \cong U \times \mathbb{C}^n$ with the canonical projection $\pi_E$. Using the projection $\pi : \mathcal{P} \to U$, we can pull back $E$ to the bundle $\pi^* E$ over $\mathcal{P}$, that is $\pi^* E = \{(x, e) \in \mathcal{P} \times E \,|\, \pi_E(e) = \pi(x)\}$ or pictorially:

$$\begin{array}{ccc} \pi^* E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi_E \\ \mathcal{P} & \xrightarrow[\pi]{} & U \end{array}$$

The pulled back bundle $\pi^* E$ is trivial on the fibers $\mathbb{CP}^1_x$ of the bundle $\pi : \mathcal{P} \to U$ and, moreover, we know that the connection one-form $\mathcal{A}$ induces the connection $D = \mathrm{d} + \mathcal{A}$ on $E$; cf. equation (II.48). We shall consider irreducible connections. By definition, the pulled back connection one-form $\pi^* \mathcal{A}$ on $\pi^* E$ is flat along the fibers $\mathbb{CP}^1$ and hence the pulled back connection $\pi^* D$ is nothing but

$$\pi^* D|_{\mathcal{U}_1} = D + \mathrm{d}\,\lambda\,\partial_\lambda + \mathrm{d}\,\bar{\lambda}\,\partial_{\bar{\lambda}} \qquad \text{and} \qquad \pi^* D|_{\mathcal{U}_2} = D + \mathrm{d}\,\zeta\,\partial_\zeta + \mathrm{d}\,\bar{\zeta}\,\partial_{\bar{\zeta}}. \tag{III.44}$$

Writing $\pi^*\mathcal{A} = \mathcal{A}_\mu \,\mathrm{d}\, x^\mu = \mathcal{A}_{z^a} \,\mathrm{d}\, z^a + \mathcal{A}_{\bar{z}^{\bar{a}}} \,\mathrm{d}\, \bar{z}^{\bar{a}} := \mathcal{B} + \bar{\mathcal{B}}$ with

$$\bar{\mathcal{B}} = \bar{\mathcal{B}}^{(1,2)}_{\bar{a}} \bar{\theta}^{\bar{a}}_{(1,2)} = \bar{\mathcal{B}}_{\bar{w}^{\bar{a}}_{1,2}} \,\mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} \qquad \text{on} \qquad \mathcal{U}_{1,2}, \tag{III.45}$$

we discover the $(0,1)$-components with respect to $\mathcal{J}$

$$\bar{\mathcal{B}}^{(1)}_{\bar{1}} := \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda \mathcal{A}_{z^2}, \quad \bar{\mathcal{B}}^{(1)}_{\bar{2}} := \mathcal{A}_{\bar{z}^{\bar{2}}} + \lambda \mathcal{A}_{z^1} \qquad \text{and} \qquad \bar{\mathcal{B}}^{(1)}_{\bar{3}} := 0 \quad \text{on} \quad \mathcal{U}_1, \tag{III.46a}$$

$$\bar{\mathcal{B}}^{(2)}_{\bar{1}} := \zeta \mathcal{A}_{\bar{z}^{\bar{1}}} - \mathcal{A}_{z^2}, \quad \bar{\mathcal{B}}^{(2)}_{\bar{2}} := \zeta \mathcal{A}_{\bar{z}^{\bar{2}}} + \mathcal{A}_{z^1} \qquad \text{and} \qquad \bar{\mathcal{B}}^{(2)}_{\bar{3}} := 0 \quad \text{on} \quad \mathcal{U}_2, \tag{III.46b}$$

where on the intersection $\mathcal{U}_{12}$ we have $\bar{\mathcal{B}}^{(1)}_{\bar{a}} = \lambda \bar{\mathcal{B}}^{(2)}_{\bar{a}}$. Similarly, for $\bar{\mathcal{B}}_{\bar{w}^{\bar{a}}_{1,2}}$ we find

$$\bar{\mathcal{B}}_{\bar{w}^{\bar{1}}_1} := \gamma_1 \bar{\mathcal{B}}^{(1)}_{\bar{1}}, \quad \bar{\mathcal{B}}_{\bar{w}^{\bar{2}}_1} := \gamma_1 \bar{\mathcal{B}}^{(1)}_{\bar{2}}, \qquad \bar{\mathcal{B}}_{\bar{w}^{\bar{3}}_1} := z^2 \gamma_1 \bar{\mathcal{B}}^{(1)}_{\bar{1}} - z^1 \gamma_1 \bar{\mathcal{B}}^{(1)}_{\bar{2}} \quad \text{on} \quad \mathcal{U}_1, \tag{III.47a}$$

$$\bar{\mathcal{B}}_{\bar{w}^{\bar{1}}_2} := \gamma_2 \bar{\mathcal{B}}^{(2)}_{\bar{1}}, \quad \bar{\mathcal{B}}_{\bar{w}^{\bar{2}}_2} := \gamma_2 \bar{\mathcal{B}}^{(2)}_{\bar{2}}, \qquad \bar{\mathcal{B}}_{\bar{w}^{\bar{3}}_2} := -\bar{z}^1 \gamma_2 \bar{\mathcal{B}}^{(2)}_{\bar{1}} - \bar{z}^2 \gamma_2 \bar{\mathcal{B}}^{(2)}_{\bar{2}} \quad \text{on} \quad \mathcal{U}_2. \tag{III.47b}$$

Using $\mathcal{B}$ we can write the pulled back connection $\pi^*D$ on $\pi^*E$ as

$$\pi^*D := \partial_{\mathcal{B}} + \bar{\partial}_{\bar{\mathcal{B}}} \qquad \text{with} \qquad \bar{\partial}_{\bar{\mathcal{B}}} = \bar{\partial} + \bar{\mathcal{B}} \tag{III.48}$$

with $\bar{\partial}$ given by (III.37).

**Self-duality vs. holomorphicity I.** Let us now consider the equations

$$\bar{\partial}_{\bar{\mathcal{B}}} s = 0 \tag{III.49}$$

for a local smooth section $s$ of $\pi^*E$. Of course, by definition local solutions to these equations are just local holomorphic sections of the complex vector bundle $\pi^*E$. The bundle $E' := \pi^*E \to \mathcal{P}$ is called holomorphic iff these equations are compatible in the sense that $\bar{\partial}^2_{\bar{\mathcal{B}}} = 0$, i.e. the $(0,2)$-components of the curvature given by $D' := \pi^*D$ are equal to zero.[2] Using $\bar{\partial}_{\bar{\mathcal{B}}} = \bar{\partial} + \bar{\mathcal{B}} = \mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} (\partial_{\bar{w}^{\bar{a}}_{1,2}} + \bar{\mathcal{B}}_{\bar{w}^{\bar{a}}_{1,2}}) = \mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} \nabla_{\bar{w}^{\bar{a}}_{1,2}}$ we verify

$$\begin{aligned}
\bar{\partial}^2_{\bar{\mathcal{B}}} &= \mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} \nabla_{\bar{w}^{\bar{a}}_{1,2}} \wedge \mathrm{d}\, \bar{w}^{\bar{b}}_{1,2} \nabla_{\bar{w}^{\bar{b}}_{1,2}} \\
&= \tfrac{1}{2} \,\mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} \wedge \mathrm{d}\, \bar{w}^{\bar{b}}_{1,2} \left[ \nabla_{\bar{w}^{\bar{a}}_{1,2}}, \nabla_{\bar{w}^{\bar{b}}_{1,2}} \right] \\
&= \tfrac{1}{2} \,\mathrm{d}\, \bar{w}^{\bar{a}}_{1,2} \wedge \mathrm{d}\, \bar{w}^{\bar{b}}_{1,2} \mathcal{F}'_{\bar{w}^{\bar{a}}_{1,2} \bar{w}^{\bar{b}}_{1,2}} \stackrel{!}{=} 0 \qquad \Longrightarrow \qquad \mathcal{F}'_{\bar{w}^{\bar{a}}_{1,2} \bar{w}^{\bar{b}}_{1,2}} = 0.
\end{aligned}$$

Employing equations (III.24), (III.35b) and (III.47), we rewrite (III.49) in the trivialization over $\mathcal{U}_1$ as

$$[(\nabla_1 + i\nabla_2) - \lambda(\nabla_3 + i\nabla_4)] s_1(x, \lambda, \bar{\lambda}) = 0, \tag{III.50a}$$

$$[(\nabla_3 - i\nabla_4) + \lambda(\nabla_1 - i\nabla_2)] s_1(x, \lambda, \bar{\lambda}) = 0, \tag{III.50b}$$

$$\partial_{\bar{\lambda}} s_1(x, \lambda, \bar{\lambda}) = 0, \tag{III.50c}$$

[2]Note that the definition of a holomorphic vector bundle with a Hermitian structure implies that the $(0,2)$-components of the curvature vanish.

with $s_1 := s|_{\mathcal{U}_1}$ and similarly for $\mathcal{U}_2$. The last of these equations implies that $s_1$ is a function of $x$ and $\lambda$ but not of $\bar{\lambda}$. The remaining two equations for $s_1(x,\lambda)$ are referred to as the linear system of the SDYM equations, since with the help of (III.35b) and (III.47) the compatibility conditions $\mathcal{F}'_{\bar{w}_1^{\bar{a}}\bar{w}_1^{\bar{b}}} = 0$ for $a = 1$ and $b = 2$ read

$$\begin{aligned}\mathcal{F}'_{\bar{w}_1^{\bar{1}}\bar{w}_1^{\bar{2}}} &= \left[\nabla_{\bar{w}_1^{\bar{1}}}, \nabla_{\bar{w}_1^{\bar{2}}}\right] \\ &= \left[\partial_{\bar{w}_1^{\bar{1}}} + \bar{\mathcal{B}}_{\bar{w}_1^{\bar{1}}}, \partial_{\bar{w}_1^{\bar{2}}} + \bar{\mathcal{B}}_{\bar{w}_1^{\bar{2}}}\right] \\ &= \gamma_1\gamma_2\left[\nabla_{\bar{z}^{\bar{1}}} - \lambda\nabla_{z^2}, \nabla_{\bar{z}^{\bar{2}}} + \lambda\nabla_{z^1}\right] \stackrel{!}{=} 0.\end{aligned}$$

This has to be true for all $\lambda$ and hence we obtain

$$\mathcal{F}_{z^1z^2} = [\nabla_{z^1}, \nabla_{z^2}] = 0, \tag{III.51a}$$

$$\mathcal{F}_{\bar{z}^{\bar{1}}\bar{z}^{\bar{2}}} = [\nabla_{\bar{z}^{\bar{1}}}, \nabla_{\bar{z}^{\bar{2}}}] = 0, \tag{III.51b}$$

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} = [\nabla_{z^1}, \nabla_{\bar{z}^{\bar{1}}}] + [\nabla_{z^2}, \nabla_{\bar{z}^{\bar{2}}}] = 0, \tag{III.51c}$$

which are exactly the SDYM equations (III.17) written in the complex coordinates (III.24). Note that all these computations have to be understood up to action on some section, e.g. $\mathcal{F}_{z^1z^2}s = [\nabla_{z^1}, \nabla_{z^2}]\, s$. In summary, if a gauge potential $\mathcal{A}_\mu$ is a smooth solution of (III.51) on $U \subset \mathbb{R}^4$ then solutions to (III.49) exist and thus, the bundle $E' \to \mathcal{P}$ is holomorphic.

The solutions $s_{1,2}$ over $\mathcal{U}_{1,2}$ to (III.49) satisfy the condition $s_1 = s_2$ on the intersection $\mathcal{U}_{12}$. We can always decompose $s_{1,2}$ into $s_{1,2} = \psi_{1,2}\eta_{1,2}$, where $\psi_{1,2}$ live in the complexified gauge group $G \otimes \mathbb{C}$, are nonsingular on $\mathcal{U}_{1,2}$ and satisfy

$$\bar{\partial}_{\bar{\mathcal{B}}}\psi_{1,2} = 0 \qquad \text{on} \qquad \mathcal{U}_{1,2}. \tag{III.52}$$

The vector-valued functions $\eta_{1,2} \in \mathbb{C}^n$ are holomorphic on $\mathcal{U}_{1,2}$ in the sense that

$$\bar{V}_{\bar{a}}^{(1,2)}\eta_{1,2} = 0 \qquad \Longrightarrow \qquad \partial_{\bar{w}_{1,2}^{\bar{a}}}\eta_{1,2} = 0. \tag{III.53}$$

From (III.52) we obtain with (III.35b) and (III.47)

$$\psi_1(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi_1^{-1} = \psi_2(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi_2^{-1} = \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda\mathcal{A}_{z^2}, \tag{III.54a}$$

$$\psi_1(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi_1^{-1} = \psi_2(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi_2^{-1} = \mathcal{A}_{\bar{z}^{\bar{2}}} + \lambda\mathcal{A}_{z^1}, \tag{III.54b}$$

$$\partial_{\bar{\lambda}}\psi_1 = \partial_{\bar{\lambda}}\psi_2 = 0 \tag{III.54c}$$

on $\mathcal{U}_{12}$. Furthermore, the vector-valued functions $\eta_{1,2}$ are related by $\eta_1 = f_{12}\eta_2$ on $\mathcal{U}_{12}$, so that

$$f_{12} = \psi_1^{-1}\psi_2 \tag{III.55}$$

is the transition function of $E'$. The equation $\eta_1 = f_{12}\eta_2$ implies that $f_{12}$ is holomorphic.

**Gauge transformations vs. holomorphic equivalence.** It is easy to see that gauge transformations (II.29) of the gauge potential $\mathcal{A}$ are induced by the transformations

$$\psi_{1,2} \mapsto \psi_{1,2}^g := g^{-1}\psi_{1,2}. \tag{III.56}$$

Obviously, the transition function (III.55) is invariant under these transformations. Moreover, by inspecting (III.54), we discover that a transformation of the form

$$\psi_{1,2} \mapsto \psi_{1,2}^{h_{1,2}} := \psi_{1,2} h_{1,2}^{-1} \tag{III.57}$$

leaves the components $\mathcal{A}_\mu$ invariant. Here, $h_{1,2}$ are regular holomorphic, i.e. $\{w_{1,2}^a\}$-dependent matrix-valued functions on $\mathcal{U}_{1,2}$ living in the complexified gauge group $G \otimes \mathbb{C}$. This means that a class of holomorphically equivalent bundles (since $\tilde{f}_{12} = h_1 f_{12} h_2^{-1}$) over the twistor space $\mathcal{P}$ corresponds to a self-dual connection on $U$.

**Reality condition.** Equations (III.54) show that in general the gauge potential $\mathcal{A}_\mu$ will take values in the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. However, we are interested in real gauge fields and therefore we need to impose additional constraints on the vector bundle $E'$ induced by the real structure (III.41). For the case $G = SU(n)$ and $\mathfrak{g} = \mathfrak{su}(n)$ we have $\mathcal{A}_\mu^\dagger = -\mathcal{A}_\mu$ implying

$$\mathcal{A}_{z^a}^\dagger = -\mathcal{A}_{\bar{z}^{\bar{a}}} \qquad \text{for} \qquad a = 1, 2. \tag{III.58}$$

Then the matrices $f_{12} \in SL(n, \mathbb{C})$ and $\psi_{1,2} \in SL(n, \mathbb{C})$ have to satisfy

$$f_{12}^\dagger(\tau(\bar{z}_1^{\bar{a}})) = f_{12}(z_1^a) \qquad \text{and} \qquad \psi_1^\dagger(\tau(x, \lambda)) = \psi_2^{-1}(x, \lambda) \tag{III.59}$$

on $\mathcal{U}_{12}$, since the substitution $(x, \lambda) \mapsto (x, -\bar{\lambda}^{-1})$ in (III.54a), Hermitian conjugation and (III.58) yield equation (III.54b).

**Riemann-Hilbert problems.** Suppose that we have a matrix-valued function $\Psi(x, \lambda) \in SL(n, \mathbb{C})$ living on $U_1 \cap U_2 \subset \mathbb{CP}^1$ and depending holomorphically on $\lambda$ and smoothly on some parameters $\{x^\mu\}$. Then we consider the following problem, known as a parametric Riemann-Hilbert problem in the literature: We are looking for two matrix-valued functions $\psi_{1,2} \in SL(n, \mathbb{C})$ defined on $U_1 \cap U_2$, such that $\psi_1$ ($\psi_2$) can be continuously extended to a regular (= holomorphic with non-vanishing determinant) matrix-valued function on $U_1$ ($U_2$) and

$$\Psi(x, \lambda) = \psi_1^{-1}(x, \lambda)\psi_2(x, \lambda) \qquad \text{on} \qquad U_1 \cap U_2. \tag{III.60}$$

The Birkhoff decomposition theorem implies that for fixed $x$ any holomorphic nonsingular matrix-valued function $\Psi$ on $U_1 \cap U_2$ admits the decomposition

$$\Psi = \psi_1^{-1}\Lambda\psi_2, \tag{III.61}$$

where the $\psi_{1,2}$ are as above and $\Lambda = \mathrm{diag}(\lambda^{k_1}, \ldots, \lambda^{k_n})$ with $k_i \in \mathbb{Z}$ and $k_1 + \cdots + k_n = 0$. The $k_i$'s are unique up to permutations and, moreover, they are Chern classes of holomorphic line bundles over $\mathbb{CP}^1$ occuring in the decomposition of the holomorphic vector bundle over $\mathbb{CP}^1$ with $\Psi$ as a transition matrix. This is known as Birkhoff-Grothendieck theorem.

If $\Lambda$ is the identity (i.e. $k_i = 0$ for all $i = 1, \ldots, n$), the decomposition (III.61) is said to be a solution to the Riemann-Hilbert problem. Note that this solution is unique up to gauge transformations (III.56).

**Self-duality vs. holomorphicity II.** Let us consider a topologically trivial holomorphic vector bundle $E'$ over the twistor space $\mathcal{P}$ with the cover $\{U_1, U_2\}$ and a transition function $f_{12}$ satisfying the unitarity condition (III.59). For fixed $x \in U$ we obtain a parametric Riemann-Hilbert problem on $\mathbb{CP}^1$. Moreover, let $f_{12} = \psi_1^{-1}\psi_2$ be a solution to this problem. From the holomorphicity of $f_{12}$ we discover

$$(\bar{V}_{\bar{a}}^{(2)}\psi_1)\psi_1^{-1} = (\bar{V}_{\bar{a}}^{(2)}\psi_2)\psi_2^{-1} \qquad \text{for} \qquad a = 1, 2 \tag{III.62}$$

on $\mathcal{U}_{12}$. Since as functions on $\mathbb{CP}^1$ the matrices $\psi_1$ and $\psi_2$ are regular on $U_1$ and $U_2$, respectively, we can expand them on $U_1 \cap U_2$ in powers of $\lambda$

$$\psi_1(x, \lambda) = \sum_{n\in\mathbb{N}_0} \lambda^n \psi_1^{(n)}(x) \qquad \text{and} \qquad \psi_2(x, \lambda) = \sum_{n\in\mathbb{N}_0} \lambda^{-n} \psi_2^{(n)}(x). \tag{III.63}$$

The substitution of these expansions into (III.62) yields the condition, that both sides of (III.62) have to be at most linear in $\lambda$ and thus,

$$\psi_1(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi_1^{-1} = \psi_2(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi_2^{-1} \;=\; \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda\mathcal{A}_{z^2}, \tag{III.64a}$$
$$\psi_1(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi_1^{-1} = \psi_2(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi_2^{-1} \;=\; \mathcal{A}_{\bar{z}^{\bar{2}}} + \lambda\mathcal{A}_{z^1}, \tag{III.64b}$$

where all $\lambda$-dependences have been made explicitly. Therefore we have

$$\mathcal{A}_{z^1} \;:=\; -\operatorname*{Res}_{\lambda=0} \lambda^{-2}(\bar{V}_{\bar{2}}^{(2)}\psi_2)\psi_2^{-1} = -\oint_c \frac{\mathrm{d}\lambda}{2\pi i\lambda^2}(\bar{V}_{\bar{2}}^{(2)}\psi_2)\psi_2^{-1} = -(\partial_{z^1}\psi_2^{(0)})(\psi_2^{(0)})^{-1}, \tag{III.65a}$$
$$\mathcal{A}_{z^2} \;:=\; \operatorname*{Res}_{\lambda=0} \lambda^{-2}(\bar{V}_{\bar{1}}^{(2)}\psi_2)\psi_2^{-1} = \oint_c \frac{\mathrm{d}\lambda}{2\pi i\lambda^2}(\bar{V}_{\bar{1}}^{(2)}\psi_2)\psi_2^{-1} = -(\partial_{z^2}\psi_2^{(0)})(\psi_2^{(0)})^{-1}, \tag{III.65b}$$
$$\mathcal{A}_{\bar{z}^{\bar{1}}} \;:=\; -\operatorname*{Res}_{\lambda=0} \lambda^{-1}(\bar{V}_{\bar{1}}^{(2)}\psi_1)\psi_1^{-1} = -\oint_c \frac{\mathrm{d}\lambda}{2\pi i\lambda}(\bar{V}_{\bar{1}}^{(2)}\psi_1)\psi_1^{-1} = -(\partial_{\bar{z}^{\bar{1}}}\psi_1^{(0)})(\psi_1^{(0)})^{-1}, \tag{III.65c}$$
$$\mathcal{A}_{\bar{z}^{\bar{2}}} \;:=\; -\operatorname*{Res}_{\lambda=0} \lambda^{-1}(\bar{V}_{\bar{2}}^{(2)}\psi_1)\psi_1^{-1} = -\oint_c \frac{\mathrm{d}\lambda}{2\pi i\lambda^2}(\bar{V}_{\bar{2}}^{(2)}\psi_1)\psi_1^{-1} = -(\partial_{\bar{z}^{\bar{2}}}\psi_1^{(0)})(\psi_1^{(0)})^{-1}, \tag{III.65d}$$

where the contour of integration is $c = \{\lambda \in \mathbb{CP}^1 \,|\, |\lambda| = 1\}$. The gauge potential $\mathcal{A}_\mu$ determined by (III.65) satisfies the SDYM equations on $U$.

In summary, we have described a one-to-one correspondence between gauge equivalence classes of self-dual connection one-forms $\mathcal{A}$ on a complex vector bundle $E$ over $U \subset \mathbb{R}^4$ and equivalence classes of holomorphic vector bundles $E'$ over the twistor space $\mathcal{P}$ trivial on $\mathbb{CP}^1 \hookrightarrow \mathcal{P}$. This is known as the twistor correspondence or the Euclidean version of Penrose-Ward's theorem.

# § III.3 Construction of Instanton Solutions

In this section we shall explain the construction of instanton solutions. However, at this stage we are not too explicit in our derivations, and postpone them to the noncommutative setup (cf. chapter VII). For a more detailed discussion on constructing instantons we refer the reader to [164] and [166], for instance.

Throughout this section, we assume that the gauge group $G$ is $SU(n)$. Firstly, we sketch the Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton solution using the method of dressing transformations and the aforementioned twistor method in the case of $SU(2)$. Both methods are closely related as we shall see momentarily. Eventually, we discuss the $N$-instanton solution and explain the generalization to $SU(n)$.

## § III.3.1 $SU(2)$ BPST Instanton from the Dressing Approach

**Linear system.** From the previous section we know that the SDYM equations (III.51) are the integrability conditions of the linear system (III.54). Let us consider the same sytem (having therefore the same integrability conditions)

$$(\nabla_{\bar{z}^{\bar{1}}} - \lambda\nabla_{z^2})\psi = 0, \qquad (\nabla_{\bar{z}^{\bar{2}}} + \lambda\nabla_{z^1})\psi = 0 \qquad \text{and} \qquad \partial_{\bar{\lambda}}\psi = 0, \tag{III.66}$$

with $\lambda \in \mathbb{CP}^1$. However, this time we assume that $\psi \in SL(2,\mathbb{C})$ is a global meromorphic function of the spectral parameter $\lambda$.

Following Belavin and Zakharov [12], we impose the following reality condition on $\psi$ (cf. also equation (III.59))

$$[\psi(\tau(x,\lambda))]^\dagger = \psi^{-1}(x,\lambda), \tag{III.67}$$

where $\tau$ is given by (III.41).

The linear system (III.66) can be rewritten in the familiar form

$$\psi(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi^\dagger = \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda\mathcal{A}_{z^2}, \tag{III.68a}$$

$$\psi(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi^\dagger = \mathcal{A}_{z^2} + \lambda\mathcal{A}_{z^1}, \tag{III.68b}$$

where $\psi^\dagger$ is shorthand notation for $[\psi(x,-\bar{\lambda}^{-1})]^\dagger$. Now, the task is to find the auxiliary field $\psi$.

**Dressing method.** Belavin and Zakharov [12] constructed solutions to the SDYM equations (III.51) using the method of dressing transformations, which is a recursive procedure for generating solutions from a given seed solution. After having derived the one-instanton BPST solution they used it as the seed solution to give a recursion relation for the construction of the $N$-instanton configuration. Let us now review the basic steps of their derivation.

Let $\psi$ be a given solution to the linear system (III.66). To generate a new solution out of $\psi$ we multiply the so-called dressing factor $\chi$ on the left, i.e. we consider

$$\tilde{\psi}(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda) = \chi(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda)\, \psi(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda). \tag{III.69}$$

The dressing factor $\chi \in SL(2, \mathbb{C})$ is assumed to be a global meromorphic function in the spectral parameter $\lambda \in \mathbb{CP}^1$, which implies the expansion

$$\chi = \lambda R_{-1} + R_0 + \sum_{i=1}^{r} \frac{R_i}{(\mu_i \lambda + \nu_i)}, \tag{III.70}$$

where $\mu_i$, $\nu_i \in \mathbb{C}$ and the coefficients $R_{-1}$, $R_0$ and $R_i$ depend on $\{z^a\}$ and $\{\bar{z}^{\bar{a}}\}$ but not on the spectral parameter $\lambda$. We have restricted ourselves to first-order poles, since one can generate higher-order poles by a successive multiplication of such expressions. Thus, $\chi$ has finitely many poles which are located at $\lambda_\infty = \infty$ and $\lambda_i = -\nu_i/\mu_i$ with $\mu_i \neq 0$ for $i = 1, \ldots, r$ .

Since $\tilde{\psi}$ is supposed to be a new solution we can write

$$\tilde{\psi}(\partial_{\bar{z}^{\bar{1}}} - \lambda \partial_{z^2})\tilde{\psi}^\dagger = \tilde{\mathcal{A}}_{\bar{z}^{\bar{1}}} - \lambda \tilde{\mathcal{A}}_{z^2}, \tag{III.71a}$$
$$\tilde{\psi}(\partial_{\bar{z}^{\bar{2}}} + \lambda \partial_{z^1})\tilde{\psi}^\dagger = \tilde{\mathcal{A}}_{\bar{z}^{\bar{2}}} + \lambda \tilde{\mathcal{A}}_{z^1}. \tag{III.71b}$$

A short calculation shows that

$$\chi(\nabla_{\bar{z}^{\bar{1}}} - \lambda \nabla_{z^2})\chi^\dagger = \tilde{\mathcal{A}}_{\bar{z}^{\bar{1}}} - \lambda \tilde{\mathcal{A}}_{z^2}, \tag{III.72a}$$
$$\chi(\nabla_{\bar{z}^{\bar{2}}} + \lambda \nabla_{z^1})\chi^\dagger = \tilde{\mathcal{A}}_{\bar{z}^{\bar{2}}} + \lambda \tilde{\mathcal{A}}_{z^1}, \tag{III.72b}$$

where $\nabla_{z^a} = \partial_{z^a} + \mathcal{A}_{z^a}$ is the covariant derivative in the background of the old gauge potential $\mathcal{A}_{z^a}$ determined by $\psi$. Since the gauge potential $\tilde{\mathcal{A}}_{z^a}$ is $\lambda$-independent, the left hand side of (III.72) is at most linear in $\lambda$. The ansatz (III.70) for $\chi$, however, contains finitely many poles in the spectral parameter. Therefore, all proper residues must vanish; cf. equations (III.65). This requirement yields differential equations for the coefficients $R_{-1}$, $R_0$ and $R_i$ for $i = 1, \ldots, r$. After solving these equations one obtains a new solution $\tilde{\mathcal{A}}_{z^a}$ to the SDYM equations. Finally, this procedure may be iterated to get new solutions from old ones.

**Geometry of dressing transformations.** As we have seen, dressing transformations map old solutions into new solutions. The set of all such transformations forms an infinite-dimensional group, the group of dressing transformations $\mathcal{G}$, which acts on the moduli space $\mathcal{M}$ of all solutions. Solutions then correspond to finite-dimensional orbits of $\mathcal{G}$. For further discussions see [7], for instance.

**BPST solution.** The trivial solution to (III.68) is obviously given by $(\mathcal{A}, \psi) = (0, 1)$. Hence, one may take this as a seed solution. Then it remains to construct an appropriate dressing factor $\psi$. Note that here we have renamed the dressing factor to be conform with the literature. As previously mentioned, we shall postpone the explicit construction of $\psi$, etc. to the noncommutative setup. Here, we just want to give the results [11, 12]

$$\psi = \frac{1}{2\Lambda}\frac{1}{\sqrt{r^2+\Lambda^2}}\begin{pmatrix} r^2+2\Lambda^2-\lambda\bar{z}^{\bar{1}}\bar{z}^{\bar{2}}-\frac{z^1z^2}{\lambda} & -\lambda(\bar{z}^{\bar{1}})^2+\frac{(z^2)^2}{\lambda} \\ \lambda(\bar{z}^{\bar{2}})^2-\frac{(z^1)^2}{\lambda} & r^2+2\Lambda^2+\lambda\bar{z}^{\bar{1}}\bar{z}^{2}+\frac{z^1z^2}{\lambda} \end{pmatrix}, \tag{III.73}$$

where $r^2 := \bar{z}^{\bar{1}}z^1+\bar{z}^{\bar{2}}z^2$ and $\Lambda\in\mathbb{R}$ with $\Lambda\geq 0$. With this expression we can compute the gauge potential from (III.68) by looking at terms which are of linear order in $\lambda$

$$\mathcal{A}_{z^1} = \frac{1}{2(r^2+\Lambda^2)}\begin{pmatrix} \bar{z}^{\bar{1}} & 0 \\ -2\bar{z}^{\bar{2}} & -\bar{z}^{\bar{1}} \end{pmatrix}, \tag{III.74a}$$

$$\mathcal{A}_{z^2} = \frac{1}{2(r^2+\Lambda^2)}\begin{pmatrix} -\bar{z}^{\bar{2}} & -2\bar{z}^{\bar{1}} \\ 0 & \bar{z}^{\bar{2}} \end{pmatrix}. \tag{III.74b}$$

Using $\mathcal{F}_{z^a\bar{z}^{\bar{b}}} = \partial_{z^a}\mathcal{A}_{\bar{z}^{\bar{b}}}-\partial_{\bar{z}^{\bar{b}}}\mathcal{A}_{z^a}+[\mathcal{A}_{z^a},\mathcal{A}_{\bar{z}^{\bar{b}}}]$ for $a,b=1,2$, we obtain for the Yang-Mills curvature

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} = -\mathcal{F}_{z^2\bar{z}^{\bar{2}}} = -\frac{\Lambda^2}{(r^2+\Lambda^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{III.75a}$$

$$\mathcal{F}_{z^1\bar{z}^{\bar{2}}} = \mathcal{F}^\dagger_{z^2\bar{z}^{\bar{1}}} = \frac{2\Lambda^2}{(r^2+\Lambda^2)^2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{III.75b}$$

Since the equations of motion are translationally invariant we may consider a constant shift of the form $z^a \mapsto z^a - z^a_0$, where $z^a_0\in\mathbb{C}$. Hence, we obtain five moduli $\Lambda$ characterizing the size of the instanton and $z^a_0$ corresponding to the position of the instanton. In real coordinates $\{x^\mu\}$ we have for the gauge potential $\mathcal{A}_\mu = g_{\mathrm{YM}}A^a_\mu X^{(f)}_a$ with the $X^{(f)}_a$'s given by (II.11)

$$A^a_\mu = \frac{2\eta^a_{\mu\nu}(x-x_0)^\nu}{(x-x_0)^2+\Lambda^2}, \tag{III.76}$$

where $(x-x_0)^2$ is a short hand notation for $(x-x_0)_\mu(x-x_0)^\mu$. The curvature $\mathcal{F}_{\mu\nu} = g_{\mathrm{YM}}F^a_{\mu\nu}X^{(f)}_a$ becomes

$$F^a_{\mu\nu} = -4\eta^a_{\mu\nu}\frac{\Lambda^2}{((x-x_0)^2+\Lambda^2)^2}, \tag{III.77}$$

where we have used the 't Hooft symbol (III.19). In the last expression the self-duality of the solutions becomes apparent. Using the identity $\eta^a_{\mu\nu}\eta^{a\mu\nu}=12$ and

$$\int_{\mathbb{R}^4} \mathrm{d}^4\mathrm{x}\,\frac{1}{(x^2+\Lambda^2)^4} = \frac{\pi^2}{6\Lambda^4}$$

one readily verifies a topological charge (III.8) of one for this configuration.

**Remark.** At first sight it seems that there are five moduli characterizing the solution. However, there are three additional parameters corresponding to the gauge orientation, i.e. the gauge group $SU(2)$ acts via the adjoint representation on the solution (III.76)

$$\mathcal{A} \mapsto \mathrm{Ad}_g\,\mathcal{A} = g\,\mathcal{A}\,g^{-1} \tag{III.79}$$

for some constant $g \in SU(2)$. One might argue that this new solution is gauge equivalent to the old one. However, having fixed a gauge one still can act adjointly [13]. Thus, we have eight moduli. As we will discuss below, for the $N$-instanton solution we have $5N$ moduli corresponding to size and position parameters and $3N$ moduli which are gauge orientations of each instanton, similar to that one discussed above. Therefore we obtain $8N$ moduli in total. To be precise, here we assume that the moduli space of solutions is the space of all self-dual configurations modulo small gauge transformations. If we mod out all gauge transformations, then we would need to subtract the dimension of the considered gauge algebra in order to get the correct dimension, i.e. $8N-3$ for the case of $\mathfrak{su}(2)$.

Note that one could also act with the Euclidean Lorentz group $SO(4)$. Jackiw and Rebbi [105], however, have demonstrated that such rotations can be undone by suitable chosen gauge transformations. They also showed that for the conformal group $SO(5,1)$ of $\mathbb{R}^4$, the $SO(5)$ subgroup consisting of rotations and special conformal transformations leaves the instanton invariant (modulo gauge transformations). Their analysis eventually led to the five-dimensional moduli space $SO(5,1)/SO(5)$ which coincides with the Euclidean five-dimensional anti-de Sitter space $AdS_5$.

### § III.3.2 $SU(2)$ BPST Instanton from the Twistor Approach

In section § III.2.3 we have learned that we can start with a holomorphic matrix $f_{12}$ defined on $U_1 \cap U_2 \subset \mathbb{CP}^1$ (parametrized by $\lambda$) in order to construct a self-dual connection. Here, $\{U_1, U_2\}$ is the canonical cover of $\mathbb{CP}^1$. Remember that $f_{12}$ depends smoothly on the coordinates $\{x^\mu\}$ (or equivalently on $\{z^a, \bar{z}^{\bar{a}}\}$), as well. Then we may try to split $f_{12}$ into $\psi_{1,2} \in SL(2,\mathbb{C})$ according to (III.55), where $\psi_{1,2}$ are holomorphic in $\lambda$ on $U_{1,2}$, respectively. If successful, we can use (III.65) to obtain a self-dual gauge potential.

**BPST solution.** Crane [40] constructed a matrix $\Upsilon \in SL(2,\mathbb{C})$ which depends meromorphically on $\{w^a\}$ and acts by right multiplication on $\psi$ given by (III.73), such that

$$\Upsilon\,:\,\psi \mapsto \psi_1 := \psi\,\Upsilon, \tag{III.80}$$

where $\psi_1$ depends holomorphically on $\lambda$ on $U_1$, as desired. We then define $\psi_2$ by

$$\psi_2(x,\lambda) := [\psi_1^{-1}(\tau(x,\lambda))]^\dagger = [\psi_1^{-1}(x,-1/\bar{\lambda})]^\dagger. \tag{III.81}$$

The matrix $\Upsilon$ is given by [40]

$$\Upsilon(w^a,\lambda)=\frac{1}{2\Lambda^2}\begin{pmatrix} -2\Lambda^2-\frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & -2\Lambda^2+\frac{w^1w^2}{\lambda}\end{pmatrix}. \tag{III.82}$$

Obviously, $\Upsilon$ is invariant with respect to the anti-holomorphic involution (III.41), i.e. $\Upsilon(x,\lambda)=[\Upsilon(\tau(x,\lambda))]^\dagger$. Using the explicit form of $\Upsilon$ and $\psi$, we obtain for $\psi_1$

$$\begin{aligned}\psi_1(x,\lambda) &= \psi(x,\lambda)\,\Upsilon(w^a,\lambda)\\ &= -\frac{1}{\Lambda}\frac{1}{\sqrt{r^2+\Lambda^2}}\begin{pmatrix} \bar z^{\bar 1}z^1+\Lambda^2-\lambda\bar z^{\bar 1}\bar z^{\bar 2} & -\bar z^{\bar 1}z^2-\lambda(\bar z^{\bar 1})^2 \\ -z^1\bar z^{\bar 2}+\lambda(\bar z^{\bar 2})^2 & \bar z^{\bar 2}z^2+\Lambda^2+\lambda\bar z^{\bar 1}\bar z^{\bar 2}\end{pmatrix}.\end{aligned} \tag{III.83}$$

Moreover, the matrix $f_{12}$ becomes

$$f_{12}(w^a,\lambda)=\psi_1^{-1}(w^a,\lambda)\,\psi_2(w^a,\lambda)=[\Upsilon^2(w^a,\lambda)]^{-1}=\frac{1}{\Lambda^2}\begin{pmatrix} \Lambda^2-\frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & \Lambda^2+\frac{w^1w^2}{\lambda}\end{pmatrix}. \tag{III.84}$$

With the help of (III.65), we can now compute the gauge potential and hence the curvature. What we find is, of course, identical to the result of the previous section.

### § III.3.3 $SU(2)$ Instantons for a General Topological Charge

Let us now discuss the generalization of the one-instanton solution (III.76) given in the previous subsections to general $N$. As already pointed out, Belavin and Zakharov [12] took the solution (III.73) as the seed solution in order to derive the two-instanton solution. Moreover, they gave the recursion relation in order to construct the $N$-instanton solution. However, already in the case of charge two, the equations for the coefficients $R$ in the ansatz (III.70) are quite complicated. Therefore, we do not follow their derivation but discuss a more elegant solution due to Corrigan, Fairlie, 't Hooft, and Wilczek (CFTW), although this way of construction lies somewhat afield from the main thread of development.

**CFTW ansatz and $N$-instanton solution.** The CFTW ansatz is given by [38]

$$A_\mu^a=-\bar\eta^a_{\mu\nu}\partial^\nu\ln(\phi), \tag{III.85}$$

where $\phi$ is at the moment an arbitrary function of $x^\mu$. Using this ansatz, one easily computes the components of the curvature to be

$$F^a_{\mu\nu}=\bar\eta^a_{\mu\sigma}\left\{\frac{\partial_\nu\partial^\sigma\phi}{\phi}-2\frac{(\partial_\nu\phi)(\partial^\sigma\phi)}{\phi^2}\right\}-\mu\leftrightarrow\nu+\bar\eta^a_{\mu\nu}\frac{\partial_\sigma\phi}{\phi}\frac{\partial^\sigma\phi}{\phi}. \tag{III.86}$$

With the help of the 't Hooft symbols (III.19) the self-duality equation (III.17) can be rewritten as

$$\bar{\eta}^a_{\mu\nu} F^{a\mu\nu} = 0, \tag{III.87}$$

since $\bar{\eta}^a_{\mu\nu}$ is anti-self dual. Using (III.86), we obtain from the above equation

$$\phi^{-1}\Box\phi = 0 \qquad \text{with} \qquad \Box := \partial_\mu \partial^\mu. \tag{III.88}$$

This equation can be solved by

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\Lambda_i^2}{(x - x_{i0})^2}. \tag{III.89}$$

The gauge potential (III.85) following from this $\phi$ has singularities at $x = x_{i0}$. However, these singularities are a pure gauge artifact, since they can be gauged away by using a singular gauge transformation. This can be seen as follows. Let us consider the case $N = 1$. Then the gauge potential reads

$$A^a_\mu = 2\bar{\eta}^a_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^2((x - x_0)^2 + \Lambda^2)} \tag{III.90}$$

Now, consider $g \in SU(2)$ with

$$g_1(x) = \frac{x^\mu}{|x|}\sigma_\mu, \tag{III.91}$$

where $\sigma^\mu$ is given by (I.5). After a gauge transformation $\mathcal{A} \mapsto g_1^{-1}\mathcal{A}g_1 + g_1^{-1}\,\mathrm{d}\,g_1$ with $g_1(x - x_{i0})$ we arrive at the solution (III.76).

In the general case we have

$$A^a_\mu = 2\bar{\eta}^a_{\mu\nu} \frac{\sum_{i=1}^N \frac{\Lambda_i^2\,(x - x^\nu_{i0})}{|x - x_{i0}|^4}}{1 + \sum_{i=1}^N \frac{\Lambda_i^2}{|x - x_{i0}|^2}}. \tag{III.92}$$

Also here, we need some singular gauge transformation in order to remove the singularites from the solution. One possible choice is

$$g_N(x) = \frac{H^\mu(x)}{|H(x)|}\sigma_\mu \qquad \text{with} \qquad H^\mu(x) = \sum_{i=1}^N \alpha_i \frac{(x - x_{i0})^\mu}{(x - x_{i0})^2}, \tag{III.93}$$

where $\alpha_i \in \mathbb{R}$ with $\alpha_i > 0$. Note that in the limit $x \to x_{i0}$ we have $g_N(x) \to g_1(x - x_{i0})$. Performing a gauge transformation with $g_N(x)$, one can remove all singularities.

In order to compute the topological charge (III.8) for this configuration we note that

$$\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\tfrac{1}{2}\Box\Box \ln(\phi), \tag{III.94}$$

as one may check. Inserting (III.89) into the above equation, integrating over the four-dimensional Euclidean space and excluding the singularities from the region of integration one obtains with the help of Gauß' law a charge of $N$. Note that taking $\eta^a_{\mu\nu}$ in the ansatz (III.85) yields eventually a configuration with charge $-N$.

The solution discussed above is referred to as the 't Hooft instanton solution. The number of moduli of this solution is $5N$ corresponding to size and position parameters. However, as we will show in a minute using the Atiyah-Singer index theorem (II.143), a configuration with topological charge $N$ must have $8N$ moduli. Therefore, the 't Hooft solution is not the most general solution. The complete solution can be derived within the Atiyah-Drinfield-Hitchin-Manin (ADHM) construction [6], which we do not discuss here. For a review, see for instance [164].

## § III.3.4 Generalization to $SU(n)$

In this subsection we shall give the idea, how one can obtain instanton solutions when the gauge group is chosen to be $SU(n)$ and demonstrate that the Atiyah-Singer index theorem (II.143) is a powerful tool for counting the number of independent solutions, i.e. the dimension of the moduli space of solutions. For a detailed discussion of this subject we refer the reader to [19].

**Embedding of the $SU(2)$ solution into $SU(n)$.** Let us consider the case $N = 1$. In order to construct instanton solutions for $SU(n)$ with $n > 2$ one can use the $SU(2)$ solution (III.76). Then one considers the $SU(2)$ group as a subgroup of $SU(n)$ and chooses different embeddings. One obvious choice of an embedding is given by the $n \times n$ matrix

$$\mathcal{A}_\mu^{SU(n)} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_\mu^{SU(2)} \end{pmatrix}, \tag{III.95}$$

which of course is not the most general solution. Now one can act adjointly on (III.95) with some $g \in SU(n)$, i.e.

$$\mathcal{A}_\mu^{SU(n)} = g \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_\mu^{SU(2)} \end{pmatrix} g^{-1}. \tag{III.96}$$

However, not all $g \in SU(n)$ yield new solutions since this action is not free. There are elements of $SU(n)$ which leave (III.95) invariant. The stabilizer of (III.95) is $SU(n-2) \times U(1) \subset SU(n)$ and hence $g$ has to be an element of the quotient space $SU(n)/(SU(n-2) \times U(1))$. The dimension of this quotient space is $4n - 5$. Together with the position and size moduli of the $SU(2)$ solution (III.76) we find in total $4n$ moduli. In the case $N > 1$, however, there are embeddings which cannot be obtained adjointly [202]. Thus, we need another method in order to find the number of moduli characterizing the solution.

**Number of moduli via the Atiyah-Singer theorem.** Remember that the Yang-Mills action and the equations of motion are conformally invariant. Since the stereographic pojection is a conformal map, the equations of motion on $\mathbb{R}^4$ and $S^4$ are of the same form. Thus, using the stereographic projection we can pull back the solution (III.76) given on $\mathbb{R}^4$ to a self-dual solution on $S^4$ subject to the SDYM equations

$$\mathcal{F}_{ij} = \tfrac{1}{2}\epsilon_{ijkl}\mathcal{F}^{kl}, \tag{III.97}$$

given in a local orthonormal basis $\{e_i\}$ (vier-bein frame). Throughout this paragraph we shall understand the theory to be formulated on $S^4$.

Let us consider a perturbation of the form $\mathcal{A} \mapsto \mathcal{A}^0 + \delta\mathcal{A}$, where $\mathcal{A}^0$ represents a self-dual solution. We are only interested in perturbations such that the resulting curvature remains self-dual. Expanding (III.97) to first order in the deformation we find the condition

$$\mathcal{D}_i\delta\mathcal{A}_j = \tfrac{1}{2}\epsilon_{ijkl}\mathcal{D}^k\delta\mathcal{A}^l, \tag{III.98}$$

where the covariant derivative is computed in the background of the instanton. Here, $\mathcal{D}_i$ is given by (II.42) when acting on one-forms. In addition we need to require that the modified solution $\mathcal{A}^0 + \delta\mathcal{A}$ is not related to the old one by a gauge transformation. This can be done by requiring that the deformations are orthogonal to any gauge transformation[3] $\mathcal{D}\alpha = e^i(\partial_i\alpha + [\mathcal{A}_i, \alpha])$ with $\alpha \in \mathfrak{su}(n) \otimes \Omega^0(S^4)$, i.e.

$$\int_{S^4} \mathrm{tr}(\mathcal{D}\alpha \wedge *\delta\mathcal{A}) = 0. \tag{III.99}$$

Integration by parts leads to the constraint

$$\mathcal{D}*\delta\mathcal{A} = 0 \qquad \Longrightarrow \qquad [\nabla_i, \delta\mathcal{A}^i] = 0. \tag{III.100}$$

In subsection § II.2.5 we have seen that the second Stiefel-Whitney class of any sphere is trivial, in particular of $S^4$. Hence, $S^4$ is a spin manifold. In this case there is no need to introduce a spin$^c$ structure rather it is enough to consider a spin structure. Therefore, the spin$^c$ bundle $W$ we have defined in subsection § II.2.4 reduces to the standard spin bundle $S$.[4] Consider a complex vector bundle of rank $n^2-1$ over the four-sphere, i.e. $E \to S^4$. The gauge group $SU(n)$ acts adjointly on $E$. Using the standard spin representation defined in § II.2.4 we may define $\Omega := \sum_{i=1}^4 \omega_i\Gamma(e_i)$, where $\{e_i\}$ forms a local orthonormal basis of $TS^4$. The twisted spin$^c$ Dirac operator (II.124) reduces to the ordinary twisted Dirac operator, i.e. one can simply neglect the $U(1)$ part in the definition (II.124). Letting $\omega_i = \delta\mathcal{A}_i$ the equations (III.98) and (III.100) can be rewritten as

$$\mathcal{D}_{\mathcal{A}}\Omega = 0, \tag{III.101}$$

where $\mathcal{D}_{\mathcal{A}}$ is the twisted Dirac operator. This can be shown with the help of the Clifford algebra (II.66). The matrix $\Omega$ can be understood as

$$\Omega = \begin{pmatrix} a & -b^\dagger \\ b & a^\dagger \end{pmatrix}, \tag{III.102}$$

[3] Note that this expression is just the infinitesimal version of (II.29).

[4] In the case of spin manifolds we have globally $W = S \otimes L_\Gamma^{1/2}$.

where $a$ and $b$ are adjoint-valued functions. Thus, equation (III.101) reduces to two Dirac equations, i.e. on the one hand we obtain

$$\mathcal{D}_{\mathcal{A}}\Phi = 0 \qquad \text{with} \qquad \Phi \in \Gamma(S^4, S^+ \otimes E) \qquad \text{and} \qquad \Phi = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{III.103}$$

On the other hand we get one equation for $\bar{\Phi}$, where bar denotes the usual spinor conjugation as defined in chapter I. However, these two equations are equivalent in a sense that, if $\Phi$ is a solution to the Dirac equation then $\bar{\Phi}$ solves also the Dirac equation. Therefore, the number of solutions for $\Omega$ is twice the number of solutions for a single chiral adjoint spinor.

Having done this derivation we have related the number of moduli characterizing a self-dual connection with the number of the zero modes of the Dirac operator and hence with its index. Remember that the index of $\mathcal{D}_{\mathcal{A}}$ is given by

$$\operatorname{ind}\mathcal{D}_{\mathcal{A}} = \dim\ker\mathcal{D}_{\mathcal{A}} - \dim\ker\mathcal{D}^{\dagger}_{\mathcal{A}}. \tag{III.104}$$

Obviously, $\ker\mathcal{D}^{\dagger}_{\mathcal{A}} \subset \ker\mathcal{D}_{\mathcal{A}}\mathcal{D}^{\dagger}_{\mathcal{A}}$. The twisted version of Weitzenböck's formula (II.125b) yields

$$\mathcal{D}_{\mathcal{A}}\mathcal{D}^{\dagger}_{\mathcal{A}}\Psi = \nabla_{\mathcal{A}}\nabla^{\dagger}_{\mathcal{A}}\Psi + \tfrac{1}{4}R\,\Psi + [\rho^{-}(\mathcal{F}_{\mathcal{A}}), \Psi] = \nabla_{\mathcal{A}}\nabla^{\dagger}_{\mathcal{A}}\Psi + \tfrac{1}{4}R\,\Psi \stackrel{!}{=} 0, \tag{III.105}$$

since $\rho^{-}(\mathcal{F}_{\mathcal{A}}) = 0$ in the instanton background. The commutator appears in Weitzenböck's formula because $\Psi$ is an adjoint spinor. The integration of this equation leads to

$$\|\nabla_{\mathcal{A}}\Psi\|^2 + \tfrac{1}{4}R\,\|\Psi\|^2 = 0. \tag{III.106}$$

Thus, we conclude that $\Psi$ has to vanish and hence $\dim\ker\mathcal{D}^{\dagger}_{\mathcal{A}} = 0$. Applying the Atiyah-Singer index theorem (II.143) we obtain

$$\dim\ker\mathcal{D}_{\mathcal{A}} = \int_{S^4}\hat{A}(TS^4)\wedge\operatorname{ch}(E) = \int_{S^4}\operatorname{ch}(E), \tag{III.107}$$

since the Dirac genus $\hat{A}$ of all even-dimensional spheres is unity. Finally, using the definition of the Chern character (II.128), we discover

$$\dim\ker\mathcal{D}_{\mathcal{A}} = -\frac{1}{8\pi^2}\int_{S^4}\operatorname{tr}(\mathcal{F}_{\mathcal{A}}\wedge\mathcal{F}_{\mathcal{A}}) = 2nN, \tag{III.108}$$

since in the adjoint representation the generators are normalized as $\operatorname{tr}(X^{(ad)}_a X^{(ad)}_b) = -n\delta_{ab}$. This means that we have in total $4nN$ moduli characterizing a self-dual solution if the gauge group is $SU(n)$.

**Remark.** Instead of choosing $SU(n)$ one may take any simple, compact Lie group $G$ and apply the Atiyah-Singer index theorem to count the number of instanton solutions. This was done by Bernard et al. in [19].

# Chapter IV

## Ward's Chiral Model on $\mathbb{R}^{2,1}$

In the introduction we have talked a little bit about the modified $SU(n)$ sigma model first studied by Ward [195]. We have pointed out that it can be obtained from self-dual Yang-Mills theory defined on $\mathbb{R}^{2,2}$ via dimensional reduction. In this chapter we shall explain explicitly how this reduction works. However, before getting into details we need to discuss the twistor space of $\mathbb{R}^{2,2}$. This is similar to that we have already discussed in the previous chapter. The last section of this chapter is devoted to the construction of static soliton solutions as well as soliton-soliton and soliton-antisoliton scattering configurations.

The metric on $\mathbb{R}^{2,2}$ is given by $(g_{\mu\nu}) = \mathrm{diag}(+1, +1, -1, -1)$. The SDYM equations (III.17) retain their form, i.e.

$$\mathcal{F} = *\mathcal{F} \qquad \Longleftrightarrow \qquad \mathcal{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}, \tag{IV.1}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol on $\mathbb{R}^{2,2}$ with $\epsilon_{1234} = 1$. Again, the problem of solving these equations is translated into an auxiliary linear problem as we shall discuss now (cf. e.g. [101, 102]).

## § IV.1 Some Twistor Geometry

### § IV.1.1 Complex Structures on $\mathbb{R}^{2,2}$ and $H^2$

**Complex structure on $\mathbb{R}^{2,2}$.** Let $M$ be an orientable pseudo-Riemannian four-manifold with signature $(+1, +1, -1, -1)$. Then all constant almost complex structures on $M$ are parametrized by $SO(2,2)/U(1,1) \cong SO(2,1)/SO(2)$. The quotient $SO(2,1)/SO(2)$ is topologically a two-sheeted hyperboloid $H^2$. Let now $M = \mathbb{R}^{2,2}$. It is well known that $\mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, where the generators in the fundamental representation are of the form

$$X_1^+ = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2^+ = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$X_3^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{IV.2a}$$

and

$$X_1^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2^- = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$X_3^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{IV.2b}$$

The generators satisfy

$$X_a^\pm X_b^\pm = g_{ab} + C_{ab}{}^c X_c^\pm, \qquad [X_a^\pm, X_b^\pm] = 2C_{ab}{}^c X_c^\pm \qquad \text{and} \qquad [X_a^+, X_b^-] = 0, \tag{IV.3}$$

where the structure constants $C_{ab}{}^c$ of $\mathfrak{so}(2,1)$ are given by

$$C_{12}{}^3 = -C_{23}{}^1 = -C_{31}{}^2 = 1 \tag{IV.4}$$

and $g_{ab} = C_{ad}{}^c C_{bc}{}^d$ is the Cartan-Killing form with $(g_{ab}) = \mathrm{diag}(+1,+1,-1)$. Then we can write a general constant almost complex structure in the form

$$J = -\xi^a X_a^-, \tag{IV.5}$$

where the real numbers $\xi^a$ parametrize $H^2$, i.e.

$$J^2 = g_{ab}\xi^a\xi^b \stackrel{!}{=} -1 \qquad \Longrightarrow \qquad (\xi^1)^2 + (\xi^2)^2 - (\xi^3)^2 = -1. \tag{IV.6}$$

Of course, the choice $\tilde{J} = -\xi^a X_a^+$ is also possible. This corresponds to the opposite orientation on $\mathbb{R}^{2,2}$ and to the transition from self-dual to anti-self-dual gauge fields. The eigenvalues of $J$ are $\pm i$ and $\mathbb{C}^4 \cong \mathbb{R}^{2,2} \otimes \mathbb{C}$ can be decomposed into $\mathbb{C}^4 \cong \mathcal{V} \oplus \bar{\mathcal{V}}$ with

$$\mathcal{V} = \{V \in \mathbb{C}^4 \,|\, J^\mu_\nu V^\nu = iV^\mu\} \qquad \text{and} \qquad \bar{\mathcal{V}} = \{V \in \mathbb{C}^4 \,|\, J^\mu_\nu V^\nu = -iV^\mu\}. \tag{IV.7}$$

As a basis of $\bar{\mathcal{V}}$ we may take

$$(\bar{V}_{\bar{1}}^{\mu}) = \left(\tfrac{1}{2}, \tfrac{i}{2}, -\tfrac{1}{2}\lambda, \tfrac{i}{2}\lambda\right) \qquad \text{and} \qquad (\bar{V}_{\bar{2}}^{\mu}) = \left(-\tfrac{1}{2}\lambda, \tfrac{i}{2}\lambda, \tfrac{1}{2}, \tfrac{i}{2}\right), \tag{IV.8}$$

where $\lambda$ is the local holomorphic coordinate on the hyperboloid $SO(2,1)/SO(2)$, i.e. $\lambda = -(\xi^1 + i\xi^2)/(1+\xi^3)$. Using $J$, we can introduce vector fields $\bar{V}_{\bar{a}} = \bar{V}_{\bar{a}}^{\mu}\partial_\mu$ of type $(0,1)$ with respect to $J$. For instance, we have

$$\bar{V}_{\bar{1}} = \partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2} \qquad \text{and} \qquad \bar{V}_{\bar{2}} = \partial_{\bar{z}^{\bar{2}}} - \lambda\partial_{z^1}, \tag{IV.9}$$

where

$$z^1 = x^1 + ix^2, \quad \bar{z}^{\bar{1}} = x^1 - ix^2, \quad z^2 = x^3 + ix^4, \quad \bar{z}^{\bar{2}} = x^3 - ix^4 \tag{IV.10}$$

are the canonical complex coordinates on $\mathbb{R}^{2,2} \cong \mathbb{C}^{1,1}$.

**Complex structure on $H^2$.** With the help of the stereographic projection $H^2 \to \mathbb{R}^2$, we can map the upper half $H^2_+$ of $H^2$ onto the interior of the unit disk in the $(\xi^1, \xi^2)$-plane, i.e.

$$\omega^1 = \frac{\xi^1}{1+\xi^3} \qquad \text{and} \qquad \omega^2 = \frac{\xi^2}{1+\xi^3}, \tag{IV.11}$$

so that

$$\xi^1 = \frac{2\omega^1}{1-r^2}, \qquad \xi^2 = \frac{2\omega^2}{1-r^2}, \qquad \xi^3 = \frac{1+r^2}{1-r^2} \qquad \text{with} \qquad r^2 = (\xi^1)^2 + (\xi^2)^2. \tag{IV.12}$$

Simultaneously, the lower half $H^2_-$ is mapped onto the exterior of the unit disk. With $\lambda = -(\omega^1 + i\omega^2)$ we can relate both regions via $\lambda \mapsto \bar{\lambda}^{-1}$. Note that for $|\lambda| \neq 1$ there are no fixed points of this mapping. In summary, $H^2$ can be viewed as $\mathbb{CP}^1 \backslash S^1$, where $S^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$.

We introduce the standard complex structure $\mathfrak{J}$ on $H^2$

$$\mathfrak{J} = (\mathfrak{J}^a{}_b), \qquad \mathfrak{J}^a{}_c\mathfrak{J}^c{}_b = -\delta^a{}_b \qquad \text{and} \qquad \mathfrak{J}^1{}_2 = -\mathfrak{J}^2{}_1 = -1. \tag{IV.13}$$

The anti-holomorphic vector field on $H^2_+$ is given by

$$\bar{V}_{\bar{3}} = -\tfrac{1}{2}(\partial_{\omega^1} + i\partial_{\omega^2}) = \partial_{\bar{\lambda}} \qquad \text{and} \qquad \mathfrak{J}^a{}_b\bar{V}_{\bar{3}}^b = -i\bar{V}_{\bar{3}}^a. \tag{IV.14}$$

## § IV.1.2 Twistor Space of $\mathbb{R}^{2,2}$

**Definition of the twistor space.** Let again $M$ be an orientable pseudo-Riemannian four-manifold with signature $(+1,+1,-1,-1)$. Similarly to the previous chapter we define the twistor space by

$$\mathcal{Z}(M) := P(M, SO(2,2)) \times_{SO(2,2)} (SO(2,2)/U(1,1)), \tag{IV.15}$$

where $P(M, SO(2,2))$ is the principal frame bundle of $M$. Let $M = \mathbb{R}^{2,2}$. Then this definition translates to

$$\mathcal{Z}(\mathbb{R}^{2,2}) \cong \mathbb{R}^{2,2} \times H^2 \qquad \text{and} \qquad H^2 = H^2_+ \cup H^2_-. \tag{IV.16}$$

Thus, the twistor space $\mathcal{Z}$ can be considered as a trivial bundle $\pi : \mathcal{Z} \to \mathbb{R}^{2,2}$ over $\mathbb{R}^{2,2}$ with fiber $H^2$ and the canonical projection $\pi : \{x^\mu, \omega^a\} \to \{x^\mu\}$.

**Complex structure on $\mathcal{Z}$.** Using the complex structures on $\mathbb{R}^{2,2}$ and on $H^2$, we define the complex structure on $\mathcal{Z}$ to be $\mathcal{J} = (J, \mathfrak{J})$. Then the vector fields $\{\bar{V}^{\bar{a}}\}$ given by (IV.8) and (IV.14) are anti-holomorphic vector fields on $\mathbb{R}^{2,2} \times H^2_+$ with respect to $\mathcal{J}$. Therefore, the appropriate coordinates on the twistor space $\mathcal{Z}$ are[1]

$$w^1 = z^1 + \lambda \bar{z}^{\bar{2}}, \qquad w^2 = z^2 + \lambda \bar{z}^{\bar{1}} \qquad \text{and} \qquad w^3 = \lambda. \tag{IV.17}$$

After some algebra we find from these expressions

$$\partial_{\bar{w}^{\bar{1}}} = \gamma \bar{V}_{\bar{1}}, \qquad \partial_{\bar{w}^{\bar{2}}} = \gamma \bar{V}_{\bar{2}} \qquad \text{and} \qquad \partial_{\bar{w}^{\bar{3}}} = \partial_{\bar{\lambda}} - \gamma z^1 \bar{V}_{\bar{2}} - \gamma z^2 \bar{V}_{\bar{1}}, \tag{IV.18}$$

with $\gamma = (1 - \lambda\bar{\lambda})^{-1}$. Moreover, we can derive a local basis of $(0,1)$-forms with respect to $\mathcal{J}$ satisfying $\bar{\theta}^{\bar{a}}(\bar{V}_{\bar{b}}) = \delta^{\bar{a}}_{\bar{b}}$. The $\bar{\theta}^{\bar{a}}$'s are given by

$$\bar{\theta}^{\bar{1}} = \gamma(\mathrm{d}\,\bar{z}^{\bar{1}} + \bar{\lambda}\,\mathrm{d}\,z^2), \qquad \bar{\theta}^{\bar{2}} = \gamma(\mathrm{d}\,\bar{z}^{\bar{2}} + \bar{\lambda}\,\mathrm{d}\,z^1) \qquad \text{and} \qquad \bar{\theta}^{\bar{3}} = \mathrm{d}\,\bar{\lambda}. \tag{IV.19}$$

### § IV.1.3 Penrose-Ward Correspondence

**Pulled back bundle over $\mathcal{P}$.** Let $U$ be a sufficiently small open ball in $\mathbb{R}^{2,2}$, $\mathcal{P} := \mathcal{Z}|_U$ the restriction of $\mathcal{Z}$ to $U$, i.e. $\mathcal{P} = U \times H^2$ and let $P = U \times G$ be some principal $G$-bundle over $U \subset \mathbb{R}^{2,2}$. Consider the associated complex rank $n$ vector bundle $E = P \times_G \mathbb{C}^n \cong U \times \mathbb{C}^n$ with the canonical projection $\pi_E$. Again, using the projection $\pi : \mathcal{P} \to U$, we can pull back $E$ to the bundle $\pi^*E$ over $\mathcal{P}$. The pulled pack bundle $\pi^*E$ is trivial on $H^2$. The connection on $E$ is given by $D = \mathrm{d} + \mathcal{A}$. By definition the pulled back connection one-form $\pi^*\mathcal{A}$ on $\pi^*E$ is flat along the $H^2$ fibers. Thus, on $\mathcal{P}_+ = U \times H^2_+$ we find

$$\pi^* D = D + \mathrm{d}\,\lambda\,\partial_\lambda + \mathrm{d}\,\bar{\lambda}\,\partial_{\bar{\lambda}}. \tag{IV.20}$$

Repeating the steps made in subsection § III.2.3, we write $\pi^*\mathcal{A} = \mathcal{A}_\mu\,\mathrm{d}\,x^\mu = \mathcal{A}_{z^a}\,\mathrm{d}\,z^a + \mathcal{A}_{\bar{z}^{\bar{a}}}\,\mathrm{d}\,\bar{z}^{\bar{a}} = \mathcal{B} + \bar{\mathcal{B}}$ with $\bar{\mathcal{B}} = \bar{\mathcal{B}}_{\bar{a}}\bar{\theta}^{\bar{a}}$ and

$$\bar{\mathcal{B}}_{\bar{1}} := \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda\mathcal{A}_{z^2}, \qquad \bar{\mathcal{B}}_{\bar{2}} := \mathcal{A}_{\bar{z}^{\bar{2}}} - \lambda\mathcal{A}_{z^1} \qquad \text{and} \qquad \bar{\mathcal{B}}_{\bar{3}} := 0. \tag{IV.21}$$

Thus, the pulled back connection $\pi^*D$ on $\pi^*E$ reads $\pi^*D = \partial_{\mathcal{B}} + \bar{\partial}_{\bar{\mathcal{B}}}$ with $\bar{\partial}_{\bar{\mathcal{B}}} = \bar{\partial} + \bar{\mathcal{B}}$.

**Linear sytem.** Again, we consider the equations

$$\bar{\partial}_{\bar{\mathcal{B}}} s = 0 \tag{IV.22}$$

for $s \in \Gamma(\mathcal{P}_+, \pi^*E)$. The bundle $E' := \pi^*E$ is said to be holomorphic iff these equations are compatible, i.e. $\bar{\partial}^2_{\bar{\mathcal{B}}} = 0$. Similarly to section § III.2.3 we obtain

$$[(\nabla_1 + i\nabla_2) - \lambda(\nabla_3 - i\nabla_4)]s(x, \lambda, \bar{\lambda}) = 0, \tag{IV.23a}$$

$$[(\nabla_3 + i\nabla_4) - \lambda(\nabla_1 - i\nabla_2)]s(x, \lambda, \bar{\lambda}) = 0, \tag{IV.23b}$$

$$\partial_{\bar{\lambda}} s(x, \lambda, \bar{\lambda}) = 0. \tag{IV.23c}$$

[1] Strictly speaking: on $\mathcal{Z}_+ = \mathbb{R}^{2,2} \times H^2_+$. However, $H^2_+$ and $H^2_-$ are releated via $\lambda \mapsto \bar{\lambda}^{-1}$ with $|\lambda| \neq 1$.

The last of these equations implies that $s$ is a function of $x$ and $\lambda$ but not of $\bar{\lambda}$. The compatibility conditions for this system read

$$\mathcal{F}_{z^1 z^2} = 0, \qquad \mathcal{F}_{\bar{z}^{\bar{1}} \bar{z}^{\bar{2}}} = 0 \qquad \text{and} \qquad \mathcal{F}_{z^1 \bar{z}^{\bar{1}}} - \mathcal{F}_{z^2 \bar{z}^{\bar{2}}} = 0, \tag{IV.24}$$

which are exactly the SDYM equations on (IV.1) written in the complex coordinates (IV.10). Thus, in the case that the equations (IV.24) hold, $s$ is holomorphic on $\mathcal{P}_+$.

Combining $n$ such sections into an $n \times n$ matrix, i.e. $\psi = (s_1, \ldots, s_n)$ with $\psi = \psi(x, \lambda)$ and $\psi \in SL(n, \mathbb{C})$ we obtain from (IV.23)

$$\psi(\partial_{\bar{z}^{\bar{1}}} - \lambda \partial_{z^2})\psi^{-1} = \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda \mathcal{A}_{z^2}, \tag{IV.25a}$$

$$\psi(\partial_{\bar{z}^{\bar{2}}} - \lambda \partial_{z^1})\psi^{-1} = \mathcal{A}_{\bar{z}^{\bar{2}}} - \lambda \mathcal{A}_{z^1}. \tag{IV.25b}$$

This works since (IV.23) exhibits $n$ linearly independent zero modes.

**Real isotropic coordinates.** The vector fields $\{\bar{V}_{\bar{1}}, \bar{V}_{\bar{2}}\}$ span a null plane in $\mathbb{R}^{2,2}$, as one may readily check. The values of $\lambda$ with $|\lambda| = 1$ do not correspond to a complex structure on $\mathbb{R}^{2,2}$. However, the vector fields $\{\bar{V}_{\bar{1}}, \bar{V}_{\bar{2}}\}$ in this case still span a null plane, as was pointed out e.g. in [123]. Using $|\lambda| = 1$, one observes that complex conjugation carries $\bar{V}_{\bar{1}}$ to a multiple of $\bar{V}_{\bar{2}}$. This means that the isotropic plane is real. With the help of the linear fractional transformation defined by

$$\lambda = \frac{\zeta - i}{\zeta + i} \qquad \Longleftrightarrow \qquad \zeta = i\frac{1 - \lambda}{1 + \lambda}, \tag{IV.26}$$

which maps the unit circle $|\lambda| = 1$ to the real axis in the $\zeta$-plane, we can introduce real isotropic coordinates, e.g. $(x^\mu) = (x, y, \tilde{t}, -t)$, where

$$u := \tfrac{1}{2}(t + y), \qquad v := \tfrac{1}{2}(t - y), \qquad \tilde{u} := \tfrac{1}{2}(\tilde{t} + x) \qquad \text{and} \qquad \tilde{v} := \tfrac{1}{2}(\tilde{t} - x) \tag{IV.27}$$

implying the derivatives

$$\partial_u = \partial_t + \partial_y, \qquad \partial_v = \partial_t - \partial_y, \qquad \partial_{\tilde{u}} = \partial_{\tilde{t}} + \partial_x \qquad \text{and} \qquad \partial_{\tilde{v}} = \partial_{\tilde{t}} - \partial_x. \tag{IV.28}$$

Note that the transformation (IV.26) carries the lower (upper) half plane in $\zeta$ to the exterior (interior) of the unit disk $\{|\lambda| > 1\}$ $(\{|\lambda| < 1\})$. Moreover, the space $\mathbb{CP}^1 \cong H^2 \cup S^1$ can be considered as the moduli space of all null two-planes in $\mathbb{R}^{2,2}$, where real null two-planes are parametrized by $S^1$ and complex ones by $H^2$. In that sense, $\mathbb{CP}^1$ is the space of extended complex structures on $\mathbb{R}^{2,2}$.

Using (IV.26), we rewrite the linear system (IV.23) as

$$(\zeta \nabla_{\tilde{u}} + \nabla_u) s(x, \zeta, \bar{\zeta}) = 0, \quad (\zeta \nabla_v - \nabla_{\tilde{u}}) s(x, \zeta, \bar{\zeta}) = 0 \quad \text{and} \quad \partial_{\bar{\zeta}} s(x, \zeta, \bar{\zeta}) = 0, \tag{IV.29}$$

i.e. $s = s(x, \zeta)$. Thus, we obtain

$$\psi(\zeta \partial_{\tilde{v}} + \partial_u)\psi^{-1} = \zeta \mathcal{A}_{\tilde{v}} + \mathcal{A}_u, \tag{IV.30a}$$

$$\psi(\zeta \partial_v - \partial_{\tilde{u}})\psi^{-1} = \zeta \mathcal{A}_v - \mathcal{A}_{\tilde{u}} \tag{IV.30b}$$

with $\zeta \in \mathbb{CP}^1 \cong \mathbb{C}\cup\{\infty\}$ and $\psi \in SL(n,\mathbb{C})$ depends meromorphically on the spectral parameter $\zeta$. The compatibility conditions in these coordinates read

$$\mathcal{F}_{u\tilde{u}} = 0, \qquad \mathcal{F}_{v\tilde{v}} = 0 \qquad \text{and} \qquad \mathcal{F}_{uv} + \mathcal{F}_{\tilde{u}\tilde{v}} = 0. \tag{IV.31}$$

**Reality condition.** In general, the gauge fields $\mathcal{A}_\mu$ will again live in the complexified Lie algebra $\mathfrak{g}\otimes\mathbb{C}$. Similarly to the previous chapter we need to impose some reality conditions. Let $\mathfrak{g} = \mathfrak{su}(n)$. Then in $\lambda$-parametrization we require

$$[\psi(x,\bar{\lambda}^{-1})]^\dagger = \psi^{-1}(x,\lambda), \tag{IV.32}$$

while in the $\zeta$-parametrization this translates to

$$[\psi(x,\bar{\zeta})]^\dagger = \psi^{-1}(x,\zeta). \tag{IV.33}$$

This will again ensure, that the gauge fields computed from the auxiliary system are real.

## § IV.2 Action and Equations of Motion

In the previous section we have translated the problem of finding solutions to the self-duality equations on $\mathbb{R}^{2,2}$ to a linear auxiliary problem whose compatibility conditions are the SDYM equations. Let us now discuss how these equations are related to the equations of motion of Ward's chiral model. In the sequel we will always assume that the gauge group is $SU(n)$.

**Equations of motion on $\mathbb{R}^{2,2}$.** Let us start from the auxiliary system (IV.30) in real isotropic coordinates. Yang pointed out [212] that if one requires

$$\psi(u,v,\tilde{u},\tilde{v},\zeta\to 0) = g_1^{-1}(u,v,\tilde{u},\tilde{v}) + \mathcal{O}(\zeta), \tag{IV.34}$$

where $g_1 \in SU(n)$ and

$$\mathcal{A}_u = g_1^{-1}\partial_u g_1 \qquad \text{and} \qquad \mathcal{A}_{\tilde{u}} = g_1^{-1}\partial_{\tilde{u}} g_1 \tag{IV.35}$$

the equations (IV.30) in the limit $\zeta \to 0$ are identically satisfied. Clearly, this choice solves the first equation of (IV.31). In the limit $\zeta \to \infty$ we demand

$$\mathcal{A}_v = g_2^{-1}\partial_v g_2 \qquad \text{and} \qquad \mathcal{A}_{\tilde{v}} = g_2^{-1}\partial_{\tilde{v}} g_2, \tag{IV.36}$$

for some $g_2 \in SU(n)$ with $\psi(u,v,\tilde{u},\tilde{v},\zeta=\infty) = g_2^{-1}(u,v,\tilde{u},\tilde{v})$, which solves the second equation of (IV.31). Moreover, we can choose a gauge in which $\mathcal{A}_v$ and $\mathcal{A}_{\tilde{v}}$ do vanish, i.e. consider a gauge transformation of the form (cf. equation (III.56))

$$\psi \mapsto \psi^{g_2} := g_2\psi, \tag{IV.37}$$

which is equivalent to $\psi^{g_2}(u, v, \tilde{u}, \tilde{v}, \zeta = \infty) = 1$. Using (IV.30) and (IV.35), we obtain

$$\mathcal{A}_u^{g_2} = \Omega^{-1}\partial_u\Omega \qquad \text{and} \qquad \mathcal{A}_{\tilde{u}}^{g_2} = \Omega^{-1}\partial_{\tilde{u}}\Omega \tag{IV.38}$$

with $\Omega^{-1} := g_2 g_1^{-1} = \psi^{g_2}(u, v, \tilde{u}, \tilde{v}, \zeta = 0)$, which is known as the Yang prepotential (see e.g. [104]). We shall refer to this particular gauge as the Yang gauge. The gauge-fixed linear system (IV.30) reads

$$\psi^{g_2}(\zeta\partial_{\tilde{v}} + \partial_u)(\psi^{g_2})^{-1} = \mathcal{A}_u^{g_2} \qquad \text{and} \qquad \psi^{g_2}(\zeta\partial_v - \partial_{\tilde{u}})(\psi^{g_2})^{-1} = -\mathcal{A}_{\tilde{u}}^{g_2}. \tag{IV.39}$$

Note that the reality condition (IV.33) under the gauge transformation (IV.37) holds true. The remaing self-duality equation, i.e. the third equation of (IV.31) takes the form

$$\partial_v(\Omega^{-1}\partial_u\Omega) + \partial_{\tilde{v}}(\Omega^{-1}\partial_{\tilde{u}}\Omega) = 0. \tag{IV.40}$$

For brevity we shall omit the superscript '$g_2$' in the following.

**Action functional on $\mathbb{R}^{2,2}$.** Consider the antisymetric tensor $\omega^{\mu\nu}$ of rank two defined by $\omega^{13} = -\omega^{31} = -1$ and $\omega^{24} = -\omega^{42} = 1$. One can interpret $\omega = \frac{1}{2}\omega_{\mu\nu}\,\mathrm{d}\,x^\mu \wedge \mathrm{d}\,x^\nu$ as the Kähler form compatible with the complex structure $\tilde{J} = -\left(\begin{smallmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2,2}$, where $\sigma^3$ denotes the third Pauli matrix (II.12). Then in $x^\mu$-coordinates equation (IV.40) reads

$$(g^{\mu\nu} - \omega^{\mu\nu})\partial_\mu(\Omega^{-1}\partial_\nu\Omega) = 0. \tag{IV.41}$$

Note that the Kähler form $\omega$ is not invariant under $SO(2,2)$ transformations. Hence, (IV.41) is not Lorentz invariant.

An action functional producing equation (IV.41) is the Nair-Schiff type action [145, 135]

$$S_{\mathrm{NS}}(\Omega) = \frac{1}{2}\int_{\mathbb{R}^{2,2}} \mathrm{tr}(\mathrm{d}\,\Omega^{-1} \wedge * \mathrm{d}\,\Omega) - \frac{1}{3}\int_{\mathbb{R}^{2,2}\times[0,1]} \omega \wedge \mathrm{tr}(\tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}}), \tag{IV.42}$$

where the gauge potential $\mathcal{A} = \Omega^{-1}\,\mathrm{d}\,\Omega$ and the Kähler form $\omega$ have nonvanishing components only on $\mathbb{R}^{2,2}$. Here $\tilde{\mathcal{A}} = \tilde{\Omega}^{-1}\,\mathrm{d}\,\tilde{\Omega}$ is defined via a homotopy $\tilde{\Omega}$ from a fixed group element $\Omega_1$ to $\Omega$, i.e. $\tilde{\Omega}(0) = \Omega_1$ and $\tilde{\Omega}(1) = \Omega$; cf. subsection § II.2.3. The second term in the action, referred to as Wess-Zumino term, gives a total divergence when varied with respect to $\tilde{\Omega}$: Let $S_{\mathrm{NS}} = S_1 + S_2$, where $S_1$ denotes the first term of (IV.42) and $S_2$ the second one. Then consider the variation $\Omega \mapsto \Omega + \delta\Omega$. We find

$$\begin{aligned} \delta S_1(\Omega) &= -\int_{\mathbb{R}^{2,2}} \mathrm{tr}(\delta\mathcal{A} \wedge *\mathcal{A}) = -\int_{\mathbb{R}^{2,2}} \mathrm{tr}(-\Omega^{-1}\delta\Omega\,\mathcal{A} \wedge *\mathcal{A} + \Omega^{-1}\,\mathrm{d}\,\delta\Omega \wedge *\mathcal{A}) \\ &= \int_{\mathbb{R}^{2,2}} \mathrm{tr}(\Omega^{-1}\delta\Omega\;\mathrm{d} *\mathcal{A}), \end{aligned}$$

where we have used $\int_{\mathbb{R}^{2,2}} \mathrm{d}\,\mathrm{tr}(\Omega^{-1}\delta\Omega \wedge *\mathcal{A}) = 0$. The variation of the second term is similar

$$\begin{aligned} \delta S_2(\Omega) &= -\int_{\mathbb{R}^{2,2}\times[0,1]} \omega \wedge \mathrm{tr}(\delta\tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}}) \\ &= -\int_{\mathbb{R}^{2,2}\times[0,1]} \omega \wedge \mathrm{tr}(-\tilde{\Omega}^{-1}\delta\tilde{\Omega}\,\tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} + \tilde{\Omega}^{-1}\,\mathrm{d}\,\delta\tilde{\Omega} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}}) \\ &= -\int_{\mathbb{R}^{2,2}\times[0,1]} \mathrm{d}\,\mathrm{tr}(\tilde{\Omega}^{-1}\delta\tilde{\Omega}\;\omega \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}}) = -\int_{\mathbb{R}^{2,2}} \mathrm{tr}(\Omega^{-1}\delta\Omega\;\omega \wedge \mathcal{A} \wedge \mathcal{A}), \end{aligned}$$

where in the last step we have used that $\tilde{\mathcal{A}}(0)=0$, since $\tilde{\Omega}(0)=\text{const.}$, $\tilde{\mathcal{A}}(1)=\mathcal{A}$ since $\tilde{\Omega}(1)=\Omega$ and $\tilde{\mathcal{A}}(|x|\to\infty,\tau)=0$ for $\tau\in[0,1]$. Using $\mathcal{A}\wedge\mathcal{A}=-\mathrm{d}\,\mathcal{A}$ and the anti-self-duality of $\omega$, i.e. $\omega=-{*\omega}$ we obtain

$$\delta S_2(\Omega)=-\int_{\mathbb{R}^{2,2}}\mathrm{tr}(\Omega^{-1}\delta\Omega\;{*\omega}\wedge\mathrm{d}\,\mathcal{A})$$

and finally

$$\delta S_{\mathrm{NS}}(\Omega)=\delta S_1(\Omega)+\delta S_2(\Omega)=\int_{\mathbb{R}^{2,2}}\mathrm{tr}\{\Omega^{-1}\delta\Omega\,(\mathrm{d}\,{*\mathcal{A}}-{*\omega}\wedge\mathrm{d}\,\mathcal{A})\}\stackrel{!}{=}0.$$

Thus, we discover the equation

$$\mathrm{d}\,{*\mathcal{A}}-{*\omega}\wedge\mathrm{d}\,\mathcal{A}=0, \tag{IV.45}$$

which reads in components as (IV.41).

**Dimensional reduction.** Now we have all ingredients in order to discuss the dimensional reduction to 2 + 1 dimensions. We impose the condition that all appearing quantities are independent of $x^3$ or equivalently $\tilde{t}$. Then the equation (IV.41) retains formally its form, i.e.

$$(g^{\mu\nu}-\omega^{\mu\nu})\partial_\mu(\Omega^{-1}\partial_\nu\Omega)=0, \tag{IV.46}$$

but this time $\mu,\nu=1,2,3$ on $\mathbb{R}^{2,1}$ with metric $(g^{\mu\nu})=\mathrm{diag}(+1,+1,-1)$ and $(x^\mu)=(x,y,t)$. Using the explicit form of $\omega$ on $\mathbb{R}^{2,2}$, we can rewrite (IV.46) in the form

$$(g^{\mu\nu}+V_\alpha\epsilon^{\alpha\mu\nu})\partial_\mu(\Omega^{-1}\partial_\nu\Omega)=0, \tag{IV.47}$$

where $(V_\alpha)=(V_x,V_y,V_t)=(1,0,0)$ and $\epsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol. Note that $V_\alpha$ breaks manifestly the Lorentz group $SO(2,1)$ to $GL(1,\mathbb{R})$ generated by boosts in $y$-direction. In terms of real isotropic coordinates (IV.46) becomes

$$\partial_x(\Omega^{-1}\partial_x\Omega)-\partial_v(\Omega^{-1}\partial_u\Omega)=0. \tag{IV.48}$$

The action (IV.42) reduces on $\mathbb{R}^{2,1}$ to

$$\begin{aligned}S_{\mathrm{NS}}(\Omega)=&-\frac{1}{2}\int_{\mathbb{R}^{2,1}}\mathrm{d}t\,\mathrm{d}^2\mathrm{x}\;g^{\mu\nu}\,\mathrm{tr}(\partial_\mu\Omega^{-1}\partial_\nu\Omega)\\&-\frac{1}{3}\int_{\mathbb{R}^{2,1}}\mathrm{d}t\,\mathrm{d}^2\mathrm{x}\int_0^1\mathrm{d}\tau\;\tilde{V}_\alpha\epsilon^{\alpha\mu\nu\sigma}\,\mathrm{tr}\{(\tilde{\Omega}^{-1}\partial_\mu\tilde{\Omega})(\tilde{\Omega}^{-1}\partial_\nu\tilde{\Omega})(\tilde{\Omega}^{-1}\partial_\sigma\tilde{\Omega})\},\end{aligned} \tag{IV.49}$$

where $(\tilde{V}_\alpha)=(V_\alpha,0)$.

**Energy momentum tensor and energy conservation.** As already mentioned, Ward's model exhibits a conserved energy functional. In order to derive the energy we consider the variation of (IV.49) with respect to the metric. Remember that

$$\delta g=g\,g^{\mu\nu}\delta g_{\mu\nu}=-g\,g_{\mu\nu}\delta g^{\mu\nu},$$

where $g$ denotes the determinant of the metric $g_{\mu\nu}$. Therefore, we find

$$\delta(\sqrt{-g}\, g^{\alpha\beta}) = \sqrt{-g}\, \delta g^{\mu\nu} (\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta}). \tag{IV.51}$$

Thus, we obtain from (IV.49)

$$\begin{aligned} \delta S_{\mathrm{NS}}(\Omega) &= -\frac{1}{2} \int_{\mathbb{R}^{2,1}} \mathrm{d}t\, \mathrm{d}^2x\, \delta g^{\mu\nu} (\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta})\, \mathrm{tr}(\partial_\alpha \Omega^{-1} \partial_\beta \Omega) \\ &=: -\frac{1}{2} \int_{\mathbb{R}^{2,1}} \mathrm{d}t\, \mathrm{d}^2x\, \delta g^{\mu\nu} T_{\mu\nu}, \end{aligned}$$

since the second term of (IV.49) does not contribute. Thus, we find for the energy momentum tensor

$$T_{\mu\nu} = (\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta})\, \mathrm{tr}(\partial_\alpha \Omega^{-1} \partial_\beta \Omega). \tag{IV.52}$$

The energy density is then simply

$$\mathcal{E} = T_{33} = \tfrac{1}{2}\, \mathrm{tr}\{(\partial_t \Omega)^\dagger \partial_t \Omega + (\partial_x \Omega)^\dagger \partial_x \Omega + (\partial_y \Omega)^\dagger \partial_y \Omega\}, \tag{IV.53}$$

where we have used the fact that $\Omega \in SU(n)$.

The divergence of $T_{\mu\nu}$ is given by

$$\partial^\mu T_{\mu\nu} = -\tfrac{1}{3} V_\nu \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}\{\Omega^\dagger(\partial_\alpha \Omega)\Omega^\dagger(\partial_\beta \Omega)\Omega^\dagger(\partial_\gamma \Omega)\}. \tag{IV.54}$$

This can be seen as follows. Let $\mathcal{A}_\mu = \Omega^\dagger \partial_\mu \Omega$. Then the energy momentum tensor (IV.52) reads

$$T_{\mu\nu} = -(\mathrm{tr}\, \mathcal{A}_\mu \mathcal{A}_\nu - \tfrac{1}{2} g_{\mu\nu}\, \mathrm{tr}\, \mathcal{A}^\alpha \mathcal{A}_\alpha)$$

and the equation of motion (IV.47) translates into

$$\partial^\mu \mathcal{A}_\mu = -V_\alpha \epsilon^{\alpha\beta\gamma} \partial_\beta \mathcal{A}_\gamma.$$

Moreover, we need the identity

$$\partial_\mu \mathcal{A}_\nu = -\mathcal{A}_\mu \mathcal{A}_\nu + \Omega^\dagger \partial_\mu \partial_\nu \Omega.$$

Using these ingredients, we immediatelly arrive at

$$\partial^\mu T_{\mu\nu} = -V_\alpha \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}(\mathcal{A}_\beta \mathcal{A}_\gamma \mathcal{A}_\nu) = -\tfrac{1}{2} V_\alpha \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}([\mathcal{A}_\beta, \mathcal{A}_\gamma]\mathcal{A}_\nu).$$

The quantity $\mathrm{tr}([\mathcal{A}_\beta, \mathcal{A}_\gamma]\mathcal{A}_\nu)$ is totally antisymmetric what follows from $\mathrm{tr}([A,B]C) = \mathrm{tr}(A[B,C])$ and hence we can write $\mathrm{tr}([\mathcal{A}_\beta, \mathcal{A}_\gamma]\mathcal{A}_\nu) = C\epsilon_{\beta\gamma\nu}$, where $C$ is some function of spacetime. Inverting this relation yields

$$C = -\tfrac{1}{6} \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}([\mathcal{A}_\alpha, \mathcal{A}_\beta]\mathcal{A}_\gamma).$$

Therefore we find

$$\partial^\mu T_{\mu\nu} = -\tfrac{1}{2} V_\alpha \epsilon^{\alpha\beta\gamma} C\, \epsilon_{\beta\gamma\nu} = V_\nu\, C = -\tfrac{1}{3} V_\nu \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}\{\Omega^\dagger(\partial_\alpha \Omega)\Omega^\dagger(\partial_\beta \Omega)\Omega^\dagger(\partial_\gamma \Omega)\}$$

and conclude, that the energy is conserved iff $V_3 = 0$. However, this is exactly what we have assumed from the very beginning.

## § IV.3 Construction of $SU(2)$ Soliton and Antisoliton Solutions

Having discussed the relation between SDYM theory on $\mathbb{R}^{2,2}$ and the chiral model on $\mathbb{R}^{2,1}$, we shall now proceed further and construct explicit soliton solutions. Note that as it was shown by Leese [132], this model exhibits besides soliton configurations also plane wave solutions. The latter ones are not of our interest and we refer the reader to [132]. For the remaining part of this chapter we restrict the gauge group to $SU(2)$. Again, we postpone the explicit derivations to the noncommutative setup (cf. chapter VIII).

**Ansatz and solution.** Starting point is the two linear partial differential equations (IV.39) reduced to $2+1$ dimensions, i.e.

$$(\zeta\partial_x - \partial_u)\psi = A\psi \qquad \text{and} \qquad (\zeta\partial_v - \partial_x)\psi = B\psi, \tag{IV.61}$$

where we have renamed $\mathcal{A}_u$ ($\mathcal{A}_x$) to $A$ ($B$). Now we consider an ansatz of the form [195]

$$\tilde{\psi}(x,y,t,\zeta) = \chi(x,y,t,\zeta)\psi(x,y,t,\zeta) \qquad \text{with} \qquad \chi = 1 + \sum_{\alpha=1}^{s}\sum_{k=1}^{m} \frac{R_{\alpha k}}{(\zeta-\mu_k)^\alpha}, \tag{IV.62}$$

where the $R_{\alpha k}$'s are $2\times 2$ matrices depending only on spacetime and $\mu_k \in \mathbb{C}$ for $k = 1,\dots,m$. Suppose, that $R_{\alpha k} = n^{\alpha k} \otimes {}^t m^{\alpha k}$, where ${}^t(n_a^{\alpha k}) = (n_1^{\alpha k}, n_2^{\alpha k})$ and ${}^t(m_a^{\alpha k}) = (m_1^{\alpha k}, m_2^{\alpha k})$. Let us consider only poles of first order and choose the trivial solution $(\psi, A, B) = (1,0,0)$ as the seed solution. Then we obtain

$$\psi_{ab} = \delta_{ab} + \sum_{k=1}^{m} \frac{n_a^k m_b^k}{\zeta - \mu_k}. \tag{IV.63}$$

Note that we have again renamed the dressing factor to be conform with the literature. Due to the reality condition (IV.33) the product $\sum_{b=1}^{2}(\psi(x,\zeta))_{ab}((\psi(x,\bar{\zeta}))^\dagger)_{bc}$ should be $\delta_{ac}$. Therefore the poles at $\bar{\mu}_k$ must be removable, i.e.

$$\begin{aligned}
\operatorname*{Res}_{\zeta=\bar{\mu}_k}\{(\psi(x,\zeta))_{ab}((\psi(x,\bar{\zeta}))^\dagger)_{bc}\} &= \operatorname*{Res}_{\zeta=\bar{\mu}_k}\left\{\sum_{p=1}^{m}\frac{\bar{m}_a^p\bar{n}_c^p}{\zeta-\bar{\mu}_p} + \sum_{b=1}^{2}\sum_{p,q=1}^{m}\frac{n_a^q m_b^q \bar{m}_b^p \bar{n}_c^p}{(\zeta-\bar{\mu}_p)(\zeta-\mu_q)}\right\}\\
&= \lim_{\zeta\to\bar{\mu}_k}(\zeta-\bar{\mu}_k)\left\{\sum_{p=1}^{m}\frac{\bar{m}_a^p\bar{n}_c^p}{\zeta-\bar{\mu}_p} + \sum_{b=1}^{2}\sum_{p,q=1}^{m}\frac{n_a^q m_b^q \bar{m}_b^p \bar{n}_c^p}{(\zeta-\bar{\mu}_p)(\zeta-\mu_q)}\right\}\\
&= \bar{m}_a^k\bar{n}_c^k + \sum_{b=1}^{2}\sum_{q=1}^{m} n_a^q \bar{n}_c^k\,(\Gamma^{-1})^{kq} \overset{!}{=} 0,
\end{aligned}$$

where we have assumed that $\bar{\mu}_k \neq \mu_p$ and $\mu_k \neq \mu_p$ for all $k, p = 1,\dots,m$ and defined

$$(\Gamma^{-1})^{kq} := \sum_{a=1}^{2}\frac{\bar{m}_a^k \bar{m}_a^q}{\bar{\mu}_k - \mu_q}. \tag{IV.64}$$

By considering the residue at $\zeta = \mu_k$, we obtain the complex conjugate equation. Thus, we discover the following relation

$$n_a^q = -\sum_{k=1}^{m} \Gamma^{qk} \bar{m}_a^k \qquad \text{with} \qquad \sum_{k=1}^{m} (\Gamma^{-1})^{pq} \Gamma^{qk} = \delta^{pk}. \tag{IV.65}$$

Similarly, since the matrices $A$ and $B$ are independent of $\zeta$, all proper residues in the equations (IV.61) have to vanish. This yields differential equations for the $m_a^k$'s, namely

$$(\mu_k \partial_x - \partial_u) m_a^k = 0 = (\mu_k \partial_v - \partial_x) m_a^k, \tag{IV.66}$$

which can be solved if the $m_a^k$'s depend meromorphically on

$$\omega_k := x + \mu_k u + \mu_k^{-1} v = x + \tfrac{1}{2}(\mu_k - \mu_k^{-1}) y + \tfrac{1}{2}(\mu_k + \mu_k^{-1}) t. \tag{IV.67}$$

There is an obvious homogeneity property. Namely, if we rescale $(m_a^k)$ by some function $\lambda_k$ which also depends meromorphically on $\omega_k$, then $\psi$ does not change. Hence, without loss of generality we may set $m_1^k = 1$ and $m_2^k = f_k$.

Thus, the solution depends on $m$ constants $\mu_k$ and $m$ meromorphic functions $f_k = f_k(\omega_k)$. The determinant of $\psi$ is given by

$$\alpha := \det(\psi) = \prod_{k=1}^{m} \frac{\bar{\mu}_k}{\mu_k}, \tag{IV.68}$$

as it was shown in [195]. For $m = 1$ this is rather obvious and for $m > 1$ one may use induction in order to prove (IV.68). Then the final solution looks like

$$(\Omega^{-1})_{ab} = (\psi(x, y, t, \zeta = 0))_{ab} = \frac{1}{\sqrt{\alpha}} \left\{ \delta_{ab} + \sum_{p,q=1}^{m} \frac{1}{\mu_k} \bar{m}_a^l m_b^k \, \Gamma^{kl} \right\}. \tag{IV.69}$$

**One-soliton solution.** As a simple example consider the case of $m = 1$. Then $\Gamma^{-1}$ is just a $1 \times 1$ matrix

$$\Gamma^{-1} = \frac{1}{\bar{\mu} - \mu} (1 + |f|^2) \tag{IV.70}$$

and $\Omega$ takes the form [195]

$$\Omega = \frac{1}{|\mu|(1 + |f|^2)} \begin{pmatrix} \mu + \bar{\mu}|f|^2 & (\mu - \bar{\mu}) f \\ (\mu - \bar{\mu}) \bar{f} & \bar{\mu} + \mu |f|^2 \end{pmatrix}, \tag{IV.71}$$

where we assume that $\mu \in \mathbb{C} \setminus \mathbb{R}$ in order to avoid potential singularities. Let $\mu = m\, e^{i\phi}$. Then the energy density (IV.53) turns into

$$\mathcal{E} = \frac{2(1 + m^2)^2 \sin^2(\phi)}{m^2} \frac{|f'|^2}{(1 + |f|^2)^2} \qquad \text{with} \qquad f' = \partial_\omega f. \tag{IV.72}$$

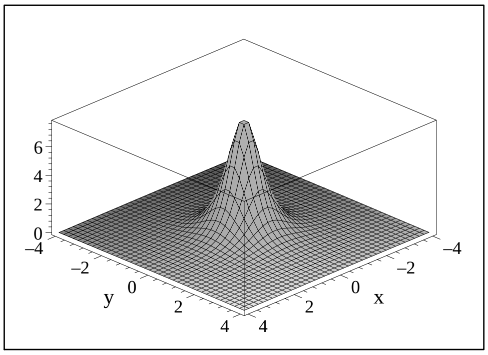


Figure IV.1. *Energy distribution $\mathcal{E}$ for a static $SU(2)$ soliton.*

If $\mu = \pm i$ the coordinate $\omega$ (IV.67) reduces to $\omega = x \pm iy$, i.e. the energy density becomes time independent and hence the solution is static. Let $f$ be an arbitrary rational function of degree $N$. Then the energy $E = \int_{\mathbb{R}^2} \mathrm{d}^2\mathrm{x}\, \mathcal{E}$ is $E = 8\pi N$.

Consider the case of $f(\omega) = \omega$ and $\mu \neq \pm i$. Then the maximum of $\mathcal{E}$ is given at $\Re\{\omega\} = \Im\{\omega\} = 0$. The real part of $\omega$ reads $\Re\{\omega\} = x + \frac{1}{2}\cos(\phi)((m - m^{-1})y + (m + m^{-1})t)$, while the imaginary part is $\Im\{\omega\} = \frac{1}{2}\sin(\phi)((m + m^{-1})y + (m - m^{-1})t)$. Therefore, we find

$$(x, y)_{\max} = \left(-\frac{2m}{m^2+1}\cos(\phi)t, \frac{1-m^2}{1+m^2}t\right), \tag{IV.73}$$

from which it follows that the soliton moves with a constant velocity given by the expression

$$(v_x, v_y) = \left(-\frac{2m}{m^2+1}\cos(\phi), \frac{1-m^2}{1+m^2}\right) \qquad \Longrightarrow \qquad v^2 = 1 - \frac{4m^2\sin^2(\phi)}{(1+m^2)^2} \tag{IV.74}$$

and hence $v < 1$, i.e. the soliton can travel at any speed less than that of light. Let $\gamma = (1 - v^2)^{-1/2}$ be the usual relativistic factor. After a few lines of algebra one obtains for the energy in the non-static case $E = 8\pi\gamma\sin^2(\phi)$. Let us examine two special cases: For $\phi = \pm\frac{1}{2}\pi$ the soliton moves along the $y$-axis and its behavior is relativistic, i.e. the total energy is rest energy $8\pi$ multiplied by $\gamma$. On the other hand, if $m = 1$ the soliton moves along the $x$-axis and its behavior is not relativistic at all, since its energy $8\pi|\sin(\phi)|$ is less than the rest energy. This was initially ment by the statement that the Lorentz group $SO(2, 1)$ is broken to $GL(1, \mathbb{R})$ generated by boosts in $y$-direction.

**Soliton-soliton scattering.** The next step is to try higher order poles in the ansatz (IV.62). The simplest scattering configuration arises from a $\psi$ with a second order pole [198], i.e.

$$\psi = 1 + \frac{R_1}{\zeta - \mu} + \frac{R_2}{(\zeta - \mu)^2}. \tag{IV.75}$$

The reality condition (IV.33) can only be satisfied for all $\zeta$ if $\psi$ takes the form [198]

$$\psi = \left(1 - \frac{(\bar\mu - \mu)}{\zeta - \mu}\frac{n^\dagger \otimes n}{|n|^2}\right)\left(1 - \frac{(\bar\mu - \mu)}{\zeta - \mu}\frac{m^\dagger \otimes m}{|m|^2}\right), \tag{IV.76}$$

where $n, m$ are two-dimensional row vectors. Moreover, the equations (IV.61) yield conditions on the vectors $n$ and $m$. One way of obtaining these conditions is a limit of the simple pole case, i.e. consider two first-order poles at $\mu + \epsilon$ and $\mu - \epsilon$ and take the limit $\epsilon \to 0$. This eventually leads to [198, 94]

$$n = (1 + |f|^2)(1, f) + (\bar{\mu} - \mu)g(\bar{f}, -1) \qquad \text{and} \qquad m = (1, f), \tag{IV.77}$$

where $f = f(\omega)$ and $g = (u - \mu^{-2}v)f'(\omega) + h(\omega)$ and $h$ is some arbitrary function depending meromorphically on $\omega$.

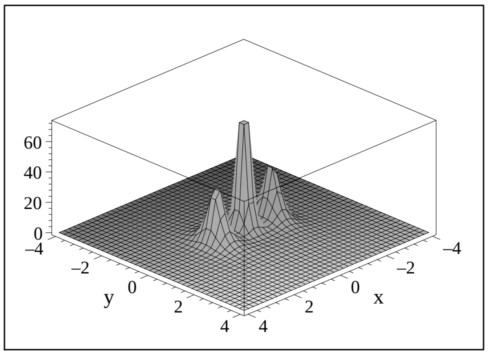


Figure IV.2. *Energy distribution $\mathcal{E}$ for three $SU(2)$ solitons showing right angle scattering at time $t = -1$.*

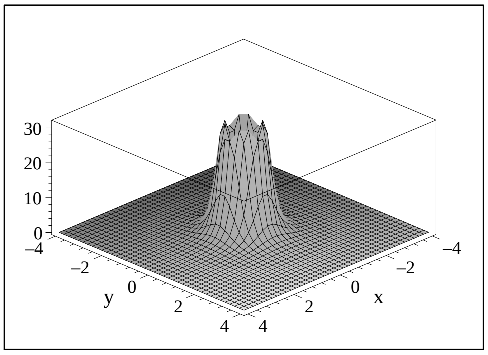


Figure IV.3. *Energy distribution $\mathcal{E}$ for three $SU(2)$ solitons showing right angle scattering at time $t = 0$.*

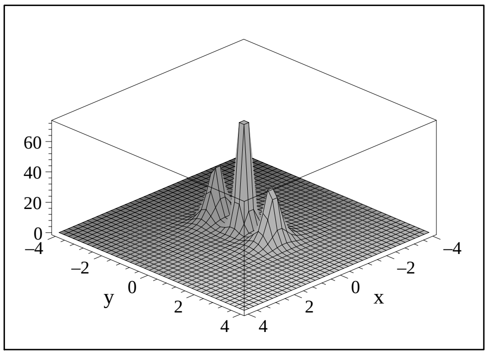


Figure IV.4. *Energy distribution $\mathcal{E}$ for three $SU(2)$ solitons showing right angle scattering at time $t = 1$.*

To exemplify our situation, we consider $\mu = i$, $f(\omega) = \omega^2$ and $h(\omega) = \omega^3$. Thus, $g = z(2t + z^2)$. The energy distribution (IV.53) for this configuration computes to [94]

$$\begin{aligned}\mathcal{E} =& 32[r^{12} + 2r^2(r^8 + r^6 + 1) + 36t^2r^8 + 4r^6 + 9r^4 + 8t^2r^4 + 4t^2 + 12t(x^{10} - y^{10}) \\ &+ 4t(x^2 - y^2)(3 + 2x^2y^2 + 6x^4y^4) + 4t(x^6 - y^6)(9x^2y^2 - 2) - r^{10}]/[r^8 + 4r^6 \\ &+ 2r^4 + 16tr^2(t + x^2 - y^2) + 1]^2,\end{aligned} \tag{IV.78}$$

where $r^2 = x^2 + y^2$. Obviously, $\mathcal{E}$ is symmetric under the interchange $t \mapsto -t$ and $x \rightleftharpoons y$. This configuration describes a right angle scattering configuration of three solitons. It consists of one static soliton sitting at the origin and two moving solitons (see figures IV.2-IV.4). The total energy is $E = 24\pi$.

In general, one may consider polynomials of type $f(\omega) = \omega^p$ and $h(\omega) = \omega^q$ with $p, q \in \mathbb{N}$. If $q > p$ the configuration consists of $(p-1)$ static solitons accompanied by $N = q - p + 1$ solitons accelerating towards the center and scattering at an angle of $\pi/N$. For more details see [94, 96].

**Soliton-antisoliton scattering.** In this last paragraph we finally discuss the case of soliton-antisoliton scattering configurations. For this, however, as the name already implies we need some quantity which distinguishes solitons from antisolitons. The following discussion is borrowed from [94].

The unmodified chiral model, i.e. (IV.47) with $(V_\alpha) = 0$ is equivalent to the $O(4)$ $\sigma$-model, as it was shown in [183] via the relation $\Omega = \Omega_\mu \sigma^\mu = i\mathbf{\Omega}\sigma + \Omega_4$, where the $\sigma^i$'s are the Pauli matrices defined by (II.12). Since $\Omega \in SU(2)$, we have $|\Omega|^2 = \Omega_\mu \Omega^\mu = 1$. Moreover, the only static finite energy solutions of the $O(4)$ $\sigma$-model will correspond to the embedding of the $O(3)$ $\sigma$-model [25]. Hence, the only static solution which we shall consider correspond to $O(3)$ embeddings, because for the static one-soliton configuration the term in (IV.47) proportional to $V_\alpha$ vanishes, so that the system behaves like the $O(4)$ model, for which the $O(3)$ embedding is totally geodesic. In order to have finite energy solution, the field must take the same value at spatial infinity, i.e. $\mathbb{R}^2$ is compactified to $S^2$. In the $O(3)$ $\sigma$-model case, the field $\mathbf{\Omega}$ is a map from $S^2$ to $S^2$ and because of the homotopy relation $\pi_2(S^2) \cong \mathbb{Z}$ we have a winding number characterizing these maps. The charge looks like [166]

$$Q = \frac{1}{8\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\mathrm{x}\, \epsilon^{ij}\, \mathbf{\Omega} \cdot (\partial_i \mathbf{\Omega} \times \partial_j \mathbf{\Omega}), \tag{IV.79}$$

where $i, j = 1, 2$. However, in the $O(4)$ case the field defines a map from $S^2 \to S^3$ and therefore $\pi_2(S^3) \cong \{e\}$ meaning that there is no winding number. But still, we can define a suitable topological quantity, if the considered soliton configuration correspond to some embedding of $O(3)$ space into $O(4)$. Consider an $O(4)$ configuration which corresponds to an $O(3)$ embedding at some time, i.e. choose $\Omega_4 = 0$. Thus, at this time the field is restricted to an $S^2$ equator of $S^3$. Moreover, we shall assume that the field never maps to $\{-1, 1\}$. Therefore we have $S^3 \setminus \{-1, 1\} \cong S^2 \times \mathbb{R}$ and hence $\pi_2(S^3 \setminus \{-1, 1\}) \cong \pi_2(S^2 \times \mathbb{R}) \cong \pi_2(S^2) \times \pi_2(\mathbb{R}) \cong \mathbb{Z}$; cf. section § II.2.3. Thus, we conclude that there is a winding number. The expression is

$$Q = \frac{1}{8\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\mathrm{x}\, \epsilon^{ij}\, \hat{\mathbf{\Omega}} \cdot (\partial_i \hat{\mathbf{\Omega}} \times \partial_j \hat{\mathbf{\Omega}}), \tag{IV.80}$$

where $|\hat{\mathbf{\Omega}}| = 1$.

Let us now discuss an example. Consider the ansatz

$$\psi = 1 + \frac{n^1 \otimes m^1}{\zeta - i} + \frac{n^2 \otimes m^2}{\zeta + i}, \tag{IV.81}$$

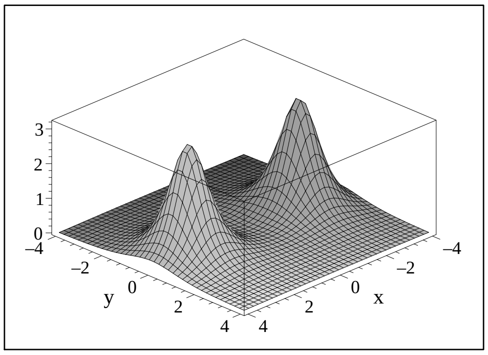


Figure IV.5. *Energy distribution $\mathcal{E}$ for one $SU(2)$ soliton and one antisoliton showing right angle scattering at time $t = -5$.*

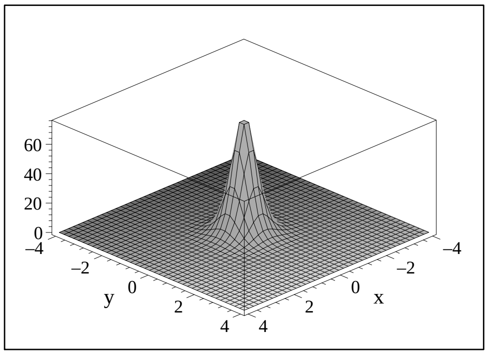


Figure IV.6. *Energy distribution $\mathcal{E}$ for one $SU(2)$ soliton and one antisoliton showing right angle scattering at time $t = 0$.*

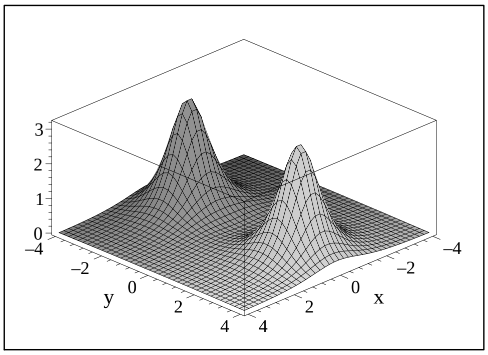


Figure IV.7. *Energy distribution $\mathcal{E}$ for one $SU(2)$ soliton and one antisoliton showing right angle scattering at time $t = 5$.*

where $n^k, m^k$ are two-vectors. In order to satisfy the reality condition (IV.33) and the equations (IV.62) we have to put all proper residues to zero, what yields again equations for $n^k$ and $m^k$. Also in this case, we can restrict ourselves to the first order pole ansatz by taking $\mu_1 = i + \epsilon$ and $\mu_2 = -i - \epsilon$ and considering the limit $\epsilon \to 0$. This eventually leads to [94]

$$n^1 = \frac{2i(1+|f|^2)}{(1+|f|^2)^2+|w|^2}\bar{m}^1 + \frac{2\bar{w}}{(1+|f|^2)^2+|w|^2}\bar{m}^2, \tag{IV.82a}$$

$$n^2 = -\frac{2w}{(1+|f|^2)^2+|w|^2}\bar{m}^1 - \frac{2i(1+|f|^2)}{(1+|f|^2)^2+|w|^2}\bar{m}^2, \tag{IV.82b}$$

where $m^1 = (1, f)$, $m^2 = (-\bar{f}, 1)$, $w = \bar{h}f^2 + 2tf'$ and $f = f(z)$ and $h = h(\bar{z})$ are arbitrary rational functions. In this case the solution takes the form

$$\begin{aligned}\Omega &= \frac{1}{(1+|f|^2)^2+|w|^2}\times\\ &\times\begin{pmatrix} |w|^2 + 2i(f\bar{w}+\bar{f}w) - (1+|f|^2)^2 & -2i(w - f^2\bar{w}) \\ -2i(\bar{w}-\bar{f}^2 w) & |w|^2 - 2i(f\bar{w}+\bar{f}w) - (1+|f|^2)^2 \end{pmatrix}.\end{aligned} \tag{IV.83}$$

Taking $f(z) = z^p$ and $h(\bar{z}) = \bar{z}^q$ for $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$, one obtains a configuration consisting of $(2p + q)$ lumps sitting at arbitrary positions in the $(x, y)$-plane. The total energy for this configuration is $E = (2p + q)8\pi$.

Let us take $f(z) = z$ and $h(\bar{z}) = 1$. This configuration consists of two lumps. The energy density (IV.53) is given by [94]

$$\mathcal{E} = 16\frac{2r^4 + 4r^2 + 4t^2(1 + r^2) - 4t(x^2 - y^2) + 1}{(2r^4 + 2r^2 + 4t(x^2 - y^2) + 4t^2 + 1)^2}. \tag{IV.84}$$

By varying $t$ we see the two lumps accelerating symmetrically towards each other along the $x$-axis, interacting at the origin and decelerating to infinity along the $y$-axis (see figures IV.5-IV.6). Thus, we have again a right angle scattering configuration. But this time, we have one soliton and one antisoliton, as one may check by computing the topological charge (IV.80), which is throughout the whole scattering process zero. The charge density $q$, however, has an almost identical distribution to that of the energy density. But instead of having two maxima at the positions of the solitons, it has one global minimum ($q < 0$) and one global maximum ($q > 0$). For a more detailed discussion see [94].

# Chapter V

## Seiberg-Witten Theory

In this chapter we discuss some mathematical aspects of Seiberg-Witten theory. We write down the monopole equations and the corresponding action functional producing the equations. Moreover, some properties of the moduli space of solutions are reviewed and we point out how one can construct topological invariants. Because of the very rich nature of this field of mathematics we can only list a very few results and for a more detailed discussion of this subject we refer the reader to [206, 179, 141, 137]. The discussion presented in this chapter is mainly borrowed from these references.

## § V.1 Action and Equations of Motion

**Preliminaries.** Let $M$ be an orientable Riemannian four-manifold with a fixed spin$^c$ structure $(W, \Gamma)$. In subsection § II.2.4 we have introduced the map $\rho : \Omega^*(M) \to \mathrm{End}(W)$ given by (II.97). We have shown that its restriction to two-forms yields the isomorphisms $\rho^\pm : \mathfrak{u}(1) \otimes \Omega^2_\pm(M) \to \mathrm{Herm}_0(W^\pm)$ between the imaginary (anti-)self-dual two-forms and Hermitian traceless endomorphisms of $W^\pm$. At this stage we want to be more explicit about the functional form of the inverse mapping

$$\sigma^+ := (\rho^+)^{-1} : \mathrm{Herm}_0(W^+) \to \mathfrak{u}(1) \otimes \Omega^2_+(M). \tag{V.1}$$

Let $\{e_i\}$ be a local orthonormal basis of $TM$ and $\{e^i\}$ its dual basis on $T^*M$. Moreover, let us denote the standard inner product on the fibers of $W^+$ by $\langle \cdot, \cdot \rangle$ which is given by (I.7). The basis $\{e_i\}$ acts on sections of $W^+$ via Clifford multiplication, i.e. via the standard spin representation given by the complexified Dirac map (II.93). For brevity we shall simply write

$e_i$ instead of $\gamma(e_i)$. Then one easily derives the following relations

$$\langle\Phi, e_1e_2\Phi\rangle = \langle\Phi, e_3e_4\Phi\rangle = \begin{pmatrix}\bar\phi_1 & \bar\phi_2\end{pmatrix}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -i(|\phi_1|^2 - |\phi_2|^2), \tag{V.2a}$$

$$\langle\Phi, e_1e_3\Phi\rangle = \langle\Phi, e_4e_2\Phi\rangle = \begin{pmatrix}\bar\phi_1 & \bar\phi_2\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \bar\phi_1\phi_2 - \bar\phi_2\phi_1, \tag{V.2b}$$

$$\langle\Phi, e_1e_4\Phi\rangle = \langle\Phi, e_2e_3\Phi\rangle = \begin{pmatrix}\bar\phi_1 & \bar\phi_2\end{pmatrix}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -i(\bar\phi_1\phi_2 + \bar\phi_2\phi_1), \tag{V.2c}$$

for $\Phi\in\Gamma(M, W^+)$. We therefore define the map $\sigma^+$ by

$$\sigma^+(\Phi\otimes\Phi^\dagger)_0 := \sigma^+(\Phi\otimes\Phi^\dagger - \tfrac12|\Phi|^2\,\mathrm{id}) := -\tfrac14\sum_{i<j}\langle\Phi, e_ie_j\Phi\rangle\, e^i\wedge e^j, \tag{V.3}$$

where $\Phi\otimes\Phi^\dagger - \frac12|\Phi|^2\,\mathrm{id}$ is a Hermitian traceless endomorphism of $W^+$.

### § V.1.1 Unperturbed Seiberg-Witten Equations

**Unperturbed Seiberg-Witten monopole equations and action functional.** Let $M$ be an orientable, compact Riemannian four-manifold equipped with a spin$^c$ structure $(W,\Gamma)$ and a spin$^c$ connection $\nabla_{\mathcal{A}}$ generated by a virtual connection one-form $\mathcal{A}\in\mathcal{A}(\Gamma)$ on $L_\Gamma^{1/2}$. Then the Seiberg-Witten equations are given by (cf. chapter I)

$$\mathcal{F}^+_{\mathcal{A}} = \sigma^+(\Phi\otimes\Phi^\dagger)_0, \tag{V.4a}$$

$$\mathcal{D}_{\mathcal{A}}\Phi = 0, \tag{V.4b}$$

where $\mathcal{D}_{\mathcal{A}}$ denotes the spin$^c$ Dirac operator given by (II.116), $\Phi\in\Gamma(M,W^+)$ and $\mathcal{F}^+_{\mathcal{A}}$ denotes the self-dual part of the curvature $\mathcal{F}_{\mathcal{A}} = \mathrm{d}\,\mathcal{A}$.

As usual, the equations of motion of a theory are related to some variational problem. The Seiberg-Witten action functional reproducing the equations (V.4) is defined by

$$SW(\mathcal{A},\Phi) = \int_M \mathrm{dVol}\left\{|\mathcal{D}_{\mathcal{A}}\Phi|^2 + 2|\mathcal{F}^+_{\mathcal{A}} - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2\right\}. \tag{V.5}$$

Obviously, the functional (V.5) is positive semi-definite, and the absolute minimum is achieved iff the Seiberg-Witten equations (V.4) are satisfied. The Euler-Lagrange equations for this functional are

$$\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi - \rho^+(\mathcal{F}_{\mathcal{A}})\Phi + \tfrac12\Phi|\Phi|^2 = 0, \tag{V.6a}$$

$$\mathrm{d}^\dagger\{\mathcal{F}^+_{\mathcal{A}} - \sigma^+(\Phi\otimes\Phi^\dagger)_0\} - \tfrac{i}{2}\Im\{\langle\mathcal{D}_{\mathcal{A}}\Phi, e_i\Phi\rangle\}e^i = 0, \tag{V.6b}$$

where $\mathrm{d}^\dagger = -{*}\,\mathrm{d}\,{*}$. It can readily be seen, that these equations are satisfied if (V.4) hold true. To derive them, however, we have to do some work. The functional (V.5) can be rewritten as

$$\begin{aligned} SW(\mathcal{A},\Phi) &= \int_M \mathrm{dVol}\,\langle\Phi,\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi\rangle - 2\int_M(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)\wedge*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)\\ &= \int_M \mathrm{dVol}\left\{\langle\Phi,\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi\rangle + 2|\mathcal{F}^+_{\mathcal{A}}|^2 + 2|\sigma^+|^2\right\} + 4\int_M \mathcal{F}^+_{\mathcal{A}}\wedge*\sigma^+. \end{aligned}$$

Using the explicit form (V.3) of $\sigma^+$, one easily verifies that

$$\int_M \mathrm{dVol}\ |\sigma^+|^2 = -\int_M \sigma^+\wedge*\sigma^+ = \frac{1}{8}\int_M \mathrm{dVol}\ |\Phi|^4.$$

Moreover, we have

$$\begin{aligned} \int_M \mathcal{F}^+_{\mathcal{A}}\wedge*\sigma^+ &= -\int_M\left\{\tfrac{1}{2}\mathcal{F}^+_{ij}e^i\wedge e^j\right\}\wedge*\left\{\tfrac{1}{8}\langle\Phi,e_ke_l\Phi\rangle e^k\wedge e^l\right\}\\ &= -\frac{1}{32}\int_M \mathrm{dVol}\ \mathcal{F}^+_{ij}\,\langle\Phi,e_ke_l\Phi\rangle\,\epsilon^{ijmn}\,\epsilon^{kl}{}_{mn}\\ &= -\frac{1}{4}\int_M \mathrm{dVol}\,\langle\Phi,\underbrace{\sum_{i<j}\mathcal{F}^+_{ij}e_ie_j}_{=\rho^+(\mathcal{F}_{\mathcal{A}})}\ \Phi\rangle = -\frac{1}{4}\int_M \mathrm{dVol}\,\langle\Phi,\rho^+(\mathcal{F}_{\mathcal{A}})\,\Phi\rangle. \end{aligned}$$

Therefore, we discover

$$SW(\mathcal{A},\Phi) = \int_M \mathrm{dVol}\left\{\langle\Phi,\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi\rangle + 2|\mathcal{F}^+_{\mathcal{A}}|^2 + \tfrac{1}{4}\langle\Phi,\Phi\rangle^2 - \langle\Phi,\rho^+(\mathcal{F}_{\mathcal{A}})\,\Phi\rangle\right\}. \tag{V.8}$$

Now we consider the variation $\Phi\mapsto\Phi+\delta\Phi$ yielding

$$\delta_\Phi SW(\mathcal{A},\Phi) = \int_M \mathrm{dVol}\left\{\langle\delta\Phi,\mathcal{D}^\dagger_{\mathcal{A}}\mathcal{D}_{\mathcal{A}}\Phi + \tfrac{1}{2}|\Phi|^2\Phi - \rho^+(\mathcal{F}_{\mathcal{A}})\,\Phi\rangle\right\} \overset{!}{=} 0$$

and obtain equation (V.6a). In order to prove equation (V.6b) we need to compute the variation $\mathcal{A}\mapsto\mathcal{A}+\delta\mathcal{A}$. This yields

$$\begin{aligned} \delta_{\mathcal{A}}\int_M \mathrm{dVol}\ |\mathcal{D}_{\mathcal{A}}\Phi|^2 &= \delta_{\mathcal{A}}\int_M \mathrm{dVol}\,\langle\mathcal{D}_{\mathcal{A}}\Phi,\mathcal{D}_{\mathcal{A}}\Phi\rangle = \delta_{\mathcal{A}}\int_M \mathrm{dVol}\,\langle e_i\nabla_i\Phi,e_j\nabla_j\Phi\rangle\\ &= \int_M \mathrm{dVol}\,\{\langle e_i\delta\mathcal{A}_i\Phi,\mathcal{D}_{\mathcal{A}}\Phi\rangle + \langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\delta\mathcal{A}_i\Phi\rangle\}\\ &= \int_M \mathrm{dVol}\,\delta\mathcal{A}_i\left\{\langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\Phi\rangle - \overline{\langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\Phi\rangle}\right\}\\ &= \int_M \delta\mathcal{A}\wedge*(2i\,\Im\{\langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\Phi\rangle\}e^i) \end{aligned}$$

as well as

$$\begin{aligned} \delta_{\mathcal{A}}\int_M \mathrm{dVol}\ 2|\mathcal{F}^+_{\mathcal{A}}-\sigma^+|^2 &= -2\delta_{\mathcal{A}}\int_M(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)\wedge*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+) = -4\int_M \delta_{\mathcal{A}}\mathcal{F}^+_{\mathcal{A}}\wedge*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)\\ &= -4\int_M \delta_{\mathcal{A}}\mathcal{F}_{\mathcal{A}}\wedge*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+) = -4\int_M \mathrm{d}\,\delta\mathcal{A}\wedge*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)\\ &= -4\int_M \delta\mathcal{A}\wedge\mathrm{d}\,{*}(\mathcal{F}^+_{\mathcal{A}}-\sigma^+) \end{aligned}$$

and finally

$$\delta_{\mathcal{A}} SW(\mathcal{A},\Phi) = 4\int_M \delta\mathcal{A}\wedge\left\{-\,\mathrm{d}*(\mathcal{F}^+_{\mathcal{A}}-\sigma^+)+\tfrac{i}{2}*(\Im\{\langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\Phi\rangle\}e^i)\right\}\stackrel{!}{=}0,$$

i.e. we obtain (V.6b). Remember that $**\omega=(-1)^{p(n-p)}\omega$ if $\omega\in\Omega^p(M)$.

The functional (V.8) can be reduced further to

$$SW(\mathcal{A},\Phi)=E(\mathcal{A},\Phi)+8\pi^2 Q, \tag{V.11}$$

with

$$E(\mathcal{A},\Phi)=\int_M \mathrm{dVol}\left\{|\nabla_{\mathcal{A}}\Phi|^2+|\mathcal{F}_{\mathcal{A}}|^2+\tfrac{1}{4}|\Phi|^2(R+|\Phi|^2)\right\}. \tag{V.12}$$

In this derivation, $\int_M \mathrm{dVol}\, 2|\mathcal{F}^+_{\mathcal{A}}|^2=\int_M \mathrm{dVol}\,|\mathcal{F}_{\mathcal{A}}|^2-\int_M\mathcal{F}_{\mathcal{A}}\wedge\mathcal{F}_{\mathcal{A}}$ and Weitzenböck's formula (II.122) have been used. The topological charge $Q$ is given by

$$Q=-\frac{1}{8\pi^2}\int_M\mathcal{F}_{\mathcal{A}}\wedge\mathcal{F}_{\mathcal{A}}. \tag{V.13}$$

This then implies the inequality

$$E(\mathcal{A},\Phi)\geq -8\pi^2 Q, \tag{V.14}$$

and the equality is only achieved on solutions to the Seiberg-Witten equations. The functional $E(\mathcal{A},\Phi)$ is called the Seiberg-Witten energy functional. In a similar manner one obtains by varying $E(\mathcal{A},\Phi)$ the following equations of motion

$$\nabla^\dagger_{\mathcal{A}}\nabla_{\mathcal{A}}\Phi+\tfrac{1}{4}R\Phi+\tfrac{1}{2}\Phi|\Phi|^2 = 0, \tag{V.15a}$$
$$\mathrm{d}^\dagger\,\mathcal{F}^+_{\mathcal{A}}-\tfrac{i}{2}\Im\{\langle\nabla_i\Phi,\Phi\rangle\}e^i = 0, \tag{V.15b}$$

which, of course, are identical to (V.6). The equivalence of (V.6a) and (V.15a) can readily be checked using Weitzenböck's formula (II.122). Showing the equivalence between (V.6b) and (V.15b) is a bit more involving, since one needs to compute $-\,\mathrm{d}^\dagger\,\sigma^+$ which is

$$-\,\mathrm{d}^\dagger\,\sigma^+=-\tfrac{1}{8}*\mathrm{d}*\langle\Phi,e_ie_j\Phi\rangle\,e^i\wedge e^j=-\tfrac{1}{8}\,\partial_i\langle\Phi,(e_ie_j-e_je_i)\Phi\rangle\,e^j.$$

The expression for $\partial_i\langle\Phi,e_ie_j\Phi\rangle$ is given by

$$\begin{aligned}\partial_i\langle\Phi,e_ie_j\Phi\rangle &= \langle\nabla_i\Phi,e_ie_j\Phi\rangle+\langle\Phi,e_ie_j\nabla_i\Phi\rangle\\ &= -\langle\mathcal{D}_{\mathcal{A}}\Phi,e_j\Phi\rangle+\langle e_j\Phi,\mathcal{D}_{\mathcal{A}}\Phi\rangle-2\langle\Phi,\nabla_j\Phi\rangle,\end{aligned}$$

where the skew-Hermiticity of $e_i$ and the Clifford algebra $e_ie_j+e_je_i=-2\delta_{ij}$ have been used. A similar expression holds for $\partial_i\langle\Phi,e_je_i\Phi\rangle$ and therefore

$$-\,\mathrm{d}^\dagger\,\sigma^+=-\tfrac{i}{2}\Im\{\langle\nabla_i\Phi,\Phi\rangle\}\,e^i+\tfrac{i}{2}\Im\{\langle\mathcal{D}_{\mathcal{A}}\Phi,e_i\Phi\rangle\}\,e^i, \tag{V.17}$$

what verifies the equivalence of (V.6b) and (V.15b).

In summary, we have introduced an energy functional $E(\mathcal{A},\Phi)$ which can be rewritten as a Bogomolny type functional $SW(\mathcal{A},\Phi)$ plus some topological term, both having equations of motion of second order. The equality between energy functional and topological term is achieved if and only if some first order differential equations, the Seiberg-Witten equations, are satisfied. These then correspond to the absolute minima of the energy functional. This situation is quite familiar, since for instance in pure Yang-Mills gauge theory we have seen that the Euler-Lagrange equations of the Yang-Mills action give rise to a second order problem, while the ASDYM/SDYM equations are of first order.

**Gauge group.** The gauge group $G$ of $L_\Gamma^{1/2}$ is well defined, although $L_\Gamma^{1/2}$ might not exist globally. The reason for this is that the definition of the gauge group is given just in terms of the transition functions. In our case, the gauge group is $G = C^\infty(M, U(1))$. The Seiberg-Witten equations, the Bogomolny type functional $SW(\mathcal{A},\Phi)$ and the energy functional $E(\mathcal{A},\Phi)$ are invariant under gauge transformations of the form

$$\mathcal{A} \mapsto \mathcal{A} + g^{-1}\,\mathrm{d}\,g \qquad \text{and} \qquad \Phi \mapsto g^{-1}\Phi, \tag{V.18}$$

where $g \in G$. This gauge action is free, unless $\Phi \equiv 0$.

## § V.1.2 Perturbed Seiberg-Witten Equations

**Perturbed Seiberg-Witten monopole equations and action functional.** Above we have seen that the gauge action (V.18) is free, unless $\Phi$ vanishes identically. Such kind of solutions will correspond to singularities in the moduli space of solutions. We can avoid them by introducing a $\mathfrak{u}(1)$-valued two-form $\chi$, such that the monopole equations (V.4) become

$$\mathcal{F}_\mathcal{A}^+ + \chi^+ \;=\; \sigma^+(\Phi\otimes\Phi^\dagger)_0, \tag{V.19a}$$

$$\mathcal{D}_\mathcal{A}\Phi \;=\; 0. \tag{V.19b}$$

Here $\chi^+$ denotes the projection onto the self-dual part of $\chi$. These equations are known as the perturbed Seiberg-Witten equations. Note that in the case of $b_+^2 = \dim H_+^2(M,\mathbb{R}) \geq 1$ one can always find a perturbation $\chi$, such that the perturbed monopole equations have no solutions of the type $(\mathcal{A},0)$. Here, $H_+^2(M,\mathbb{R})$ denotes the harmonic self-dual two-forms on $M$.

The corresponding Bogomolny type action functional reads

$$SW_\chi(\mathcal{A},\Phi) = \int_M \mathrm{dVol}\left\{|\mathcal{D}_\mathcal{A}\Phi|^2 + 2|\mathcal{F}_\mathcal{A}^+ + \chi^+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2\right\}, \tag{V.20}$$

which can be rewritten as

$$SW_\chi(\mathcal{A},\Phi) = E_\chi(\mathcal{A},\Phi) + 16\pi^2 K_\chi + 8\pi^2 Q \tag{V.21}$$

with

$$E_\chi(\mathcal{A},\Phi) = \int_M \mathrm{dVol}\left\{|\nabla_\mathcal{A}\Phi|^2 + |\mathcal{F}_\mathcal{A}|^2 + \tfrac{1}{4}|\Phi|^2 R + |\chi^+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2)\right\}. \tag{V.22}$$

The charge $Q$ is given by (V.13) and the Chern-Simons type term $K_\chi$ reads

$$K_\chi = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \mathcal{F}^+_{\mathcal{A}} \wedge \chi^+. \tag{V.23}$$

## § V.2 Moduli Space of Solutions

### § V.2.1 Properties of the Moduli Space

**Definition and properties of the moduli space.** The moduli space of solutions to the perturbed Seiberg-Witten equations (V.19) is defined as the space of all solutions modulo gauge transformations, i.e.

$$\mathcal{M}_\chi := \{\text{Seiberg-Witten solutions } (\mathcal{A}, \Phi)\}/\mathcal{G}. \tag{V.24}$$

The following theorems tell us some important properties of the moduli space of solutions:

(i) Let $M$ be simply connected. Then for every choice of a self-dual two-form $\chi^+$, the moduli space $\mathcal{M}_\chi$ of solutions to the perturbed Seiberg-Witten equations (V.19) is compact.

(ii) Let $M$ be a compact, simply connected smooth four-manifold endowed with a spin$^c$ structure $(W, \Gamma)$. If $b^2_+ \geq 1$, then for a generic choice of a self-dual two-form $\chi^+$, the moduli space $\mathcal{M}_\chi$ is an orientable, smooth manifold of dimension

$$\dim \mathcal{M}_\chi = 2 \operatorname{Ind} \mathcal{D}_{\mathcal{A}} - b^2_+ - 1, \tag{V.25}$$

or equivalently

$$\dim \mathcal{M}_\chi = \langle c_1(L_\Gamma)^2, [M]\rangle - \frac{2\chi(M) + 3\sigma(M)}{4}, \tag{V.26}$$

where $\mathcal{D}_{\mathcal{A}} : \Gamma(M, W^+) \to \Gamma(M, W^-)$ is the spin$^c$ Dirac operator given by (II.116), $\chi(M)$ the Euler characteristic and $\sigma(M)$ the signature of $M$. The intersection pairing $\langle c_1(L_\Gamma)^2, [M]\rangle$ is given by (cf. subsection § II.2.5)

$$\langle c_1(L_\Gamma)^2, [M]\rangle = \int_M c_1(L_\Gamma) \wedge c_1(L_\Gamma) = -\frac{1}{(2\pi)^2}\int_M \mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{A}}. \tag{V.27}$$

In order to prove these assertions, however, one has to do some work, which would fill at least ten pages. We therefore refrain from writing down the proofs and refer the interested reader to [141, 137], where these theorems are extensively discussed.

**Seiberg-Witten invariant.** If the number $\langle c_1(L_\Gamma)^2, [M]\rangle - \frac{1}{4}(2\chi(M) + 3\sigma(M))$ is negative, there are no solutions to the Seiberg-Witten equations. If the dimension is zero, the moduli space $\mathcal{M}_\chi$ consists of a finite union of points. With each such point we can associate a sign $\pm 1$. In this case we define the Seiberg-Witten invariant $\mathrm{SW}(M, \Gamma)$ by

$$\mathrm{SW}(M, \Gamma) := \sum_{\text{points of } \mathcal{M}_\chi} \text{signs} \qquad \in \mathbb{Z}, \tag{V.28}$$

which is invariant under orientation-preserving diffeomorphisms $f$ of $M$, i.e. $\mathrm{SW}(M,\Gamma) = \mathrm{SW}(f(M), f^*\Gamma)$. Note that if $M$ is an orientable Riemannian four-manifold with a Riemannian metric of positive scalar curvature, all of its Seiberg-Witten invariants vanish [206].

**Scale transformations.** Note that the Seiberg-Witten equations (V.4) are not conformally invariant, i.e. they are not invariant under a rescaling of the underlying Riemannian metric $g$. To be precise, there is a one-to-one correspondence between

$$\{\text{Seiberg-Witten solutions } (\mathcal{A},\Phi) \text{ for metric } g\} \tag{V.29a}$$

and

$$\{\text{Seiberg-Witten solutions } (\mathcal{A},\lambda^{-1}\Phi) \text{ for metric } \lambda^2 g\}, \tag{V.29b}$$

where $\lambda \in \mathcal{F}(M)$ and $\lambda > 0$. Moreover, we have for the energy functional (V.12)

$$E_g(\mathcal{A},\Phi) = E_{\lambda^2 g}(\mathcal{A},\lambda^{-1}\Phi). \tag{V.30}$$

## § V.2.2 Vanishing Theorems on $\mathbb{R}^4$

The Seiberg-Witten equations are formulated on an orientable, compact Riemannian four-manifold. However, they can also be discussed on Euclidean four-dimensional space $\mathbb{R}^4$ with the restriction of having a finite energy functional $E(\mathcal{A},\Phi)$. This effectively corresponds to a one-point compactification of $\mathbb{R}^4$ to $S^4$. In this subsection we shall show, that there are no regular solutions to the unperturbed Seiberg-Witten equations (V.4) on $\mathbb{R}^4$ with finite energy $E(\mathcal{A},\Phi)$.

**Vanishing of $\Phi$ and flat $A$.** If $(\mathcal{A},\Phi)$ is a solution to the monopole equations (V.4), where the maximum value of $|\Phi|$ is assumed at a point $p \in M$ then either $\Phi$ does vanish everywhere or

$$|\Phi|^2(p) \leq -\tfrac{1}{2}R(p), \tag{V.31}$$

where $R$ is the scalar curvature [121]. The proof of this assertion is rather simple. First of all, remember that

$$\int_M \mathrm{dVol}\, |\nabla_\mathcal{A}\Phi|^2 = \int_M \mathrm{dVol}\, \langle\Phi,\nabla^\dagger_\mathcal{A}\nabla_\mathcal{A}\Phi\rangle = \int_M \mathrm{dVol}\, \Re\{\langle\Phi,\nabla^\dagger_\mathcal{A}\nabla_\mathcal{A}\Phi\rangle\} \geq 0 \tag{V.32}$$

and hence $\Re\{\langle\Phi,\nabla^\dagger_\mathcal{A}\nabla_\mathcal{A}\Phi\rangle\}(p) \geq 0$. Take now the inner product of $\Phi$ with equation (V.15a) to get

$$\langle\Phi,\nabla^\dagger_\mathcal{A}\nabla_\mathcal{A}\Phi\rangle(p) = -\tfrac{1}{4}R|\Phi|^2(p) - \tfrac{1}{2}|\Phi|^4(p) \quad\Longrightarrow\quad \tfrac{1}{4}R|\Phi|^2(p) + \tfrac{1}{2}|\Phi|^4(p) \leq 0 \tag{V.33}$$

which completes the proof. Therefore, if the scalar curvature of $M$ is non-negative, in particular on $\mathbb{R}^4$, we conclude that $\Phi$ has to vanish everywhere. This will also follow from our next considerations.

Let us now investigate the behavior of the gauge field $\mathcal{A}$ on $\mathbb{R}^4$. The functional (V.11) can be rewritten as

$$SW(\mathcal{A},\Phi)=\int_M \mathrm{dVol}\left\{|\nabla_{\mathcal{A}}\Phi|^2+2|\mathcal{F}^+_{\mathcal{A}}|^2+\tfrac{1}{4}|\Phi|^2(R+|\Phi|^2)\right\}. \qquad \text{(V.34)}$$

The vanishing of $SW(\mathcal{A},\Phi)$ implies in the case of non-negative scalar curvature $R$, particularly on $\mathbb{R}^4$, that $\mathcal{F}^+_{\mathcal{A}}=0$, i.e. the curvature has to be anti-self-dual. From this it follows, that $\mathcal{F}_{\mathcal{A}}$ is a harmonic two-form since

$$\Delta\mathcal{F}_{\mathcal{A}}=(\mathrm{d}\,\mathrm{d}^\dagger+\mathrm{d}^\dagger\,\mathrm{d})\mathcal{F}_{\mathcal{A}}=-(\mathrm{d}*\mathrm{d}*+*\mathrm{d}*\mathrm{d})\,\mathrm{d}\,\mathcal{A}=-\,\mathrm{d}*\mathrm{d}*\mathrm{d}\,\mathcal{A}=\mathrm{d}*\mathrm{d}\,\mathrm{d}\,\mathcal{A}=0.$$

However, it is a well known fact that on $\mathbb{R}^4$ there are no harmonic two-forms. Therefore, we conclude that $\mathcal{F}_{\mathcal{A}}$ has to vanish everywhere and hence $\mathcal{A}$ is a flat connection. In summary, we have shown that on $\mathbb{R}^4$ there are no (regular) solutions to the Seiberg-Witten equations besides the trivial one $(\mathcal{A},\Phi)=(0,0)$ (modulo gauge transformations).

For the sake of completeness let us state the following theorem, which can be found in [137], for instance. Let $M$ be an orientable, compact Riemannian four-manifold with non-negative scalar curvature. Furthermore, assume that $b^2_+>0$. Then for a generic choice of the metric, the only solutions will be $\Phi=0$ and $\mathcal{A}$ flat.

## § V.3 Seiberg-Witten Equations on Kähler Manifolds

**Seiberg-Witten monopole equations.** Of particular interest is the discussion of the Seiberg-Witten equations on Kähler manifolds. Let now $(M,g,J,\omega)$ be an orientable, compact Kähler manifold of (complex) dimension two. In subsection § II.2.4 we have introduced the canonical spin$^c$ structure $(W_{\mathrm{can}},\Gamma_{\mathrm{can}})$ on Kähler manifolds and stated that all other spin$^c$ structures can be obtained from the canonical one via tensoring with some line bundle $E\to M$. Since now we are in the two-dimensional case we have

$$W^+_E\cong W^+_{\mathrm{can}}\otimes E\cong\Lambda^{(0,0)}(E)\oplus\Lambda^{(0,2)}(E)\qquad\text{and}\qquad W^-_E\cong W^-_{\mathrm{can}}\otimes E\cong\Lambda^{(0,1)}(E) \qquad \text{(V.35)}$$

and the characteristical line bundle reads

$$L_{\Gamma_E}\cong K^*\otimes E^2. \qquad \text{(V.36)}$$

We also discussed the spin$^c$ Dirac operator which is given by (II.121)

$$\mathcal{D}_{\mathcal{A}}=\sqrt{2}(\bar\partial_{\bar{\mathcal{B}}}+\bar\partial^\dagger_{\bar{\mathcal{B}}}) \qquad \text{(V.37)}$$

Moreover, on Kähler manifolds we have the decomposition

$$\Lambda^2_+\otimes\mathbb{C}\cong\Lambda^{(2,0)}\oplus\mathbb{C}[\omega]\oplus\Lambda^{(0,2)}. \qquad \text{(V.38)}$$

Let $U \subset M$ be an open neighborhood of $p \in M$. Then we introduce the following local complex coordinates on $U \subset M$

$$z^1 = x^1 + ix^2 \qquad \text{and} \qquad z^2 = x^3 + ix^4. \tag{V.39}$$

As we have already mentioned, on Kähler manifolds we can choose them in such a way that they are normal at $p \in U$. Remember that a local basis of self-dual two-forms is given by (II.99) by setting $\mathrm{d}\, x^i = e^i$ for $i = 1, \dots, 4$. Using (V.39) we can rewrite (II.99) as

$$\tfrac{i}{2}(\mathrm{d}\, z^1 \wedge \mathrm{d}\, \bar{z}^{\bar{1}} + \mathrm{d}\, z^2 \wedge \mathrm{d}\, \bar{z}^{\bar{2}}), \qquad \tfrac{1}{2}(\mathrm{d}\, z^1 \wedge \mathrm{d}\, z^2 + \mathrm{d}\, \bar{z}^{\bar{1}} \wedge \mathrm{d}\, \bar{z}^{\bar{2}}) \quad \text{and} \quad -\tfrac{i}{2}(\mathrm{d}\, z^1 \wedge \mathrm{d}\, z^2 - \mathrm{d}\, \bar{z}^{\bar{1}} \wedge \mathrm{d}\, \bar{z}^{\bar{2}}). \tag{V.40}$$

The Kähler form on $M$ is locally given by

$$\omega = \tfrac{i}{2}(\mathrm{d}\, z^1 \wedge \mathrm{d}\, \bar{z}^{\bar{1}} + \mathrm{d}\, z^2 \wedge \mathrm{d}\, \bar{z}^{\bar{2}}), \tag{V.41}$$

what makes the self-duality of $\omega$ apparent. Therefore, on Kähler manifolds we may rewrite the map $\sigma^+$ defined by (V.3) as

$$\sigma^+(\Phi \otimes \Phi^\dagger)_0 = \tfrac{i}{4}(|\phi_1|^2 - |\phi_2|^2)\omega + \tfrac{1}{4}\phi_1\bar{\phi}_2 - \tfrac{1}{4}\bar{\phi}_1\phi_2, \tag{V.42}$$

for $\Phi \in \Gamma(M, W_E^+)$, i.e. ${}^t\Phi = (\phi_1, \phi_2) \in \Lambda^{(0,0)}(E) \oplus \Lambda^{(0,2)}(E)$. Using all these ingredients, the perturbed Seiberg-Witten equations (V.19) become

$$\begin{aligned} \mathcal{F}_{\mathcal{B}}^{(0,2)} + \chi^{(0,2)} &= -\tfrac{1}{4}\bar{\phi}_1\phi_2, && \text{(V.43a)}\\ \mathcal{F}^{\omega}_{\mathcal{A}_{\text{can}}} + \mathcal{F}^{\omega}_{\mathcal{B}} + \chi^{\omega} &= \tfrac{i}{4}(|\phi_1|^2 - |\phi_2|^2), && \text{(V.43b)}\\ \bar{\partial}_{\mathcal{B}}\phi_1 + \bar{\partial}^\dagger_{\mathcal{B}}\phi_2 &= 0. && \text{(V.43c)} \end{aligned}$$

**Restriction on solutions.** Now, the question is which particular form solutions to the perturbed Seiberg-Witten equations (V.43) may take. Let $\chi$ be a form of type $(1,1)$. Apply the $\bar{\partial}^\dagger_{\mathcal{B}}$ to (V.43c)

$$\bar{\partial}^\dagger_{\mathcal{B}}\bar{\partial}_{\mathcal{B}}\phi_1 + \bar{\partial}^\dagger_{\mathcal{B}}\bar{\partial}^\dagger_{\mathcal{B}}\phi_2 = \bar{\partial}^\dagger_{\mathcal{B}}\bar{\partial}_{\mathcal{B}}\phi_1 + \mathcal{F}_{\mathcal{B}}^{(2,0)}\phi_2 = \bar{\partial}^\dagger_{\mathcal{B}}\bar{\partial}_{\mathcal{B}}\phi_1 + \tfrac{1}{4}\phi_1|\phi_2|^2 = 0,$$

where in the second step we have used the definition of $\mathcal{F}_{\mathcal{B}}^{(2,0)}$ and in the last one (V.43a). Taking the inner product with $\phi_1$ and integrating over $M$, we obtain

$$\|\bar{\partial}_{\mathcal{B}}\phi_1\|^2 + \tfrac{1}{4}\|\phi_1\phi_2\|^2 = 0, \tag{V.45}$$

and hence we conclude that either $\phi_1$ or $\phi_2$ has to vanish identically. In order to decide which component must be zero we integrate equation (V.43b). Consider

$$\begin{aligned} \frac{i}{4}\int_M (|\phi_1|^2 - |\phi_2|^2)\omega \wedge \omega &= \int_M (\mathcal{F}_{\mathcal{A}_{\text{can}}} + \mathcal{F}_{\mathcal{B}} + \chi^+) \wedge \omega \\ &= -2\pi i(c_1(K^*) + 2c_1(E)) \cdot [\omega] + \int_M \chi^+ \wedge \omega \\ &= -2\pi i(-c_1(K) + 2c_1(E)) \cdot [\omega] + \int_M \chi^+ \wedge \omega, \end{aligned}$$

where we have used (II.127b). Note that

$$\frac{1}{4}\int_M (|\phi_1|^2 - |\phi_2|^2)\omega \wedge \omega = \frac{1}{2}(\|\phi_1\|^2 - \|\phi_2\|^2).$$

Considering the case of vanishing perturbation, i.e. $\chi \equiv 0$, we obtain

$$\|\phi_2\|^2 - \|\phi_1\|^2 = 4\pi(2c_1(E) - c_1(K)) \cdot [\omega]. \tag{V.47}$$

Therefore we find the following conditions:

(i) If $c_1(E)\cdot[\omega] > \frac{1}{2}c_1(K)\cdot[\omega]$, then $\phi_1 \equiv 0$ and $\phi_2 \neq 0$.

(ii) If $c_1(E)\cdot[\omega] < \frac{1}{2}c_1(K)\cdot[\omega]$, then $\phi_1 \neq 0$ and $\phi_2 \equiv 0$.

**The case of trivial $E$.** Let us now consider the case when the line bundle $E \to M$ is trivial. The canonical choice of the perturbation is

$$\chi^+ = -\mathcal{F}^+_{\mathcal{A}_{\rm can}} + \tfrac{i\lambda}{4}\omega, \tag{V.48}$$

where $\lambda \in \mathbb{R}$ with $\lambda > 0$. Then equation (V.43b) implies

$$4i\mathcal{F}^\omega_{\mathcal{B}} = |\phi_2|^2 - |\phi_1|^2 + \lambda. \tag{V.49}$$

After integration over $M$ we obtain

$$\|\phi_2\|^2 - \|\phi_1\|^2 + \lambda \mathrm{Vol}(M) = 0 \tag{V.50}$$

implying that $\phi_2$ is identically zero. Therefore the equations (V.43) reduce to

$$\mathcal{F}^{(0,2)}_{\mathcal{B}} = 0, \tag{V.51a}$$
$$\mathcal{F}^\omega_{\mathcal{B}} = \tfrac{i}{4}(|\phi_1|^2 - \lambda), \tag{V.51b}$$
$$\bar\partial_{\bar{\mathcal{B}}}\phi_1 = 0. \tag{V.51c}$$

Since $E$ is chosen to be trivial, these equations have only the trivial solution

$$\mathcal{B} \equiv 0, \qquad \phi_1 \equiv \sqrt{\lambda} \qquad \text{and} \qquad \phi_2 \equiv 0, \tag{V.52}$$

which is unique up to the gauge transformations (V.18). Moreover, in this case the Seiberg-Witten invariant (V.28) is

$$\mathrm{SW}(M, \Gamma_{\rm can}) = 1. \tag{V.53}$$

In order to prove that in the case of trivial $E$ the only solution to the perturbed Seiberg-Witten equations (V.43) is the trivial solution (modulo gauge transformations) we have used the fact of compactness of the underlying Kähler manifold $M$. However, we shall deal with $\mathbb{R}^4$, which is trivially Kähler but non-compact. In this case, the discussion differs essentially from the compact case. The following properties of solutions to the perturbed Seiberg-Witten equations are due to Taubes [186].

Let $M$ be $\mathbb{R}^4$ equipped with the standard metric $g = (\delta_{\mu\nu})$, the standard almost complex structure $J$ and the induced Kähler form $\omega$. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and consider the trivial bundle $E \to \mathbb{C}^2$ over $\mathbb{C}^2$. We know from the discussion above that at least one component of the spinor $\Phi$ has to vanish. Using scaling properties of the Seiberg-Witten equations one can argue that the choice ${}^t\Phi = (\phi_1, \phi_2) = (\phi_1, 0)$ is correct. Taking $\lambda = 1$ the equations (V.43) yield the following a priori estimations

$$|\phi_1| \leq 1 \qquad \text{and} \qquad |\nabla_{\mathcal{B}}\phi_1| \leq C(1 - |\phi_1|^2), \tag{V.54a}$$

$$|\mathcal{F}^-_{\mathcal{B}}| \leq |\mathcal{F}^+_{\mathcal{B}}| = \tfrac{1}{4}(1 - |\phi_1|^2), \tag{V.54b}$$

$$\int_{\mathbb{R}^4} \mathrm{dVol}\,(|\mathcal{F}^+_{\mathcal{B}}|^2 - |\mathcal{F}^-_{\mathcal{B}}|^2) \leq C < \infty, \tag{V.54c}$$

where $C \in \mathbb{R}$ with $C > 0$ is some constant. With the help of these inequalities one can prove the following properties [186]:

(i) Either $|\phi_1| \equiv 1$ or $|\phi_1| < 1$ everywhere on $\mathbb{C}^2$ and

  (ia) if $|\phi_1| \equiv 1$ then the only solution to (V.43) is the trivial one modulo gauge transformations (V.18).

  (ib) if $|\phi_1| < 1$ then the zero set $\phi_1^{-1}(0)$ coincides with a zero set of a complex polynomial on $\mathbb{C}^2$, where its degree is controlled by the constant $C$ given in (V.54).

(ii) Either $|\mathcal{F}^-_{\mathcal{B}}| \equiv |\mathcal{F}^+_{\mathcal{B}}|$ or $|\mathcal{F}^-_{\mathcal{B}}| < |\mathcal{F}^+_{\mathcal{B}}|$ everywhere on $\mathbb{C}^2$. If $|\mathcal{F}^-_{\mathcal{B}}| \equiv |\mathcal{F}^+_{\mathcal{B}}|$, then there exists a $\mathbb{C}$-linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ such that a solution $(\mathcal{B}, \phi_1)$ is gauge equivalent to the $\pi$ pull back of a vortex solution on $\mathbb{C}$ with finite energy.

(iii) The integral $\frac{1}{4\pi}\int_{\mathbb{R}^4} \mathrm{dVol}\,(|\mathcal{F}^+_{\mathcal{B}}|^2 - |\mathcal{F}^-_{\mathcal{B}}|^2)$ yields a non-negative integer.

(iv) The quantity $(1 - |\phi_1|^2)$ decreases exponentially fast with the distance from the zero set $\phi_1^{-1}(0)$.

Note that the property (iv) from above ensures the finiteness of $\mathcal{F}^+_{\mathcal{B}}$.

# Chapter VI

## Mathematical Preliminaries - Part II

Having presented the commutative setup of our theories, we are now ready to discuss their noncommutative extension. Subject of this chapter is to review the mathematics needed for the understanding of the subsequent chapters. As we have already mentioned in the introduction, we give first a little digression on category theory and define $C^*$-algebras - the essential objects in making topological notions algebraic. We shall then discuss the Gelfand-Naimark contravariant functors and the $\Gamma$-functor which yield explicit transitions between topolgical notions and algebraic formulations. We explain the noncommutative pendants of vector bundles and introduce connection, curvature, etc. Eventually, we will specify our discussion to noncommutative $\mathbb{R}^n$-spaces and formulate Yang-Mills theory on them. Again, most of our assertions will not be proven and therefore we refer the interested reader to [86, 191, 30, 169, 33, 32, 210, 118, 192, 116, 117, 110, 49, 148, 83, 188, 139, 175, 153, 85, 120]. The ideas presented below are borrowed from these references.

## § VI.1 A Little Digression on Noncommutative Geometry

### § VI.1.1 Algebraic Preliminaries

**Categories and functors.** Categories and functors are very important objects in modern mathematics. They give a very general way to characterize mathematical structures. Let us start with the textbook definition of a category. A category $\mathcal{C}$ consists of the following:

(i) a collection $\mathrm{Ob}(\mathcal{C})$ of objects,

(ii) sets $\mathrm{Mor}(X, Y)$ of morphisms for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, including a distinguished identity morphism $\mathrm{id} = \mathrm{id}_X \in \mathrm{Mor}(X, X)$ for each $X$,

(iii) a composition of morphism function $\circ\, :\, \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ satisfying $f \circ \mathrm{id} = f = \mathrm{id} \circ f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

There are many examples of categories, such as:

(i) The category of affine spaces over some field $\mathbb{F}$ consists of all affine spaces over $\mathbb{F}$ (= objects). The morphisms are linear maps.

(ii) The category of topological spaces consists of all topological spaces (= objects). The morphism are continuous maps.

(iii) The category of $C^k$-manifolds consists of all $C^k$-manifolds (= objects). The morphisms are $C^k$-maps.

(iv) The category of Lie algebras over some field $\mathbb{F}$ consists of all Lie algebras over $\mathbb{F}$ (= objects) and the morphisms are those linear maps respecting the Lie bracket.

(v) The category of Lie groups consists of all Lie groups (= objects). The morphisms are the so-called Lie morphisms, which are group morphisms and $C^\infty$.

This is a fairly short list. A lot of other examples can be found in the literature cited above. Besides the notion of categories we need so-called functors which relate different categories. A functor $F$ from a category $\mathcal{C}$ to another category $\mathcal{D}$ assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$ and to each morphism $f \in \mathrm{Mor}(X, Y)$ a morphism $F(f) \in \mathrm{Mor}(F(X), F(Y))$, such that $F(\mathrm{id}) = \mathrm{id}$ and $F(f \circ g) = F(f) \circ F(g)$. Pictorially, we have

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

What we have just defined is a covariant functor. A contravariant functor differs from the covariant functor by assigning to $f \in \mathrm{Mor}(X, Y)$ the morphism $F(f) \in \mathrm{Mor}(F(Y), F(X))$ with $F(\mathrm{id}) = \mathrm{id}$ and $F(f \circ g) = F(g) \circ F(f)$. The standard example is the so-called dual vector space functor. This functor assigns to a vector space $V$ over a field $\mathbb{F}$ the dual vector space $F(V) = V^*$ and to each linear map $f : V \to W$ the dual map $F(f) = f^* : W^* \to V^*$ in the reverse direction. Hence, it is a contravariant functor.

**$C^*$-algebras.** To define what a $C^*$-algebra is we need first the notion of a Banach algebra. A Banach algebra over some field $\mathbb{F}$ is a Banach space $(\mathcal{A}, \|\cdot\|)$ together with a multiplication rule such that

(i) $(AB)C = A(BC)$ for all $A, B, C \in \mathcal{A}$,

(ii) $C(A + B) = CA + CB$ and $(A + B)C = AC + BC$ for all $A, B, C \in \mathcal{A}$,

(iii) $(\lambda A)B = A(\lambda B) = \lambda(AB)$ for all $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{F}$,

(iv) $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathcal{A}$,

(v) there exists $\mathrm{id} \in \mathcal{A}$ such that $\mathrm{id}\, A = A\, \mathrm{id} = A$ for all $A \in \mathcal{A}$ and $\|\, \mathrm{id}\, \| = 1$.

Property (v) is not necessary, but if it holds one says that $\mathcal{A}$ is unital. A Banach algebra $\mathcal{A}$ is said to be commutative if $AB = BA$ for all $A, B \in \mathcal{A}$. The standard example of a Banach algebra is the set $L(X, X)$ of all linear continuous operators $A : X \to X$ with respect to the operator norm, where $X$ is some Banach space over $\mathbb{F}$.

A $C^*$-algebra $\mathcal{A}$ is nothing but a Banach algebra endowed with an involution $* : \mathcal{A} \to \mathcal{A}$ such that the following holds:

(i) $(A^*)^* = A$, $(AB)^* = B^* A^*$ and $(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*$ for all $A, B \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{F}$,

(ii) $\|AA^*\| = \|A\|^2$ for all $A \in \mathcal{A}$.

An example of a commutative $C^*$-algebra is the set $C(X) = \{f : X \to \mathbb{F} \mid \text{continuous}\}$, where $X$ is a nonempty compact topological space, together with $\|f\| := \max_{x \in X} |f(x)|$ and $f^*(x) := \overline{f(x)}$ for all $x \in X$.

## § VI.1.2 Gelfand-Naimark Contravariant Functors

Originally, in geometry and topology spaces are considered as sets of points with some particular structure. However, geometrical objects are often best studied not as sets of points but rather by examining functions living on these sets. For instance, let $X$ be a locally compact or compact topological space. Consider the algebra $C(X)$ as defined above. Then points $x$ of $X$ can be viewed as maximal ideals of $C(X)$, i.e.

$$f \in \mathcal{I}_x \qquad \Longleftrightarrow \qquad f(x) = 0. \tag{VI.2}$$

In a lot of cases the algebra of functions completely determines the set of points. Then one obtains all information from the functions alone. To exemplify this, let $X$ be as before. By Urysohn's lemma there exist sufficiently many continuous functions $f \in C(X)$ on $X$ to determine the topology of $X$ uniquely. Also, in many geometrical situations the associated set is very pathological. Thus, in such cases one investigates the matter from the algebraic point of view. Often one discovers perfectly good operator algebras which, however, are in general noncommutative.

**Gelfand-Naimark contravariant functors.** Gelfand's and Naimark's paper [66] has become a cornerstone of noncommutative geometry. They characterized the involutive algebras of operators by dropping commutativity from the axiomatization for algebras of continuous functions on a locally compact Hausdorff space. Their theorem can be understood as the construction of two contravariant functors which act between the category of locally compact Hausdorff spaces and the category of $C^*$-algebras.

Let $X$ be some second countable, compact Hausdorff space. The first contravariant functor $C$ takes $X$ to $C(X)$, the $C^*$-algebra of continuous $\mathbb{C}$-valued functions on $X$. Moreover, it takes a continuous map $f : X \to Y$ to $C(f) = f^* : C(Y) \to C(X)$ defined by $f^*(g)(x) = (g \circ f)(x)$ for $g \in C(Y)$. Obviously, $f^*$ is an algebra homomorphism

$$f^*(g_1 g_2) = f^*(g_1) f^*(g_2) \qquad \text{and} \qquad f^*(g_1 + g_2) = f^*(g_1) + f^*(g_2) \tag{VI.3}$$

for $g_{1,2} \in C(Y)$. Note that if $X$ is only locally compact, the corresponding $C^*$-algebra is $C_0(X)$. The elements of $C_0(X)$ are continuous functions vanishing at infinity. Furthermore, we assume that the preimage of a compact set is compact for a given continuous map $f : X \to Y$ such that $f^* : C_0(Y) \to C_0(X)$.

The other contravariant functor $M$ goes in the opposite direction. It takes a $C^*$-algebra $\mathcal{A}$ onto its space of characters. The latter one consists of the nonzero homomorphisms $\mu : \mathcal{A} \to \mathbb{C}$. One can show that, if $\mathcal{A}$ is an unital algebra, $M(\mathcal{A})$ is closed in the weak$^*$ topology of the unit ball of the dual space $\mathcal{A}^*$ and hence is compact. If $\phi : \mathcal{A} \to \mathcal{B}$ is an unital $*$-homomorphism (i.e. $\phi(a^*) = \phi(a)^*$ for $a \in \mathcal{A}$), the functor $M$ takes $\phi$ to $M(\phi) = \phi^* : M(\mathcal{B}) \to M(\mathcal{A})$ with $\phi^*(\mu) = \mu \circ \phi$, as before.

In order to see the correspondence between spaces and $C^*$-algebras we need to introduce several maps. For $x \in X$ we define the evaluation map $\epsilon_x : f \mapsto \epsilon_x(f) = f(x)$. Thus, $\epsilon_x \in M(C(X))$. The map $\epsilon_X : X \to M(C(X))$ given by $x \mapsto \epsilon_x$ is a homeomorphism. Let $a \in \mathcal{A}$. Then we define its Gelfand transform by

$$\hat{a} : M(\mathcal{A}) \to \mathbb{C} \qquad \text{with} \qquad \mu \mapsto \mu(a). \tag{VI.4}$$

The Gelfand transform is a continuous function on $M(\mathcal{A})$. The map $\mathcal{G} : \mathcal{A} \to C(M(\mathcal{A}))$ which sends $a \mapsto \hat{a}$ is $*$-isomorphism for commutative $C^*$-algebras $\mathcal{A}$. In case that $\mathcal{A}$ is unital, $\mathcal{G}$ preserves the identity element. To summarize, we have the following commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \epsilon_X}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_Y} \\ M(C(X)) & \xrightarrow[M(C(f))]{} & M(C(Y)) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{B} \\ {\scriptstyle \mathcal{G}_\mathcal{A}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}_\mathcal{B}} \\ C(M(\mathcal{A})) & \xrightarrow[C(M(\phi))]{} & C(M(\mathcal{B})) \end{array}$$

There are several consequences following from this equivalence of categories:

(i) Two commutative $C^*$-algebras are isomorphic iff their character spaces are homeomorphic.

(ii) The automorphism group $\mathrm{Aut}(\mathcal{A})$ of a commutative $C^*$-algebra is isomorphic to the group of homeomorphisms of its character space. Since $\mathcal{A}$ is commutative, there are no nontrivial inner automorphisms.

(iii) The topology of $X$ may be dissected in terms of algebraic properties of $C_0(X)$. For instance, any ideal of $C_0(X)$ is of the form $C_0(U)$ where $U \subseteq X$ is an open subset.

Let $Y \subseteq X$ be a closed subset of a second countable, compact Hausdorff space $X$. Then the induced map $C(i) : C(X) \to C(Y)$ of the inclusion map $i : Y \to X$ is the restriction homomorphism, which is surjective. Note that $f : Y \to X$ is injective iff $C(f) : C(X) \to C(Y)$ is surjective.

The Gelfand-Naimark contravariant functors allow us to write down the following translation table:

| Topology | Algebra |
|---|---|
| locally compact space | $C^*$-algebra |
| compact space | unital $C^*$-algebra |
| compactification | unitization |
| continuous proper map | $*$-homomorphism |
| homeomorphism | automorphism |
| open subset | ideal |
| closed subset | quotient algebra |
| second countable | separable |

Note that for commutative $C^*$-algebras to each point of the associated space corresponds one character. This is in contrast to noncommutative $C^*$-algebras where this will not be longer the true.

To summarize, we have described a correspondence which allows us to express topological notions in terms of algebraic formulations. Imposing noncommutativity then simply corresponds to the replacement of the commutative algebra by a noncommutative one. We will continue on this "algebraization" in the next subsection by concentrating on vector bundles over some smooth manifold.

### § VI.1.3 Noncommutative Vector Bundles

**Γ-functor.** For convenience we assume that our manifold $M$ is compact. So, let $M$ be a smooth, compact $n$-dimensional manifold. Note that we implicitly assume that $M$ is Hausdorff and second countable. The $C^*$-algebra $C(M)$ of continuous functions must be replaced by the algebra $\mathcal{A} = \mathcal{F}(M)$ of $\mathbb{C}$-valued smooth functions on $M$. However, $\mathcal{A}$ is not a $C^*$-algebra. But $\mathcal{A}$ is a dense subalgebra of $C(M)$. Hence, we think of $\mathcal{A}$ as the subspace of sufficiently regular elements of $C(M)$.

Consider the category of complex vector bundles $E \to M$ with the bundle projection $\pi$. The morphisms belonging to this category are the bundle maps $\varphi : E \to E'$ satisfying $\pi' \circ \varphi = \pi$. These morphisms define fiberwise maps $\varphi_x : E_x \to E'_x$ for $x \in M$ which are required to be linear. The $\Gamma$-functor takes the category of vector bundles to the category of smooth sections, i.e.

$$\Gamma(E) := C^\infty(M, E) \tag{VI.7}$$

Clearly, if $a \in \mathcal{A}$ and $s \in \Gamma(E)$ then $sa \in \Gamma(E)$. This makes $\Gamma(E)$ a representation of $\mathcal{A}$, i.e. a right $\mathcal{A}$-module. If $\varphi : E \to E'$ is a bundle map, the map $\Gamma(\varphi) : \Gamma(E) \to \Gamma(E')$ given by $s \mapsto \varphi \circ s$ satisfies

$$(\Gamma(\varphi)(sa))(x) = \varphi_x(s(x)a(x)) = \varphi_x(s(x))a(x) = (\Gamma(\varphi(s))a)(x) \tag{VI.8}$$

and so $\Gamma(\varphi)(sa) = \Gamma(\varphi(s))a$ for $s \in \Gamma(E)$, $a \in \mathcal{A}$ and $x \in M$. This means, that $\Gamma(\varphi) : \Gamma(E) \to \Gamma(E')$ is a morphism of right $\mathcal{A}$-modules.

Note that the $\Gamma$-functor carries operations, like duality, direct sum and tensor product of vector bundles to analogous operations on $\mathcal{A}$-modules. For example, for $E, E'$ two vector bundles over $M$ we have

$$\Gamma(E \otimes E') \cong \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E'). \tag{VI.9}$$

One can show that any $\mathcal{A}$-linear map from $\Gamma(E)$ to $\Gamma(E')$ is of the form $\Gamma(\varphi)$ for a unique bundle map $\varphi : E \to E'$.

Now the questions is, what is the image of the $\Gamma$-functor. Consider the trivial bundle $E = M \times \mathbb{C}^r$. Then

$$\Gamma(E) = \mathcal{A}^{\oplus r} = \mathcal{A} \oplus \cdots \oplus \mathcal{A} \cong \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^r \tag{VI.10}$$

is a free $\mathcal{A}$-module of complex rank $r$. Let now $E$ be nontrivial. Since $M$ is Hausdorff and second countable we have a partition of unity, i.e. we can find nonnegative functions $\psi_j \in \mathcal{A}$ for $j = 1, \ldots, q$ with $\psi_1^2 + \cdots + \psi_q^2 = 1$ such that $E$ is trivial over the set $U_j \subset M$. Let $\phi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ be the transition functions. Then the functions $P_{ij} = \psi_i \phi_{ij} \psi_j$, which are defined to be zero outside $U_i \cap U_j$, satisfy $\sum_k P_{ik} P_{kj} = P_{ij}$, since

$$\sum_k P_{ik} P_{kj} = \sum_k \psi_i \phi_{ik} \psi_k \, \psi_k \phi_{kj} \psi_j = \sum_k \psi_i \underbrace{\phi_{ik} \phi_{kj}}_{= \phi_{ij}} \psi_k^2 \psi_j = \psi_i \phi_{ij} \psi_j = P_{ij}$$

and hence $P_{ij}$ defines a $P \in \mathrm{Mat}(qr, \mathcal{A})$ satisfying $P^2 = P$. An element of $\Gamma(E)$ locally given by smooth vector-valued functions $s_j : U_j \to \mathbb{C}^r$ such that $s_i = \phi_{ij} s_j$ on the intersection $U_i \cap U_j$, can be regarded as a column vector

$$s = \begin{pmatrix} \psi_1 s_1 \\ \vdots \\ \psi_q s_q \end{pmatrix} \in \mathcal{A}^{\oplus qr} \tag{VI.12}$$

satisfying $Ps = s$

$$Ps = \begin{pmatrix} \psi_1\phi_{11}\psi_1 & \psi_1\phi_{12}\psi_2 & \cdots & \psi_1\phi_{1q}\psi_q \\ \psi_2\phi_{21}\psi_1 & \psi_2\phi_{22}\psi_2 & \cdots & \psi_2\phi_{2q}\psi_q \\ \vdots & \vdots & \ddots & \vdots \\ \psi_q\phi_{q1}\psi_1 & \psi_q\phi_{q2}\psi_q & \cdots & \psi_q\phi_{qq}\psi_q \end{pmatrix} \begin{pmatrix} \psi_1 s_1 \\ \psi_2 s_2 \\ \vdots \\ \psi_q s_q \end{pmatrix} = \begin{pmatrix} \psi_1 s_1 \\ \psi_2 s_2 \\ \vdots \\ \psi_q s_q \end{pmatrix} = s$$

as one may readily check. Hence, we identify $\Gamma(E)$ with $P\mathcal{A}^{\oplus qr}$.

Due to the Serre-Swan theorem we can go the other direction, i.e. any right $\mathcal{A}$-module of the form $P\mathcal{A}^{\oplus m}$, for an idempotent $P \in \mathrm{Mat}(m, \mathcal{A})$, is of the form

$$\Gamma(E) = C^\infty(M(\mathcal{A}), E). \tag{VI.14}$$

The fiber of $E$ at the point $\mu \in M(\mathcal{A})$ is the vector space $P\mathcal{A}^{\oplus m} \otimes_{\mathcal{A}} (\mathcal{A}/\ker\mu)$ whose dimension is trace of the matrix $\mu(P) \in \mathrm{Mat}(m, \mathbb{C})$. A right $\mathcal{A}$-module of the form $P\mathcal{A}^{\oplus m}$ for $P^2 = P$ is called a finitely generated projective module, or finite projective module for short. Phrasing it differently, from the Serre-Swan theorem we know that for any vector bundle $E$ there exists another vector bundle $E'$ such that $E \oplus E' \cong M \times \mathbb{C}^m$ for sufficiently large $m$ and therefore

$$\Gamma(E) \oplus \Gamma(E') \cong \mathcal{A}^{\oplus m} \tag{VI.15}$$

with $Ps = s$ and $Ps' = 0$ for $s \in \Gamma(E)$ and $s' \in \Gamma(E')$.[1]

To summarize our discussion, we can say that $\Gamma$ is a covariant functor from the category of vector bundles over $M$ to the category of finite projective modules over $\mathcal{A} = \mathcal{F}(M)$. The Serre-Swan theorem yields the inverse functor, so that these categories are equivalent. Note that we could also have introduced left modules. However, the discussion is essentially the same. The distinction between left and right modules has no meaning in the commutative setup but, of course, makes sense in the noncommutative case. Now we can say what the noncommutative generalization of a vector bundle $E$ should be. It is simply a finite projective right and/or left module $\mathcal{E}$ over a noncommutative algebra $\mathcal{A}$ which will generally be a dense subalgebra of a $C^*$-algebra.

**Hermitian structures.** Remember that we can endow complex vector bundles $E \to M$ with Hermitian metrics, that is to put a positive sesquilinear form $\langle\cdot|\cdot\rangle_x$ on each fiber $E_x$ which varies smoothly with $x$. The noncommutative viewpoint is to drop $x$. What remains is a pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ on a finite projective right $\mathcal{A}$-module which is $\mathcal{A}$-valued. Moreover, it satisfies the following properties

$$\langle r|s+t\rangle = \langle r|s\rangle + \langle r|t\rangle, \tag{VI.16a}$$
$$\langle ra|sb\rangle = a^*\langle r|s\rangle b, \tag{VI.16b}$$
$$\langle r|s\rangle = \langle s|r\rangle^*, \tag{VI.16c}$$
$$\langle s|s\rangle > 0 \quad \text{for} \quad s \neq 0, \tag{VI.16d}$$

[1] To be precise: $P = \left(\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

where $r, s, t \in \mathcal{E}$ and $a, b \in \mathcal{A}$.

With this structure, $\mathcal{E}$ is called a pre-$C^*$-module. More precisely: A pre-$C^*$-module over a dense subalgebra $\mathcal{A}$ of a $C^*$-algebra is a right $\mathcal{A}$-module $\mathcal{E}$ with a sesquilinear pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying (VI.16). Note that $\mathcal{E}$ is not necessarily finitely generated or projective. One can complete $\mathcal{E}$ with a norm

$$|||s||| := \sqrt{\|\langle s|s\rangle\|}, \tag{VI.17}$$

where $\|\cdot\|$ is the $C^*$-norm of the $C^*$-algebra. The resulting Banach space is then called a $C^*$-module. In the case $\mathcal{E} = C^\infty(M, E)$ the completion is the Banach space of continuous sections $C(M, E)$. Note that in general this completion is not a Hilbert space. Obviously, the free $\mathcal{A}$-module $\mathcal{A}^{\oplus m}$ becomes a pre-$C^*$-module after choosing

$$\langle r|s\rangle := \sum_{j=1}^{m} r_j^* s_j. \tag{VI.18}$$

This kind of scalar product also works for modules of the form $P\mathcal{A}^{\oplus m}$ if $P \in \mathrm{Mat}(m, \mathcal{A})$ is idempotent and self-adjoint. Again, a similar discussion goes through for left modules. In fact, for any right $\mathcal{A}$-module, the conjugate space $\bar{\mathcal{E}}$ is a left $\mathcal{A}$-module, i.e. $\bar{\mathcal{E}} = \{\bar{s} \,|\, s \in \mathcal{E}\}$ with

$$a\bar{s} := \overline{(sa^*)}. \tag{VI.19}$$

For $\mathcal{E} = P\mathcal{A}^{\oplus m}$ we obtain $\bar{\mathcal{E}} = \bar{\mathcal{A}}^{\oplus m}P$, where one regards the entries of $\bar{\mathcal{A}}^{\oplus m}$ as row vectors. Note that we have assumed that $P$ is self-adjoint, what we always want to do, in the sequel.

### § VI.1.4 Connection and Curvature

Having introduced noncommutative vector bundles, we shall now discuss the notions of connection and curvature in the noncommutative setup.

**Connection.** In noncommutative geometry there are several ways to define a connection. The simplest way is via a $C^*$-dynamical system. Such system consists of the triple $(A, H, \alpha)$, where $A$ is a $C^*$-algebra and $H$ is the Lie group of automorphisms of $A$. We shall say that $a \in A$ is of $C^\infty$ class iff the map $h \mapsto \alpha_h(a)$ from $H$ to $A$ is $C^\infty$. Remember that the involutive algebra $\mathcal{A} := \{a \in A \,|\, a \text{ of class } C^\infty\}$ is norm dense in $A$. Let $\mathcal{E}$ be a finite projective (right) pre-$C^*$-module over $\mathcal{A}$, i.e. $\mathcal{E} = P\mathcal{A}^{\oplus m}$. In the following we denote the Lie algebra of $H$ by $\mathfrak{h}$. Then we define by $\delta$ a representation of $\mathfrak{h}$ in the Lie algebra of derivations of $\mathcal{A}$ by

$$\delta_X(a) = \lim_{t\to 0} \tfrac{1}{t}(\alpha_{h_t}(a) - a), \tag{VI.20}$$

where $\dot{h}_0 = X \in \mathfrak{h}$, '$\cdot$' denotes the derivative with respect to $t$ and $a \in \mathcal{A}$. Note that $\delta_{[X,Y]} = [\delta_X, \delta_Y]$. A connection $\nabla$ on $\mathcal{E}$ is a linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathfrak{h}^* \qquad \text{with} \qquad (\nabla s)(X) := \nabla_X(s) \tag{VI.21}$$

which satisfies a Leibnitz rule

$$\nabla_X(sa) = \nabla_X(s)a + s\,\delta_X(a) \tag{VI.22}$$

for $s \in \mathcal{E}$ and $a \in \mathcal{A}$. A connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathfrak{h}^*$ is said to be compatible with the Hermitian structure if

$$\langle \nabla_X r | s \rangle + \langle r | \nabla_X s \rangle = \delta_X \langle r | s \rangle \tag{VI.23}$$

for all $r, s \in \mathcal{E}$ and $X \in \mathfrak{h}$. Note that every finite projective module admits a connection. For instance, the following formula defines the so-called Graßmannian connection

$$\nabla^0_X(s) = P\delta_X P(s) \tag{VI.24}$$

Note that the difference $\nabla - \nabla'$ is an endomorphism of $\mathcal{E}$, since

$$(\nabla - \nabla')(sa) = \nabla(sa) - \nabla'(sa) = (\nabla - \nabla')(s)(a).$$

Hence, fixing some fiducial connection $\tilde{\nabla}$ on $\mathcal{E}$ allows us to write an arbitrary connection in the form

$$\nabla_X = \tilde{\nabla}_X + \mathcal{A}_X, \tag{VI.26}$$

where $\mathcal{A}_X \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. If $\tilde{\nabla}$ and $\nabla$ are compatible with the Hermitian structure, one obviously has to require that $\mathcal{A}_X^\dagger = -\mathcal{A}_X$. The involution '$*$' on $\mathcal{A}$ extends to an involution '$\dagger$' on $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

To the representation $\delta$ of $\mathfrak{h}$ in the Lie algebra of derivations of $\mathcal{A}$ we can assign the complex

$$\Omega^*(\mathcal{A}) = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Lambda^*(\mathfrak{h}^*) \cong \bigoplus_{p \geq 0} \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Lambda^p(\mathfrak{h}^*) =: \bigoplus_{p\geq 0} \Omega^p(\mathcal{A}) \tag{VI.27}$$

which consists of $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$-valued left-invariant differential forms on $H$. Here we endow $\Omega^*(\mathcal{A})$ with the algebra structure coming from the exterior algebra $\Lambda^*(\mathfrak{h}^*)$. The exterior derivative is given by:

(i) for $a \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ and $X \in \mathfrak{h}$ we have $\mathrm{d}\,a(X) = \delta_X(a)$,

(ii) $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\,\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\,\omega_2$ for $\omega_1 \in \Omega^p(\mathcal{A})$ and $\omega_2 \in \Omega^*(\mathcal{A})$,

(iii) $\mathrm{d}^2\,\omega = 0$ for $\omega \in \Omega^*(\mathcal{A})$.

Note that $\Omega^*(\mathcal{A})$ is an $\mathcal{A}$-bimodule since $\mathcal{A}$ is contained in $\Omega^*(\mathcal{A})$.

**Curvature.** Let $\nabla$ be a connection on a finite projective $\mathcal{A}$-module $\mathcal{E}$. Then the curvature $\mathcal{F}$ is given by

$$\mathcal{F} : \mathfrak{h} \times \mathfrak{h} \to \Omega^2(\mathcal{A}) \tag{VI.28}$$

with

$$\mathcal{F} : (X, Y) \mapsto \mathcal{F}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}). \tag{VI.29}$$

Note that $\mathcal{F}$ measures the deviation of the mapping $X \mapsto \nabla_X$ from being a Lie algebra homomorphism. For instance, the curvature $\mathcal{F}^0$ of the Graßmannian connection (VI.24) is given by

$$\mathcal{F}^0 = P\,\mathrm{d}\,P \wedge \mathrm{d}\,P, \tag{VI.30}$$

since

$$\begin{aligned}
\mathcal{F}^0(X,Y)(s) &= (P\,\mathrm{d}\,P \wedge \mathrm{d}\,P)(X,Y)(s) = P(\mathrm{d}\,P(X)\,\mathrm{d}\,P(Y) - \mathrm{d}\,P(Y)\,\mathrm{d}(X))(s) \\
&= P(\delta_X(P)\delta_Y(Ps) - \delta_Y(P)\delta_X(Ps)) \\
&= P(\delta_X(P\delta_Y(Ps)) - \delta_Y(P\delta_X(Ps)) - \delta_X(\delta_Y(Ps)) + \delta_Y(\delta_X(Ps))) \\
&= P\delta_X(P\delta_Y(Ps)) - P\delta_Y(P\delta_X(Ps)) - P\delta_{[X,Y]}(Ps)
\end{aligned}$$

for $s \in \mathcal{E}$.

Similarly, we can compute the curvature $\mathcal{F}$ of[2] $\nabla = \nabla^0 + \mathcal{A}$. What we find is

$$\mathcal{F} = \mathcal{F}^0 + P(\mathrm{d}\,\mathcal{A} + \mathcal{A} \wedge \mathcal{A})P. \tag{VI.31}$$

This can be seen as follows. First of all, one may easily check that the substitution $\nabla = \nabla^0 + \mathcal{A}$ into the definition (VI.29) yields

$$\mathcal{F}(X,Y)(s) = \mathcal{F}^0(X,Y)(s) + (\nabla^0_X \mathcal{A}_Y - \nabla^0_Y \mathcal{A}_X + [\mathcal{A}_X, \mathcal{A}_Y] - \mathcal{A}_{[X,Y]})(s),$$

for $s \in \mathcal{E}$. Remember that $\nabla^0_X = P\delta_X P$ and $Ps = s$. Thus, we only need to show that

$$(\nabla^0_X \mathcal{A}_Y - \nabla^0_Y \mathcal{A}_X + [\mathcal{A}_X, \mathcal{A}_Y] - \mathcal{A}_{[X,Y]})(s) = (P(\mathrm{d}\,\mathcal{A} + \mathcal{A} \wedge \mathcal{A})P)(X,Y)(s).$$

Note that we have $\mathrm{d}\,\mathcal{A}(X,Y)P(s) = ((\delta_a P\mathcal{A}_b)(x^a y^b - y^a x^b))P(s)$, where $\{X_a\}$ is some basis of $\mathfrak{h}$ and $X = x^a X_a$ and $Y = y^a Y_a$, respectively. Moreover, we have used the abbreviations $\delta_a := \delta_{X_a}$ and $\mathcal{A}_a := \mathcal{A}_{X_a}$. Therefore we may write

$$(\delta_a P\mathcal{A}_b)(x^a y^b) = x^a \delta_a (y^b P\mathcal{A}_b) - P\mathcal{A}_b x^a \delta_a y^b = \delta_X(P\mathcal{A}_Y) - P\mathcal{A}_{(x^a X_a y^b)X_b}$$

and hence

$$\mathrm{d}\,\mathcal{A}(X,Y)P(s) = (\delta_X(P\mathcal{A}_Y) - \delta_Y(P\mathcal{A}_X) - P\mathcal{A}_{[X,Y]})P(s).$$

Finally we note that $(\mathcal{A} \wedge \mathcal{A})P(X,Y)(s) = [\mathcal{A}_X, \mathcal{A}_Y](s)$. This then proves our assertion.

Using the above notation one can show that

$$P(\mathrm{d}\,\mathcal{F})P + \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A} = 0 \tag{VI.32}$$

substituting the Bianchi identity (II.41).

[2] Do not confuse $\mathcal{A}$ with the underlying algebra. Here it denotes some endomorphism of $\mathcal{E}$. Sorry for the bad notation.

## § VI.1.5 Deformation Quantization

**Noncommutative deformations of commutative algebras.** Let $V$ be some vector space over a field $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Again, we denote by $\mathcal{T}(V)$ the tensor algebra of $V$, i.e.

$$\mathcal{T}(V) := \mathbb{F} \oplus V \oplus (V \otimes V) \oplus \cdots . \tag{VI.33}$$

Moreover, let $\langle R \rangle$ be a two-sided ideal in $\mathcal{T}(V)$ generated by a subspace of relations $R \subset \mathcal{T}(V)$. Then the algebra $\mathcal{A}$, assumed to be generated by a set of generators and relations, is given by the quotient

$$\mathcal{A} = \mathcal{T}(V)/\langle R \rangle. \tag{VI.34}$$

Suppose that $R_\theta \subset \mathcal{T}(V)$ is a one-parameter deformation of $\mathcal{A}$. In case that $\mathcal{A}$ is graded, we assume that $R$ is a graded subspace and that the deformation preserves the grading. Then

$$\mathcal{A}_\theta = \mathcal{T}(V)/\langle R_\theta \rangle \tag{VI.35}$$

is a one-parameter deformation of the algebra $\mathcal{A}$.

After these generalities, let us now discuss some examples. Let $V$ be $\mathbb{R}^n$ which is endowed with the canonical metric $g = (\delta_{\mu\nu})$. Then the noncommutative Euclidean space is the algebra

$$\mathcal{A}_\theta(\mathbb{R}^n) := \mathcal{T}(\hat{x}^1, \ldots, \hat{x}^n)/\langle [\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \rangle_{1 \leq \mu,\nu \leq n}, \tag{VI.36}$$

where $\theta = (\theta^{\mu\nu})$ is a real skew-symmetric $n \times n$-matrix. For brevity we shall denote $\mathcal{A}_\theta(\mathbb{R}^n)$ simply by $\mathbb{R}^n_\theta$. The hats on the coordinates are written in order to emphasize the noncommutative character. Moreover, the coordinates can be regarded as operators acting on some Hilbert space $\mathcal{H}$, which will be specified below. Other examples are the Lie algebra case, where the commutation relations are

$$[\hat{x}^\mu, \hat{x}^\nu] = i\lambda^{\mu\nu}_\sigma \hat{x}^\sigma \tag{VI.37}$$

with $\lambda^{\mu\nu}_\sigma \in \mathbb{R}$, or the $q$-deformed case with the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = (\tfrac{1}{q} R^{\mu\nu}_{\sigma\rho} - \delta^\mu_\rho \delta^\nu_\sigma)\hat{x}^\sigma \hat{x}^\rho, \tag{VI.38}$$

where $R^{\mu\nu}_{\sigma\rho}, q \in \mathbb{C}$ and, of course, there are much more. The latter examples, however, are not of our interest. In the sequel we shall only deal with the noncommutative Euclidean space.

**Star products.** Let $M$ be some smooth $n$-dimensional manifold endowed with a Poisson structure, i.e. we have a bilinear map

$$\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M), \tag{VI.39}$$

known as the Poisson bracket, which is skew-symmetric, satisfies a Jacobi identity and a Leibnitz rule. Then the definition of a star product is as follows. Let $C_i : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ be local bidifferential operators, i.e. bilinear maps which are differential operators with respect to each argument of globally bounded order. Then we define the star product

$$\star : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)[[\theta]] \tag{VI.40}$$

by

$$f \star g = \sum_{n\in\mathbb{N}_0} \theta^n C_n(f,g), \tag{VI.41}$$

where $f, g \in \mathcal{F}(M)$. We require that this product is associative. Moreover, we assume that $C_0(f,g) = fg$, i.e. the ordinary product, $C_1(f,g) - C_1(g,f) = -i\{f,g\}$ gives the Poisson bracket, $f \star 1 = f = 1 \star f$ and $(f \star g)^\dagger = g^\dagger \star f^\dagger$. From now on we shall denote the involution by '$\dagger$'. Note that the formal parameter $\theta$ might be regareded the scale where noncommutativity becomes important. Our definition can be generalized to a $\mathbb{C}[[\theta]]$-linear map in $\mathcal{F}(M)[[\theta]]$ by

$$\left(\sum_{n\in\mathbb{N}_0} \theta^n f_n\right) \star \left(\sum_{n\in\mathbb{N}_0} \theta^n g_n\right) = \sum_{k,l\in\mathbb{N}_0} \theta^{k+l} f_k g_l \quad + \sum_{k,l\in\mathbb{N}_0;\ m\in\mathbb{N}} \theta^{k+l+m} C_m(f_k, g_l). \tag{VI.42}$$

To summarize, the algebra $\mathcal{F}(M)$ becomes noncommutative when the ordinary product is replaced by the star product.

Changing the ordering in the noncommutative algebra leads to gauge equivalent star products. Hence, there is a natural gauge group acting on star products. This group consists of automorphisms of $\mathcal{F}(M)[[\theta]]$ of the following form:

$$f \mapsto f + \sum_{n\in\mathbb{N}} \theta^n D_n(f) \qquad \text{for} \qquad f \in \mathcal{F}(M) \tag{VI.43a}$$

and

$$f(\theta) := \sum_{n\in\mathbb{N}_0} \theta^n f_n \mapsto f(\theta) \quad + \sum_{n\in\mathbb{N}_0;\ m\in\mathbb{N}} \theta^{n+m} D_m(f_n) \quad \text{for} \quad f(\theta) \in \mathcal{F}(M)[[\theta]], \tag{VI.43b}$$

where the $D_i$'s are differential operators of order $i$. If

$$D(\theta) = 1 + \sum_{n\in\mathbb{N}} \theta^n D_n \tag{VI.44}$$

is such an automorphism, it acts as

$$\star \mapsto \star' \qquad \text{with} \qquad D(f) \star D(g) = D(f \star' g). \tag{VI.45}$$

In this case the two star products $\star$ and $\star'$ are said to be equivalent.

To exemplify our discussion we define the Moyal product. Let $f, g \in \mathcal{F}(\mathbb{R}^n)$. Then we define the Moyal product by the formula

$$(f \star g)(x) := f(x) \exp\left\{ \tfrac{i}{2} \overleftarrow{\partial_\mu} \theta^{\mu\nu} \overrightarrow{\partial_\nu} \right\} g(x), \tag{VI.46}$$

where $\theta = (\theta^{\mu\nu})$ is again some skew-symmetric $n \times n$-matrix. One easily finds that

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \tag{VI.47}$$

Thus, we recover the commutation relations of the noncommutative Euclidean space $\mathbb{R}^n_\theta$. In fact, in the next paragraph we will establish an algebra isomorphism

$$(\mathcal{F}(\mathbb{R}^n), \star) \cong (\mathbb{R}^n_\theta, \cdot) \tag{VI.48}$$

where '$\cdot$' means simply the operator product.

To close this paragraph let us state a theorem due to Kontsevich [118]: Let $M = \mathbb{R}^n$. Then star products exist for any Poisson structure.

**Weyl transform.** Consider the noncommutative Euclidean space $\mathbb{R}^n_\theta$ with the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{VI.49}$$

By an orthogonal change of coordinates one can map $\theta^{\mu\nu}$ onto its canonical form

$$(\theta^{\mu\nu}) = \mathrm{diag}(\Theta_1, \ldots, \Theta_r, \mathbf{0}_{(n-2r)\times(n-2r)}) \qquad \text{with} \qquad \Theta_a = \begin{pmatrix} 0 & \theta^{2a-1,a} \\ -\theta^{2a-1,a} & 0 \end{pmatrix}. \tag{VI.50}$$

Thus, only $2r$ coordinates are mutually noncommuting, while $n-2r$ coordinates remain commutative. Operators $\hat{\mathcal{O}}(\hat{x}^1, \ldots, \hat{x}^r)$, which act on a Hilbert space $\mathcal{H}$ discussed momentarily, can then be regarded as operators depending on $n-2r$ commuting parameters. For instance, later on we shall consider the $2+1$ dimensional case where the two spatial coordinates do not commute while the time remains commutative. For the moment we therefore restrict our considerations to the even dimensional case, i.e. $\mathbb{R}^{2n}_\theta$, and simply do not care about the commuting coordinates.

In order to construct an isomorphism $\mathcal{W} : (\mathcal{F}(\mathbb{R}^{2n}), \star) \to (\mathbb{R}^{2n}_\theta, \cdot)$ we have to choose a basis in $\mathbb{R}^{2n}_\theta$ which satisfies the Poincaré-Birkhoff-Witt property. This means when considered as a graded algebra the subspace of polynomials of fixed degree has the same dimension as the corresponding subspace of the polynomials of commuting variables. For instance, the basis of symmetrically ordered monomials will do the job

$$1, \qquad \hat{x}^\mu, \qquad \tfrac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu), \qquad \ldots \tag{VI.51}$$

Now we are looking for a map $\mathcal{W} : \mathcal{F}(\mathbb{R}^{2n}) \to \mathbb{R}^{2n}_\theta$ such that

$$\mathcal{W} : x^\mu \mapsto \hat{x}^\mu, \qquad \mathcal{W} : x^\mu x^\nu \mapsto \tfrac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu), \qquad \ldots \tag{VI.52}$$

Moreover, we want to have

$$\mathcal{W}(f \star g) = \mathcal{W}(f) \cdot \mathcal{W}(g) = \hat{f} \cdot \hat{g}. \tag{VI.53}$$

The maps $\mathcal{W}$ and $\mathcal{W}^{-1}$ satisfying these properties are

$$\mathcal{W} : \tilde{f}(k) \mapsto \hat{f}(\hat{x}) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \mathrm{d^{2n}k}\, \tilde{f}(k)\, e^{-ik\hat{x}}, \tag{VI.54a}$$

$$\mathcal{W}^{-1} : \hat{f}(\hat{x}) \mapsto \tilde{f}(k) = |\mathrm{Pf}(2\pi\theta)|\, \mathrm{Tr}_{\mathcal{H}}\{e^{ik\hat{x}}\hat{f}(\hat{x})\}, \tag{VI.54b}$$

where $\tilde{f}(k)$ stands for the Fourier transform of $f(x)$,

$$f(x) \mapsto \tilde{f}(k) = \int_{\mathbb{R}^{2n}} \mathrm{d^{2n}x}\, f(x) e^{ikx}. \tag{VI.55}$$

Here, $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace in the operator representation of the noncommutative algebra and $\mathrm{Pf}(2\pi\theta)$ is the Pfaffian of $(2\pi\theta^{\mu\nu})$, i.e.

$$\mathrm{Pf}(2\pi\theta) = \frac{\pi^n}{n!} \epsilon_{\mu_1\cdots\mu_{2n}} \theta^{\mu_1\mu_2} \cdots \theta^{\mu_{2n-1}\mu_{2n}}. \tag{VI.56}$$

Also in these equations $kx$ is a shorthand notation for $k_\mu x^\mu$. Note that the exponential takes care of the symmetrical ordering. Obviously, we have

$$\int_{\mathbb{R}^{2n}} \mathrm{d^{2n}x}\, f(x) = |\mathrm{Pf}(2\pi\theta)|\, \mathrm{Tr}_{\mathcal{H}}\, \hat{f}(\hat{x}). \tag{VI.57}$$

Moreover, one can show that

$$\int_{\mathbb{R}^{2n}} \mathrm{d^{2n}x}\, (f \star g)(x) = \int_{\mathbb{R}^{2n}} \mathrm{d^{2n}x}\, f(x) g(x). \tag{VI.58}$$

The above procedure is known as Weyl quantization.

## § VI.1.6 Noncommutative $\mathbb{R}^{2n}$-Spaces

**Algebras $\mathbb{R}^{2n}_\theta$, $\mathcal{S}(\mathbb{R}^{2n}_\theta)$ and $\Gamma^m(\mathbb{R}^{2n}_\theta)$.** Above we have introduced the Moyal product for functions which are elements of $\mathcal{F}(\mathbb{R}^{2n})$ and constructed an isomorphism between $(\mathcal{F}(\mathbb{R}^{2n}), \star)$ and $(\mathbb{R}^{2n}_\theta, \cdot)$. Note that in the sequel we shall not distinguish between these two viewpoints and simply write $\mathbb{R}^{2n}_\theta$.

For purposes of describing physical objects concentrated on a finite domain we need functions decreasing at infinity in a suitable sense. Moreover, also from a mathematical point of view we need to specify the allowed class of functions on $\mathbb{R}^{2n}$ since, for instance, the integrals (VI.57) and (VI.58) do obviously not exist for any $\mathcal{F}(\mathbb{R}^{2n})$-function.

One interesting class of functions is the space $\mathcal{S}(\mathbb{R}^{2n})$ - the Schwartz class functions. Let

$$p_{k,r}(f) := \sup_{x\in\mathbb{R}^{2n}} (1 + \|x\|^k) \sum_{|\alpha|\le r} |\partial^\alpha f(x)|, \tag{VI.59}$$

where $k, r \in \mathbb{N}_0$, $\alpha$ is a multiindex and $f \in \mathcal{F}(\mathbb{R}^{2n})$. Here $\|\cdot\|$ denotes the standard norm on $\mathbb{R}^{2n}$. The space $\mathcal{S}(\mathbb{R}^{2n})$ consists of all functions $f \in \mathcal{F}(\mathbb{R}^{2n})$ which satisfy

$$p_{k,r}(f) < \infty \qquad \text{for all} \qquad k, r. \tag{VI.60}$$

Then $(\mathcal{S}(\mathbb{R}^{2n}), \star)$, where the star product is the Moyal product given by (VI.46), forms an algebra which we denote by $\mathcal{S}(\mathbb{R}^{2n}_\theta)$.

Another class of functions one can consider is the space $\Gamma^m_\rho(\mathbb{R}^{2n})$. By definition it consists of all functions $f \in \mathcal{F}(\mathbb{R}^{2n})$ having the property

$$|\partial^\alpha f(x)| \leq C_\alpha(1 + \|x\|^2)^{\frac{1}{2}(m-\rho|\alpha|)}. \tag{VI.61}$$

Here $\alpha$ is again a multiindex, $\rho \in (0, 1]$, $m \in \mathbb{R}$ and $C_\alpha \in \mathbb{R}$ with $C_\alpha > 0$. This inequality says that if for large $x$ a smooth function $f(x)$ has an asymptotics of a polynomial of degree less then $m$ then it belongs to $\Gamma^m_{\rho=1}(\mathbb{R}^{2n})$. Note that if $m$ is negative the function falls off at infinity and each derivative increases the power of the falloff by $\rho$. One can prove that if $f \in \Gamma^{m_1}_\rho(\mathbb{R}^{2n})$ and $g \in \Gamma^{m_2}_\rho(\mathbb{R}^{2n})$ then their star product (VI.46) $f \star g$ belongs to $\Gamma^{m_1+m_2}_\rho(\mathbb{R}^{2n})$. Thus, the set for negative $m$ forms an algebra. In the sequel we shall denote the class of all functions $\Gamma^m_{\rho=1}(\mathbb{R}^{2n})$ with $m < 0$ equipped with the Moyal product by $\Gamma^m(\mathbb{R}^m_\theta)$. The algebra $\mathcal{S}(\mathbb{R}^{2n}_\theta)$ can be regarded as the intersection

$$\mathcal{S}(\mathbb{R}^{2n}_\theta) = \bigcap_{m\in\mathbb{N}} \Gamma^{-m}(\mathbb{R}^{2n}_\theta). \tag{VI.62}$$

One can show that if $f \in \Gamma^m(\mathbb{R}^{2n}_\theta)$ for $m \leq 0$ the corresponding Weyl transform $\hat{f}$ will be a bounded operator acting on $\mathcal{H}$. We denote the space of bounded operators by $\mathcal{B}(\mathcal{H})$. If $m$ is non-positive then $\hat{f}$ will be a compact operator. Their space is denoted by $\mathcal{K}(\mathcal{H})$. The space $\mathcal{B}(\mathcal{H})$ forms a $C^*$-algebra. Obviously, the compact operators form a subset of the bounded operators and it consists of the norm completion of finite rank operators if $\mathcal{H}$ is separable. Note that $\mathcal{K}(\mathcal{H})$ is the only norm closed and two-sided ideal in $\mathcal{B}(\mathcal{H})$.

**Fock space realization.** In this paragraph we want to think a bit about representations of the algebra (VI.49) and talk about the aforementioned $\mathcal{H}$. Consider $\mathbb{R}^{2n}$ and let $\theta$ be given by its canonical form. For convenience we introduce complex coordinates on $\mathbb{R}^{2n} \cong \mathbb{C}^n$,

$$\hat{z}^a := \hat{x}^{2a-1} + i\hat{x}^{2a} \qquad \text{and} \qquad \hat{\bar{z}}^{\bar{a}} := \hat{x}^{2a-1} - i\hat{x}^{2a} \qquad \text{for} \qquad a = 1, \ldots, n. \tag{VI.63}$$

Then the commutation relations (VI.49) translate into

$$[\hat{z}^a, \hat{\bar{z}}^{\bar{a}}] = \theta^{a\bar{a}} \qquad \text{with} \qquad \theta^{a\bar{a}} = 2\theta^{2a-1,2a}, \tag{VI.64}$$

and all other commutators are equal to zero. The latter form of the algebra is of Heisenberg type. Thus, it suggests that the algebra $\mathbb{R}^{2n}_\theta$ may be represented by $n$ harmonic oscillators in the $n$-oscillator Fock space

$$\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \cong \bigoplus_{m_i\in\mathbb{N}_0} \mathbb{C}\,|m_1, m_2, \ldots, m_n\rangle. \tag{VI.65}$$

We introduce, as usual, annihilation and creation operators $\{c_a, c_a^\dagger\}_{a=1,\ldots,n}$, satisfying $[c_a, c_a^\dagger] = 1$. They act on $\mathcal{H}$ and are defined by the relations

$$c_a|m_1, \ldots, m_a, \ldots, m_n\rangle = \sqrt{m_a}\,|m_1, \ldots, m_a - 1, \ldots, m_n\rangle, \tag{VI.66a}$$

$$c_a^\dagger|m_1, \ldots, m_a, \ldots, m_n\rangle = \sqrt{m_a + 1}\,|m_1, \ldots, m_a + 1, \ldots, m_n\rangle, \tag{VI.66b}$$

where $\{|m_1,\ldots,m_n\rangle\,|\,m_a\in\mathbb{N}_0, a=1,\ldots,n\}$ forms an orthonormal basis in $\mathcal{H}$. The commutation relations (VI.64) imply that the operators $\{c_a,c_a^\dagger\}_{a=1,\ldots,n}$ have the form

$$c_a := \frac{\hat{z}^a}{\sqrt{\theta^{a\bar{a}}}}\frac{1+\mathrm{sgn}(\theta^{a\bar{a}})}{2}+\frac{\hat{z}^{\bar{a}}}{\sqrt{\theta^{\bar{a}a}}}\frac{1-\mathrm{sgn}(\theta^{a\bar{a}})}{2} \quad \text{and} \quad c_a^\dagger := \frac{\hat{z}^{\bar{a}}}{\sqrt{\theta^{a\bar{a}}}}\frac{1+\mathrm{sgn}(\theta^{a\bar{a}})}{2}+\frac{\hat{z}^{a}}{\sqrt{\theta^{\bar{a}a}}}\frac{1-\mathrm{sgn}(\theta^{a\bar{a}})}{2}. \tag{VI.67}$$

Note that here no summation over the index $a$ is implied.

The Fock space $\mathcal{H}$ is an example of a finite projective left $\mathbb{R}^{2n}_\theta$-module. Indeed, consider the projector $\hat{P}_0$ which projects onto the ground state of $\mathcal{H}$, i.e. $\hat{P}_0=|0,\ldots,0\rangle\langle 0,\ldots,0|$. Consider the rank one free $\mathbb{R}^{2n}_\theta$-module $\mathcal{E}=\mathbb{R}^{2n}_\theta$ and its left sub-module spanned by the elements of the form $\hat{f}\cdot\hat{P}_0$. Clearly, as a module it is isomorphic to $\mathcal{H}$ via $|m_1,\ldots,m_n\rangle\mapsto|m_1,\ldots,m_n\rangle\langle 0,\ldots,0|$. Moreover, it is a projective module, the complementary module being $\mathcal{E}(1-\hat{P}_0)$.

**Coordinate derivatives.** Having introduced the noncommutative Euclidean space, we now need to find the explicit form of the coordinate derivatives $\hat{\partial}_\mu$. In subsection § VI.1.4 we have discussed the representation $\delta$ of the Lie algebra $\mathfrak{h}$ of the automorphism group in the Lie algebra of derivations of some algebra $\mathcal{A}$. Let $\mathcal{A}=\mathbb{R}^{2n}_\theta$. An obvious symmetry of the commutation relations (VI.49) is the translational symmetry given by $\hat{x}^\mu\mapsto\hat{x}^\mu+\epsilon^\mu$ with $\epsilon^\mu\in\mathbb{R}$. Let us assume that the noncommutativity matrix $\theta^{\mu\nu}$ is invertible with $\theta^{-1}=(\theta_{\mu\nu})$ and $\theta_{\mu\sigma}\theta^{\sigma\nu}=\delta_\mu^{\ \nu}$. Then this symmetry is generated by the inner automorphism of $\mathbb{R}^{2n}_\theta$

$$\hat{f}\mapsto e^{-i\theta_{\mu\nu}\epsilon^\mu\hat{x}^\nu}\ \hat{f}\ e^{i\theta_{\mu\nu}\epsilon^\mu\hat{x}^\nu}. \tag{VI.68}$$

The Lie algebra of the group generated by (VI.68) is obviously $\mathfrak{h}\cong\mathbb{R}^{2n}$. Expanding formula (VI.68) to linear order in $\epsilon$, we find

$$\hat{f}\mapsto\hat{f}-\epsilon^\mu i\theta_{\mu\nu}[\hat{x}^\nu,\hat{f}]+\mathcal{O}(\epsilon^2)=\hat{f}+\epsilon^\mu\hat{\partial}_\mu\hat{f}+\mathcal{O}(\epsilon^2), \tag{VI.69}$$

where we have introduced

$$\hat{\partial}_\mu\hat{f}:=-i\theta_{\mu\nu}[\hat{x}^\nu,\hat{f}]. \tag{VI.70}$$

In the language of the $\delta$ operator from subsection § VI.1.4 we may set $\delta_\mu=\hat{\partial}_\mu$. Obviously, (VI.70) implies that $\hat{\partial}_\mu\hat{x}^\nu=\delta_\mu^{\ \nu}$. Moreover, one can easily check that this definition of coordinate derivatives satisfies the necessary properties of a derivative, i.e.

$$\hat{\partial}_\mu(\alpha\hat{f}+\beta\hat{g})=\alpha\hat{\partial}_\mu\hat{f}+\beta\hat{\partial}_\mu\hat{g} \qquad \text{and} \qquad \hat{\partial}_\mu(\hat{f}\hat{g})=(\hat{\partial}_\mu\hat{f})\hat{g}+\hat{f}(\hat{\partial}_\mu\hat{g}), \tag{VI.71}$$

where $\alpha,\beta\in\mathbb{C}$ and $\hat{f},\hat{g}\in\mathbb{R}^{2n}_\theta$. Note that we still have

$$[\hat{\partial}_\mu,\hat{\partial}_\nu]=0. \tag{VI.72}$$

The definition (VI.70) can also be seen from a slightly different point of view. Namely, consider

$$\begin{aligned}[][\hat{x}^\nu,\hat{f}] &= \hat{x}^\nu\hat{f}-\hat{f}\hat{x}^\nu=\mathcal{W}(x^\nu)\mathcal{W}(f)-\mathcal{W}(f)\mathcal{W}(x^\nu)\\ &= \mathcal{W}(x^\nu\star f-f\star x^\nu)=\mathcal{W}(i\theta^{\nu\mu}\partial_\mu f)=i\theta^{\nu\mu}\hat{\partial}_\mu\hat{f},\end{aligned}$$

where the fourth step immediately follows from the definition of the Moyal product (VI.46).

In complex coordinates (VI.63) equation (VI.70) becomes

$$\hat{\partial}_{z^a}\hat{f} = \theta_{a\bar{a}}[\hat{\bar{z}}^{\bar{a}}, \hat{f}] \qquad \text{and} \qquad \hat{\partial}_{\bar{z}^{\bar{a}}}\hat{f} = \theta_{\bar{a}a}[\hat{z}^a, \hat{f}], \tag{VI.73}$$

where

$$\hat{\partial}_{z^a} = \tfrac{1}{2}(\hat{\partial}_{2a-1} - i\hat{\partial}_{2a}) \qquad \text{and} \qquad \hat{\partial}_{\bar{z}^{\bar{a}}} = \tfrac{1}{2}(\hat{\partial}_{2a-1} + i\hat{\partial}_{2a}) \tag{VI.74}$$

for $a = 1, \ldots, n$.

**A little exercise.** Let us apply the above machinery to the simplest case of two noncommuting coordinates, namely $\mathbb{R}^2_\theta$. Then the commutation relations (VI.49) become $[\hat{x}, \hat{y}] = i\theta$ or equivalently $[\hat{z}, \hat{\bar{z}}] = 2\theta$. Furthermore, we have $[c, c^\dagger] = 1$ with $c = \hat{z}/\sqrt{2\theta}$ if $\theta > 0$. An arbitrary operator $\hat{\mathcal{O}}$ acting on $\mathcal{H} = \bigoplus_{m\in\mathbb{N}_0} \mathbb{C}|m\rangle$ is of the form

$$\hat{\mathcal{O}} = \sum_{n,m\in\mathbb{N}_0} \mathcal{O}_{nm}|n\rangle\langle m|. \tag{VI.75}$$

Later on, we will need the Weyl transform of the operator $|n\rangle\langle m|$ which we denote by $f_{nm}(x, y)$. Thus, as a little exercise it is illuminating to compute it explicitly. Note that the commutation relation $[\hat{x}, \hat{y}] = i\theta$ is just as the familiar commutation relation between position and momentum in ordinary one-dimensional quantum mechanics. It is therefore convenient to introduce $\hat{x}$-eigenvectors $|x\rangle$ and $\hat{y}$-eigenvectors $|y\rangle$ which are normalized as

$$\langle x|x'\rangle = \delta(x - x') \qquad \text{and} \qquad \langle y|y'\rangle = 2\pi\delta(y - y'), \tag{VI.76}$$

where $\delta$ denotes Dirac's delta distribution. Remember that $e^{-i\alpha\hat{y}}$ acts on states $|x\rangle$ as a translation operator, i.e.

$$e^{-i\alpha\hat{y}}|x\rangle = |x + \theta\alpha\rangle. \tag{VI.77}$$

Using the fact that $|\mathrm{Pf}(2\pi\theta)| = 2\pi\theta$, the Weyl transform (VI.54) can be rewritten as

$$f(x, y) = 2\pi\theta \int_{\mathbb{R}^2} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^2}\, \mathrm{Tr}_{\mathcal{H}}\{e^{i(k_1\hat{x}+k_2\hat{y})}\hat{f}(\hat{x}, \hat{y})\}e^{-i(k_1x+k_2y)} \tag{VI.78}$$

With the help of the Campbell-Baker-Hausdorff formula (II.4) and $\mathrm{Tr}_{\mathcal{H}}\{\cdot\} = \int_{\mathbb{R}}\langle x|\cdot|x\rangle$ we find after some simple algebraic manipulations

$$f(x, y) = \theta \int_{\mathbb{R}} \mathrm{d}k\; e^{-iky}\langle x + \tfrac{\theta}{2}k|\, \hat{f}(\hat{x}, \hat{y})\, |x - \tfrac{\theta}{2}k\rangle. \tag{VI.79}$$

This formula then implies that

$$f_{mn}(x, y) = \theta \int_{\mathbb{R}} \mathrm{d}k\; e^{-iky}\langle x + \tfrac{\theta}{2}k|n\rangle\langle m|x - \tfrac{\theta}{2}k\rangle. \tag{VI.80}$$

Let us introduce the unitary operator $\hat{U} := e^{-i(k_1\hat{x}+k_2\hat{y})}$. Using complex coordinates (VI.63) and defining $p = \frac{1}{2}(k_1 + ik_2)$, we can rewrite $\hat{U}$ as

$$\hat{U} = e^{-i(p\hat{\bar{z}}+\bar{p}\hat{z})} = e^{-i\sqrt{2\theta}(pc^\dagger+\bar{p}c)} = e^{-\frac{1}{2}|\lambda|^2}e^{\lambda c^\dagger}e^{\lambda c},$$

where we have set $\lambda := -i\sqrt{2\theta}p$. Then we define the coherent state $|\lambda\rangle := \hat{U}|0\rangle$ and therefore,

$$|\lambda\rangle\langle\lambda| = e^{-|\lambda|^2}\sum_{n,m\in\mathbb{N}_0}\frac{\lambda^n}{\sqrt{n!}}\frac{\bar{\lambda}^m}{\sqrt{m!}}|n\rangle\langle m|. \tag{VI.81}$$

The easiest way to get $f_{nm}(x,y)$ is via the generating function

$$\begin{aligned} G(\lambda,\bar{\lambda},x,y) &:= \theta\, e^{-|\lambda|^2}\int_{\mathbb{R}}\mathrm{dk}\; e^{-iky}\langle x+\tfrac{\theta}{2}k|\hat{U}|0\rangle\langle 0|\hat{U}^\dagger|x-\tfrac{\theta}{2}k\rangle \\ &= \theta\, e^{-|\lambda|^2}\int_{\mathbb{R}}\mathrm{dk}\; e^{-iky}\langle x+\tfrac{\theta}{2}k|\lambda\rangle\langle\lambda|x-\tfrac{\theta}{2}k\rangle \\ &= \sum_{n,m\in\mathbb{N}_0}\frac{\lambda^n}{\sqrt{n!}}\frac{\bar{\lambda}^m}{\sqrt{m!}}f_{nm}(x,y). \end{aligned} \tag{VI.82}$$

The action of $\hat{U}^\dagger$ on the state $|x-\frac{\theta}{2}k\rangle$ computes to

$$\hat{U}^\dagger|x-\tfrac{\theta}{2}k\rangle = e^{-\frac{i}{2}\theta k_1k_2}e^{ik_1(x-\frac{\theta}{2}k)}|x-\tfrac{\theta}{2}k-\theta k_2\rangle.$$

Remembering that $\langle x|0\rangle = (\pi\theta)^{-\frac{1}{4}}e^{-\frac{1}{2\theta}x^2}$, we obtain after Gaußian integration

$$G = 2e^{|\lambda|^2}e^{-\frac{1}{\theta}((x-\theta k_2)^2+(y+\theta k_2)^2)}. \tag{VI.83}$$

Taking our definitions of $p$ and $\lambda$ and introducing $x+iy=re^{i\phi}$, we can rewrite this formula

$$G = 2e^{-\frac{1}{\theta}r^2}e^{-|\lambda|^2}e^{\sqrt{\frac{2}{\theta}}r\bar{\lambda}e^{i\phi}}e^{\sqrt{\frac{2}{\theta}}r\lambda e^{-i\phi}} = 2e^{-\frac{1}{2}w}\sum_{n\in\mathbb{N}_0}\frac{1}{n!}(\bar{\lambda}\sqrt{w}e^{i\phi})^n(1+K)^ne^{-wK},$$

with the abbreviations $w := 2r^2/\theta$ and $K := -\lambda e^{-i\phi}/\sqrt{w}$. Remember that the Laguerre polynomials

$$L^k_m(x) = \frac{e^x x^{-k}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}(e^{-x}x^{n+k}) \tag{VI.84}$$

satisfy the identity

$$\sum_{m\in\mathbb{N}_0}L^{n-m}_m(w)K^m = e^{-wK}(1+K)^n \tag{VI.85}$$

and hence

$$G = 2e^{-\frac{1}{2}w}\sum_{n,m\in\mathbb{N}_0}\frac{1}{n!}\bar{\lambda}^n\lambda^m(-1)^m w^{\frac{1}{2}(n-m)}e^{i\phi(n-m)}L^{n-m}_m(w).$$

Comparing this result with (VI.82), we finally obtain

$$f_{mn} = 2(-1)^m\sqrt{\tfrac{m!}{n!}}e^{-\frac{1}{\theta}r^2}(\tfrac{2}{\theta}r^2)^{\frac{1}{2}(n-m)}\,e^{i\phi(n-m)}L^{n-m}_m(\tfrac{2}{\theta}r^2). \tag{VI.86}$$

**Shift operators.** A special class of operators, which turn out to be useful in our later discussions, are so-called shift operators. They are partially isometric operators sending $\mathcal{H}$ to the subspace $(1-\hat{P}_0)\mathcal{H}$, where we denote by $\hat{P}_0$ the orthogonal projector onto the ground state of $\mathcal{H}$. Then

$$\hat{S}\,:\,\mathcal{H}\to(1-\hat{P}_0)\mathcal{H}\qquad\text{with}\qquad \hat{S}^\dagger\hat{S}=1\qquad\text{and}\qquad \hat{S}\hat{S}^\dagger = 1-\hat{P}_0. \tag{VI.87}$$

Consider the case $\mathbb{R}^2_\theta$. Then the operator $\hat{S}$ may be given by the explicit formula

$$\hat{S} = \sum_{n\in\mathbb{N}_0} |n+1\rangle\langle n|. \tag{VI.88}$$

Obviously, $\hat{S}$ is a bounded operator, i.e. $\hat{S} \in \mathcal{B}(\mathcal{H})$. Consider now the four-dimensional case, i.e. $\mathbb{R}^4_\theta$. Hence, we have $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$. The naive idea to take simply the tensor product $\hat{S}^{(1)} \otimes \hat{S}^{(2)}$, where each $\hat{S}^{(a)}$ is given by (VI.88), does not work. One possible realization of the required operator $\hat{S}$ is given by the formula [62]

$$\hat{S} = 1 + \sum_{n\in\mathbb{N}_0} (|n+1\rangle\langle n| - |n\rangle\langle n|) \otimes \hat{P}_0^{(2)} = 1 + (\hat{S}^{(1)} - 1^{(1)}) \otimes \hat{P}_0^{(2)}. \tag{VI.89}$$

Introducing the integer ordering [167]

$$|k\rangle = |n_1, n_2\rangle \qquad \text{with} \qquad k = n_1 + \tfrac{1}{2}(n_1+n_2)(n_1+n_2+1), \tag{VI.90}$$

we can give another representation of $\hat{S}$, namely

$$\hat{S} = \sum_{k\in\mathbb{N}_0} |k+1\rangle\langle k|. \tag{VI.91}$$

A third possibility is as follows [129]. The index $k$ in (VI.90) introduces the ordering

$$\{|k\rangle\} = \{|0,0\rangle, |0,1\rangle, |1,0\rangle, |0,2\rangle, \ldots\}$$

Let $N \in \mathbb{N}$ with $N > 1$. Then we move the states $|0,i\rangle$ with $0 < i \leq N-1$ in the above sequence to the left and enumerate this new order by $\hat{k}$

$$\{|\hat{k}\rangle\} = \{|0,0\rangle, |0,1\rangle, \ldots, |0,N-1\rangle, |1,0\rangle \ldots\}.$$

The shift operator defined by

$$\hat{S} = \sum_{k\in\mathbb{N}_0} |\hat{k}+1\rangle\langle\hat{k}| \tag{VI.92}$$

has also the desired properties. We could now continue in order to find such shift operators for higher dimensional cases. This, however, will not be too illuminating and moreover, they are not needed in later discussions.

## § VI.2 Yang-Mills Field Theory on Free $\mathbb{R}^{2n}_\theta$-Modules

Having given the mathematical basics which are needed for discussing noncommutative field theories, we shall finally come to Yang-Mills theory defined on free modules.

### § VI.2.1 Action and Equations of Motion

**Noncommutative Yang-Mills action and equations of motion.** Since all vector bundles over $\mathbb{R}^{2n}$ are trivial, the corresponding $\mathbb{R}^{2n}_\theta$-modules are free by the above discussion. Thus, let $\mathcal{E}$ be a free $\mathbb{R}^{2n}_\theta$-module of complex rank $r$. Assume that the gauge group is $G$, which acts on $\mathcal{E}$ by some representation $\mathfrak{R}$. The gauge algebra is denoted by $\mathfrak{g}$. Note that as we shall see momentarily not all (classical) Lie groups are allowed. Let us define the connection $\nabla = \nabla^0 + \mathcal{A}$ on $\mathcal{E}$ with respect to $\mathbb{R}^{2n}$, the Lie algebra of the automorphism group generated by (VI.68). Here $\nabla^0$ is the Graßmannian connection (VI.24) and $\mathcal{A} \in \Omega^1(\mathcal{A})$. Thus, the connection takes the form $\nabla_\mu = \partial_\mu + \mathcal{A}_\mu$ in some fixed basis of $\mathbb{R}^{2n}$, since $P = \mathbb{1}_r$ for a free module. Then the Yang-Mills action functional may be written in the form

$$S^{\mathrm{E}}_{\mathrm{YM}}(\mathcal{A}) = -\frac{1}{g^2_{\mathrm{YM}}} \int_{\mathbb{R}^{2n}} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}), \tag{VI.93}$$

where ‘tr’ denotes the trace over the gauge algebra $\mathfrak{g}$. Here the wedge product is understood such that

$$\omega_\mu \, \mathrm{d}\, x^\mu \wedge \rho_\nu \, \mathrm{d}\, x^\nu = \omega_\mu \star \rho_\nu \, \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu, \tag{VI.94}$$

for instance. The Graßmannian curvature (VI.30) is identically zero. Therefore, the curvature (VI.31) reduces to

$$\mathcal{F} = \mathrm{d}\, \mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{VI.95}$$

or in components

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\mu]_\star. \tag{VI.96}$$

Equivalently, by using the Weyl transform (VI.54) we may write

$$S^{\mathrm{E}}_{\mathrm{YM}}(\hat{\mathcal{A}}) = -\frac{1}{2g^2_{\mathrm{YM}}} |\mathrm{Pf}(2\pi\theta)| \, \mathrm{Tr}_{\mathcal{H}} \, \mathrm{tr}\{\hat{\mathcal{F}}^{\mu\nu} \hat{\mathcal{F}}_{\mu\nu}\}. \tag{VI.97}$$

The Yang-Mills equations following from these action functionals remain in their familiar form, i.e.

$$[\nabla_\mu, \mathcal{F}^{\mu\nu}]_\star = 0 \qquad \text{or} \qquad [\hat{\nabla}_\mu, \hat{\mathcal{F}}^{\mu\nu}] = 0, \tag{VI.98}$$

as one may readily check by varying (VI.93) and (VI.97), respectively. Note that for $n = 2$ the SDYM/ASDYM equations retain their familiar form, i.e.

$$\mathcal{F}_{\mu\nu} = \pm\tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma} \qquad \text{or} \qquad \hat{\mathcal{F}}_{\mu\nu} = \pm\tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} \hat{\mathcal{F}}^{\rho\sigma}. \tag{VI.99}$$

In this case, the instanton number is given by

$$Q = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}) = -\frac{1}{16\pi^2} |\mathrm{Pf}(2\pi\theta)| \, \mathrm{Tr}_{\mathcal{H}} \, \mathrm{tr}\{\hat{\mathcal{F}}^{\mu\nu} * \hat{\mathcal{F}}_{\mu\nu}\}. \tag{VI.100}$$

Also the gauge transformations, which leave the Yang-Mills action invariant, take their usual form. Namely, we have

$$\mathcal{A} \mapsto \mathcal{A}^g = g^{-1} \mathcal{A} g + g^{-1} \, \mathrm{d}\, g \qquad \Longrightarrow \qquad \mathcal{F} \mapsto \mathcal{F}^g = g^{-1} \mathcal{F} g \tag{VI.101}$$

for $g \in GL_{\mathbb{R}^{2n}_\theta}(\mathcal{E})$, the invertible endomorphisms of $\mathcal{E}$.

**Which gauge groups are allowed?** As already mentioned, not all matrix groups are allowed as possible candidates for gauge groups [188, 139]. The reason for that lies in the noncommutativity. Let us discuss a few examples. Remember that the Moyal product satisfies by construction $(f \star g)^\dagger = g^\dagger \star f^\dagger$. Let $\mathfrak{g}$ be some Lie algebra. Then for any $X, Y \in \mathfrak{g}$ we have

$$([X, Y]_\star)^\dagger = -[X^\dagger, Y^\dagger]_\star. \tag{VI.102}$$

This shows that $\mathfrak{u}(n)$ closes under the Moyal commutator and hence $\mathfrak{u}(n)$ is a good condidate for a gauge algebra. But what about $\mathfrak{su}(n)$? Consider $\mathfrak{su}(2)$ and choose the following two matrices

$$X = \begin{pmatrix} ix^\mu & 0 \\ 0 & -ix^\mu \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} ix^\nu & 0 \\ 0 & -ix^\nu \end{pmatrix}. \tag{VI.103}$$

Clearly, they are elements of $\mathfrak{su}(2)$. The Moyal commutator is easily computed to be

$$[X, Y]_\star = i\theta^{\mu\nu} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{VI.104}$$

This shows that $\mathfrak{su}(2)$ and hence $\mathfrak{su}(n)$ do not close under the Moyal commutator. Therefore $\mathfrak{su}(n)$ can not be realized as a gauge algebra on noncommutative Euclidean space.

Similar considerations go through for $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$. For these two Lie algebras one may again first consider the two-dimensional case and choose the matrices

$$X = \begin{pmatrix} 0 & x^\mu \\ -x^\mu & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & x^\nu \\ -x^\nu & 0 \end{pmatrix}. \tag{VI.105}$$

One again finds that the Moyal commutator does not close. Thus, $\mathfrak{so}(n)$, $\mathfrak{o}(n)$ and $\mathfrak{sp}(n)$ do not work in the noncommuative regime.

### § VI.2.2 Seiberg-Witten Map

In this last subsection we want to answer the question to what extent the above discussed noncommutative generalization of an ordinary gauge theory is truly novel meaning to what extent we can understand this generalization as a conventional gauge theory. In their paper about *String theory and noncommutative geometry* [175], Seiberg and Witten introduced an explicit map between the noncommuative and commutative gauge potentials.

**Seiberg-Witten map.** Consider again a complex rank $r$ free module $\mathcal{E}$ over $\mathbb{R}^{2n}_\theta$. Let $G$ be the gauge group with gauge algebra $\mathfrak{g}$ and $\alpha \in \mathrm{End}_{\mathbb{R}^{2n}_\theta}(\mathcal{E})$. Then the infinitesimal form of the gauge transformations (VI.101) is given by

$$\delta_\alpha \mathcal{A}_\mu = \partial_\mu \alpha + [\mathcal{A}_\mu\, \alpha]_\star \qquad \text{and} \qquad \delta_\alpha \mathcal{F}_{\mu\nu} = [\mathcal{F}_{\mu\nu}, \alpha]_\star. \tag{VI.106}$$

The Seiberg-Witten map is derived from the condition that

$$\mathcal{A}(\mathcal{A}(\theta);\theta+\delta\theta)+\delta_{\alpha(\theta+\delta\theta)}\mathcal{A}(\mathcal{A}(\theta);\theta+\delta\theta)=\mathcal{A}(\mathcal{A}(\theta)+\delta_{\alpha(\theta)}\mathcal{A}(\theta);\theta+\delta\theta), \tag{VI.107}$$

i.e. it relates the gauge potential $\mathcal{A}(\theta)$ at some value $\theta$ with another gauge potential $\mathcal{A}(\theta+\delta\theta)$ at some nearby value $\theta+\delta\theta$. In the following we denote the Moyal product (VI.46) at scale $\theta$ by '$\star_\theta$'. The particular choice (VI.107) ensures that a gauge transformation of $\mathcal{A}(\theta)$ by $\alpha(\theta)$ is equivalent to a gauge transformation of $\mathcal{A}(\theta+\delta\theta)$ by $\alpha(\theta+\delta\theta)$, so that gauge equivalent fields are mapped to gauge equivalent fields. Note that the gauge parameter $\alpha(\theta+\delta\theta)$ is allowed to depend not only on $\alpha(\theta)$ but also on the gauge potential $\mathcal{A}(\theta)$. The condition (VI.107) does not determine the map uniquely. However, it can be shown that different maps are related by gauge transformations and field redefinitions [3].

Let us work at linear order in $\delta\theta$ and write

$$\mathcal{A}_\mu(\mathcal{A}(\theta);\theta+\delta\theta)=\mathcal{A}_\mu(\mathcal{A}(\theta);\theta)+\delta\mathcal{A}_\mu(\mathcal{A}(\theta);\theta)\quad\text{and}\quad\alpha(\theta+\delta\theta)=\alpha(\theta)+\delta\alpha(\theta). \tag{VI.108}$$

The expansion of (VI.107) yields

$$\begin{aligned}\delta\mathcal{A}_\mu(\mathcal{A}(\theta)+\delta_{\alpha(\theta)}\mathcal{A}(\theta);\theta))-\delta\mathcal{A}_\mu(\mathcal{A}(\theta);\theta)-\partial_\mu\delta\alpha(\theta)-[\mathcal{A}_\mu(\mathcal{A}(\theta);\theta),\delta\alpha(\theta)]_{\star_\theta}\\ -[\delta\mathcal{A}_\mu(\mathcal{A}(\theta);\theta)]_{\star_\theta}=\tfrac{i}{2}\delta\theta^{\sigma\rho}\{\partial_\sigma\mathcal{A}_\mu(\mathcal{A}(\theta);\theta),\partial_\rho\alpha(\theta)\}_{\star_\theta}+\mathcal{O}(\delta\theta^2),\end{aligned} \tag{VI.109}$$

where $\{\ ,\ \}_{\star_\theta}$ denotes the anticommutator. Here we have used (VI.106) and remembered that

$$f\star_{\theta+\delta\theta}g=f\star_\theta g+\tfrac{i}{2}\delta\theta^{\mu\nu}\partial_\mu f\star_\theta\partial_\nu g+\cdots \tag{VI.110}$$

For brevity let us write $\mathcal{A}(\theta)$ for $\mathcal{A}(\mathcal{A}(\theta);\theta)$. Equation (VI.109) is solved by [175]

$$\mathcal{A}_\mu(\theta+\delta\theta)=\mathcal{A}_\mu(\theta)+\delta\mathcal{A}_\mu(\theta)=\mathcal{A}_\mu(\theta)-\tfrac{i}{4}\delta\theta^{\sigma\rho}\{\mathcal{A}_\sigma(\theta),\partial_\rho\mathcal{A}_\mu(\theta)+\mathcal{F}_{\rho\mu}(\theta)\}_{\star_\theta}, \tag{VI.111a}$$

$$\alpha(\theta+\delta\theta)=\alpha(\theta)+\delta\alpha(\theta)=\alpha(\theta)+\tfrac{i}{4}\delta\theta^{\sigma\rho}\{\partial_\sigma\alpha(\theta),\mathcal{A}_\rho(\theta)\}_{\star_\theta}, \tag{VI.111b}$$

as one may check by substituting these equations into (VI.109). These equations are known as the Seiberg-Witten map. Thus, we have a relation which expresses noncommutative gauge fields in terms of commutative ones. Formula (VI.111a) implies for the Yang-Mills curvature $\mathcal{F}_{\mu\nu}(\theta+\delta\theta)=\mathcal{F}_{\mu\nu}(\theta)+\delta\mathcal{F}_{\mu\nu}(\theta)$ that

$$\delta\mathcal{F}_{\mu\nu}(\theta)=\tfrac{i}{4}\delta\theta^{\sigma\rho}(2\{\mathcal{F}_{\mu\sigma}(\theta),\mathcal{F}_{\nu\rho}(\theta)\}_{\star_\theta}-\{\mathcal{A}_\sigma(\theta),\nabla_\rho\mathcal{F}_{\mu\nu}(\theta)+\partial_\rho\mathcal{F}_{\mu\nu}(\theta)\}_{\star_\theta}). \tag{VI.112}$$

The authors of [153, 143, 134] gave an explicit solution (to all orders of $\delta\theta$) to (VI.107) for the $U(1)$ case.

# Chapter VII

## Noncommutative Yang-Mills Instantons

In this chapter we will generalize the BPST instanton solution discussed in chapter III to Yang-Mills theory which is defined on the noncommutative Euclidean space $\mathbb{R}^4_\theta$. We first discuss its construction using the dressing and splitting approach, and then we explain the construction of multi-instantons à la 't Hooft. The ideas presented below are based on [129, 91]. In the sequel we will mainly work in the operator formalism. We therefore omit the hats over the operators except for situations where confusion may arise. Throughout this chapter we consider a complex rank two free $\mathbb{R}^4_\theta$-module $\mathcal{E}$ and assume that the gauge group is $U(2)$ which acts on $\mathcal{E}$ in the fundamental representation. Thus, the connection on $\mathcal{E}$ reads $\nabla_\mu = \partial_\mu + \mathcal{A}_\mu$, where $\partial_\mu$ is given by (VI.70) and $\mathcal{A}_\mu \in \mathrm{End}_{\mathbb{R}^4_\theta}(\mathcal{E})$ with $\mathcal{A}^\dagger_\mu = -\mathcal{A}_\mu$.

**Remark on Abelian instantons.** As already mentioned, $\mathbb{R}^4_\theta$ allows for Abelian instantons which do not survive in the commutative limit [147]. In this chapter we will focus on non-Abelian instanton solutions. The discussion of Abelian instantons is postponed to chapter IX on noncommutative Seiberg-Witten theory, since as we have already seen the ASDYM equations are just a special case of the Seiberg-Witten monopole equations.

## § VII.1 Noncommutative $U(2)$ BPST Instanton

Consider the noncommutative Euclidean space $\mathbb{R}^4_\theta$ with the noncommutativity tensor $\theta^{\mu\nu}$ restricted to be self-dual, i.e. we choose

$$\theta^{12} = -\theta^{21} = \theta^{34} = -\theta^{43} =: \theta > 0 \tag{VII.1}$$

and all other components being identically zero. As in chapter III, we introduce complex coordinates[1] on $\mathbb{R}^4_\theta \cong \mathbb{C}^2_\theta$

$$z^1 = x^1 + ix^2, \quad \bar{z}^{\bar{1}} = x^1 - ix^2, \quad z^2 = x^3 - ix^4, \quad \bar{z}^{\bar{2}} = x^3 + ix^4 \tag{VII.2}$$

[1]Note that this choice differs from equation (VI.63) by a sign. This is due to the particular orientation on $\mathbb{R}^4_\theta$. See chapter III.

implying the commutation relations

$$[z^1, \bar{z}^{\bar{1}}] = 2\theta \qquad \text{and} \qquad [z^2, \bar{z}^{\bar{2}}] = -2\theta. \tag{VII.3}$$

Therefore, the annihilation and creation operators (VI.67) are given by

$$c_1 = \tfrac{1}{\sqrt{2\theta}} z^1, \qquad c_1^\dagger = \tfrac{1}{\sqrt{2\theta}} \bar{z}^{\bar{1}} \qquad \text{and} \qquad c_2 = \tfrac{1}{\sqrt{2\theta}} \bar{z}^{\bar{2}}, \qquad c_2^\dagger = \tfrac{1}{\sqrt{2\theta}} z^2. \tag{VII.4}$$

The coordinate derivatives (VI.70) become

$$\partial_{z^1} f = -\tfrac{1}{2\theta}[\bar{z}^{\bar{1}}, f] \qquad \text{and} \qquad \partial_{\bar{z}^{\bar{1}}} f = \tfrac{1}{2\theta}[z^1, f] \tag{VII.5a}$$

as well as

$$\partial_{z^2} f = \tfrac{1}{2\theta}[\bar{z}^{\bar{2}}, f] \qquad \text{and} \qquad \partial_{\bar{z}^{\bar{2}}} f = -\tfrac{1}{2\theta}[z^2, f] \tag{VII.5b}$$

for $f \in \mathbb{R}^4_\theta$. Again, we define

$$\mathcal{A}_{z^1} = \tfrac{1}{2}(\mathcal{A}_1 - i\mathcal{A}_2) \qquad \text{and} \qquad \mathcal{A}_{z^2} = \tfrac{1}{2}(\mathcal{A}_3 + i\mathcal{A}_4) \tag{VII.6a}$$

as well as

$$\mathcal{A}_{\bar{z}^{\bar{1}}} = \tfrac{1}{2}(\mathcal{A}_1 + i\mathcal{A}_2) \qquad \text{and} \qquad \mathcal{A}_{\bar{z}^{\bar{2}}} = \tfrac{1}{2}(\mathcal{A}_3 - i\mathcal{A}_4), \tag{VII.6b}$$

such that the SDYM equations (VI.99) are again

$$\mathcal{F}_{z^1 z^2} = 0, \qquad \mathcal{F}_{\bar{z}^{\bar{1}} \bar{z}^{\bar{2}}} = 0 \qquad \text{and} \qquad \mathcal{F}_{z^1 \bar{z}^{\bar{1}}} + \mathcal{F}_{z^2 \bar{z}^{\bar{2}}} = 0. \tag{VII.7}$$

### § VII.1.1 $U(2)$ BPST Instanton from the Dressing Appraoch

**Linear system.** From chapter III we know that the SDYM equations are the integrability conditions of some linear system. In subsection § III.3.1 we have considered

$$(\nabla_{\bar{z}^{\bar{1}}} - \lambda \nabla_{z^2})\psi = 0, \qquad (\nabla_{\bar{z}^{\bar{2}}} + \lambda \nabla_{z^1})\psi = 0 \qquad \text{and} \qquad \partial_{\bar{\lambda}}\psi = 0, \tag{VII.8}$$

with $\lambda \in \mathbb{CP}^1$. Also in the noncommutative setup we can use this system. But this time, $\psi$ is regarded as an operator living on the noncommutative twistor space

$$\mathcal{Z}_\theta = \mathbb{R}^4_\theta \times \mathbb{CP}^1, \tag{VII.9}$$

and depending meromorphically on the (commutative) spectral parameter $\lambda$.

Again we impose the following reality condition on $\psi$

$$[\psi(\tau(x, \lambda))]^\dagger = \psi^{-1}(x, \lambda), \tag{VII.10}$$

where $\tau$ is given by (III.41). Of course, we have to be careful by imposing this condition since now $\psi$ is a matrix of infinite dimension (with respect to $\mathcal{H}$) and the inverse may not exist.

We rewrite (VII.8) in its familiar form

$$\psi(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\psi^\dagger = \mathcal{A}_{\bar{z}^{\bar{1}}} - \lambda\mathcal{A}_{z^2}, \tag{VII.11a}$$

$$\psi(\partial_{\bar{z}^{\bar{2}}} + \lambda\partial_{z^1})\psi^\dagger = \mathcal{A}_{z^2} + \lambda\mathcal{A}_{z^1}. \tag{VII.11b}$$

Remember that $\psi^\dagger$ is a shorthand notation for $[\psi(x, -\bar{\lambda}^{-1})]^\dagger$.

Also the dressing method discussed in § III.3.1 remains the same, i.e. we consider

$$\tilde{\psi}(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda) = \chi(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda)\, \psi(z^1, \bar{z}^{\bar{1}}, z^2, \bar{z}^{\bar{2}}, \lambda). \tag{VII.12}$$

and

$$\chi = \lambda R_{-1} + R_0 + \sum_{i=1}^{r} \frac{R_i}{(\mu_i\lambda + \nu_i)}, \tag{VII.13}$$

where $\mu_i$, $\nu_i \in \mathbb{C}$ and $R_{-1}$, $R_0$ and $R_i$ depend on $\{z^a\}$ and $\{\bar{z}^{\bar{a}}\}$ but not on the spectral parameter $\lambda$.

**Ansatz and noncommutative BPST instanton.** Let us obtain a one-instanton solution by way of dressing. We again take the trivial solution $(\psi, \mathcal{A}_\mu) = (1, 0)$ as the seed solution and choose the dressing factor of the form [12]

$$\psi(x, \lambda) = G\left(1 + 2H + \lambda S^\dagger + \frac{1}{\lambda}S\right), \tag{VII.14}$$

where all $\lambda$ dependence is made explicit and $G$ and $H$ are taken to be Hermitian diagonal $2 \times 2$ matrix functions of $\{z^a, \bar{z}^{\bar{a}}\}$, i.e.

$$G =: \mathrm{diag}(g_-, g_+) \qquad \text{and} \qquad H =: \mathrm{diag}(h_-, h_+). \tag{VII.15}$$

From equation (VII.14) we obtain

$$[\psi(x, -\bar{\lambda}^{-1})]^\dagger = \left(1 + 2H - \lambda S^\dagger - \frac{1}{\lambda}S\right) G. \tag{VII.16}$$

In order to simplify further calculations we put some restrictions on $H$ and $G$, namely we require that

$$[G, H] = [G, S] = [H, S] = 0. \tag{VII.17}$$

Then the reality condition (VII.10) implies that

$$S^2 = 0 \qquad \text{and} \qquad (S^\dagger)^2 = 0 \tag{VII.18a}$$

as well as

$$G^2(1 + 2H)^2 = 1 + G^2\{S, S^\dagger\}, \tag{VII.18b}$$

since

$$\begin{aligned} \psi\psi^\dagger &= G\left(1 + 2H + \lambda S^\dagger + \frac{1}{\lambda}S\right)\left(1 + 2H - \lambda S^\dagger - \frac{1}{\lambda}S\right) G \\ &= G^2\Big((1 + 2H)^2 - \{S, S^\dagger\} - \lambda^2(S^\dagger)^2 - \frac{1}{\lambda^2}S^2\Big) \overset{!}{=} 1. \end{aligned}$$

The last equality has to be true for all $\lambda$ and hence by comparison of coefficients we conclude (VII.18). The $\lambda$ dependence of the differential equations (VII.11) leads to the following equations

$$G\big\{S^\dagger\partial_{z^a}S^\dagger\big\}G = 0 \tag{VII.19a}$$

as well as

$$G\big\{(1+2H)\partial_{z^a}S^\dagger - S^\dagger\partial_{z^a}(1+2H) + \epsilon_{ab}S^\dagger\partial_{\bar{z}^{\bar{b}}}S^\dagger\big\}G = 0 \tag{VII.19b}$$

for $a = 1, 2$. To arrive at these equations, we note that (VII.11) has to hold for all $\lambda$ and hence by a similar computation we obtain (VII.19). Moreover, the nilpotency of $S^\dagger$ has been used. We shall find matrix functions $H$ and $S$ for which the brackets above will vanish by themselves.

In order to construct a $2\times 2$ matrix $S$ satisfying (VII.18a) and (VII.19a) (which do not involve $H$), we consider the two vectors

$$T_1 = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} \qquad \text{and} \qquad T_2 = \begin{pmatrix} v_2^\dagger \\ v_1^\dagger \end{pmatrix} \tag{VII.20}$$

built from functions $v_1$ and $v_2$ and introduce some real function $K = K(r^2)$ which depends only on the combination

$$r^2 := \bar{z}^{\bar{1}}z^1 + \bar{z}^{\bar{2}}z^2. \tag{VII.21}$$

With these ingredients we parametrize $S^\dagger$ as follows,

$$S^\dagger = T_1\frac{1}{K}T_2^\dagger = \begin{pmatrix} v_1\frac{1}{K}v_2 & v_1\frac{1}{K}v_1 \\ -v_2\frac{1}{K}v_2 & -v_2\frac{1}{K}v_1 \end{pmatrix}, \tag{VII.22}$$

and expect conditions on $v_1$, $v_2$, and $K$. Due to the nilpotency property (VII.18a) of $S$ and $S^\dagger$ we find $[v_1, v_2] = 0$, since

$$(S^\dagger)^2 = \begin{pmatrix} v_1\frac{1}{K}[v_2,v_1]\frac{1}{K}v_2 & v_1\frac{1}{K}[v_2,v_1]\frac{1}{K}v_1 \\ -v_2\frac{1}{K}[v_2,v_1]\frac{1}{K}v_2 & -v_2\frac{1}{K}[v_2,v_1]\frac{1}{K}v_1 \end{pmatrix} \stackrel{!}{=} 0$$

and hence $T_1^\dagger T_2 = T_2^\dagger T_1 = 0$. The condition (VII.19a) then tells us that $v_1$ and $v_2$ are anti-holomorphic functions depending on $\bar{z}^{\bar{1}}$ and $\bar{z}^{\bar{2}}$ only. A simple choice is $v_1 = \bar{z}^{\bar{1}}$ and $v_2 = \bar{z}^{\bar{2}}$, which specifies

$$S^\dagger = \begin{pmatrix} \bar{z}^{\bar{1}}\frac{1}{K}\bar{z}^{\bar{2}} & \bar{z}^{\bar{1}}\frac{1}{K}\bar{z}^{\bar{1}} \\ -\bar{z}^{\bar{2}}\frac{1}{K}\bar{z}^{\bar{2}} & -\bar{z}^{\bar{2}}\frac{1}{K}\bar{z}^{\bar{1}} \end{pmatrix}. \tag{VII.23}$$

The commutation property (VII.17) of $G$ and $H$ with this form of $S$ can be achieved by choosing $g_\pm$ and $h_\pm$ in (VII.15) to depend only on $r^2$ and by relating them via

$$g_\pm(r^2) = g(r^2 \pm 2\theta) \qquad \text{and} \qquad h_\pm(r^2) = h(r^2 \pm 2\theta) \tag{VII.24}$$

to yet unknown scalar functions $g$ and $h$. To see this we note that in verifying (VII.17) one meets

$$\big[\bar{z}^{\bar{1}}\tfrac{1}{K}\bar{z}^{\bar{2}}, g_-\big] = 0 \qquad \text{and} \qquad \bar{z}^{\bar{1}}\tfrac{1}{K}\bar{z}^{\bar{1}}g_+ - g_-\bar{z}^{\bar{1}}\tfrac{1}{K}\bar{z}^{\bar{1}} = 0,$$

for instance, what is clearly satisfied for the above choice. It is useful to define the notation $f_\pm(r^2) := f(r^2 \pm 2\theta)$ for arbitrary functions $f \in \mathbb{R}^4_\theta$.

Next we address equations (VII.19b). Remembering that partial derivatives are given by the inner derivations (VII.5) we reformulate:

$$\begin{aligned} G\big\{(1+2H)\,\bar z^{\bar 1} S^\dagger - S^\dagger \bar z^{\bar 1}(1+2H) + S^\dagger z^2 S^\dagger\big\}G &= 0, && \text{(VII.25a)}\\ G\big\{(1+2H)\,\bar z^{\bar 2} S^\dagger - S^\dagger \bar z^{\bar 2}(1+2H) - S^\dagger z^1 S^\dagger\big\}G &= 0. && \text{(VII.25b)} \end{aligned}$$

Demanding that the brackets vanish, gives us two relations for the functions $h$ and $K$. We claim that these are fulfilled for

$$h(r^2) = -\tfrac{1}{2}\Big(1 + \frac{K(r^2)+2C}{K(r^2)}\Big) \qquad \text{and} \qquad K(r^2+2\theta) = K(r^2) + 2\theta \tag{VII.26}$$

with $C \in \mathbb{R}$ being some (almost) arbitrary constant. The proof of this assertion is rather straightforward and we therefore show this only for one component of equation (VII.25a), i.e. for $\{(1+2H)\,\bar z^{\bar 1} S^\dagger - S^\dagger \bar z^{\bar 1}(1+2H) + S^\dagger z^2 S^\dagger\}_{ij}$ with $i=j=1$. Before proving it, however, we note that

$$\begin{aligned} z^1 f(r^2) &= f(r^2+2\theta) z^1 && \text{and} && \bar z^{\bar 1} f(r^2) = f(r^2-2\theta)\bar z^{\bar 1}, && \text{(VII.27a)}\\ z^2 f(r^2) &= f(r^2-2\theta) z^2 && \text{and} && \bar z^{\bar 2} f(r^2) = f(r^2+2\theta)\bar z^{\bar 2}, && \text{(VII.27b)} \end{aligned}$$

where $f \in \mathbb{R}^4_\theta$ depends only on $r^2$. Then we find

$$\begin{aligned} \{(1+2H)\bar z^{\bar 1} S^\dagger\}_{11} &= (1+2h_-)\bar z^{\bar 1}\bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 2} = -\bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1}(K_+ + 2C)\tfrac{1}{K}\bar z^{\bar 2},\\ -\{S^\dagger \bar z^{\bar 1}(1+2H)\}_{11} &= -\bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 2}\bar z^{\bar 1}(1+2h_-) = \bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1}(K+2C)\tfrac{1}{K}\bar z^{\bar 2},\\ \{S^\dagger z^2 S^\dagger\}_{11} &= \bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1}\bar z^{\bar 2} z^2 \tfrac{1}{K}\bar z^{\bar 2} - \bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1} z^2 \bar z^{\bar 2}\tfrac{1}{K}\bar z^{\bar 2}. \end{aligned}$$

Summing up these terms yields

$$\bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1}\{-(K_+ + 2C) + (K+2C) + \underbrace{(\bar z^{\bar 2} z^2 - z^2\bar z^{\bar 2})}_{=2\theta}\}\tfrac{1}{K}\bar z^{\bar 2} = \bar z^{\bar 1}\tfrac{1}{K}\bar z^{\bar 1}\{K - K_+ + 2\theta\}\tfrac{1}{K}\bar z^{\bar 2}$$

and hence this expression vanishes if (VII.26) does hold. Similarly, by considering the other components and the other equation (VII.25b) one always discovers (VII.26). The general solution to the functional equation in (VII.26) is

$$K(r^2) = r^2 + \tilde K(r^2) \tag{VII.28}$$

with an arbitrary $2\theta$-periodic function $\tilde K$. By imposing a smooth $\theta \to 0$ limit, we force $\tilde K$ to be constant, i.e.

$$K(r^2) = r^2 + 2\Lambda^2 - 2C \tag{VII.29}$$

with some new real constant $\Lambda^2 \geq 0$. The positivity of $\Lambda^2$ will follow from the Hermiticity of $g$. We are going to see that $\Lambda$ corresponds to the size of the instanton. The corresponding $h$ reads

$$h(r^2) = -\frac{r^2 + 2\Lambda^2 - C}{r^2 + 2\Lambda^2 - 2C}. \tag{VII.30}$$

A particularly simple choice is $C = 0$ which results in $h = -1$. More generally, we should only exclude the values $C = \Lambda^2 + n\theta$ for $n \in \mathbb{N}$, where $H$ becomes singular.

It remains to impose the condition (VII.18b) which serves to determine the function $g$. With the help of

$$\{S, S^\dagger\} = \begin{pmatrix} \frac{(r^2)_-^2}{K_-^2} & 0 \\ 0 & \frac{(r^2)_+^2}{K_+^2} \end{pmatrix} \tag{VII.31}$$

equation (VII.18b) reduces to

$$g^2(1+2h)^2 = 1 + g^2\frac{(r^2)^2}{K^2}$$

which yields

$$g(r^2) = \pm\frac{1}{2\Lambda}\frac{r^2+2\Lambda^2-2C}{\sqrt{r^2+\Lambda^2}}. \tag{VII.32}$$

During this computation we tacitly assumed that all quantities are well behaved such that we could legally perform all necessary operations. However, this needs not always be the case. As a prime example, $(r^2)_-^{-1} \equiv (r^2-2\theta)^{-1}$ is ill behaved when acting on the ground state $|0,0\rangle$ of $\mathcal{H}$. Luckily, the potentially dangerous $K_-^{-1}$ is regulated by $\Lambda^2$. Only the limit $\Lambda^2 \to 0$ is apparently singular. The results above may be recombined to compose the final expression for $\psi$

$$\psi = \frac{1}{2\Lambda}\begin{pmatrix} \frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0 \\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}} \end{pmatrix} \begin{pmatrix} r^2+2\Lambda^2-2\theta-\lambda\bar{z}^{\bar{1}}\bar{z}^{\bar{2}}-\frac{z^1z^2}{\lambda} & -\lambda(\bar{z}^{\bar{1}})^2+\frac{(\bar{z}^{\bar{2}})^2}{\lambda} \\ \lambda(\bar{z}^{\bar{2}})^2-\frac{(z^1)^2}{\lambda} & r^2+2\Lambda^2+2\theta+\lambda\bar{z}^{\bar{1}}\bar{z}^{\bar{2}}+\frac{z^1z^2}{\lambda} \end{pmatrix}, \tag{VII.33}$$

where we have chosen the negative sign in (VII.32). The $\theta$ dependence of this noncommutative solution is very simple. It is also quite remarkable that it differs from the commutative solution (as given by Belavin and Zakharov [12]; cf. equation (III.73)) only in a few spots. Moreover, the final expression is independent of the parameter $C$, in accordance with the latter's interpretation as a regulator of intermediate singularities in the course of the calculation.

**Connection, curvature and charge.** We are now in the position to construct the gauge potential using our solution (VII.33). By looking at the terms of (VII.11) which are linear in $\lambda$, we can compute $\mathcal{A}_{z^1}$ and $\mathcal{A}_{z^2}$. The calculation is a little tedious but straightforward. We therefore omit here the explicit derivation and postpone it to subsection § VII.1.2. The results are

$$\mathcal{A}_{z^1} = \begin{pmatrix} -\frac{\bar{z}^{\bar{1}}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right) & 0 \\ -\bar{z}^{\bar{2}}\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}} & -\frac{\bar{z}^{\bar{1}}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right) \end{pmatrix} \tag{VII.34a}$$

and

$$\mathcal{A}_{z^2} = \begin{pmatrix} \left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right)\frac{\bar{z}^{\bar{2}}}{2\theta} & -\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}}\bar{z}^{\bar{1}} \\ 0 & \left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right)\frac{\bar{z}^{\bar{2}}}{2\theta} \end{pmatrix} \tag{VII.34b}$$

which in the commutative limit $\theta \to 0$ coincide with the Belavin-Zakharov solution [12] given by (III.74). At first sight, it seems that the components of the gauge potential are singular at $\theta \to 0$. But the small $\theta$-expansion yields,

$$-\frac{\bar{z}^{\bar{1}}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right) = -\frac{\bar{z}^{\bar{1}}}{2\theta}\left[\left(1-\frac{2\theta}{2(r^2+\Lambda)^2}\right)-1\right]+\mathcal{O}(\theta^2) \xrightarrow{\theta\to 0} \frac{\bar{z}^{\bar{1}}}{2(r^2+\Lambda^2)},$$

for instance. Comparing this expression with (III.74), we recognize full agreement. The remaining components are $\mathcal{A}_{\bar{z}^{\bar{a}}} = -\mathcal{A}^\dagger_{z^a}$ for $a = 1, 2$.

Furthermore, we compute the nonvanishing components of the Yang-Mills curvature using $\mathcal{F}_{z^a\bar{z}^{\bar{b}}} = \partial_{z^a}\mathcal{A}_{\bar{z}^{\bar{b}}} - \partial_{\bar{z}^{\bar{b}}}\mathcal{A}_{z^a} + [\mathcal{A}_{z^a}, \mathcal{A}_{\bar{z}^{\bar{b}}}]$ for $a, b = 1, 2$. Again, this task is lengthy but not difficult, just matrix multiplication and computing commutators. Ultimately we find the self-dual configuration

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} = -\mathcal{F}_{z^2\bar{z}^{\bar{2}}} = \begin{pmatrix} -\frac{\Lambda^2}{(r^2+\Lambda^2)(r^2+\Lambda^2-2\theta)} & 0 \\ 0 & \frac{\Lambda^2}{(r^2+\Lambda^2)(r^2+\Lambda^2+2\theta)} \end{pmatrix} \tag{VII.35a}$$

and

$$\mathcal{F}_{z^1\bar{z}^{\bar{2}}} = \mathcal{F}^\dagger_{z^2\bar{z}^{\bar{1}}} = \begin{pmatrix} 0 & 0 \\ \frac{2\Lambda^2}{(r^2+\Lambda^2)\sqrt{r^2+\Lambda^2-2\theta}\sqrt{r^2+\Lambda^2+2\theta}} & 0 \end{pmatrix}. \tag{VII.35b}$$

Comparing this solution, constructed by the dressing method, with the one obtained via the ADHM approach [63] we recognize complete agreement if we identify $\Lambda$ with the size of the instanton. The zero-size limit produces a pure-gauge configuration as it should for self-dual $\theta^{\mu\nu}$. This is different for anti-self-dual $\theta^{\mu\nu}$ [147].

Recall that the calculation of the topological charge produces a surface integral at infinity, where the noncommutativity goes to zero.[2] Hence, the instanton number is classified by $\pi_3(U(2)) \cong \mathbb{Z}$. Since our solution coincides with the standard BPST instanton in the commutative limit, we conclude that it has topological charge $Q = 1$. This result can, of course, also be obtained in a direct evaluation of $Q$ from the expressions (VII.35). This was done by Furuuchi [63]. Let us repeat the steps. The charge $Q$ given by (VI.100) reads in complex coordinates (VII.2) as

$$Q = -(2\theta)^2\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}\big\{\mathcal{F}_{z^1\bar{z}^{\bar{1}}}\mathcal{F}_{z^2\bar{z}^{\bar{2}}} - \tfrac{1}{2}\{\mathcal{F}_{z^1\bar{z}^{\bar{2}}}, \mathcal{F}_{z^2\bar{z}^{\bar{1}}}\}\big\}, \tag{VII.36}$$

where the instanton equations (VII.7) have partly been used. Then we have

$$\mathrm{tr}\, \mathcal{F}_{z^1\bar{z}^{\bar{1}}}\mathcal{F}_{z^2\bar{z}^{\bar{2}}} = -\frac{1}{(2\theta)^2}\frac{s^2}{(N+1+s)^2}\left[\frac{1}{(N+s)^2}+\frac{1}{(N+2+s)^2}\right]$$

and

$$-\,\mathrm{tr}\,\tfrac{1}{2}\{\mathcal{F}_{z^1\bar{z}^{\bar{2}}}, \mathcal{F}_{z^2\bar{z}^{\bar{1}}}\} = -\frac{1}{(2\theta)^2}\frac{4s^2}{(N+s)(N+1+s)^2(N+2+s)}.$$

[2]However, when $U(1)$ is gauged this is not true [62].

Here we have introduced the abbreviation $s := \Lambda^2/(2\theta)$ and $N$ is the number operator, i.e. $N = c_1^\dagger c_1 + c_2^\dagger c_2$. Therefore the charge $Q$ is given by

$$\begin{aligned}
Q &= \mathrm{Tr}_{\mathcal{H}}\left\{\frac{s^2}{(N+1+s)^2}\left[\frac{1}{(N+s)^2}+\frac{1}{(N+2+s)^2}+\frac{4}{(N+s)(N+2+s)}\right]\right\}\\
&= \sum_{n_1,n_2\in\mathbb{N}_0}\left\{\frac{s^2}{(n_1+n_2+1+s)^2}\left[\frac{1}{(n_1+n_2+s)^2}\right.\right.\\
&\qquad\qquad\left.\left.+\frac{1}{(n_1+n_2+2+s)^2}+\frac{4}{(n_1+n_2+s)(n_1+n_2+2+s)}\right]\right\}\\
&= \sum_{n\in\mathbb{N}_0}\left\{\frac{s^2(n+1)}{(n+1+s)^2}\left[\frac{1}{(n+s)^2}+\frac{1}{(n+2+s)^2}+2\left(\frac{1}{(n+s)}-\frac{1}{(n+2+s)}\right)\right]\right\}\\
&= \sum_{n\in\mathbb{N}_0}\left\{\frac{s^2(n+1)}{(n+1+s)^2}\left[\frac{2(n+s)+1}{(n+s)^2}-\frac{2(n+s+1)+1}{(n+2+s)^2}\right]\right\}\\
&= \sum_{n\in\mathbb{N}_0}\frac{s^2}{(n+1+s)^2}\frac{2(n+s)+1}{(n+s)^2}\\
&\quad+\sum_{n\in\mathbb{N}_0}s^2\left\{n\frac{2(n+s)+1}{(n+s)^2(n+1+s)^2}-(n+1)\frac{2(n+1+s)+1}{(n+1+s)^2(n+2+s)^2}\right\}\\
&= \sum_{n\in\mathbb{N}_0}\frac{s^2}{(n+1+s)^2}\frac{2(n+s)+1}{(n+s)^2}=\sum_{n\in\mathbb{N}_0}\left\{\frac{s^2}{(n+s)^2}-\frac{s^2}{(n+1+s)^2}\right\}=1.
\end{aligned}\tag{VII.37}$$

## § VII.1.2 $U(2)$ BPST Instanton from the Twistor Approach

In the section § III.2 we gave the geometric picture of the linear system (VII.11) in the commutative case in terms of holomorphic vector bundles over the twistor space $\mathcal{Z} = \mathbb{R}^4 \times \mathbb{CP}^1$ for the Euclidean space $\mathbb{R}^4$. This eventually led us to the Penrose-Ward correspondence. As we have seen in subsection § VI.1.3, in the noncommutative extension the notion of vector bundles is replaced by the notion of projective modules. Still, the assertion of Penrose-Ward's theorem remains the same on noncommutative $\mathbb{R}^4_\theta \times \mathbb{CP}^1$ in the following sense. Let $U_{1,2}$ be the two canonical coordinate patches covering $\mathbb{CP}^1$. Taking an operator-valued holomorphic matrix $f_{12}$ restricted to $U_1 \cap U_2 \subset \mathbb{CP}^1$, one may try to split $f_{12} = \psi_1^{-1}\psi_2$ into $\psi_1$ and $\psi_2$ having no singularities in the spectral parameter $\lambda$ on $U_1$ and $U_2$, respectively. If successful then one can use the equations

$$\mathcal{A}_{\bar z^{\bar a}} = \psi_1 \partial_{\bar z^{\bar a}} \psi_1^{-1}\big|_{\lambda=0} \qquad \text{for} \qquad a = 1,2 \tag{VII.38}$$

to find a self-dual gauge potential.

Remember that the linear system (VII.11) is equivalent to the equations (cf. subsection § III.3.2)

$$\psi_1(\partial_{\bar z^{\bar 1}} - \lambda\partial_{z^2})\psi_1^{-1} = \psi_2(\partial_{\bar z^{\bar 1}} - \lambda\partial_{z^2})\psi_2^{-1} \;=\; \mathcal{A}_{\bar z^{\bar 1}} - \lambda\mathcal{A}_{z^2}, \tag{VII.39a}$$

$$\psi_1(\partial_{\bar z^{\bar 2}} + \lambda\partial_{z^1})\psi_1^{-1} = \psi_2(\partial_{\bar z^{\bar 2}} + \lambda\partial_{z^1})\psi_2^{-1} \;=\; \mathcal{A}_{\bar z^{\bar 2}} + \lambda\mathcal{A}_{z^1}, \tag{VII.39b}$$

since using the solution (VII.33) for $\psi$ one can construct a matrix $\Upsilon$ which depends meromorphically on

$$w^1 = z^1 - \lambda\bar{z}^{\bar{2}}, \qquad w^2 = z^2 + \lambda\bar{z}^{\bar{1}} \qquad \text{and} \quad w^3 = \lambda \tag{VII.40}$$

and acts by right multiplication so that

$$\psi_1 = \psi\,\Upsilon. \tag{VII.41}$$

Then $\psi_2$ was defined by $\psi_2(\lambda) := [\psi_1^{-1}(-1/\bar{\lambda})]^\dagger$.

**One-instanton solution.** For notational simplicity let us rewrite the solution (VII.33) for $\psi$ as

$$\psi = X + \lambda Y^\dagger + \frac{1}{\lambda}Y\ , \tag{VII.42}$$

i.e. we abbreviate the known expressions by $X = G(1{+}2H)$ and $Y = GS = GT_2\frac{1}{K}T_1^\dagger$, where the matrices $G$, $H$, $T_i$ and $K$ are given in the previous subsection. Moreover, we expand $\psi_1$ and $\Upsilon$ on $U_1 \cap U_2$ as

$$\psi_1(x,\lambda) = \sum_{n\in\mathbb{N}_0} \psi_{1n}(x)\,\lambda^n \qquad \text{and} \qquad \Upsilon(w^1,w^2,\lambda) = \sum_{n\in\mathbb{Z}} \Upsilon_n(w^1,w^2)\,\lambda^n, \tag{VII.43}$$

since $w^3 = \lambda$. Comparing coefficients of $\lambda^n$ in $\psi_1 = \psi\,\Upsilon$ then yields the equations

$$\psi_{1n} \;=\; X\Upsilon_n + Y^\dagger\Upsilon_{n-1} + Y\Upsilon_{n+1} \qquad \text{for} \qquad n \geq 0, \tag{VII.44a}$$

$$0 \;=\; X\Upsilon_n + Y^\dagger\Upsilon_{n-1} + Y\Upsilon_{n+1} \qquad \text{for} \qquad n < 0, \tag{VII.44b}$$

since

$$\psi\,\Upsilon = \sum_{n\in\mathbb{Z}} \lambda^n(X\Upsilon_n + Y^\dagger\Upsilon_{n-1} + Y\Upsilon_{n+1}) = \sum_{n\in\mathbb{N}_0} \lambda^n\psi_{1n}.$$

The holomorphicity conditions $\partial_{\bar{w}^{\bar{a}}}\Upsilon = 0$ imply the recursion relations

$$\partial_{\bar{z}^{\bar{1}}}\Upsilon_n = \partial_{z^2}\Upsilon_{n-1} \qquad \text{and} \qquad \partial_{\bar{z}^{\bar{2}}}\Upsilon_n = -\partial_{z^1}\Upsilon_{n-1}, \tag{VII.45}$$

since, for instance, $\partial_{\bar{w}^{\bar{1}}}\Upsilon = 0$ leads to

$$\sum_{n\in\mathbb{Z}} \lambda^n(\partial_{\bar{z}^{\bar{1}}} - \lambda\partial_{z^2})\Upsilon_n = 0 \qquad \Longleftrightarrow \qquad \sum_{n\in\mathbb{Z}} \lambda^n\partial_{\bar{z}^{\bar{1}}}\Upsilon_n = \sum_{n\in\mathbb{Z}} \lambda^n\partial_{z^2}\Upsilon_{n-1}.$$

Bearing in mind the commutative limit (III.82), we would like to truncate to

$$\psi_1(\lambda) = \psi_{10} + \psi_{11}\lambda \qquad \text{and} \qquad \Upsilon(\lambda) = \Upsilon_{-1}\lambda^{-1} + \Upsilon_0 + \Upsilon_1\lambda. \tag{VII.46}$$

We claim that it is indeed consistent to require $\Upsilon_{n<-1} = \Upsilon_{n>1} = 0$ and $\psi_{1n<0} = \psi_{1n>1} = 0$, which reduces the infinite set (VII.44) to

$$0 \;=\; Y\Upsilon_{-1}, \tag{VII.47a}$$

$$0 \;=\; X\Upsilon_{-1} + Y\Upsilon_0, \tag{VII.47b}$$

$$\psi_{10} \;=\; X\Upsilon_0 + Y^\dagger\Upsilon_{-1} + Y\Upsilon_1, \tag{VII.47c}$$

$$\psi_{11} \;=\; X\Upsilon_1 + Y^\dagger\Upsilon_0, \tag{VII.47d}$$

$$0 \;=\; Y^\dagger\Upsilon_1. \tag{VII.47e}$$

The truncated recursion relations (VII.45) imply that

$$
\begin{aligned}
\partial_{\bar z^{\bar a}}\Upsilon_{-1} &= 0 = \partial_{z^a}\partial_{z^b}\partial_{z^c}\Upsilon_{-1}, && \text{(VII.48a)}\\
\partial_{\bar z^{\bar a}}\partial_{\bar z^{\bar b}}\Upsilon_0 &= 0 = \partial_{z^a}\partial_{z^b}\Upsilon_0 \quad \text{and} \quad \Box\Upsilon_0 = 0, && \text{(VII.48b)}\\
\partial_{\bar z^{\bar a}}\partial_{\bar z^{\bar b}}\partial_{\bar z^{\bar c}}\Upsilon_1 &= 0 = \partial_{z^a}\Upsilon_1, && \text{(VII.48c)}
\end{aligned}
$$

so that the $2\times 2$ matrices $\Upsilon_0$ and $\Upsilon_{-1,1}$ are quadratic functions of $\{z^a\}$ and $\{\bar z^{\bar a}\}$. Demanding invariance under reflection at the origin and additionally imposing $\Upsilon(\lambda) = [\Upsilon(-1/\bar\lambda)]^\dagger$ the functional dependence takes the form

$$
\begin{aligned}
\Upsilon_{-1} &= -\tau_-(z^1)^2 + \tau_+(z^2)^2 - \tau_3\, z^1 z^2, && \text{(VII.49a)}\\
\Upsilon_0 &= -\tau_3\,(z^1\bar z^{\bar 1} - z^2\bar z^{\bar 2}) + 2\tau_- z^1\bar z^{\bar 2} + 2\tau_+ z^2\bar z^{\bar 1} - \tau_4, && \text{(VII.49b)}\\
\Upsilon_1 &= \tau_+(\bar z^{\bar 1})^2 - \tau_-(\bar z^{\bar 2})^2 + \tau_3\,\bar z^{\bar 1}\bar z^{\bar 2}, && \text{(VII.49c)}
\end{aligned}
$$

with constant matrices $\tau_- = \tau_+^\dagger$ as well as $\tau_3$ and $\tau_4$ (the latter two being Hermitian).

Equations (VII.47a) and (VII.47e) with $\Upsilon_{-1}^\dagger = -\Upsilon_1$ are clearly solved by putting

$$
\Upsilon_{-1} = -\tfrac{1}{2\Lambda}T_2T_1^\dagger \qquad \text{and} \qquad \Upsilon_1 = \tfrac{1}{2\Lambda}T_1T_2^\dagger, \tag{VII.50}
$$

where a convenient normalization has been chosen. This fixes $\tau_\pm = \frac{1}{2\Lambda}(\sigma_1 \pm i\sigma_2)$ and $\tau_3 = \frac{1}{2\Lambda}\sigma_3$, where $\sigma_i$ denotes the Pauli matrices (II.12). The remaining condition (VII.47b) then determines the matrix $\tau_4 = \mathbb{1}_2$. The equations (VII.47c) and (VII.47d) finally serve to compute $\psi_1$. Expressed in terms of $\{w^a\}$ coordinates, we thus arrive at

$$
\Upsilon(w^1,w^2,\lambda) = \frac{1}{2\Lambda^2}\begin{pmatrix} -2\Lambda^2 - \frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & -2\Lambda^2 + \frac{w^1w^2}{\lambda}\end{pmatrix}. \tag{VII.51}
$$

Note that this solution for $\Upsilon$ formally coincides with the commutative solution (III.82). This is due to the effect that for our choice of $\theta^{\mu\nu}$ the $\{w^a\}$ are commuting operators on $\mathcal{H}$, i.e. $[w^1,w^2]=0$, as one may readily check. The matrix $\psi_1$ is then given by

$$
\begin{aligned}
\psi_1(x,\lambda) &= \psi(x,\lambda)\,\Upsilon(w^1,w^2,\lambda)\\
&= -\frac{1}{\Lambda}\begin{pmatrix}\frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0 \\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}}\end{pmatrix}\times\\
&\quad\times\begin{pmatrix}\bar z^{\bar 1}z^1 + \Lambda^2 - \lambda\bar z^{\bar 1}\bar z^{\bar 2} & -\bar z^{\bar 1}z^2 - \lambda(\bar z^{\bar 1})^2 \\ -z^1\bar z^{\bar 2} + \lambda(\bar z^{\bar 2})^2 & \bar z^{\bar 2}z^2 + \Lambda^2 + \lambda\bar z^{\bar 1}\bar z^{\bar 2}\end{pmatrix}
\end{aligned} \tag{VII.52}
$$

which coincides with Crane's solution (III.83) in the commutative limit. The matrix $f_{12}$ then becomes

$$
f_{12}(w^a,\lambda) = \psi_1^{-1}(w^a,\lambda)\,\psi_2(w^a,\lambda) = \Upsilon^2(w^a,\lambda) = \frac{1}{\Lambda^2}\begin{pmatrix}\Lambda^2 - \frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & \Lambda^2 + \frac{w^1w^2}{\lambda}\end{pmatrix}. \tag{VII.53}
$$

In order to construct the gauge potential we need the inverse of $\psi_1$ at $\lambda = 0$ which is

$$\psi_1^{-1}\big|_{\lambda=0} = -\frac{1}{\Lambda}\begin{pmatrix} \frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0 \\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}} \end{pmatrix}\begin{pmatrix} \bar{z}^{\bar{2}}z^2+\Lambda^2-2\theta & \bar{z}^{\bar{1}}z^2 \\ z^1\bar{z}^{\bar{2}} & \bar{z}^{\bar{1}}z^1+\Lambda^2+2\theta \end{pmatrix}. \qquad \text{(VII.54)}$$

With these ingredients we can use equation (VII.38) to reconstruct the gauge potential and hence the curvature. Let us exemplify this derivation by looking at the $\mathcal{A}_{\bar{z}^{\bar{1}}}$ component. We have

$$\mathcal{A}_{\bar{z}^{\bar{1}}} = \psi_1\partial_{\bar{z}^{\bar{1}}}\psi_1^{-1}\big|_{\lambda=0} = \tfrac{1}{2\theta}\,\psi_1[z^1,\psi_1^{-1}]\big|_{\lambda=0} = \tfrac{1}{2\theta}\,\psi_1 z^1\psi_1^{-1}\big|_{\lambda=0} - \tfrac{1}{2\theta}z^1. \qquad \text{(VII.55)}$$

Consider the (11)-component, i.e. $\{\mathcal{A}_{\bar{z}^{\bar{1}}}\}_{11}$. Thus,

$$\begin{aligned}
\{\psi_1 z^1\psi_1^{-1}\big|_{\lambda=0}\}_{11} &= \frac{1}{\Lambda^2}\bigg(\frac{1}{\sqrt{r^2+\Lambda^2-2\theta}}(\bar{z}^{\bar{1}}z^1+\Lambda^2)z^1\frac{1}{\sqrt{r^2+\Lambda^2-2\theta}}(\bar{z}^{\bar{2}}z^2+\Lambda^2-2\theta) \\
&\qquad - \frac{1}{\sqrt{r^2+\Lambda^2-2\theta}}\bar{z}^{\bar{1}}z^2z^1\frac{1}{\sqrt{r^2+\Lambda^2+2\theta}}z^1\bar{z}^{\bar{2}}\bigg) \\
&= \frac{1}{\Lambda^2}\frac{1}{\sqrt{r^2+\Lambda^2-2\theta}}\frac{(\bar{z}^{\bar{1}}z^1+\Lambda^2)(z^2\bar{z}^{\bar{2}}+\Lambda^2)-\bar{z}^{\bar{1}}z^1z^2\bar{z}^{\bar{2}}}{\sqrt{r^2+\Lambda^2}}z^1 \\
&= \sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}z^1
\end{aligned}$$

and hence

$$\{\mathcal{A}_{\bar{z}^{\bar{1}}}\}_{11} = \frac{1}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right)z^1.$$

Using $\mathcal{A}_{z^1} = -\mathcal{A}^\dagger_{\bar{z}^{\bar{1}}}$, we finally obtain

$$\{\mathcal{A}_{z^1}\}_{11} = -\frac{\bar{z}^{\bar{1}}}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right).$$

A similar calculation applies for the other components. We leave it as an exercise.

## § VII.2 Noncommutative $U(2)$ 't Hooft Instantons

As we have already mentioned in the commutative setup, the construction of multi-instantons via the dressing method becomes quite complicated due to the algebraic constraints which have to be satisfied. For the sake of completeness, however, we also want to show how one can find noncommutative multi-instantons by way of splitting, i.e. we consider again an operator-valued holomorphic function $f_{12}$ restricted to $U_1 \cap U_2 \subset \mathbb{CP}^1$, where the $U_{1,2}$ are the canonical coordinate patches covering $\mathbb{CP}^1$. This time, however, we choose a particular ansatz for $f_{12}$ - the so-called Atiyah-Ward ansatz. We will concentrate on the self-dual multi-instanton solution à la 't Hooft. However, using this method we are only able to give an explicit formula for the curvature. The following discussion is taken from [129].

**Noncommutative Atiyah-Ward ansatz and gauge potential.** In the commutative situation the infinite hierarchy of Atiyah-Ward ansätze reads

$$f_{12}^{(k)} = \begin{pmatrix} \lambda^k & 0 \\ \rho & \lambda^{-k} \end{pmatrix}, \tag{VII.56}$$

where $k = 1, 2, \dots$ and $\rho$ denotes a holomorphic function on $U_1 \cap U_2$. We note that the ansätze (VII.56) can be used to construct solutions to the noncommutative SDYM equations simply by promoting $\rho$ to an operator acting on the Fock space $\mathcal{H}$.

Let us specialize to $k = 1$ and introduce the matrix

$$f_{12} := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} f_{12}^{(1)} = \begin{pmatrix} \rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}. \tag{VII.57}$$

Note that $f_{12}$ is related to $f_{12}^{(1)}$ by a transformation of the form (III.57), with $h_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, it leads to the same gauge potential. Yet, $f_{12}$ has the advantage of satisfying the reality condition (III.59). Here, $\rho$ fulfills the properties

$$\partial_{\bar{w}^{\bar{a}}} \rho = 0 \qquad \text{and} \qquad \rho^\dagger(x, -\bar{\lambda}^{-1}) = \rho(x, \lambda). \tag{VII.58}$$

It will be helpful to expand $\rho$ in a Laurent series in $\lambda$, i.e.

$$\rho = \sum_{n \in \mathbb{Z}} \rho_n \, \lambda^n = \rho_- + \phi + \rho_+, \tag{VII.59a}$$

where

$$\rho_+ := \sum_{n \in \mathbb{N}} \rho_n \, \lambda^n, \qquad \rho_- := \sum_{n \in \mathbb{N}} \rho_{-n} \, \lambda^{-n} \qquad \text{and} \qquad \phi := \rho_0. \tag{VII.59b}$$

The reality condition (III.59) then becomes

$$\phi^\dagger(x) = \phi(x) \qquad \text{and} \qquad \rho_+^\dagger(x, -\bar{\lambda}^{-1}) = \rho_-(x, \lambda). \tag{VII.60}$$

As one may readily check, $f_{12}$ can be split as $f_{12} = \psi_1^{-1}\psi_2$, where

$$\psi_2 = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \phi + \rho_- & \lambda^{-1} \\ \lambda\rho_- & 1 \end{pmatrix} \qquad \text{and} \qquad \psi_1^{-1} = \begin{pmatrix} \phi + \rho_+ & -\lambda^{-1}\rho_+ \\ -\lambda & 1 \end{pmatrix} \frac{1}{\sqrt{\phi}}. \tag{VII.61a}$$

Consequently, we have

$$\psi_2^{-1} = \begin{pmatrix} 1 & -\lambda^{-1} \\ -\lambda\rho_- & \phi + \rho_- \end{pmatrix} \frac{1}{\sqrt{\phi}} \qquad \text{and} \qquad \psi_1 = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & \lambda^{-1}\rho_+ \\ \lambda & \phi + \rho_+ \end{pmatrix} \tag{VII.61b}$$

which obviously satisfy the reality condition (III.59). Therefore the first Atiyah-Ward ansatz has easily been generalized to the noncommutative case.

Again, the holomorphicity conditions $\partial_{\bar{w}^{\bar{a}}}\rho = 0$ imply the recursion relations

$$\partial_{\bar{z}^{\bar{1}}}\rho_n = \partial_{z^2}\rho_{n-1} \qquad \text{and} \qquad \partial_{\bar{z}^{\bar{2}}}\rho_n = -\partial_{z^1}\rho_{n-1}. \tag{VII.62}$$

Finally, using (VII.38) the gauge potential takes the following form

$$\mathcal{A}_{\bar{z}^{\bar{1}}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{z}^{\bar{1}}}\phi^{\frac{1}{2}} & -\phi^{-\frac{1}{2}}(\partial_{z^2}\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{1}^{\bar{z}}}\phi^{-\frac{1}{2}} \end{pmatrix}, \quad \mathcal{A}_{\bar{z}^{\bar{2}}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\partial_{\bar{z}^{\bar{2}}}\phi^{\frac{1}{2}} & \phi^{-\frac{1}{2}}(\partial_{z^1}\phi)\phi^{-\frac{1}{2}} \\ 0 & \phi^{\frac{1}{2}}\partial_{\bar{z}^{\bar{2}}}\phi^{-\frac{1}{2}} \end{pmatrix}. \tag{VII.63}$$

Remember that $\mathcal{A}_{z^a} = -\mathcal{A}^\dagger_{\bar{z}^{\bar{a}}}$. In real coordinates these expressions look like

$$\mathcal{A}_\mu = \bar{\eta}^a_{\mu\nu}\tfrac{\sigma_a}{2i}\big(\phi^{\frac{1}{2}}\partial_\nu\phi^{-\frac{1}{2}} - \phi^{-\frac{1}{2}}\partial_\nu\phi^{\frac{1}{2}}\big) + \tfrac{\mathbb{1}_2}{2}\big(\phi^{-\frac{1}{2}}\partial_\mu\phi^{\frac{1}{2}} + \phi^{\frac{1}{2}}\partial_\mu\phi^{-\frac{1}{2}}\big). \tag{VII.64}$$

This is known as the noncommutative generalization of the CFTW ansatz given by (III.85). Here, $\bar{\eta}^a_{\mu\nu}$ is the 't Hooft symbol (III.19) and the $\sigma_a$'s are the Pauli matrices (II.12).

**Instanton solutions.** The calculation of the Yang-Mills curvature for the noncommutative CFTW ansatz (VII.64) results in

$$\mathcal{F}_{z^1z^2} = \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix}, \qquad \mathcal{F}_{\bar{z}^{\bar{1}}\bar{z}^{\bar{2}}} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \tag{VII.65}$$

with

$$X := \phi^{-\frac{1}{2}}\left(\partial_{z^1}\partial_{\bar{z}^{\bar{1}}}\phi + \partial_{z^2}\partial_{\bar{z}^{\bar{2}}}\phi\right)\phi^{-\frac{1}{2}}. \tag{VII.66}$$

Hence, for the ansatz (VII.64) we only need to consider the equation

$$\phi^{-\frac{1}{2}}\left(\partial_{z^1}\partial_{\bar{z}^{\bar{1}}}\phi + \partial_{z^2}\partial_{\bar{z}^{\bar{2}}}\phi\right)\phi^{-\frac{1}{2}} = 0. \tag{VII.67}$$

Let us assume that a solution $\phi_N$ looks like the standard 't Hooft solution, i.e.

$$\phi_N = 1 + \sum_{i=1}^N \frac{\Lambda_i^2}{(x_\mu - b_{i\mu})(x^\mu - b_i^\mu)} =: 1 + \sum_{i=1}^N \frac{\Lambda_i^2}{r_i^2}, \tag{VII.68}$$

where now the $x^\mu$ are operators and

$$\frac{1}{x_\mu x^\mu} := \frac{1}{2\theta}\sum_{n_1,n_2\in\mathbb{N}_0} \frac{1}{n_1+n_2+1}\,|n_1,n_2\rangle\langle n_1,n_2|, \tag{VII.69a}$$

$$r_i^2 = \delta_{\mu\nu}(x^\mu - b_i^\mu)(x^\nu - b_i^\nu) = \bar{z}_i^{\bar{1}}z_i^1 + \bar{z}_i^{\bar{2}}z_i^2 \tag{VII.69b}$$

and

$$z_i^a := z^a - b_i^{z^a} \qquad \text{with} \qquad b_i := (b_i^\mu) = (b_i^{z^1}, b_i^{z^2}). \tag{VII.69c}$$

The real parameters $b_i^\mu$ and $\Lambda_i$ are position and scale moduli.

Firstly, let us look at the one-instanton solution, i.e.

$$\phi_1 = 1 + \frac{\Lambda^2}{r^2}, \tag{VII.70}$$

with $b_1^\mu \equiv 0$. Then we find

$$X_1 := \phi_1^{-\frac{1}{2}} (\partial_{z^1}\partial_{\bar z^{\bar 1}}\phi_1 + \partial_{z^2}\partial_{\bar z^{\bar 2}}\phi_1)\, \phi_1^{-\frac{1}{2}} = -\frac{\Lambda^2}{2\theta(\Lambda^2+2\theta)}\, P_0, \tag{VII.71}$$

where $P_0 = |0,0\rangle\langle 0,0|$. This can be seen as follows. Let $|n_1,n_2\rangle\langle n_1,n_2| = P^{(1)}_{n_1} \otimes P^{(2)}_{n_2}$. Therefore we have

$$\begin{aligned}
\partial_{z^1}\partial_{\bar z^{\bar 1}} \frac{1}{r^2} &= -\frac{1}{(2\theta)^2} \sum_{n_1,n_2\in\mathbb{N}_0} \frac{1}{n_1+n_2+1} \left[c_1^\dagger, \left[c_1, P^{(1)}_{n_1}\otimes P^{(2)}_{n_2}\right]\right] \\
&= -\frac{1}{(2\theta)^2} \sum_{n_1,n_2\in\mathbb{N}_0} \frac{n_1+1}{(n_1+n_2+1)(n_1+n_2+2)} \big\{P^{(1)}_{n_1} - P^{(1)}_{n_1+1}\big\} \otimes P^{(2)}_{n_2}
\end{aligned}$$

and similarly

$$\partial_{z^2}\partial_{\bar z^{\bar 2}} \frac{1}{r^2} = -\frac{1}{(2\theta)^2} \sum_{n_1,n_2\in\mathbb{N}_0} \frac{n_2+1}{(n_1+n_2+1)(n_1+n_2+2)} P^{(1)}_{n_1} \otimes \big\{P^{(2)}_{n_2} - P^{(2)}_{n_2+1}\big\}.$$

Summing up both terms yields after a few lines of algebra

$$(\partial_{z^1}\partial_{\bar z^{\bar 1}} + \partial_{z^2}\partial_{\bar z^{\bar 2}}) \frac{1}{r^2} = -\frac{1}{(2\theta)^2} P^{(1)}_0 \otimes P^{(2)}_0 = -\frac{1}{(2\theta)^2} P_0.$$

Note that $\frac{1}{r^2}|0,0\rangle = \frac{1}{2\theta}|0,0\rangle$. Finally, we obtain

$$\phi_1^{-\frac{1}{2}} (\partial_{z^1}\partial_{\bar z^{\bar 1}}\phi_1 + \partial_{z^2}\partial_{\bar z^{\bar 2}}\phi_1)\, \phi_1^{-\frac{1}{2}} = \frac{1}{\sqrt{1+\frac{\Lambda^2}{2\theta}}} \frac{-\Lambda^2}{(2\theta)^2}\, P_0\, \frac{1}{\sqrt{1+\frac{\Lambda^2}{2\theta}}} = -\frac{\Lambda^2}{2\theta(\Lambda^2+2\theta)}\, P_0.$$

More generally, one can show that ($b_i^{z^a} \neq 0$)

$$(\partial_{z^1}\partial_{\bar z^{\bar 1}} + \partial_{z^2}\partial_{\bar z^{\bar 2}})\phi_N = -\frac{1}{(2\theta)^2} \sum_{i=1}^{N} \Lambda_i^2\, |b_i\rangle\langle b_i|, \tag{VII.72}$$

where

$$|b_i\rangle := |b_i^{z^1}, b_i^{z^2}\rangle = e^{-\frac{1}{2}|b_i|^2}\, e^{b_i^{z^1} \frac{1}{\sqrt{2\theta}} \bar z^{\bar 1}}\, e^{b_i^{z^2} \frac{1}{\sqrt{2\theta}} z^2}\, |0,0\rangle \tag{VII.73}$$

denotes a coherent state (shifted ground state). The vectors $|b_i\rangle$ with $i = 1,\ldots,N$ span an $N$-dimensional subspace of $\mathcal{H}$. Note that they are not orthonormalized. Therefore we introduce an orthonormal basis $\{|h_i\rangle\}_{i=1,\ldots,N}$ of this subspace by

$$(|h_1\rangle,\ldots,|h_N\rangle) := T\,(T^\dagger T)^{-\frac{1}{2}} \qquad \text{with} \qquad T := (|b_1\rangle,\ldots,|b_N\rangle). \tag{VII.74}$$

This basis can be extended to an orthonormal basis of the whole Fock space $\mathcal{H}$ by simply adjoining further vectors $|h_{j+1}\rangle$ with $j = N, N+1, \ldots$. The explicit form of the function $\phi_N$

implies that the matrix elements $\langle b_i|\phi_N^{-\frac{1}{2}}|b_j\rangle$ are nonzero only for $i,j \leq N$. Thus, we conclude that the operator $\phi_N^{-\frac{1}{2}}$ preserves the subspace of $\mathcal{H}$ which is spanned by $\{|b_i\rangle\}_{i=1,\dots,N}$, i.e.

$$\phi_N^{-\frac{1}{2}}|b_i\rangle = \sum_{j=1}^{N} M_{ij}\,|b_j\rangle \qquad \text{and} \qquad \langle b_i|\,\phi_N^{-\frac{1}{2}} = \sum_{k=1}^{N}\langle b_k|\,\bar{M}_{ki}, \tag{VII.75}$$

where $M_{ij}$ are some constants. Therefore we arrive at

$$X_N := \phi_N^{-\frac{1}{2}}(\partial_{z^1}\partial_{\bar{z}^1}\phi_N + \partial_{z^2}\partial_{\bar{z}^2}\phi_N)\,\phi_N^{-\frac{1}{2}} = -\frac{1}{(2\theta)^2}\sum_{i,j,k=1}^{N}|b_j\rangle\,\bar{M}_{ki}\,\Lambda_i^2\,M_{ij}\,\langle b_k|, \tag{VII.76}$$

which is not zero if at least one $\Lambda_i^2$ does not vanish. Hence, we are led to the conclusion that this configuration is not self-dual on the whole Fock space $\mathcal{H}$. Is it possible to repair this deficiency? We shall answer this question in the subsequent paragraphs.

**Projected field configurations.** As we have seen above, the ansatz (VII.68) does not solve the SDYM equation. Remember that the components $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are operators acting on the free module $\mathcal{E} = \mathcal{H}\otimes\mathbb{C}^2 \cong \mathcal{H}\oplus\mathcal{H}$. It is easy to verify that for the ansatz (VII.70) each term of the operators $\mathcal{A}_\mu$ in (VII.64) annihilates the state $|0,0\rangle\otimes\mathbb{C}^2$ or $\mathbb{C}^2\otimes\langle 0,0|$. This obviously shows, that $\mathcal{A}_\mu$ is well defined only on the finite projective module $(1-P_0)\mathcal{H}\otimes\mathbb{C}^2 \subset \mathcal{H}\otimes\mathbb{C}^2$. Furthermore, equation (VII.71) implies that the curvature is self-dual only on the same subspace since $X_1(1-P_0) = 0$. Analogously, one may check that $X_N$ given by (VII.76) is annihilated by the projector

$$1 - P_{N-1} := 1 - \sum_{i,j=1}^{N}|b_i\rangle\frac{1}{\langle b_i|b_j\rangle}\langle b_j| = 1 - \sum_{i=1}^{N}|h_i\rangle\langle h_i| \tag{VII.77}$$

projecting onto the orthogonal complement of the subspace spanned by $\{|b_i\rangle\}_{i=1,\dots,N}$. Therefore, the solutions based on (VII.68) are self-dual on $(1-P_{N-1})\mathcal{H}\otimes\mathbb{C}^2$. Note that the deletion of a subspace generated by $N$ states $|b_i\rangle$ for $i = 1,\dots,N$ from the Fock space $\mathcal{H}$ corresponds to the exclusion of $N$ points $b_i$ for $i = 1,\dots,N$ (in which the gauge potential is singular) from the commutative space $\mathbb{R}^4$. Hence, the ansatz (VII.68) produces a singular solution.

**Murray-von Neumann transformations.** To be more general, let $P$ be some finite-rank projector. If the deviation from self-duality is localized on $P\mathcal{H}\otimes\mathbb{C}^2$, then we can obviously produce a self-dual curvature $\mathcal{F}^p_{\mu\nu}$ by projecting the gauge field onto $(1-P)\mathcal{H}\otimes\mathbb{C}^2$,

$$\mathcal{F}_{\mu\nu} \mapsto \mathcal{F}^p_{\mu\nu} = (1-P)\mathcal{F}_{\mu\nu}(1-P) = (1-P)(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu,\mathcal{A}_\nu])(1-P). \tag{VII.78}$$

Thus, by virtue of equations (VI.30) and (VI.31) the components $\mathcal{F}^p_{\mu\nu}$ of the projected curvature in terms of the components $\mathcal{A}^p_\mu$ of the projected gauge potential are given by

$$\mathcal{F}^p_{\mu\nu} = (1-P)(\partial_\mu P\partial_\nu P - \partial_\nu P\partial_\mu P) + (1-P)(\partial_\mu\mathcal{A}^p_\nu - \partial_\nu\mathcal{A}^p_\mu + [\mathcal{A}^p_\mu,\mathcal{A}^p_\nu])(1-P). \tag{VII.79}$$

Note that here the projector is $(1-P)$ instead of $P$. Therefore together, equations (VII.78) and (VII.79) yield an implicit relation to determine $\mathcal{A}^p_\mu$ from $\mathcal{A}_\mu$. Finding an explicit solution seems to be very difficult. However, below we shall give an explicit formula for the self-dual curvature, without knowing the particular form of $\mathcal{A}^p_\mu$.

After projecting our gauge field $\mathcal{F} \mapsto \mathcal{F}^p$ via (VII.78) and finding a solution $\mathcal{A}^p$, we obtain a self-dual configuration $(\mathcal{A}^p, \mathcal{F}^p)$ which may be termed the noncommutative 't Hooft multi-instanton $(\mathcal{A}^p, \mathcal{F}^p)$ in a singular gauge. As we have discussed in subsection § III.3.3, in the commutative situation the singularity in the gauge potential can be removed by a singular gauge transformation, leading for instance to the BPST form of the one-instanton solution. The noncommutative analogues of such singular gauge transformations are the so-called Murray-von Neumann transformations.

Given our finite-rank projector $P$, we consider a special kind of Murray-von Neumann transformations, namely

$$\mathcal{A}^p_\mu \mapsto \mathcal{A}'_\mu = V^\dagger \mathcal{A}^p_\mu V + V^\dagger \partial_\mu V \qquad \text{and} \qquad \mathcal{F}^p_{\mu\nu} \mapsto \mathcal{F}'_{\mu\nu} = V^\dagger \mathcal{F}^p_{\mu\nu} V, \tag{VII.80}$$

where the intertwining operators

$$V \,:\, \mathcal{H} \otimes \mathbb{C}^2 \to (1-P)\mathcal{H} \otimes \mathbb{C}^2 \qquad \text{and} \qquad V^\dagger \,:\, (1-P)\mathcal{H} \otimes \mathbb{C}^2 \to \mathcal{H} \otimes \mathbb{C}^2 \tag{VII.81a}$$

are subject to

$$V^\dagger\, V = 1 \qquad \text{while} \qquad V\, V^\dagger = 1 - P \tag{VII.81b}$$

upon extension to $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^2$. The configuration $(\mathcal{A}^p, \mathcal{F}^p)$ lives on the projective module $(1-P)\mathcal{H} \otimes \mathbb{C}^2$, but $(\mathcal{A}', \mathcal{F}')$ exists on the free module $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^2$. Therefore, the gauge fields are related to their potentials by (VII.79) and

$$\mathcal{F}'_{\mu\nu} = \partial_\mu \mathcal{A}'_\nu - \partial_\nu \mathcal{A}'_\mu + [\mathcal{A}'_\mu, \mathcal{A}'_\nu] \tag{VII.82}$$

The natural embedding of the reduced Fock space into the full Fock space allows us to extend $V$ and $V^\dagger$ to endomorphisms of $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^2$ by declaring $V^\dagger P = 0 = VP$. Note that the extended $V$ is not unitary, but the transformations (VII.80) can still be regarded as a noncommutative version of singular gauge transformations. Furthermore, we may actually bypass the projection as far as the field strength is concerned and directly compute

$$\mathcal{F}'_{\mu\nu} = V^\dagger \mathcal{F}_{\mu\nu} V. \tag{VII.83}$$

Pictorially, we may summarize

$$(\mathcal{A}_\mu, \mathcal{F}_{\mu\nu}) \quad \overset{1-P_{N-1}}{\longmapsto} \quad (\mathcal{A}^p_\mu, \mathcal{F}^p_{\mu\nu}) \qquad \overset{V_N}{\longmapsto} \qquad (\mathcal{A}'_\mu, \mathcal{F}'_{\mu\nu}).$$

**One-instanton solution.** Let us first consider the one-instanton configuration (VII.70) in (VII.64), project it onto $(1-P_0)\mathcal{H}\otimes\mathbb{C}^2$ and construct its Murray-von Neumann transformation matrix $V \equiv V_1$. Because of $(1-P_0)X_1(1-P_0) = 0$, the projected field strength

$$\mathcal{F}^p_{\mu\nu} = (1-P_0)\mathcal{F}_{\mu\nu}(1-P_0) \tag{VII.84}$$

is obviously self-dual, but it is singular since it is ill-defined on $|0,0\rangle\otimes\mathbb{C}^2$.

As justified momentarily, the subsequent Murray-von Neumann transformation applied to $(\mathcal{A}^p,\mathcal{F}^p)$ can be factorized as $V_1 = \check{S}_1\,U_1$, with

$$\check{S}_1 = \begin{pmatrix} 1 & 0 \\ 0 & S_1 \end{pmatrix} \qquad\text{and}\qquad U_1 = \begin{pmatrix} z^2 & \bar{z}^{\bar{1}} \\ z^1 & -\bar{z}^{\bar{2}} \end{pmatrix}\frac{i}{r}, \tag{VII.85}$$

where

$$S_1^\dagger S_1 = 1 \qquad\text{and}\qquad S_1 S_1^\dagger = 1-P_0, \tag{VII.86}$$

and $S \equiv S_1$ is a shift operator; cf. subsection § VI.1.6.

Now we will show that

$$\mathcal{A}'_\mu = V_1^\dagger \mathcal{A}^p_\mu V_1 + V_1^\dagger\partial_\mu V_1 \qquad\text{and}\qquad \mathcal{F}'_{\mu\nu} = V_1^\dagger\mathcal{F}^p_{\mu\nu}V_1 \equiv V_1^\dagger\mathcal{F}_{\mu\nu}V_1 \tag{VII.87}$$

brings us to the nonsingular noncommutative one-instanton solution. By using

$$r_0 := (r^2-2\theta)^{\frac12} = (\bar{z}^{\bar{1}}z^1 + z^2\bar{z}^{\bar{2}})^{\frac12}, \tag{VII.88a}$$

$$r_0^{-1} := (1-P_0)(\bar{z}^{\bar{1}}z^1 + z^2\bar{z}^{\bar{2}})^{-\frac12}(1-P_0) = \frac{1}{\sqrt{2\theta}}\sum_{n_1,n_2\in\mathbb{N}}\frac{|n_1n_2\rangle\langle n_1n_2|}{\sqrt{n_1+n_2}}, \tag{VII.88b}$$

one may readily check that

$$V_1^\dagger V_1 \;=\; 1\otimes\mathbb{1}_2 \qquad\text{but}\qquad V_1V_1^\dagger = \begin{pmatrix} 1-P_0 & 0 \\ 0 & 1-P_0 \end{pmatrix} = (1-P_0)\otimes\mathbb{1}_2. \tag{VII.89}$$

Moreover, we have

$$X_1V_1 = -\frac{\Lambda^2}{2\theta(\Lambda^2+2\theta)}P_0V_1 = 0$$

which assures that the anti-self-dual part of $\mathcal{F}'_{\mu\nu}$ vanishes. Note that we have not used at all the projection $\mathcal{F}\mapsto\mathcal{F}^p$ in this computation, but nevertheless it is required if we want $\mathcal{F}'$ to come from a gauge potential $\mathcal{A}'$ which derives from $\mathcal{A}^p$ by a Murray-von Neumann transformation.

Clearly, the Murray-von Neumann transformation (VII.87) extends the self-duality of $\mathcal{F}^p_{\mu\nu}$ to the free module $\mathcal{E} = \mathcal{H}\otimes\mathbb{C}^2$. This is exactly what was desired to cure the incompleteness of the configuration (VII.64) with $\phi$ from (VII.70). The final expression of the curvature of the one-instanton solution reads

$$\mathcal{F}'_{\mu\nu} = \frac{1}{r}\begin{pmatrix} \bar{z}^{\bar{2}} & \bar{z}^{\bar{2}}S_1^\dagger \\ z^1 & -z^2S_1^\dagger \end{pmatrix}\Big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu,\mathcal{A}_\nu]\Big)\begin{pmatrix} z^{\bar{2}} & \bar{z}^{\bar{1}} \\ S_1z^1 & -S_1\bar{z}^{\bar{2}} \end{pmatrix}\frac{1}{r}. \tag{VII.90}$$

Here $\mathcal{A}_\mu$ is obtained by substituting (VII.70) into (VII.64).

In the commutative limit, our configuration $(\mathcal{A}', \mathcal{F}')$ clearly reduces to the commutative BPST instanton, since $\check{S}_1 = \mathrm{diag}(1, S_1)$ turns into a unitary matrix and the remaining part of the Murray-von Neumann transformation (mediated by $U_1$) turns exactly into the singular gauge transformation (III.91) from the 't Hooft to the BPST gauge. For illustration, the operator $S_1$ from (VI.89) can be rewritten as [89]

$$S_1 = \bar{z}^{\bar{1}}(\bar{z}^{\bar{1}}z^1 + 2\theta)^{-1}z^1 + (1 - \bar{z}^{\bar{1}}(\bar{z}^{\bar{1}}z^1 + 2\theta)^{-1}z^1)z^2(z^2\bar{z}^{\bar{2}} + 2\theta)^{-\frac{1}{2}}, \tag{VII.91}$$

from which one easily sees that in the commutative limit it approaches the identity.

**Multi-instanton solution.** Finally, let us consider instantons with higher topological charges, i.e. $Q > 1$. Again, a projection mediated by $1 - P_{N-1}$ plus a Murray-von Neumann transformation given by $V_N$ will enable us to extend the projected self-dual solution to the free module $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^2$. We may again decompose $V_N$ as $V_N = \check{S}_N U_N$ and take

$$\check{S}_N = \begin{pmatrix} 1 & 0 \\ 0 & S_N \end{pmatrix} \qquad \text{and} \qquad U_N = \mathcal{U}\, U_1^N\, \mathcal{U}^\dagger, \tag{VII.92}$$

where

$$\mathcal{U} := \sum_{i \in \mathbb{N}_0} |h_{i+1}\rangle\langle\hat{i}| \tag{VII.93}$$

is the unitary transformation from the basis $\{|\hat{i}\rangle\}$ to the basis $\{|h_{i+1}\rangle\}$ in $\mathcal{H}$, and $S_N : \mathcal{H} \to (1 - P_{N-1})\mathcal{H}$ is subject to

$$S_N^\dagger S_N = 1 \qquad \text{and} \qquad S_N S_N^\dagger = 1 - P_{N-1} = 1 - \sum_{i=1}^N |h_i\rangle\langle h_i|. \tag{VII.94}$$

In the orthonormal basis one can choose $S_N$ and $S_N^\dagger$ in the form

$$S_N = \sum_{i \in \mathbb{N}\setminus\{1\}} |h_{i+N}\rangle\langle h_i| \qquad \text{and} \qquad S_N^\dagger = \sum_{i \in \mathbb{N}\setminus\{1\}} |h_i\rangle\langle h_{i+N}|. \tag{VII.95}$$

Since this basis was constructed from the states $\{|b_i\rangle\}_{i=1,\ldots,N}$, the operator $S_N$ contains all the information about the position parameters $b_1, \ldots, b_N$. It is easy to see that

$$S_N = \mathcal{U}\, S_1^N\, \mathcal{U}^\dagger \tag{VII.96}$$

with $S_1$ from (VI.92).

Let us look more detailed at equation (VII.92). Employing the relations

$$U_1^N (U_1^\dagger)^N = 1 \otimes \mathbb{1}_2 - \sum_{i=0}^{N-1} \begin{pmatrix} |\hat{i}\rangle\langle\hat{i}| & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$\mathcal{U}\Big(\sum_{i=0}^{N-1}|\hat{i}\rangle\langle\hat{i}|\Big)\mathcal{U}^\dagger = \sum_{i=0}^{N-1}|h_{i+1}\rangle\langle h_{i+1}| = P_{N-1},$$

one easily finds that

$$U_N^\dagger\, U_N = 1\otimes \mathbb{1}_2 \qquad \text{and} \qquad U_N\, U_N^\dagger = \begin{pmatrix} 1-P_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{VII.97}$$

and eventually with (VII.94)

$$V_N^\dagger V_N = 1\otimes \mathbb{1}_2 \quad \text{and} \quad V_N V_N^\dagger = \begin{pmatrix} 1-P_{N-1} & 0 \\ 0 & 1-P_{N-1} \end{pmatrix} = (1-P_{N-1})\otimes \mathbb{1}_2. \tag{VII.98}$$

Finally, substituting (VII.68) into (VII.64), computing its curvature $\mathcal{F}_{\mu\nu}$ and performing the Murray-von Neumann transformation (VII.83) via the matrices given by (VII.92), we realize that the anti-self-dual part of $\mathcal{F}'_{\mu\nu}$ vanishes everywhere on the free module $\mathcal{E}=\mathcal{H}\otimes\mathbb{C}^2$. This is due to $X_N V_N = 0 = V_N^\dagger X_N$. Thus, the final result

$$\mathcal{A}'_\mu = V_N^\dagger \mathcal{A}^p_\mu V_N + V_N^\dagger \partial_\mu V_N \qquad \text{and} \qquad \mathcal{F}'_{\mu\nu} = V_N^\dagger \mathcal{F}^p_{\mu\nu} V_N \equiv V_N^\dagger \mathcal{F}_{\mu\nu} V_N \tag{VII.99}$$

yields a proper noncommutative generalization of the 't Hooft $N$-instanton solution. Obviously, in the limit $\theta\to 0$ this solution coincides with the standard 't Hooft solution and hence we conclude that it has a topological charge of $Q=N$.

**Remark.** In constructing the noncommutative self-dual multi-instanton solution à la 't Hooft, we have restricted to self-dual background geometry, i.e. $\theta = *\theta$. However, this is not necessary [129].

# Chapter VIII

## Noncommutative Ward Model

In this chapter we discuss the noncommutative generalization of Ward's model introduced in chapter IV. Here we are only concerned with the construction of Abelian and non-Abelian soliton and soliton-antisoliton configurations. As already mentioned, this model also allows for plane wave solutions. Their noncommutative generalization is given in [20]. The discussion presented in the sequel is based on [127, 128, 211].

## § VIII.1 Noncommutative Model and Dressing Approach

**Noncommutative action and energy functionals.** Let us consider the noncommutative space $\mathbb{R}^{2,1}_\theta$ with the canonical metric $g = \mathrm{diag}(1, 1, -1)$ and the commutation relations

$$[x, y] = i\theta, \qquad [x, t] = 0 \qquad \text{and} \qquad [y, t] = 0 \tag{VIII.1}$$

with $\theta > 0$. Again, we introduce the complex coordinate $z = x + iy$ such that $[z, \bar{z}] = 2\theta$. Hence, the annihilation and creation operators (VI.67) are given by

$$c = \tfrac{1}{\sqrt{2\theta}} z \qquad \text{and} \qquad c^\dagger = \tfrac{1}{\sqrt{2\theta}} \bar{z} \tag{VIII.2}$$

and act on $\mathcal{H} = \bigoplus_{m \in \mathbb{N}_0} \mathbb{C}|m\rangle$. The coordinate derivatives (VI.70) are

$$\partial_z f = -\tfrac{1}{2\theta}[\bar{z}, f] \qquad \text{and} \qquad \partial_{\bar{z}} f = \tfrac{1}{2\theta}[z, f] \tag{VIII.3}$$

for $f \in \mathbb{R}^{2,1}_\theta$.

The noncommutative extension of the action functional (IV.49) then reads

$$\begin{aligned} S_{\mathrm{NS}}(\Omega) = &- \pi\theta \int_{\mathbb{R}} \mathrm{dt}\; g^{\mu\nu}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}(\partial_\mu \Omega^\dagger \partial_\nu \Omega) \\ &- \frac{2\pi\theta}{3} \int_{\mathbb{R}} \mathrm{dt} \int_0^1 \mathrm{d}\tau\, \tilde{V}_\alpha \epsilon^{\alpha\mu\nu\sigma}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}\{(\tilde{\Omega}^\dagger \partial_\mu \tilde{\Omega})(\tilde{\Omega}^\dagger \partial_\nu \tilde{\Omega})(\tilde{\Omega}^\dagger \partial_\sigma \tilde{\Omega})\}, \end{aligned} \tag{VIII.4}$$

where $(\tilde{V}_\alpha) = (V_\alpha, 0) = (1, 0, 0, 0)$. However, this time the field $\Omega$ is regarded as an $U(n)$-valued operator acting on the Fock space $\mathcal{H}$. Also in the noncommutative setup we have a conserved energy functional

$$E = 2\pi\theta\, \mathrm{Tr}_{\mathcal{H}}\, \mathcal{E} = \pi\theta\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}\big\{\partial_t \Omega^\dagger \partial_t \Omega + \partial_x \Omega^\dagger \partial_x \Omega + \partial_y \Omega^\dagger \partial_y \Omega\big\}. \tag{VIII.5}$$

Note that the energy density $\mathcal{E}$ defined in (VIII.5) is no longer a positive semidefinite functional of the field $\Omega$. In terms of the operators (VIII.2) $E$ reads as

$$E = \pi\theta \, \mathrm{Tr}_{\mathcal{H}} \, \mathrm{tr}\big\{\partial_t\Omega^\dagger\partial_t\Omega + \tfrac{1}{\theta}([c,\Omega^\dagger][c,\Omega^\dagger]^\dagger + [c^\dagger,\Omega^\dagger][c^\dagger,\Omega^\dagger]^\dagger)\big\}. \tag{VIII.6}$$

**Noncommutative equations of motion.** The equations of motion following from (VIII.4) retain their familiar form, i.e.

$$(g^{\mu\nu} + V_\alpha\epsilon^{\alpha\mu\nu})\partial_\mu(\Omega^\dagger\partial_\nu\Omega) = 0 \tag{VIII.7}$$

or by introducing the coordinates

$$u := \tfrac{1}{2}(t+y) \qquad \text{and} \qquad v := \tfrac{1}{2}(t-y) \tag{VIII.8}$$

we can rewrite them as

$$\partial_x(\Omega^\dagger\partial_x\Omega) - \partial_v(\Omega^\dagger\partial_u\Omega) = 0. \tag{VIII.9}$$

**Dressing approach and ansatz.** In section § IV.3 we discussed the linear auxiliary problem (IV.61) whose compatibility conditions are the equations of motion. Also in the noncommutative regime we can consider this system, i.e.

$$(\zeta\partial_x - \partial_u)\psi = A\psi \qquad \text{and} \qquad (\zeta\partial_v - \partial_x)\psi = B\psi, \tag{VIII.10}$$

and impose the same reality condition

$$[\psi(x,\bar{\zeta})]^\dagger = \psi^{-1}(x,\zeta) \tag{VIII.11}$$

as before. This will again ensure that the field $\Omega$ is real. Note that $\psi$ is operator-valued and lives on $\mathbb{R}^{2,1}_\theta \times \mathbb{CP}^1$. Also $A$ and $B$ are operator-valued $n\times n$ matrices which, of course, do not depend on the spectral parameter $\zeta$. Again we consider an ansatz of the form

$$\tilde{\psi}(x,y,t,\zeta) = \chi(x,y,t,\zeta)\psi(x,y,t,\zeta) \qquad \text{with} \qquad \chi = 1 + \sum_{\alpha=1}^{s}\sum_{k=1}^{m}\frac{R_{\alpha k}}{(\zeta-\mu_k)^\alpha}, \tag{VIII.12}$$

where the $R_{\alpha k}$'s are $n\times n$ matrices depending only on spacetime and $\mu_k \in \mathbb{C}$ for $k = 1,\ldots,m$. We take the vacuum seed solution $A = B = 0$, $\psi = 1$ and restrict the ansatz to first-order poles in $\zeta$

$$\psi = 1 + \sum_{k=1}^{m}\frac{R_k}{\zeta-\mu_k}, \tag{VIII.13}$$

where $\mu_k \in \mathbb{C}$ with $\Im\{\mu_k\} < 0$. Moreover, we take the matrices $R_k$ to be of the form[1]

$$R_k = \sum_{l=1}^{m} T_l\,\Gamma^{lk}\,T_k^\dagger, \tag{VIII.14}$$

[1]Note that this definition differs slightly from the one used in section § IV.3.

where the $T_k(x,y,t)$ are $n{\times}r$ matrices and $\Gamma^{lk}(x,y,t)$ are $r\times r$ matrices with $r\geq 1$. Note that in the Abelian case ($n=1$) one may think of $T_k$ as a time-dependent row vector $(|b_k^1,t\rangle,\dots,|b_k^r,t\rangle)$ in $\mathcal{H}$.

We must make sure to satisfy the reality condition (VIII.11) as well as our linear system (VIII.10). In particular, since $A$ and $B$ do not depend on $\zeta$ all proper residues must vanish. Inserting the ansatz (VIII.13) with (VIII.14) and putting to zero the corresponding residues, we learn from (VIII.11) that

$$\left(1-\sum_{p=1}^{m}\frac{R_p}{\mu_p-\bar{\mu}_k}\right)T_k=0, \tag{VIII.15}$$

while from (VIII.10) we obtain the differential equations

$$(A\text{ or }B)=-\psi(\zeta)\,\bar{L}_k\,\psi(\bar{\zeta})^\dagger\Big|_{\zeta\to\bar{\mu}_k}\qquad\Longrightarrow\qquad\left(1-\sum_{p=1}^{m}\frac{R_p}{\mu_p-\bar{\mu}_k}\right)\bar{L}_k\,R_k^\dagger=0, \tag{VIII.16}$$

where

$$\bar{L}_k:=\bar{\mu}_k\partial_x-\partial_u\qquad\text{or}\qquad\bar{L}_k:=\bar{\mu}_k\partial_v-\partial_x. \tag{VIII.17}$$

The algebraic conditions (VIII.15) imply that the $\Gamma^{lp}$ invert the matrices

$$\tilde{\Gamma}_{pk}=\frac{1}{\mu_p-\bar{\mu}_k}\,T_p^\dagger\,T_k\qquad\text{with}\qquad\sum_{p=1}^{m}\Gamma^{lp}\,\tilde{\Gamma}_{pk}=\delta^l{}_k. \tag{VIII.18}$$

The final expression of our solution is then given by

$$\Omega^\dagger=\psi(x,y,t,\zeta=0)=1-\sum_{k,l=1}^{m}\frac{1}{\mu_k}\,T_l\,\Gamma^{lk}\,T_k^\dagger. \tag{VIII.19}$$

**Moving-frame coordinates.** At this stage it is convenient to introduce so-called co-moving coordinates

$$w_k:=\nu_k\big(x+\bar{\mu}_k u+\bar{\mu}_k^{-1}v\big)=\nu_k\big(x+\tfrac{1}{2}(\bar{\mu}_k-\bar{\mu}_k^{-1})y+\tfrac{1}{2}(\bar{\mu}_k+\bar{\mu}_k^{-1})t\big), \tag{VIII.20a}$$

$$\bar{w}_k:=\bar{\nu}_k\big(x+\mu_k u+\mu_k^{-1}v\big)=\bar{\nu}_k\big(x+\tfrac{1}{2}(\mu_k-\mu_k^{-1})y+\tfrac{1}{2}(\mu_k+\mu_k^{-1})t\big), \tag{VIII.20b}$$

where the $\nu_k$ are expressed in terms of the $\mu_k$ via

$$\nu_k=\sqrt{\frac{4i}{\mu_k-\bar{\mu}_k-\mu_k^{-1}+\bar{\mu}_k^{-1}}\,\frac{\mu_k-\mu_k^{-1}-2i}{\bar{\mu}_k-\bar{\mu}_k^{-1}+2i}}. \tag{VIII.21}$$

Note that the coordinates (VIII.20) lie in the kernel of the differential operators (VIII.17). Using the moving-frame coordinates we can rewrite the operators defined in (VIII.17) as

$$\bar{L}_k=\bar{\nu}_k\,(\bar{\mu}_k-\mu_k)\,\partial_{\bar{w}_k}\qquad\text{or}\qquad\bar{L}_k=\bar{\nu}_k\,\mu_k^{-1}(\bar{\mu}_k-\mu_k)\,\partial_{\bar{w}_k}, \tag{VIII.22}$$

respectively, so that we have only one equation (VIII.16) for each pole. Since

$$[w_k, \bar{w}_k] = \tfrac{i}{2}\theta\, \nu_k \bar{\nu}_k \,(\mu_k - \bar{\mu}_k - \mu_k^{-1} + \bar{\mu}_k^{-1}) = 2\theta, \tag{VIII.23}$$

we may define the co-moving creation and annihilation operators

$$b_k = \tfrac{1}{\sqrt{2\theta}} w_k \qquad \text{and} \qquad b_k^\dagger = \tfrac{1}{\sqrt{2\theta}} \bar{w}_k \qquad \text{with} \quad [b_k, b_k^\dagger] = 1, \tag{VIII.24}$$

which act on $\mathcal{H}_k = \bigoplus_{m \in \mathbb{N}_0} \mathbb{C}|m\rangle_k$.

It is necessary to relate the co-moving oscillators with the static one. Expressing $w_k$ by $z$ via (VIII.20) and (VIII.21) and using (VIII.2), we find after some algebra

$$b_k = \cosh(\tau_k)\, c - e^{i\vartheta_k} \sinh(\tau_k)\, c^\dagger - \beta_k\, t \;=\; U_k(t)\, c\, U_k^\dagger(t). \tag{VIII.25}$$

This is an inhomogeneous $SU(1,1)$ transformation[2] mediated by the operator

$$U_k(t) = e^{\frac{1}{2}\alpha_k\, c^{\dagger 2} - \frac{1}{2}\bar{\alpha}_k\, c^2}\, e^{(\beta_k\, c^\dagger - \bar{\beta}_k\, c)t}. \tag{VIII.26}$$

Here, we have abbreviated

$$\alpha_k = e^{i\vartheta_k}\, \tau_k \qquad \text{and} \qquad \beta_k = -\tfrac{1}{2\sqrt{2\theta}} \nu_k\, (\bar{\mu}_k + \bar{\mu}_k^{-1}) \tag{VIII.27}$$

with the relations

$$\xi_k := e^{i\vartheta_k} \tanh(\tau_k) = \frac{\bar{\mu}_k - \bar{\mu}_k^{-1} - 2i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2i} \qquad \text{and} \qquad \nu_k = \cosh(\tau_k) - e^{i\vartheta_k} \sinh(\tau_k), \tag{VIII.28}$$

so that $\mu_k$ may be expressed in terms of $\xi_k$ via $\bar{\mu}_k = i\frac{(1+\sqrt{\xi_k})^2}{1-\xi_k}$. Obviously, we have

$$U_k(t) \,:\, \mathcal{H} \to \mathcal{H}_k \qquad \text{with} \qquad |m\rangle_k = U_k(t)|m\rangle.$$

Thus, all co-moving oscillators are unitarily equivalent to the static one. The unitary transformation (VIII.25) is known as squeezing by $\alpha_k$ plus shifting by $\beta_k\, t$. Note that the static case is recovered by setting $\mu_k = -i$.

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[2] Remember that for $[c, c^\dagger] = 1$ a representation of $\mathfrak{su}(1,1)$ is given by

$$K_+ := \tfrac{1}{2} c^{\dagger 2}, \qquad K_- := \tfrac{1}{2} c^2 \qquad \text{and} \qquad K_0 := \tfrac{1}{4}(cc^\dagger + c^\dagger c)$$

which satisfy the $\mathfrak{su}(1,1)$ algebra

$$[K_0, K_\pm] = \pm K_\pm \qquad \text{and} \qquad [K_+, K_-] = -2K_0.$$

# § VIII.2 Soliton Solutions - Part A

**Solutions.** We are now in shape to write down explicit solutions. By the above discussion, the remaining equations of motion (VIII.16) turn into

$$\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) b_k\, T_k = 0. \tag{VIII.29}$$

Clearly, a sufficient condition for a solution is

$$b_k\, T_k = T_k\, Z_k \tag{VIII.30}$$

with some $r \times r$ matrix $Z_k$.

Firstly, let us look at the Abelian case, i.e. $n = 1$. Remember that the row vector $T_k$ is given by $T_k = (|b_k^1, t\rangle, \ldots, |b_k^r, t\rangle)$. Let us take

$$Z_k = \mathrm{diag}(b_k^1, \ldots, b_k^r) \tag{VIII.31}$$

with $b_k^i \in \mathbb{C}$ and $b_k\, |b_k^i, t\rangle = b_k^i\, |b_k^i, t\rangle$. This also explains our labeling of the kets and qualifies them as coherent states based on the co-moving ground state $|0\rangle_k$,

$$|b_k^i, t\rangle = e^{b_k^i b_k^\dagger - \bar{b}_k^i b_k} |0\rangle_k = e^{b_k^i b_k^\dagger - \bar{b}_k^i b_k}\, U_k(t)\, |0\rangle = U_k(t)\, e^{b_k^i c^\dagger - \bar{b}_k^i c}\, |0\rangle. \tag{VIII.32}$$

Summarizing, the Abelian solutions $|b_k^i, t\rangle$ are nothing but squeezed states. The squeezing parameter is $\alpha_k$ while the shift parameter is $b_k^i + \beta_k t$.

Secondly, we consider the case where $n \geq 2$. Then we may choose

$$Z_k = \mathbb{1}_r\, b_k \qquad \Longrightarrow \qquad [b_k, T_k] = 0, \tag{VIII.33}$$

which can be interpreted as a holomorphicity condition on $T_k$. Therefore any $n \times r$ matrix $T_k$ whose entries are arbitrary functions of $b_k$ (but independent of $b_k^\dagger$) provides a solution $\psi$ to the linear system (VIII.10). As we shall see momentarily, the complex moduli $\mu_k$ turn out to parametrize the velocities $\vec{v}_k$ of individual asymptotic lumps of energy. The shape and location of these lumps is encoded in further moduli hidden in the functions $T_k(b_k)$.

## § VIII.2.1 Static Configurations

**Projectors.** In this subsection we want to stick to the static case, i.e. we consider $m = 1$ and $\mu = -i$. Therefore, the ansatz (VIII.13) simplifies to

$$\psi = 1 + \frac{R}{\zeta + i} =: 1 - \frac{2i}{\zeta + i} P \tag{VIII.34a}$$

and hence

$$\Omega^\dagger = 1 - 2P. \tag{VIII.34b}$$

The reality condition (VIII.11) directly yields

$$P^\dagger = P \qquad \text{and} \qquad P^2 = P. \tag{VIII.35}$$

Thus, $P$ can be regarded as a Hermitian projector and we have $\Omega^\dagger = \Omega$. Actually, (VIII.11) degenerates to $(1-P)P = 0$, while (VIII.16) simplifies to

$$(1-P)\bar{L}P = 0 \qquad \Longrightarrow \qquad (1-P)cP = 0. \tag{VIII.36}$$

It is illuminating to specialize also our ansatz (VIII.14) for the residue, with (VIII.18), to the static situation

$$\Gamma = \frac{-2i}{T^\dagger T} \qquad \Longrightarrow \qquad P \equiv \frac{R}{-2i} = T\,\frac{\Gamma}{-2i}\,T^\dagger = T\,\frac{1}{T^\dagger T}\,T^\dagger. \tag{VIII.37}$$

Hence, the projector $P$ is parametrized by an $n \times r$ matrix $T$, where in the non-Abelian case $r \leq n$ can be identified with the rank of the projector in the $U(n)$ group space (but not in $\mathcal{H}$). The equation (VIII.29) then simplifies to

$$(1-P)c\,T = 0. \tag{VIII.38}$$

**Abelian static solutions.** Abelian solutions $n = 1$ are now easily generated. Since $\mu = -i$ implies $\alpha = 0$ and $\beta = 0$, the squeezing disappears and $r$ lumps are simply sitting at fixed positions $b^i$ for $i = 1, \ldots, r$. Therefore we have

$$T = (|b^1\rangle, \ldots, |b^r\rangle) \qquad \text{with} \qquad |b^i\rangle = e^{b^i c^\dagger - \bar{b}^i c}|0\rangle. \tag{VIII.39}$$

This is obvious because $w \to z$ in the static case. The coherent states $|b^i\rangle$ are normalized to unity but we know that they are not orthogonal. One reads off the rank $r$ projector

$$P = \sum_{i,j=1}^{r} |b^i\rangle\,\frac{1}{\langle b^i|b^j\rangle}\,\langle b^j| = \mathcal{U}\sum_{n=0}^{r-1}|n\rangle\langle n|\,\mathcal{U}^\dagger, \tag{VIII.40}$$

where the unitary operator $\mathcal{U}$ is given by

$$\mathcal{U} := \sum_{i\in\mathbb{N}_0}|h^{i+1}\rangle\langle i| \qquad \text{with} \qquad (|h^1\rangle, \ldots, |h^r\rangle) := T\,(T^\dagger T)^{-\frac{1}{2}}. \tag{VIII.41}$$

Note that the row vector $T = (|0\rangle, \ldots, |r{-}1\rangle)$ directly solves (VIII.38), and $P$ projects onto the space spanned by the $r$ lowest oscillator states. One easily verifies that these solutions have an energy (VIII.6) of $E = 8\pi r$, since for instance

$$\begin{aligned} E &= 8\pi\,\mathrm{Tr}_{\mathcal{H}}\big\{[c,P][c,P]^\dagger\big\} = 8\pi\,\mathrm{Tr}_{\mathcal{H}}\big\{c^\dagger P c - P c^\dagger + 2c^\dagger(1-P)cP\big\} \\ &= 8\pi\,\mathrm{Tr}_{\mathcal{H}}\big\{c^\dagger P c - P c^\dagger c\big\} = 8\pi\sum_{n=0}^{r-1}(n+1) - 8\pi\sum_{n=0}^{r-1} n = 8\pi r, \end{aligned}$$

where equation (VIII.36) and $P = \sum_{n=0}^{r-1}|n\rangle\langle n|$ have been used.

Having constructed noncommutative solitons in operator language, it is natural to wonder how these configurations look like in star product formulation. As an example, consider the solution

$$\Omega = 1 - 2|0\rangle\langle 0| \tag{VIII.42a}$$

which can be translated into star product language by using (VI.86)

$$\Omega_\star = 1 - 4e^{-\frac{1}{\theta}z\bar{z}} = 1 - 4e^{-\frac{1}{\theta}r^2}. \tag{VIII.42b}$$

Note that there is no smooth $\theta \to 0$ limit.

**Non-Abelian static solutions.** Another special case are non-Abelian solutions with $r = 1$. They are provided by column vectors $T$ whose entries are functions only of $c$ (or equivalently of $z$). For notational simplicity we define

$$K := \frac{1}{1 + \sum_{i=1}^{n-1} \bar{z}^{m_i} z^{m_i}} \qquad \text{with} \qquad m_i \in \mathbb{N}_0. \tag{VIII.43}$$

Then $P$ takes the following form

$$P = T\frac{1}{T^\dagger T}T^\dagger = \begin{pmatrix} K & K\bar{z}^{m_1} & \cdots & K\bar{z}^{m_{n-1}} \\ z^{m_1}K & z^{m_1}K\bar{z}^{m_1} & \cdots & z^{m_1}K\bar{z}^{m_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ z^{m_{n-1}}K & z^{m_{n-1}}K\bar{z}^{m_1} & \cdots & z^{m_{n-1}}K\bar{z}^{m_{n-1}} \end{pmatrix} \tag{VIII.44}$$

with $T^\dagger = (1, \bar{z}^{m_1}, \ldots, \bar{z}^{m_{n-1}})$. One can show that this ansatz describes a field configuration consisting of

$$Q = \max_i \{m_i\}$$

static lumps. The total energy is given by

$$E = 8\pi Q = 8\pi \max_i \{m_i\}. \tag{VIII.45}$$

To exemplify this, let us consider the case $n = 2$ and $m_1 = 1$. Then we have

$$P = \begin{pmatrix} \frac{s}{N+s} & \frac{\sqrt{s}}{N+s}c^\dagger \\ c\frac{\sqrt{s}}{N+s} & c\frac{1}{N+s}c^\dagger \end{pmatrix} \qquad \text{with} \qquad N = c^\dagger c \qquad \text{and} \qquad s = \tfrac{1}{2\theta}.$$

The energy computes to

$$\begin{aligned} E &= 8\pi\,\mathrm{Tr}_{\mathcal{H}}\big\{[c,P][c,P]^\dagger\big\} = 8\pi\,\mathrm{Tr}_{\mathcal{H}}\big\{c^\dagger P c - P c^\dagger c\big\} \\ &= 8\pi\,\mathrm{Tr}_{\mathcal{H}}\left\{\frac{Ns}{(N+s)(N+s-1)} - \frac{Ns}{(N+s)(N+s+1)}\right\} \\ &= 8\pi \sum_{n\in\mathbb{N}_0} \frac{(n+1)s}{(n+s+1)(n+s)} - 8\pi \sum_{n\in\mathbb{N}_0} \frac{ns}{(n+s)(n+s+1)} \\ &= 8\pi \sum_{n\in\mathbb{N}_0} \frac{s}{(n+s)(n+s+1)} = 8\pi. \end{aligned}$$

### § VIII.2.2 Non-Static Configurations

Let us now construct time-dependent solutions. We still want to consider only first order poles in the ansatz (VIII.12). Similarly to the commutative case, we will not obtain scattering configurations, since for these we need higher order poles. Such configurations will be the subject of the next section.

To find a time-dependent configuration, we consider our first-order pole ansatz (VIII.13) with $m = 1$, such that

$$\psi = 1 + \frac{R}{\zeta - \mu} =: 1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P \qquad \text{with} \qquad P = T \frac{1}{T^\dagger T} T^\dagger, \tag{VIII.46}$$

where $P$ and $T$ satisfy

$$(1 - P) b P = 0 \qquad \text{and} \qquad b\, T = T\, Z. \tag{VIII.47}$$

Here

$$b = \cosh(\tau)\, c - e^{i\vartheta} \sinh(\tau)\, c^\dagger - \beta\, t = U(t)\, c\, U^\dagger(t) \tag{VIII.48}$$

is the annihilation operator in the moving frame. This equation then implies that

$$\begin{pmatrix} w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} \cosh(\tau) & -e^{i\vartheta} \sinh(\tau) \\ -e^{-i\vartheta} \sinh(\tau) & \cosh(\tau) \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \nu\, (\bar{\mu} + \bar{\mu}^{-1}) \\ \bar{\nu}\, (\mu + \mu^{-1}) \end{pmatrix} t \tag{VIII.49}$$

and is in fact a symplectic coordinate transformation, preserving $\mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}$. Even though the inhomogeneous part of (VIII.49) is linear in time, this does not mean that the solution

$$\Omega \equiv \Omega_{\vec{v}} = 1 - \bar{\rho}\, P \qquad \text{with} \qquad \rho = 1 - \frac{\bar{\mu}}{\mu} \tag{VIII.50}$$

is simply obtained by subjecting the corresponding static solution $\Phi_{\vec{0}}$ to such a symplectic boost. To understand this situation better, consider the moving frame with the coordinates $w, \bar{w}$ and $t' = t$. The related change of derivatives is given by

$$\partial_x = \nu\, \partial_w + \bar{\nu}\, \partial_{\bar{w}}, \tag{VIII.51a}$$

$$\partial_t = \tfrac{1}{2} \nu\, (\bar{\mu} + \bar{\mu}^{-1})\, \partial_w + \tfrac{1}{2} \bar{\nu}\, (\mu + \mu^{-1})\, \partial_{\bar{w}} + \partial_{t'}, \tag{VIII.51b}$$

$$\partial_y = \tfrac{1}{2} \nu\, (\bar{\mu} - \bar{\mu}^{-1})\, \partial_w + \tfrac{1}{2} \bar{\nu}\, (\mu - \mu^{-1})\, \partial_{\bar{w}}. \tag{VIII.51c}$$

Note that in the moving frame our solution (VIII.50) will be static, i.e. $\partial_{t'} \Omega = 0$. However, the field equation (VIII.9) is not invariant with respect to the coordinate transformation (VIII.49). This one may readily check by a direct calculation. From this then follows that the solution (VIII.50), which is static in the moving frame, has a functional form different from that of the static solution (VIII.34a) - the coefficient in front of the projector is altered from 2 to $\bar{\rho}$. Due to this non-invariance, the coordinate frame $(z, \bar{z}, t)$ is distinguished because in it the field equation takes the simplest form.

The (constant) velocity $\vec{v}$ in the $(x, y)$-plane is derived by writing down the map (VIII.49) in real coordinates ($w =: x' + iy'$)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh(\tau) - \cos(\vartheta)\sinh(\tau) & -\sin(\vartheta)\sinh(\tau) \\ -\sin(\vartheta)\sinh(\tau) & \cosh(\tau) + \cos(\vartheta)\sinh(\tau) \end{pmatrix} \left[ \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} v_x \\ v_y \end{pmatrix} t \right], \tag{VIII.52}$$

where we find

$$\vec{v} = (v_x, v_y) = -\Big(\frac{\cos(\varphi)}{\cosh(\eta)}, \frac{\sinh(\eta)}{\cosh(\eta)}\Big) \qquad \text{for} \qquad \mu =: e^{\eta - i\varphi}. \tag{VIII.53}$$

The velocity dependence of the energy (VIII.5) is obtained by substituting (VIII.51) and (VIII.50) into (VIII.5). The calculation is straightforward. We find

$$\begin{aligned} E[\Omega_{\vec{v}}] &= \pi\theta\, \nu\bar{\nu}(\sqrt{\mu\bar{\mu}} + \sqrt{\mu\bar{\mu}}^{-1})^2\ \mathrm{Tr}_{\mathcal{H}}\big\{\partial_w \Omega_{\vec{v}}^\dagger\, \partial_{\bar{w}} \Omega_{\vec{v}}\big\} \\ &= \pi\theta\, 16 f(\vec{v})\ \mathrm{Tr}_{\mathcal{H}}\big\{\partial_w P\, \partial_{\bar{w}} P\big\} \\ &= 8\pi f(\vec{v})\ \mathrm{Tr}_{\mathcal{H}}\big\{[b, P]\, [b, P]^\dagger\big\} \\ &= 8\pi f(\vec{v})\, Q = f(\vec{v})\, E[\Omega_{\vec{0}}], \end{aligned} \tag{VIII.54}$$

because the final trace is, by unitary equivalence, independent of the motion. Here, we have defined the velocity factor

$$f(\vec{v}) := \tfrac{1}{16}\, \rho\bar{\rho}\, \nu\bar{\nu}\, (\sqrt{\mu\bar{\mu}} + \sqrt{\mu\bar{\mu}}^{-1})^2 = \frac{1}{\nu\bar{\nu}} = \cosh(\eta)\, \sin(\varphi) = \frac{\sqrt{1 - v^2}}{1 - v_y^2}. \tag{VIII.55}$$

Note that for motion in $y$-direction ($\varphi = \frac{\pi}{2}$) $f(\vec{v})$ agrees with the usual relativistic contraction factor $1/\sqrt{1 - v^2} \geq 1$. However, it is smaller than one for motion with $|v_x| > |v_y|\sqrt{1 - v_y^2}$, and the energy may be made arbitrarily close to zero by a large boost, since $v^2 \to 1$ is equivalent to $\varphi \to 0$ or $\eta \to \pm\infty$.

**Abelian non-static solutions.** Let us exemplify our above discussion. Since $|0\rangle' := U(t)|0\rangle$ denotes the moving ground state, the $r = 1$ Abelian soliton solution reads

$$\Omega = 1 - \bar{\rho}\, U(t)\, |0\rangle\langle 0|\, U(t)^\dagger \tag{VIII.56a}$$

or in star product language

$$\Omega_\star = 1 - 2\bar{\rho}\, e^{-\frac{1}{\theta} w\bar{w}} = 1 - 2\bar{\rho}\, e^{-\frac{1}{\theta}\, {}^t[\vec{r} - \vec{v}t]\, {}^t\Lambda\Lambda\, [\vec{r} - \vec{v}t]}. \tag{VIII.56b}$$

Here, the matrix $\Lambda$ is the squeezing matrix appearing in (VIII.52).

**Non-Abelian non-static solutions.** In a similar manner, we can write down an $r = 1$ non-Abelian solution. Consider the ansatz (VIII.44) with $n = 2$ and $m_1 = 1$. Then we have

$$\Omega = 1 - \bar{\rho}\, U(t) \begin{pmatrix} \frac{1}{1 + \bar{z}z} & \frac{1}{1 + \bar{z}z}\bar{z} \\ z\frac{1}{1 + \bar{z}z} & z\frac{1}{1 + \bar{z}z}\bar{z} \end{pmatrix} U(t)^\dagger = 1 - \bar{\rho} \begin{pmatrix} \frac{1}{1 + \bar{w}w} & \frac{1}{1 + \bar{w}w}\bar{w} \\ w\frac{1}{1 + \bar{w}w} & w\frac{1}{1 + \bar{w}w}\bar{w} \end{pmatrix}. \tag{VIII.57a}$$

The time dependence is hidden in

$$w = w(x, y, t) = (\cosh(\tau) - e^{i\vartheta}\sinh(\tau))(x - v_x t) + i(\cosh(\tau) + e^{i\vartheta}\sinh(\tau))(y - v_y t).$$

## § VIII.3 Soliton Solutions - Part B

In this last section we want to discuss scattering configurations. We first look at pure soliton scattering processes and then turn our attention to soliton-antisoliton scattering.

### § VIII.3.1 Soliton Scattering Configurations

**Dressing approach.** To construct scattering configurations we need to consider higher order poles in the ansatz (VIII.12). To achieve this, let us take the static solution (VIII.34a) as the seed solution and consider

$$\psi \mapsto \tilde{\psi} = \chi\psi = \Big(1 - \frac{2i}{\zeta+i}\tilde{P}\Big)\Big(1 - \frac{2i}{\zeta+i}P\Big) = 1 - \frac{2i}{\zeta+i}(P+\tilde{P}) - \frac{4}{(\zeta+i)^2}\tilde{P}P, \qquad \text{(VIII.58)}$$

where $\tilde{P}$ is some new matrix. This leads to a configuration of the form

$$\Omega = \tilde{\psi}^{-1}(x,y,t,\zeta=0) = (1-2P)(1-2\tilde{P}). \qquad \text{(VIII.59)}$$

Since both $\psi$ and $\chi$ possess a first-order pole at $\zeta = -i$ our ansatz for $\tilde{\psi}$ contains a second-order pole. Substituting (VIII.58) into the reality condition (VIII.11) directly yields

$$\tilde{P}^\dagger = \tilde{P} \qquad \text{and} \qquad \tilde{P}^2 = \tilde{P}, \qquad \text{(VIII.60)}$$

qualifying $\tilde{P}$ as a Hermitian projector, i.e.

$$\tilde{P} = \tilde{T}\,\frac{1}{\tilde{T}^\dagger\tilde{T}}\,\tilde{T}^\dagger \qquad \text{(VIII.61)}$$

with some $n \times \tilde{r}$ matrix $\tilde{T}$.

**Linear equations.** Demanding that $\tilde{\psi}$ is again a solution to the linear equations (VIII.10) with some $\tilde{A}$ and $\tilde{B}$, we obtain

$$\begin{aligned}\tilde{A}(x,y,t) &= -\tilde{\psi}(x,y,t,\zeta)(\zeta\partial_x - \partial_u)[\tilde{\psi}(x,y,t,\bar{\zeta})]^\dagger \\ &= \Big(1 - \frac{2i}{\zeta+i}\,\tilde{P}\Big)\Big(A - (\zeta\partial_x - \partial_u)\Big)\Big(1 + \frac{2i}{\zeta-i}\,\tilde{P}\Big),\end{aligned} \qquad \text{(VIII.62a)}$$

$$\begin{aligned}\tilde{B}(x,y,t) &= -\tilde{\psi}(x,y,t,\zeta)(\zeta\partial_v - \partial_x)[\tilde{\psi}(x,y,t,\bar{\zeta})]^\dagger \\ &= \Big(1 - \frac{2i}{\zeta+i}\,\tilde{P}\Big)\Big(B - (\zeta\partial_v - \partial_x)\Big)\Big(1 + \frac{2i}{\zeta-i}\,\tilde{P}\Big).\end{aligned} \qquad \text{(VIII.62b)}$$

The poles at $\zeta = \pm i$ on the right hand side of these equations have to be removable since $\tilde{A}$ and $\tilde{B}$ are independent of $\zeta$. Putting to zero the corresponding residues, we obtain the equations

$$(1-\tilde{P})\,\Big\{\partial_{\bar{z}}\tilde{P} \;+\; (\partial_{\bar{z}}P)\tilde{P}\Big\} = 0 \qquad \text{and} \qquad (1-\tilde{P})\,\Big\{\tfrac{i}{2}\,\partial_t\tilde{P} + (\partial_z P)\tilde{P}\Big\} = 0. \qquad \text{(VIII.63)}$$

With the help of the identities

$$(1-\tilde{P})\tilde{P} \equiv 0 \qquad \text{and} \qquad (1-\tilde{P})\tilde{T} \equiv 0,$$

the equations (VIII.63) reduce to

$$(1-\tilde{P})\left\{c\tilde{T} + [c,P]\,\tilde{T}\right\}\frac{1}{\tilde{T}^\dagger\tilde{T}}\tilde{T}^\dagger \;=\; 0, \tag{VIII.64a}$$

$$(1-\tilde{P})\;\left\{\partial_t\tilde{T} + i\gamma\,[c^\dagger,P]\,\tilde{T}\right\}\frac{1}{\tilde{T}^\dagger\tilde{T}}\tilde{T}^\dagger \;=\; 0, \tag{VIII.64b}$$

where $\gamma = \sqrt{\frac{2}{\theta}}$. Clearly, a sufficient condition for a solution is

$$c\tilde{T} + [c,P]\,\tilde{T} = \tilde{T}\,Z_1 \qquad \text{and} \qquad \partial_t\tilde{T} + i\gamma[c^\dagger,P]\,\tilde{T} = \tilde{T}\,Z_2, \tag{VIII.65}$$

with some functions $Z_1(c,c^\dagger,t)$ and $Z_2(c,c^\dagger,t)$.

**Non-Abelian multi-soliton solutions.** Let $P$ be given by (VIII.44). We build the dressing factor $\chi$ with a matrix $\tilde{T}^\dagger = (u_1^\dagger,\ldots,u_n^\dagger)$, where $u_i(z,\bar{z},t)$ are operator-valued functions to be determined. Choosing $Z_1 = c$ and the ansatz for $\tilde{T}$, the first equation of (VIII.65) turns into

$$[z,u_1] \;=\; [K,z]\Big(u_1 + \sum_{i=1}^{n-1}\bar{z}^{m_i}u_{i+1}\Big) - 2\theta K\sum_{i=1}^{n-1}m_i\bar{z}^{m_i-1}u_{i+1}, \tag{VIII.66a}$$

$$[z,u_{i+1}] \;=\; z^{m_i}[K,z]\Big(u_1 + \sum_{i=1}^{n-1}\bar{z}^{m_i}u_{i+1}\Big) - z^{m_i}2\theta K\sum_{i=1}^{n-1}m_i\bar{z}^{m_i-1}u_{i+1}, \tag{VIII.66b}$$

for $i = 1,\ldots,n-1$. These equations immediately tell us that

$$[z, z^{m_i}u_1 - u_{i+1}] = 0 \qquad \Longrightarrow \qquad u_{i+1} = z^{m_i}u_1 - f_{i+1}(t,z), \tag{VIII.67}$$

where $f_{i+1}(z,t)$ is arbitrary, but independent of $\bar{z}$. By imposing $Z_2 = 0$ the second equation of (VIII.65) reduces to

$$i\theta\partial_t u_1 = [\bar{z},K]\Big(u_1 + \sum_{i=1}^{n-1}\bar{z}^{m_i}u_{i+1}\Big). \tag{VIII.68}$$

Plugging relation (VIII.67) into (VIII.68) and defining

$$u_1 = 1 + K\sum_{i=1}^{n-1}\bar{z}^{m_i}f_{i+1}(z,t) \tag{VIII.69}$$

yields

$$i\theta\partial_t u_1 = [\bar{z},K]K^{-1}. \tag{VIII.70}$$

It is not difficult to see that

$$f_{i+1}(z,t) = -2i(m_i z^{m_i-1}t + h_{i+1}(z)) \tag{VIII.71}$$

solves our equations consistently, where the $h_{i+1}$'s are arbitrary holomorphic operator-valued functions which depend only on $z$. Hence,

$$\tilde{T} = \begin{pmatrix} 1 \\ z^{m_1} \\ \vdots \\ z^{m_{n-1}} \end{pmatrix} + \begin{pmatrix} 1 \\ z^{m_1} \\ \vdots \\ z^{m_{n-1}} \end{pmatrix} \frac{1}{1+\sum_{i=1}^{n-1} \bar{z}^{m_i} z^{m_i}} \sum_{i=1}^{n-1} \bar{z}^{m_i} f_{i+1} - \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} . \qquad \text{(VIII.72)}$$

Note that for locations given by $m_i z^{m_i-1} t + h_{i+1}(z) = 0$ for all $i = 1, \ldots, n-1$ our solution degenerates, since $\tilde{T} = T$ and therefore $\Omega = (1-2P)^2 = 1$.

**Examples.** Firstly, consider the case of $n = 2$. Then we have

$$\tilde{T} = \begin{pmatrix} 1 \\ z^m \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \end{pmatrix} \frac{1}{1+\bar{z}^m z^m} z^m f - \begin{pmatrix} 0 \\ f \end{pmatrix} , \qquad \text{(VIII.73)}$$

with $m := m_1$ and $f := f_2$ and hence

$$f = -2i(m z^{m-1} t + h(z)). \qquad \text{(VIII.74)}$$

Performing the inverse Weyl transform then yields

$$\tilde{T} \mapsto \tilde{T}_\star = \begin{pmatrix} 1 \\ z^m \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \end{pmatrix} \star \frac{1}{1+\bar{z}^m \star z^m} \star \bar{z}^m \star f_\star - \begin{pmatrix} 0 \\ f_\star \end{pmatrix} \qquad \text{(VIII.75)}$$

with commutative coordinates $z, \bar{z}, t$. Using these expressions we can calculate the projectors $P_\star$ and $\tilde{P}_\star$ as well as the field $\Omega_\star = (1-2P_\star) \star (1-2\tilde{P}_\star)$ and the energy density $\mathcal{E}_\star$ for this family of solutions. Moreover, for large $r^2 \equiv z\bar{z}$ the field $\Omega_\star$ and the energy density $\mathcal{E}_\star$ will approach their commutative limits $\Omega$ and $\mathcal{E}$, respectively.

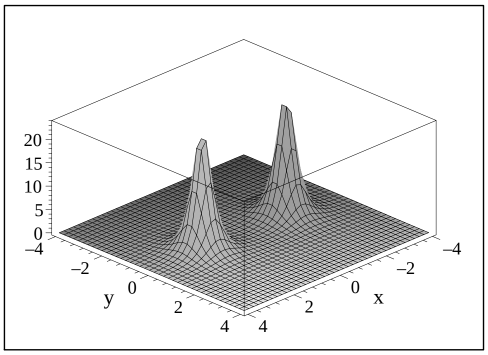


Figure VIII.1. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing right angle scattering at time $t = -4$ for $\theta \ll 1$.*

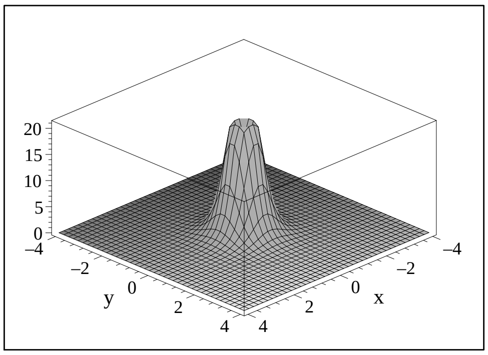


Figure VIII.2. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing right angle scattering at time $t = 0$ for $\theta \ll 1$.*

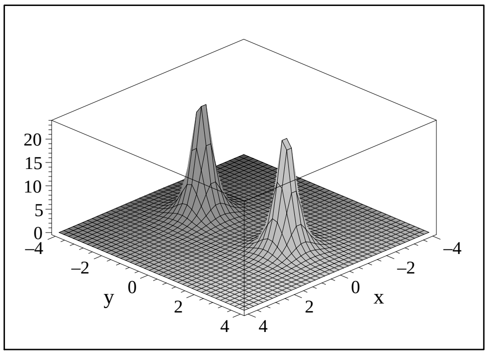


Figure VIII.3. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing right angle scattering at time $t=4$ for $\theta \ll 1$.*

Let $m=1$. The simplest scattering case occurs for $h=z^2$. One can show that the energy density for $\theta \ll 1$

$$\mathcal{E}_\star = 16\frac{1+10r^2+5r^4+4t^2(1+2r^2)-8t(x^2-y^2)}{(1+2r^2+5r^4+4t^2+8t(x^2-y^2))^2}\big[1+\mathcal{O}(\theta/r^2)\big] \tag{VIII.76}$$

peaks near the degeneration locus $z^2+t=0$. Hence, for a fixed large (positive or negative) time $t$, two lumps are centered at $z=\pm\sqrt{-t}$. Varying $t$ we see the two lumps accelerating symmetrically towards each other along the $x$-axis, interacting at the origin near $t=0$ and decelerating to infinity along the $y$-axis. Thus, a head-on collision of these solitons results in 90° scattering. Figures VIII.1-VIII.3 visualize this case.

To exhibit a configuration with ring-like structure, one can take e.g. $h=0$. In this case it turns out that the energy density

$$\mathcal{E}_\star = 16\frac{1+2r^2+r^4+4t^2(1+2r^2)}{(1+2r^2+r^4+4t^2)^2}\big[1+\mathcal{O}(\theta/r^2)\big] \tag{VIII.77}$$

is invariant under rotations around the origin. For fixed large time $t$, its degenerate maximum is found at $r^2 \sim 2|t|$. Varying $t$, this ring shrinks until $t=0$ and then expands again. One can also show that $\Omega_\star(t\to\pm\infty)\to -1$, i.e. no scattering occurs.

Consider now the case of $n=3$. For this case $\tilde{T}$ is given by

$$\tilde{T} = \begin{pmatrix} 1 \\ z^m \\ z^n \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \\ z^n \end{pmatrix} \frac{1}{1+\bar{z}^m z^m + \bar{z}^n z^n}\,(\bar{z}^m f + \bar{z}^n g) - \begin{pmatrix} 0 \\ f \\ g \end{pmatrix}, \tag{VIII.78}$$

with $m := m_1$ and $n := m_2$, $f := f_2$ and $g := f_3$ and hence

$$f = -2i(mz^{m-1}t + h_2(z)) \qquad \text{and} \qquad g = -2i(nz^{n-1}t + h_3(z)). \tag{VIII.79}$$

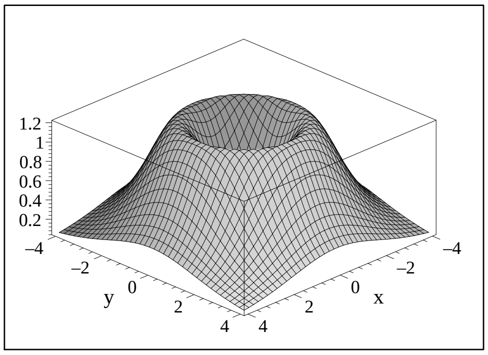


Figure VIII.4. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing a ring like structure at time $t=-4$ for $\theta \ll 1$.*

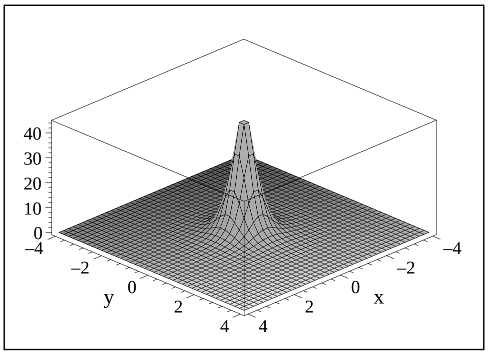


Figure VIII.5. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing a ring like structure at time $t=0$ for $\theta \ll 1$.*

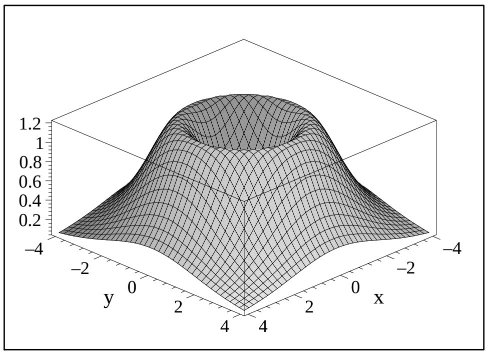


Figure VIII.6. *Energy distribution $\mathcal{E}_\star$ for two $U(2)$ solitons, showing a ring like structure at time $t=4$ for $\theta \ll 1$.*

Eventually, this leads to

$$\tilde{T} \mapsto \tilde{T}_\star = \begin{pmatrix} 1 \\ z^m \\ z^n \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \\ z^n \end{pmatrix} \star \frac{1}{1+\bar{z}^m \star z^m + \bar{z}^n \star z^n} \star (\bar{z}^m \star f_\star + \bar{z}^n \star g_\star) - \begin{pmatrix} 0 \\ f_\star \\ g_\star \end{pmatrix}. \quad \text{(VIII.80)}$$

One may easily check, that the solution (VIII.80) coincides in the limit $\theta \to 0$ with the one given by Ioannidou and Zakrzewski in [95].

Choosing $m=n=1$, $h_2(z)=z^2$, $h_3(z)=3$ and therefore $f=-2i(t+z^2)$ and $g=-2i(t+3)$, the solution (VIII.80) represents a three soliton configuration, which for large negative and positive times has a ring-like structure. Figures VIII.7 - VIII.10 show the energy density for various times in the commutative limit.

In order to obtain scattering solutions we consider $m=2$, $n=1$, $h_2(z)=z^3$ and $h_3(z)=0$. Therefore we have $f=-2i(2zt+z^3)$ and $g=-2it$. Figures VIII.11 - VIII.13 visualize the scattering configuration for small $\theta$. It consists of two static solitons sitting at the origin

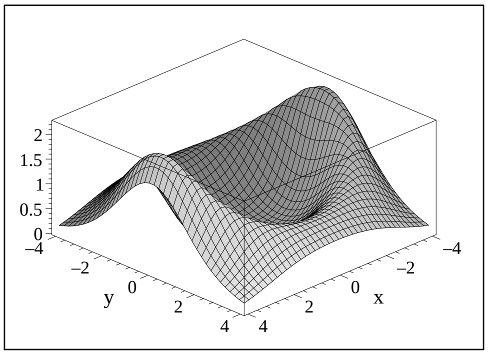


Figure VIII.7. *Energy distribution $\mathcal{E}_\star$ for three $U(3)$ solitons at time $t = -40$ for $\theta \ll 1$.*

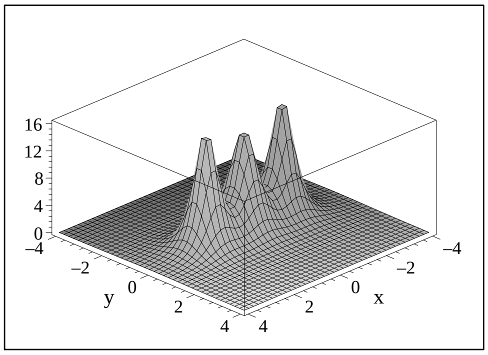


Figure VIII.8. *Energy distribution $\mathcal{E}_\star$ for three $U(3)$ solitons at time $t = -1$ for $\theta \ll 1$.*

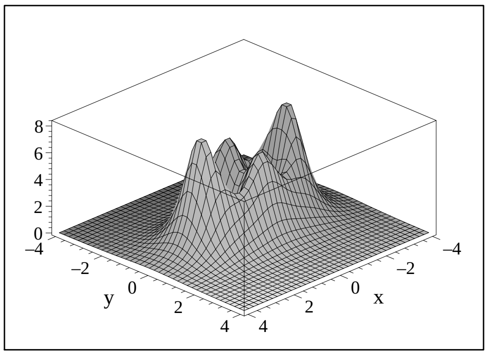


Figure VIII.9. *Energy distribution $\mathcal{E}_\star$ for three $U(3)$ solitons at time $t = 1$ for $\theta \ll 1$.*

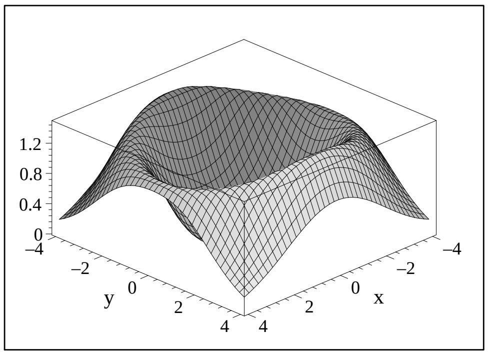


Figure VIII.10. *Energy distribution $\mathcal{E}_\star$ for three $U(3)$ solitons at time $t = 40$ for $\theta \ll 1$.*

accompanied by two moving solitons. The latter accelerate along the $x$-axis towards the origin, perform a 90° angle scattering and decelerate along the $y$-axis towards infinity.

We refrain from writing down the lengthy explicit expressions for the energy density for these two cases. Note that the limit $r \to \infty$, with $r^2 = z\bar{z}$, is equivalent to the limit $\theta \to 0$, since noncommuativity goes to zero for any fixed $\theta < \infty$. Hence, Ioannidou's and Zakrzewski's [95] asymptotic analysis can be applied without alteration.

## § VIII.3.2 Soliton-Antisoliton Scattering Configurations

**Linear equations.** In this subsection we shall contruct soliton-antisoliton solutions. For this we choose the dressing factor $\chi$ to be

$$\chi = 1 + \frac{2i}{\zeta - i}\tilde{P}, \tag{VIII.81}$$

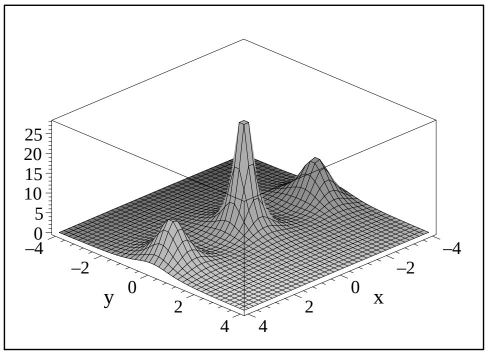


Figure VIII.11. *Energy distribution $\mathcal{E}_\star$ for four $U(3)$ solitons, showing right angle scattering at time $t = -10$ for $\theta \ll 1$.*

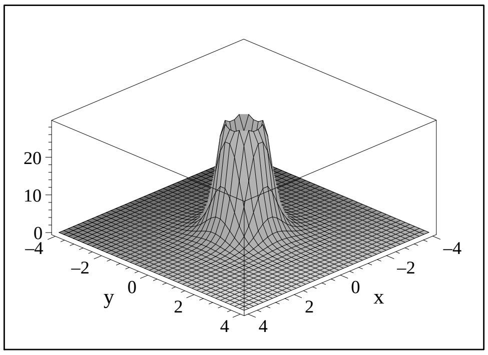


Figure VIII.12. *Energy distribution $\mathcal{E}_\star$ for four $U(3)$ solitons, showing right angle scattering at time $t = 0$ for $\theta \ll 1$.*

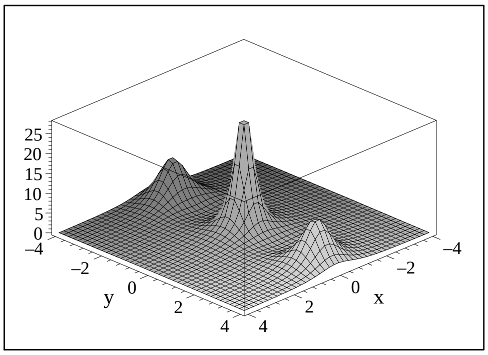


Figure VIII.13. *Energy distribution $\mathcal{E}_\star$ for four $U(3)$ solitons, showing right angle scattering at time $t = 10$ for $\theta \ll 1$.*

since if one uses $\Im\{\mu_k\} < 0$ for solitons then antisolitons correspond to $\mu_k$'s with $\Im\{\mu_k\} > 0$ . Once again, the reality condition (VIII.11) teaches us that

$$\tilde{P}^\dagger = \tilde{P} \qquad \text{and} \qquad \tilde{P}^2 = \tilde{P}, \tag{VIII.82}$$

so that $\tilde{P}$ can be considered to be a Hermitian projector $\tilde{P} = \tilde{T}(\tilde{T}^\dagger\tilde{T})^{-1}\tilde{T}^\dagger$. Since $\tilde{\psi}$ is again a solution of (VIII.10) with some $\tilde{A}$ and $\tilde{B}$, we discover

$$\begin{aligned} \tilde{A}(x,y,t) &= -\tilde{\psi}(x,y,t,\zeta)(\zeta\partial_x - \partial_u)[\tilde{\psi}(x,y,t,\bar{\zeta})]^\dagger \\ &= \Big(1 + \frac{2i}{\zeta - i}\tilde{P}\Big)\Big(A - (\zeta\partial_x - \partial_u)\Big)\Big(1 - \frac{2i}{\zeta + i}\tilde{P}\Big), \end{aligned} \tag{VIII.83a}$$

$$\begin{aligned} \tilde{B}(x,y,t) &= -\tilde{\psi}(x,y,t,\zeta)(\zeta\partial_v - \partial_x)[\tilde{\psi}(x,y,t,\bar{\zeta})]^\dagger \\ &= \Big(1 + \frac{2i}{\zeta - i}\tilde{P}\Big)\Big(B - (\zeta\partial_v - \partial_x)\Big)\Big(1 - \frac{2i}{\zeta + i}\tilde{P}\Big). \end{aligned} \tag{VIII.83b}$$

Remember that the poles at $\zeta = \pm i$ on the right hand side have to be removable since $\tilde{A}$ and $\tilde{B}$ do not depend on $\zeta$. Putting to zero the corresponding residues, we obtain the following pair of equations

$$(1-\tilde{P})\left\{\partial_z\tilde{P}-(\partial_z P)\tilde{P}\right\}=0 \qquad \text{and} \qquad (1-\tilde{P})\left\{\tfrac{i}{2}\partial_t\tilde{P}+(\partial_{\bar{z}}P)\tilde{P}\right\}=0. \tag{VIII.84}$$

Using the identities

$$(1-\tilde{P})\tilde{P}\equiv 0 \qquad \text{and} \qquad (1-\tilde{P})\tilde{T}\equiv 0,$$

we reduce the equations (VIII.84) to the form

$$(1-\tilde{P})\left\{c^\dagger\tilde{T}-[c^\dagger,P]\tilde{T}\right\}\frac{1}{\tilde{T}^\dagger\tilde{T}}\tilde{T}^\dagger \;=\; 0, \tag{VIII.85a}$$

$$(1-\tilde{P})\left\{\partial_t\tilde{T}-i\gamma[c,P]\tilde{T}\right\}\frac{1}{\tilde{T}^\dagger\tilde{T}}\tilde{T}^\dagger \;=\; 0 \tag{VIII.85b}$$

with $\gamma=\sqrt{\frac{2}{\theta}}$. Obviously, these equations are solved by

$$c^\dagger\tilde{T}-[c^\dagger,P]\tilde{T}=\tilde{T}Z_1 \qquad \text{and} \qquad \partial_t\tilde{T}-i\gamma[c,P]\tilde{T}=\tilde{T}Z_2, \tag{VIII.86}$$

where $Z_{1,2}(c,c^\dagger,t)$ are some operator-valued functions.

**Non-Abelian soliton-antisoliton configurations.** Again, we build the dressing factor $\chi$ with a matrix $\tilde{T}^\dagger=(u_1^\dagger,\dots,u_n^\dagger)$ and choose the ansatz for $P$ to be of the form (VIII.44). This implies for the first equation of (VIII.86)

$$\bar{z}u_1-[\bar{z},K]\Big(u_1+\sum_{i=1}^{n-1}\bar{z}^{m_i}u_{i+1}\Big) \;=\; u_1Z_1, \tag{VIII.87}$$

$$\bar{z}u_{j+1}-[\bar{z},z^{m_j}K]\Big(u_1+\sum_{i=1}^{n-1}\bar{z}^{m_i}u_{i+1}\Big) \;=\; u_{j+1}Z_1, \tag{VIII.88}$$

after a rescaling of $Z_1$. Multiplying the second of these equations by $\bar{z}^{m_j}$, summing over $j$ and adding to the first one yields

$$\bar{z}\Big(u_1+\sum_{j=1}^{n-1}\bar{z}^{m_j}u_{j+1}\Big)=\Big(u_1+\sum_{j=1}^{n-1}\bar{z}^{m_j}u_{j+1}\Big)Z_1. \tag{VIII.89}$$

Let us therefore choose $Z_1$ to be $\bar{z}$, i.e.

$$\Big[\bar{z},u_1+\sum_{j=1}^{n-1}\bar{z}^{m_j}u_{j+1}\Big]=0 \qquad \Longrightarrow \qquad u_1=-\sum_{j=1}^{n-1}\bar{z}^{m_j}u_{j+1}+f(\bar{z},t), \tag{VIII.90}$$

where $f(\bar{z},t)$ is arbitrary, but independent of $z$. Putting $Z_2$ identically to zero, the second equation of (VIII.86) reduces to

$$i\theta\partial_t\tilde{T}+[z,P]\tilde{T}=0 \tag{VIII.91}$$

and hence

$$i\theta\partial_t u_1=[K,z]f-2\theta K\sum_{j=1}^{n-1}m_j\bar{z}^{m_j-1}u_{j+1}. \tag{VIII.92}$$

Defining

$$u_{i+1} = -1 + z^{m_i} K f, \tag{VIII.93}$$

it is not hard to verify that

$$f(\bar{z}, t) = -2i\bigg(t \sum_{j=1}^{n-1} m_j \bar{z}^{m_j - 1} + h(\bar{z})\bigg) \tag{VIII.94}$$

solves our equations consistently with $h$ some operator-valued holomorphic function of $\bar{z}$. Hence,

$$\tilde{T} = \begin{pmatrix} \sum_{i=1}^{n-1} \bar{z}^{m_i} \\ -1 \\ \vdots \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ z^{m_1} \\ \vdots \\ z^{m_{n-1}} \end{pmatrix} \frac{-2i}{1 + \sum_{i=1}^{n-1} \bar{z}^{m_i} z^{m_i}} \bigg(t \sum_{j=1}^{n-1} m_j \bar{z}^{m_j - 1} + h(\bar{z})\bigg). \tag{VIII.95}$$

**Examples.** Let us now spezialize to the $n = 2$ case. For this case our solution reduces to

$$\tilde{T} = \begin{pmatrix} \bar{z}^m \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \end{pmatrix} \frac{-2i}{1 + \bar{z}^m z^m} \{m \bar{z}^{m-1} t + h(\bar{z})\}, \tag{VIII.96}$$

with $m := m_1$. Depending on the explicit form of $h(\bar{z})$, the solutions (VIII.96) describe different kinds of soliton-antisoliton configurations. Let us take, for instance, $h(\bar{z}) = \frac{1}{2}\bar{z}^{2m+n}$ with $n \in \mathbb{N}_0$. This leads to a configuration consisting of $2m+n$ lumps. In the commutative limit, the solution (VIII.96) coincides with Ioannidou's given by (IV.83).

To visualize the field configurations we perform the inverse Weyl transform and find

$$\tilde{T} \mapsto \tilde{T}_\star = \begin{pmatrix} \bar{z}^m \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ z^m \end{pmatrix} \star \frac{-2i}{1 + \bar{z}^m \star z^m} \star \{m \bar{z}^{m-1} t + h_\star(\bar{z})\}. \tag{VIII.97}$$

Hence, we know the field $\Omega_\star$ and can calculate its energy density $\mathcal{E}_\star$. Furthermore, for large distances $r^2 = z\bar{z}$ the field $\Omega_\star$ and the energy density $\mathcal{E}_\star$ will approach their commutative limits $\Omega$ and $\mathcal{E}$, such that the analysis given in [94] can be applied in the same way.

The simplest scattering case occurs for $m = 1$ and $n = 0$, i.e. $h(\bar{z}) = \frac{1}{2}\bar{z}^2$. After a few lines of algebra one finds that for large $r$ the energy density is given by

$$\mathcal{E}_\star = 16 \frac{2r^4 + 4r^2 + 4t^2(1 + 2r^2) - 4t(x^2 - y^2) + 1}{(2r^4 + 2r^2 + 4t(x^2 - y^2) + 4t^2 + 1)^2} \big[1 + \mathcal{O}(\theta/r^2)\big], \tag{VIII.98}$$

i.e. it peaks near the locus $\bar{z}^2 + 2t = 0$. To be more precise, by varying $t$ we see the two lumps accelerating symmetrically towards each other along the $x$-axis, interacting at the origin (near $t = 0$) and decelerating to infinity along the $y$-axis. This configuration has the topological charge $Q = 0$ since as argued above it coincides with Ioannidou's solution [94] in the limit $\theta \to 0$ and

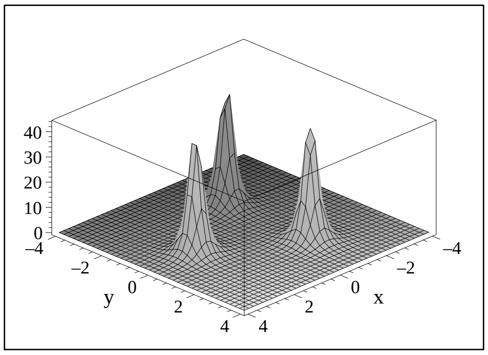


Figure VIII.14. *Energy distribution $\mathcal{E}_\star$ for one $U(2)$ soliton and two antisolitons, showing $60°$ angle scattering at time $t = -5$ for $\theta \ll 1$.*

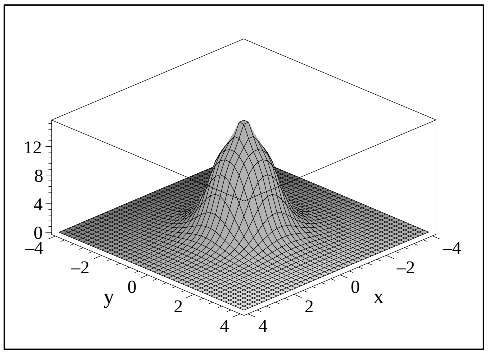


Figure VIII.15. *Energy distribution $\mathcal{E}_\star$ for one $U(2)$ soliton and two antisolitons, showing $60°$ angle scattering at time $t = 0$ for $\theta \ll 1$.*

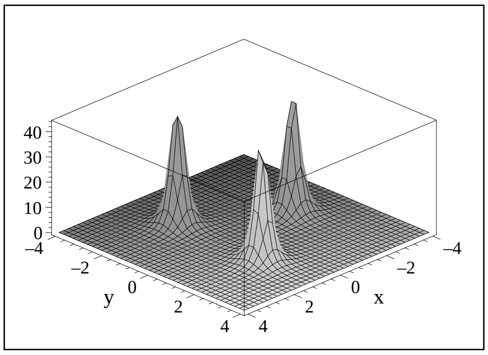


Figure VIII.16. *Energy distribution $\mathcal{E}_\star$ for one $U(2)$ soliton and two antisolitons, showing $60°$ angle scattering at time $t = 5$ for $\theta \ll 1$.*

the topological charge is independent of $\theta$. Thus, a head-on collision of one soliton and one antisoliton results in a $90°$ angle scattering.

To exhibit a scattering configuration with one soliton and two antisolitons we take $m = 1$ and $n = 1$, i.e. $h(\bar{z}) = \frac{1}{2}\bar{z}^3$ (i.e. $Q = -1$). In that case the energy density turns out to be

$$\begin{aligned}\mathcal{E}_\star =& 8[r^8 + 8r^6 + 11r^4 + 4r^2 - 8x^5 t + 16ty^2(x^3 + t) + 8t^2 + 48xy^2 t - 16x^2 t(x - t) \\ &+ 24xty^4 + 2]/[r^6 + r^4 + 2r^2 + 4t^2 + 4tx^3 - 12xy^2 t + 1]^2[1 + \mathcal{O}(\theta/r^2)].\end{aligned} \tag{VIII.99}$$

Finding its extremas we realize that a head-on collision of one soliton and two antisolitons results in a $60°$ angle scattering, as can be seen from figures VIII.14 - VIII.16.

# Chapter IX

## Noncommutative Seiberg-Witten Theory

In this last chapter we shall generalize the Abelian Seiberg-Witten monopole equations defined on commutative $\mathbb{R}^4$ to the noncommutative case. As we have seen in the previous chapters, Yang-Mills theory on noncommutative Euclidean spaces is obtained by a naive replacement of the ordinary pointwise product with the Moyal product. In case of pure Yang-Mills theory, this prescription is useful and in fact unique. However, it is well known that this method of translating a commutative theory into a noncommutative one breaks down when matter fields are involved. The reason for this lies in the noncommutativity itself, since matter fields can live in different representations which become equivalent in the commutative limit. For instance, scalar fields in noncommutative $U(1)$ gauge theory can be regarded as elements of a left module, or as elements of a right module, or they can transform in the adjoint representation. For this reason we propose to derive the noncommutative Seiberg-Witten equations from noncommutative self-duality type Yang-Mills equations in eight dimensions, which are uniquely defined. Eventually, we will discover the equations corresponding to the above mentioned naive substitution rule by a formal reduction of more general equations. In the commutative case a similar idea has been worked out by the authors of [10]. The ideas below are based on [163].

## § IX.1 Seiberg-Witten Monopole Equations on $\mathbb{R}^4_\theta$

### § IX.1.1 Non-Abelian Seiberg-Witten Equations

**Self-duality type equations in eight dimensions.** Let us consider a complex rank two free $\mathbb{R}^8_\theta$-module $\mathcal{E}$, where $\theta$ is given by its canonical form (VI.50), and assume that the gauge group $U(2)$ is acting on $\mathcal{E}$ in the fundamental representation. Therefore the connection on $\mathcal{E}$ reads as $\nabla_\mu = \partial_\mu + \mathcal{A}_\mu$, with $\partial_\mu$ given by (VI.70) and $\mathcal{A}_\mu \in \mathrm{End}_{\mathbb{R}^8_\theta}(\mathcal{E})$ with $\mathcal{A}_\mu^\dagger = -\mathcal{A}_\mu$. Let us first work in star product formulation. Remember that the components $\mathcal{F}_{\mu\nu}$ of the Yang-Mills curvature $\mathcal{F}_\mathcal{A}$ read

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_\star \qquad \text{(IX.1)}$$

and take values in $\mathfrak{u}(2)$. Here $\mu, \nu, \ldots$ run from 1 to 8. Consider the generalized self-duality equations for Yang-Mills fields in eight dimensions,

$$\mathcal{F}_{\mu\nu} = \tfrac{1}{2} T_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}, \tag{IX.2}$$

where the totally antisymmetric tensor $T_{\mu\nu\rho\sigma}$ is determined by the octonionic structure constants $f_{ijk}$ as[1]

$$T_{ijkl} = \tfrac{1}{6} \epsilon_{ijklmnq} f_{mnq} \qquad \text{and} \qquad T_{8ijk} = f_{ijk} \qquad \text{for} \qquad i, j, k, ... = 1, \ldots, 7. \tag{IX.3}$$

In the commutative case these equations were introduced in [39] and discussed e.g. in [194, 52, 58, 99, 100, 53]. Note that the tensor $T_{\mu\nu\rho\sigma}$, and therefore the equations (IX.2) are invariant with respect to the group Spin(7) rather than $SO(8)$. In fact, it is impossible to construct a totally antisymmetric tensor of rank four in eight dimensions which is invariant under $SO(8)$ rotations.

Using the definition (IX.3), we can write down the generalized self-duality equations (IX.2) in components as follows:

$$\begin{aligned}
\mathcal{F}_{12} + \mathcal{F}_{34} + \mathcal{F}_{56} + \mathcal{F}_{78} &= 0, \\
\mathcal{F}_{13} + \mathcal{F}_{42} + \mathcal{F}_{57} + \mathcal{F}_{86} &= 0, \\
\mathcal{F}_{14} + \mathcal{F}_{23} + \mathcal{F}_{76} + \mathcal{F}_{85} &= 0, \\
\mathcal{F}_{15} + \mathcal{F}_{62} + \mathcal{F}_{73} + \mathcal{F}_{48} &= 0, \\
\mathcal{F}_{16} + \mathcal{F}_{25} + \mathcal{F}_{38} + \mathcal{F}_{47} &= 0, \\
\mathcal{F}_{17} + \mathcal{F}_{82} + \mathcal{F}_{35} + \mathcal{F}_{64} &= 0, \\
\mathcal{F}_{18} + \mathcal{F}_{27} + \mathcal{F}_{63} + \mathcal{F}_{54} &= 0.
\end{aligned} \tag{IX.4}$$

With the help of

$$z^a = x^{2a-1} + i x^{2a} \qquad \text{and} \qquad \bar{z}^{\bar{a}} = x^{2a-1} - i x^{2a} \tag{IX.5}$$

implying the commutation relations

$$[z^a, \bar{z}^{\bar{a}}]_\star = \theta^{a\bar{a}} \qquad \text{with} \qquad \theta^{a\bar{a}} = 2\theta^{2a-1,2a}, \tag{IX.6}$$

and the definitions

$$\mathcal{A}_{z^a} = \tfrac{1}{2}(\mathcal{A}_{2a-1} - i\mathcal{A}_{2a}) \qquad \text{and} \qquad \mathcal{A}_{\bar{z}^{\bar{a}}} = \tfrac{1}{2}(\mathcal{A}_{2a-1} + i\mathcal{A}_{2a}) \tag{IX.7}$$

for $a = 1, \ldots, 4$ we rewrite (IX.4) as

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} + \mathcal{F}_{z^3\bar{z}^{\bar{3}}} + \mathcal{F}_{z^4\bar{z}^{\bar{4}}} = 0, \tag{IX.8a}$$

$$\mathcal{F}_{z^1z^2} + \mathcal{F}_{\bar{z}^{\bar{3}}\bar{z}^{\bar{4}}} = 0, \tag{IX.8b}$$

$$\mathcal{F}_{z^2z^4} - \mathcal{F}_{\bar{z}^{\bar{1}}\bar{z}^{\bar{3}}} = 0, \tag{IX.8c}$$

$$\mathcal{F}_{z^1z^4} + \mathcal{F}_{\bar{z}^{\bar{2}}\bar{z}^{\bar{3}}} = 0. \tag{IX.8d}$$

[1] Note that the octonions $\mathbb{O}$ are the largest of the four normed division algebras. They form a nonassociative algebra and their defining relation is

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k.$$

We use $f_{127} = f_{347} = f_{567} = f_{163} = f_{246} = f_{253} = f_{154} = 1$.

Remember that $\mathcal{A}_{\bar{z}^{\bar{a}}} = -\mathcal{A}^\dagger_{z^a}$.

**Reduction to four dimensions.** Following Baulieu et al. [10], we assume that the gauge potential components $\mathcal{A}_{z^a}$ for $a = 1, \ldots, 4$ do not depend on the coordinates $z^3, z^4, \bar{z}^{\bar{3}}, \bar{z}^{\bar{4}}$ and define ${}^t\Psi := (\Psi_1, \Psi_2)$ with $\Psi_1 := \mathcal{A}_{\bar{z}^{\bar{3}}}$ and $\Psi_2 := \mathcal{A}_{z^4}$. Then the equations (IX.8) dimensionally reduce to

$$
\begin{aligned}
\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} &= -([\Psi_1, \Psi_1^\dagger]_\star - [\Psi_2, \Psi_2^\dagger]_\star), && \text{(IX.9a)}\\
\mathcal{F}_{z^1 z^2} &= [\Psi_1, \Psi_2^\dagger]_\star, && \text{(IX.9b)}\\
\nabla_{\bar{z}^{\bar{1}}}\Psi_1 - \nabla_{z^2}\Psi_2 &= 0, && \text{(IX.9c)}\\
\nabla_{\bar{z}^{\bar{2}}}\Psi_1 + \nabla_{z^1}\Psi_2 &= 0, && \text{(IX.9d)}
\end{aligned}
$$

where the covariant derivative acts as $\nabla_{z^a}\Psi = \partial_{z^a}\Psi + [\mathcal{A}_{z^a}, \Psi]_\star$. In the commutative limit these equations were introduced as a non-Abelian generalization of the Seiberg-Witten equations [10]. The logic is as follows. After reduction to four dimensions the fields $\Psi_{1,2}$ live in the adjoint representation of the gauge group and behave as scalars under $SO(4)$ rotations in $\mathbb{R}^4$. Now one twists the theory by imposing on the column vector $\Psi$ the $SO(4)$ transformation properties of a Weyl (chiral) spinor. To better interpret this twisting we allude to the connection of the non-Abelian Seiberg-Witten equations with the twisted version of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions [10]. The rotation group is locally given by $SU(2) \times SU(2)$. Depending on the embedding of $SU(2) \times SU(2)$ into the $\mathcal{R}$ symmetry group $SU(4)$ one can derive different twisted theories. More precisely, there are three ways of twisting this theory. For instance, the defining representation of $SU(4)$ decomposes with respect to $SU(2) \times SU(2)$ as $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1})$. This eventually leads to non-Abelian Seiberg-Witten equations (IX.9) with a hypermultiplet living in the adjoint representation of the gauge group [10]. Note that the special case $\Psi_2 \equiv 0$ was discussed in [189, 24].

Along with the unperturbed equations (IX.9) we shall also consider the perturbed equations. For that we introduce an $\mathfrak{u}(2)$-valued two-form $\chi$ and add its self-dual part $\chi^+$ to $\mathcal{F}_A^+$ ,

$$
\begin{aligned}
\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} + \chi_{z^1\bar{z}^{\bar{1}}} + \chi_{z^2\bar{z}^{\bar{2}}} &= -([\Psi_1, \Psi_1^\dagger]_\star - [\Psi_2, \Psi_2^\dagger]_\star), && \text{(IX.10a)}\\
\mathcal{F}_{z^1 z^2} + \chi_{z^1 z^2} &= [\Psi_1, \Psi_2^\dagger]_\star, && \text{(IX.10b)}\\
\nabla_{\bar{z}^{\bar{1}}}\Psi_1 - \nabla_{z^2}\Psi_2 &= 0, && \text{(IX.10c)}\\
\nabla_{\bar{z}^{\bar{2}}}\Psi_1 + \nabla_{z^1}\Psi_2 &= 0. && \text{(IX.10d)}
\end{aligned}
$$

### § IX.1.2 Abelian Seiberg-Witten Equations

In order to get Abelian Seiberg-Witten equations from the non-Abelian ones, we shall consider solutions of a particular type given by a suitable ansatz. This will reduce the gauge group $U(2)$ to $U(1) \times U(1)$ and then, further down to $U(1)$.

**Noncommutative $U_+(1)\times U_-(1)$ Seiberg-Witten monopole equations.** Let us consider the $U(1)\times U(1)$ subgroup of $U(2)$ with the generators

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{IX.11}$$

and assume that the components of the gauge potential $\mathcal{A}_{z^a}$ for $a=1,2$ take the form

$$\mathcal{A}_{z^a} := \begin{pmatrix} \mathcal{A}_{+z^a} & 0 \\ 0 & \mathcal{A}_{-z^a} \end{pmatrix} \qquad \text{with} \qquad \mathcal{A}_{\pm z^a} = \tfrac{1}{2}(\mathcal{B}_{z^a} \pm \mathcal{A}_{z^a}) \tag{IX.12}$$

and $\mathcal{A}_{z^a}, \mathcal{B}_{z^a} \in \mathcal{F}(\mathbb{R}^4)$. If not stated differently, $a, b, \ldots$ run always from 1 to 2 in the sequel. Furthermore, we restrict $\Psi_1$ and $\Psi_2$ to the form

$$\Psi_{1,2} := \begin{pmatrix} 0 & \frac{1}{\sqrt{8}}\phi_{1,2} \\ 0 & 0 \end{pmatrix}, \tag{IX.13}$$

where $\phi_{1,2} \in \mathcal{F}(\mathbb{R}^4)$.

Substituting (IX.12) and (IX.13) into the equations (IX.9), after a straightforward calculation we obtain the equations

$$\mathcal{F}_{+z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{+z^2\bar{z}^{\bar{2}}} = -\tfrac{1}{8}(\phi_1 \star \phi_1^\dagger - \phi_2 \star \phi_2^\dagger) \qquad \text{and} \qquad \mathcal{F}_{+z^1z^2} = \tfrac{1}{8}\phi_1 \star \phi_2^\dagger, \tag{IX.14a}$$

$$\mathcal{F}_{-z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{-z^2\bar{z}^{\bar{2}}} = \tfrac{1}{8}(\phi_1^\dagger \star \phi_1 - \phi_2^\dagger \star \phi_2) \qquad \text{and} \qquad \mathcal{F}_{-z^1z^2} = -\tfrac{1}{8}\phi_2^\dagger \star \phi_1 \tag{IX.14b}$$

as well as

$$\nabla_{\bar{z}^{\bar{1}}}\phi_1 - \nabla_{z^2}\phi_2 = 0 \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{2}}}\phi_1 + \nabla_{z^1}\phi_2 = 0, \tag{IX.15}$$

where we have used the definition

$$\nabla_{z^a}\phi := \partial_{z^a}\phi + \mathcal{A}_{+z^a} \star \phi - \phi \star \mathcal{A}_{-z^a}. \tag{IX.16}$$

Here $\mathcal{F}_{\pm z^a\bar{z}^{\bar{a}}}$ are the components of the curvature associated with $\mathcal{A}_{\pm z^a}$, i.e.

$$\mathcal{F}_{\pm z^a\bar{z}^{\bar{a}}} = \partial_{z^a}\mathcal{A}_{\pm\bar{z}^{\bar{a}}} - \partial_{\bar{z}^{\bar{a}}}\mathcal{A}_{\pm z^a} + [\mathcal{A}_{\pm z^a}, \mathcal{A}_{\pm\bar{z}^{\bar{a}}}]_\star. \tag{IX.17}$$

Analogously, by assuming that $\chi = \mathrm{diag}(\chi_+, \chi_-)$ with $\chi_\pm \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$ in (IX.10), we obtain the perturbed equations

$$\begin{aligned}
\mathcal{F}_{+z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{+z^2\bar{z}^{\bar{2}}} + \chi_{+z^1\bar{z}^{\bar{1}}} + \chi_{+z^2\bar{z}^{\bar{2}}} &= -\tfrac{1}{8}(\phi_1 \star \phi_1^\dagger - \phi_2 \star \phi_2^\dagger), && \text{(IX.18a)}\\
\mathcal{F}_{+z^1z^2} + \chi_{+z^1z^2} &= \tfrac{1}{8}\phi_1 \star \phi_2^\dagger, && \text{(IX.18b)}\\
\mathcal{F}_{-z^1\bar{z}^{\bar{2}}} + \mathcal{F}_{-z^2\bar{z}^{\bar{2}}} + \chi_{-z^1\bar{z}^{\bar{1}}} + \chi_{-z^2\bar{z}^{\bar{2}}} &= \tfrac{1}{8}(\phi_1^\dagger \star \phi_1 - \phi_2^\dagger \star \phi_2), && \text{(IX.18c)}\\
\mathcal{F}_{-z^1z^2} + \chi_{-z^1z^2} &= -\tfrac{1}{8}\phi_2^\dagger \star \phi_1. && \text{(IX.18d)}
\end{aligned}$$

We consider the equations (IX.14), (IX.15) and (IX.18), (IX.15) as a noncommutative extension of the unperturbed and perturbed Abelian Seiberg-Witten equations, respectively. Since there are two gauge potentials in the equations, we call them the noncommutative unperturbed and perturbed $U_+(1) \times U_-(1)$ Seiberg-Witten equations.

It remains to find out what kind of gauge transformations leave the $U_+(1) \times U_-(1)$ Seiberg-Witten equations invariant. It is obvious from the explicit form of these equations that $\phi_1$ and $\phi_2$ are in the bi-fundamental representation of $U_+(1) \times U_-(1)$. Hence, the equations (IX.14), (IX.15) and (IX.18) are invariant under gauge transformations of the form

$$\mathcal{A}_\pm \mapsto g_\pm^\dagger \star \mathcal{A}_\pm \star g_\pm + g_\pm^\dagger \star \mathrm{d}\, g_\pm, \qquad \chi_\pm \mapsto g_\pm^\dagger \star \chi_\pm \star g_\pm \qquad \text{and} \qquad \Phi \mapsto g_+^\dagger \star \Phi \star g_-, \tag{IX.19}$$

where $g_\pm \in C^\infty(\mathbb{R}^4, U_\pm(1))$ and ${}^t\Phi = (\phi_1, \phi_2)$.

In the commutative limit the covariant derivative (IX.16) turns into

$$\nabla_{z^a}\phi = \partial_{z^a}\phi + (\mathcal{A}_{+z^a} - \mathcal{A}_{-z^a})\phi = \partial_{z^a}\phi + \mathcal{A}_{z^a}\phi, \tag{IX.20}$$

i.e. the gauge potential $\mathcal{B}$ disappears from the equations (IX.15). In other words, one copy of $U(1)$ decouples from $U(2)$ and the matter field $\Phi$ interact only with the $SU(2)$ part. Hence, in the commutative case $\Phi$ is charged with respect to the diagonal $U(1)$ subgroup of $U_+(1)\times U_-(1)$ corresponding to the gauge potential $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_-$. Furthermore, the commutator in the expression for the (Abelian) curvature vanishes and hence as a corollary we have

$$\mathcal{F}_{\pm z^a \bar{z}^{\bar{a}}} = \tfrac{1}{2}(\mathcal{F}_{\mathcal{B}\, z^a \bar{z}^{\bar{a}}} \pm \mathcal{F}_{\mathcal{A}\, z^a \bar{z}^{\bar{a}}}).$$

In the commutative limit one may choose the perturbations $\chi_\pm$ so that

$$\chi_{\pm z^a \bar{z}^{\bar{a}}} = -\tfrac{1}{2}\mathcal{F}_{\mathcal{B}\, z^a \bar{z}^{\bar{a}}} \pm \tfrac{1}{2}\chi_{z^a \bar{z}^{\bar{a}}} \qquad \text{and} \qquad \chi_{\pm z^1 z^2} = -\tfrac{1}{2}\mathcal{F}_{\mathcal{B}\, z^1 z^2} \pm \tfrac{1}{2}\chi_{z^1 z^2}, \tag{IX.21}$$

where $\chi \in \Omega^2(\mathbb{R}^4, \mathfrak{u}(1))$ is some other perturbation. Then from equations (IX.18) we obtain

$$\mathcal{F}_{\mathcal{A}\, z^1 \bar{z}^{\bar{1}}} + \mathcal{F}_{\mathcal{A}\, z^2 \bar{z}^{\bar{2}}} + \chi_{z^1 \bar{z}^{\bar{1}}} + \chi_{z^2 \bar{z}^{\bar{2}}} = -\tfrac{1}{4}(\phi_1\bar{\phi}_1 - \phi_2\bar{\phi}_2) \quad \text{and} \quad \mathcal{F}_{\mathcal{A}\, z^1 z^2} + \chi_{z^1 z^2} = \tfrac{1}{4}\phi_1\bar{\phi}_2. \tag{IX.22}$$

Note that in the case of complex valued-functions '$^\dagger$' goes to '$^-$' in the commutative limit. Thus, we recover the perturbed Seiberg-Witten equations (V.19) and (V.43), respectively. Of course, the choice $\chi \equiv 0$ corresponds to the unperturbed equations.

**Remark.** Consider the unperturbed equations (IX.14) and (IX.15). In the commutative limit we arrive at the standard unperturbed Seiberg-Witten equations for configurations $(\mathcal{A}, \Phi)$ plus the Abelian ASDYM equations for $\mathcal{B}$, i.e.

$$\mathcal{F}_{\mathcal{B}\, z^1 \bar{z}^{\bar{1}}} + \mathcal{F}_{\mathcal{B}\, z^2 \bar{z}^{\bar{2}}} = 0 \qquad \text{and} \qquad \mathcal{F}_{\mathcal{B}\, z^1 z^2} = 0. \tag{IX.23}$$

Taking the trivial solution $\mathcal{B} = 0$ (recall that there are no Abelian instantons on $\mathbb{R}^4$) we remain with the standard unperturbed Seiberg-Witten equations. More generally, any pure gauge configuration for $\mathcal{B}$ will do the same. The noncommutative version of the latter statement is, however, nontrivial. If we choose $\mathcal{B}_{z^a}$ in (IX.12) of the form

$$\mathcal{B}_{z^a} = \tfrac{1}{2}(b^\dagger \star \partial_{z^a} b - b \star \partial_{z^a} b^\dagger) \tag{IX.24}$$

with $b \in C^\infty(\mathbb{R}^4, U(1))$ then it will correspond in the commutative limit to a pure gauge configuration but, of course, it is not pure gauge in the noncommutative case. Only in the commutative limit the curvature $\mathcal{F}_\mathcal{B}$ disappears and we arrive at the unperturbed Seiberg-Witten equations (V.4).

**Noncommutative $U_\pm(1)$ Seiberg-Witten monopole equations.** In the equations (IX.14) - (IX.19) the field $\Phi$ is regarded as an element of a complex rank two $\mathbb{R}^4_\theta$-bimodule (spinor module of positive chirality) transforming in the bi-fundamental representation of the gauge group $U_+(1) \times U_-(1)$. However, in the noncommutative setup the matter field $\Phi$ can also be thought of either as an element of a right $\mathbb{R}^4_\theta$-module (the $U_+(1)$ case) or as an element of a left $\mathbb{R}^4_\theta$-module (the $U_-(1)$ case). These two cases can easily be read off the equations (IX.18a), (IX.18b) and (IX.18c), (IX.18d), respectively. Namely, consider the equations

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} + \chi_{z^1\bar{z}^{\bar{1}}} + \chi_{z^2\bar{z}^{\bar{2}}} = -\tfrac{1}{4}(\phi_1 \star \phi_1^\dagger - \phi_2 \star \phi_2^\dagger) \quad \text{and} \quad \mathcal{F}_{z^1z^2} + \chi_{z^1z^2} = \tfrac{1}{4}\phi_1 \star \phi_2^\dagger, \tag{IX.25a}$$

as well as

$$\nabla_{\bar{z}^{\bar{1}}}\phi_1 - \nabla_{z^2}\phi_2 = 0 \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{2}}}\phi_1 + \nabla_{z^1}\phi_2 = 0, \tag{IX.25b}$$

where $\chi \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$ and the (right) covariant derivative reads

$$\nabla_{z^a}\phi = \partial_{z^a}\phi + \mathcal{A}_{z^a} \star \phi. \tag{IX.25c}$$

Note that the curvature $\mathcal{F}_\mathcal{A}$ is now computed from $\mathcal{A}$, i.e. $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_\star$. Formally, the above equations can be obtained from (IX.15) - (IX.18) by choosing $\mathcal{B}_{z^a} = \mathcal{A}_{z^a}$ (i.e. $\mathcal{A}_{-z^a} = 0$), taking $\chi_\pm$ such that $\chi_{+z^a\bar{z}^{\bar{a}}} = \chi_{z^a\bar{z}^{\bar{a}}}$, $\chi_{+z^1z^2} = \chi_{z^1z^2}$ and

$$\chi_{-z^1\bar{z}^{\bar{1}}} + \chi_{-z^2\bar{z}^{\bar{2}}} = \tfrac{1}{8}(\phi_1^\dagger \star \phi_1 - \phi_2^\dagger \star \phi_2) \qquad \text{and} \qquad \chi_{-z^1z^2} = -\tfrac{1}{8}\phi_2^\dagger \star \phi_1,$$

where $\chi \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$, and rescaling $\Phi \mapsto \sqrt{2}\Phi$.

Similarly, we may introduce the equations

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} + \chi_{z^1\bar{z}^{\bar{1}}} + \chi_{z^2\bar{z}^{\bar{2}}} = \tfrac{1}{4}(\phi_1^\dagger \star \phi_1 - \phi_2^\dagger \star \phi_2) \quad \text{and} \quad \mathcal{F}_{z^1z^2} + \chi_{z^1z^2} = -\tfrac{1}{4}\phi_2^\dagger \star \phi_1 \tag{IX.26a}$$

and

$$\nabla_{\bar{z}^{\bar{1}}}\phi_1 - \nabla_{z^2}\phi_2 = 0 \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{2}}}\phi_1 + \nabla_{z^1}\phi_2 = 0, \tag{IX.26b}$$

where $\chi \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$ and

$$\nabla_{z^a}\phi = \partial_{z^a}\phi - \phi \star \mathcal{B}_{z^a} \tag{IX.26c}$$

is the (left) derivative. Now the curvature $\mathcal{F}_\mathcal{B}$ is associated with the gauge potential $\mathcal{B}$, i.e. $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + [\mathcal{B}_\mu, \mathcal{B}_\nu]_\star$. These equations can also be obtained from (IX.15) - (IX.18) by choosing $\mathcal{B}_{z^a} = -\mathcal{A}_{z^a}$ (i.e. $\mathcal{A}_{+z^a} = 0$) and putting

$$\chi_{+z^1\bar{z}^{\bar{1}}} + \chi_{+z^2\bar{z}^{\bar{2}}} = -\tfrac{1}{8}(\phi_1 \star \phi_1^\dagger - \phi_2 \star \phi_2^\dagger) \qquad \text{and} \qquad \chi_{+z^1z^2} = \tfrac{1}{8}\phi_1 \star \phi_2^\dagger$$

as well as $\chi_{-z^a\bar{z}^{\bar{a}}} = \chi_{z^a\bar{z}^{\bar{a}}}$ and $\chi_{-z^1z^2} = \chi_{z^1z^2}$, where $\chi \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$. Again we should rescale $\Phi \mapsto \sqrt{2}\Phi$.

We shall call (IX.25) and (IX.26) the perturbed noncommutative $U_+(1)$ and $U_-(1)$ Seiberg-Witten equations. Obviously, the unperturbed equations appear for $\chi \equiv 0$. Note that the systems (IX.25) and (IX.26) are totally equivalent and the only difference between them is an artefact of noncommutativity. The commutative limits of both cases are, of course, identical and produce (V.19) and (V.43), respectively. Moreover, in the commutative case the gauge transformations (IX.19) reduce to the standard ones, i.e. one may choose the identity either for $g_-$ or for $g_+$.

**Summary.** Summarizing we may write down the following diagram:

$$\begin{tikzcd}
 & \begin{array}{c}U(2)\text{ self-duality}\\ \text{type YM on }\mathbb{R}^8_\theta\end{array} \arrow[d] & & \\
 & \begin{array}{c}U(2)\text{ non-Abelian}\\ \text{SW equations on }\mathbb{R}^4_\theta\end{array} \arrow[dl] \arrow[dr] & & \\
U(2)\text{ ASDYM on }\mathbb{R}^4_\theta \arrow[d] & & \begin{array}{c}U_+(1)\times U_-(1)\text{ SW}\\ \text{equations on }\mathbb{R}^4_\theta\end{array} \arrow[d] \arrow[r, dashed] & \begin{array}{c}U_\pm(1)\text{ SW}\\ \text{equations on }\mathbb{R}^4_\theta\end{array} \arrow[d] \\
\begin{array}{c}\text{various integrable}\\ \text{models on }\mathbb{R}^2_\theta\end{array} & & \begin{array}{c}U_+(1)\times U_-(1)\text{ vortex}\\ \text{equations on }\mathbb{R}^2_\theta\end{array} \arrow[r, dashed] & \begin{array}{c}U_\pm(1)\text{ vortex}\\ \text{equations on }\mathbb{R}^2_\theta\end{array}
\end{tikzcd}$$

This picture shows the connection between the theories discussed above. Note that the noncommutative vortex equations in two dimensions can easily be obtained via a dimensional reduction from the noncommutative Seiberg-Witten equations with properly chosen perturbations.

## § IX.1.3 Operator Form of the Abelian Seiberg-Witten Equations

From now on we shall understand our equations not in star product formalism but in operator language by means of the Weyl transform (VI.54).

$U_+(1)\times U_-(1)$ **Seiberg-Witten action functional.** Having introduced the $U_+(1)\times U_-(1)$ noncommutative extension of the Seiberg-Witten equations, we shall define an appropriate action functional. For this purpose we switch back to real coordinates.

Let ${}^t\Phi=(\phi_1,\phi_2)$ and ${}^t\Phi^*:=(\phi_2^\dagger,-\phi_1^\dagger)$. Then the noncommutative deformation of the action functional (V.20) will have the form

$$\begin{aligned} SW_\chi(\mathcal{A}_+,\mathcal{A}_-,\Phi;\theta) =& \tfrac{1}{2}|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\Big\{|\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2+|(\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 \\ &+8|\mathcal{F}^+_{\mathcal{A}_+}+\chi^+_+-\sigma^+(\Phi\otimes\Phi^\dagger)_0|^2+8|\mathcal{F}^+_{\mathcal{A}_-}+\chi^+_--\sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2\Big\}, \end{aligned} \tag{IX.27}$$

where

$$|\psi|^2:=\psi_1\psi_1^\dagger+\psi_2\psi_2^\dagger \qquad \text{and} \qquad |\psi^\dagger|^2:=\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2 \tag{IX.28}$$

for any ${}^t\psi=(\psi_1,\psi_2)$. Here, $\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}$ denotes the Dirac operator depending on the two gauge potentials $\mathcal{A}_+$ and $\mathcal{A}_-$, i.e.

$$\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi=\begin{pmatrix}-\nabla_4+i\nabla_3 & \nabla_2+i\nabla_1\\ -\nabla_2+i\nabla_1 & -\nabla_4-i\nabla_3\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{IX.29}$$

where the covariant derivatives $\nabla_\mu$ are given by

$$\nabla_\mu\phi=\partial_\mu\phi+\mathcal{A}_{+\mu}\phi-\phi\mathcal{A}_{-\mu}. \tag{IX.30}$$

Note that the prefactors in (IX.27) are adjusted in such a way that we recover (V.20) in the commutative limit. It is not difficult to see that the (perturbed) $U_+(1)\times U_-(1)$ Seiberg-Witten equations following from (IX.27) are given by the equations

$$\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi=0,\quad \mathcal{F}^+_{\mathcal{A}_+}+\chi^+_+=\sigma^+(\Phi\otimes\Phi^\dagger)_0 \quad\text{and}\quad \mathcal{F}^+_{\mathcal{A}_-}+\chi^+_-=\sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0 \tag{IX.31}$$

whose solutions minimize the action functional (IX.27).

Assuming that $\phi_{1,2}$ and $\mathcal{F}_{\pm\mu\nu}$ are of proper trace-class, e.g.

$$|\,\mathrm{Tr}_{\mathcal{H}}\,\phi_{1,2}|<\infty \qquad\text{and}\qquad |\,\mathrm{Tr}_{\mathcal{H}}\,\mathcal{F}_{\pm\mu\nu}|<\infty,$$

we shall show momentarily that

$$SW_\chi(\mathcal{A}_+,\mathcal{A}_-,\Phi;\theta)=E_\chi(\mathcal{A}_+,\mathcal{A}_-,\Phi;\theta)+16\pi^2\,K_\chi-|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\,\mathcal{T}, \tag{IX.32}$$

where the functionals $E_\chi$ and $K_\chi$ are given by

$$\begin{aligned} E_\chi(\mathcal{A}_+,\mathcal{A}_-,\Phi;\theta) =& |\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\Big\{2|\mathcal{F}_{\mathcal{A}_+}|^2+2|\mathcal{F}_{\mathcal{A}_-}|^2+\tfrac{1}{2}|\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2+\tfrac{1}{2}|(\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 \\ &+4|\chi^+_+-\sigma^+(\Phi\otimes\Phi^\dagger)_0|^2+4|\chi^+_--\sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2\Big\} \end{aligned} \tag{IX.33}$$

and

$$K_\chi=-\frac{1}{8\pi^2}|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\sum_{i=\pm}\{\mathcal{F}^+_{i\mu\nu},\chi_i^{+\mu\nu}\}. \tag{IX.34}$$

The topological term $\mathcal{T}$ reads

$$\mathcal{T} = \sum_{i=\pm} (\mathcal{F}_{i\mu\nu} *\mathcal{F}_i^{\mu\nu} + D_{i\mu}\mathcal{J}_i^\mu)\,, \tag{IX.35a}$$

Here

$$D_{\pm\mu}\cdot := \partial_\mu\,\cdot\, + [\mathcal{A}_{\pm\mu},\,\cdot\,], \tag{IX.35b}$$

'$*$' denotes the Hodge operator. The currents $\mathcal{J}_{\pm\mu}$ depend in a particular fashion on the fields $\phi_1$ and $\phi_2$, their derivatives and on the gauge potentials $\mathcal{A}_{\pm\mu}$. Let us work explicitly the functional (IX.32). First of all, we need the trivial identities

$$(\nabla_\mu\phi)\psi^\dagger = -\phi(\nabla_\mu\psi)^\dagger + D_{+\mu}(\phi\psi^\dagger) \qquad \text{and} \qquad (\nabla_\mu\phi)^\dagger\psi = -\phi^\dagger(\nabla_\mu\psi) + D_{-\mu}(\phi^\dagger\psi), \tag{IX.36}$$

where the covariant derivatives $D_{\pm\mu}$ are given by (IX.35b). Furthermore, we have

$$[\nabla_\mu,\nabla_\nu]\phi = \mathcal{F}_{+\mu\nu}\,\phi - \phi\,\mathcal{F}_{-\mu\nu}. \tag{IX.37}$$

Let us consider

$$\begin{aligned}
4|\mathcal{F}^+_{\mathcal{A}_+} + \chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2 =& -2(\mathcal{F}^+_{+\mu\nu} + \chi^+_{+\mu\nu} - \sigma^+_{\mu\nu}(\Phi\otimes\Phi^\dagger)_0)^2\\
=& -2(\mathcal{F}^+_{+\mu\nu})^2 - 2\{\mathcal{F}^+_{+\mu\nu},\chi_+^{+\mu\nu}\} + 2\{\mathcal{F}^+_{+\mu\nu},\sigma^{+\mu\nu}(\Phi\otimes\Phi^\dagger)_0\}\\
&- 2(\chi^+_{+\mu\nu} - \sigma^+_{\mu\nu}(\Phi\otimes\Phi^\dagger)_0)^2.
\end{aligned}$$

This expression contains

$$\begin{aligned}
(\mathcal{F}^+_{+\mu\nu})^2 &= \tfrac14(\mathcal{F}_{+\mu\nu} + *\mathcal{F}_{+\mu\nu})^2 = \tfrac12(\mathcal{F}_{+\mu\nu})^2 + \tfrac14\{\mathcal{F}_{+\mu\nu},*\mathcal{F}_+^{\mu\nu}\},\\
\{\mathcal{F}^+_{+\mu\nu},\sigma^+_{\mu\nu}(\Phi\otimes\Phi^\dagger)_0\} &= \tfrac{i}{2}\{\mathcal{F}^+_{+12},\phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger\} - \tfrac12\{\mathcal{F}^+_{+13},\phi_2\phi_1^\dagger - \phi_1\phi_2^\dagger\}\\
&\quad + \tfrac{i}{2}\{\mathcal{F}^+_{+14},\phi_2\phi_1^\dagger + \phi_1\phi_2^\dagger\},\\
-(\chi^+_{+\mu\nu} - \sigma^+_{\mu\nu}(\Phi\otimes\Phi^\dagger)_0)^2 &= 2|\chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
4|\mathcal{F}^+_{\mathcal{A}_-} + \chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2 =& -2(\mathcal{F}^+_{-\mu\nu} + \chi^+_{-\mu\nu} - \sigma^+_{\mu\nu}(\Phi^*\otimes(\Phi^*)^\dagger)_0)^2\\
=& -2(\mathcal{F}^+_{-\mu\nu})^2 - 2\{\mathcal{F}^+_{-\mu\nu},\chi_-^{+\mu\nu}\} + 2\{\mathcal{F}^+_{-\mu\nu},\sigma^{+\mu\nu}(\Phi^*\otimes(\Phi^*)^\dagger)_0\}\\
&- 2(\chi^+_{-\mu\nu} - \sigma^+_{\mu\nu}(\Phi^*\otimes(\Phi^*)^\dagger)_0)^2
\end{aligned}$$

with

$$\begin{aligned}
(\mathcal{F}^+_{-\mu\nu})^2 &= \tfrac12(\mathcal{F}_{-\mu\nu})^2 + \tfrac14\{\mathcal{F}_{-\mu\nu},*\mathcal{F}_-^{\mu\nu}\},\\
\{\mathcal{F}^+_{-\mu\nu},\sigma^+_{\mu\nu}(\Phi^*\otimes(\Phi^*)^\dagger)_0\} &= -\tfrac{i}{2}\{\mathcal{F}^+_{-12},\phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\} + \tfrac12\{\mathcal{F}^+_{-13},\phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1\}\\
&\quad - \tfrac{i}{2}\{\mathcal{F}^+_{-14},\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1\},\\
-(\chi^+_{-\mu\nu} - \sigma^+_{\mu\nu}(\Phi^*\otimes(\Phi^*)^\dagger)_0)^2 &= 2|\chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2.
\end{aligned}$$

A lengthy but straightforward calculation exploiting (IX.36) and (IX.37) yields for $\frac{1}{2}|\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2$ the expression

$$\begin{aligned}\tfrac{1}{2}|\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 &= \tfrac{1}{2}|i\nabla_3\phi_1 - \nabla_4\phi_1 + i\nabla_1\phi_2 + \nabla_2\phi_2|^2 + \tfrac{1}{2}|i\nabla_1\phi_1 - \nabla_2\phi_1 - i\nabla_3\phi_2 - \nabla_4\phi_2|^2 \\ &= -\tfrac{i}{2}\{\mathcal{F}^+_{+12}, \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger\} + \tfrac{1}{2}\{\mathcal{F}^+_{+13}, \phi_2\phi_1^\dagger - \phi_1\phi_2^\dagger\} - \tfrac{i}{2}\{\mathcal{F}^+_{+14}, \phi_2\phi_1^\dagger + \phi_1\phi_2^\dagger\} \\ &\qquad + \tfrac{1}{2}|\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 - D_{+\mu}\mathcal{J}^\mu_+ + C_+,\end{aligned}$$

where

$$\mathcal{J}_{+\mu} = \tfrac{i}{4}(\epsilon_{\mu\nu12} + \epsilon_{\mu\nu34})\mathcal{J}^{(1)\nu}_+ + \tfrac{1}{4}(\epsilon_{\mu\nu31} + \epsilon_{\mu\nu24})\mathcal{J}^{(2)\nu}_+ + \tfrac{i}{4}(\epsilon_{\mu\nu23} + \epsilon_{\mu\nu14})\mathcal{J}^{(3)\nu}_+ \tag{IX.40a}$$

and

$$\mathcal{J}^{(1)}_{+\nu} = \phi_1(\nabla_\nu\phi_1)^\dagger - (\nabla_\nu\phi_1)\phi_1^\dagger - \phi_2(\nabla_\nu\phi_2)^\dagger + (\nabla_\nu\phi_2)\phi_2^\dagger, \tag{IX.40b}$$
$$\mathcal{J}^{(2)}_{+\nu} = -\phi_1(\nabla_\nu\phi_2)^\dagger + (\nabla_\nu\phi_1)\phi_2^\dagger + \phi_2(\nabla_\nu\phi_1)^\dagger - (\nabla_\nu\phi_2)\phi_1^\dagger, \tag{IX.40c}$$
$$\mathcal{J}^{(3)}_{+\nu} = \phi_2(\nabla_\nu\phi_1)^\dagger - (\nabla_\nu\phi_2)\phi_1^\dagger + \phi_1(\nabla_\nu\phi_2)^\dagger - (\nabla_\nu\phi_1)\phi_2^\dagger. \tag{IX.40d}$$

The term $C_+$ is given by

$$C_+ = -i\phi_2\mathcal{F}^+_{-12}\phi_2^\dagger + i\phi_1\mathcal{F}^+_{-12}\phi_1^\dagger - \phi_2\mathcal{F}^+_{-13}\phi_1^\dagger + i\phi_2\mathcal{F}^+_{-14}\phi_1^\dagger + i\phi_1\mathcal{F}^+_{-14}\phi_2^\dagger + \phi_1\mathcal{F}^+_{-13}\phi_2^\dagger.$$

In a similar manner we also have

$$\begin{aligned}\tfrac{1}{2}|(\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 &= \tfrac{i}{2}\{\mathcal{F}^+_{-12}, \phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\} - \tfrac{1}{2}\{\mathcal{F}^+_{-13}, \phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1\} + \tfrac{i}{2}\{\mathcal{F}^+_{-14}, \phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1\} \\ &\qquad + \tfrac{1}{2}|(\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 - D_{-\mu}\mathcal{J}^\mu_- + C_-,\end{aligned}$$

where

$$\mathcal{J}_{-\mu} = -\tfrac{i}{4}(\epsilon_{\mu\nu12} + \epsilon_{\mu\nu34})\mathcal{J}^{(1)\nu}_- - \tfrac{1}{4}(\epsilon_{\mu\nu31} + \epsilon_{\mu\nu24})\mathcal{J}^{(2)\nu}_- - \tfrac{i}{4}(\epsilon_{\mu\nu23} + \epsilon_{\mu\nu14})\mathcal{J}^{(3)\nu}_- \tag{IX.44a}$$

and

$$\mathcal{J}^{(1)}_{-\nu} = \phi_1^\dagger(\nabla_\mu\phi_1) - (\nabla_\mu\phi_1)^\dagger\phi_1 - \phi_2^\dagger(\nabla_\mu\phi_2) + (\nabla_\mu\phi_2)^\dagger\phi_2, \tag{IX.44b}$$
$$\mathcal{J}^{(2)}_{-\nu} = -\phi_2^\dagger(\nabla_\mu\phi_1) + (\nabla_\mu\phi_2)^\dagger\phi_1 + \phi_1^\dagger(\nabla_\mu\phi_2) - (\nabla_\mu\phi_1)^\dagger\phi_2, \tag{IX.44c}$$
$$\mathcal{J}^{(3)}_{-\nu} = \phi_1^\dagger(\nabla_\mu\phi_2) - (\nabla_\mu\phi_1)^\dagger\phi_2 + \phi_2^\dagger(\nabla_\mu\phi_1) - (\nabla_\mu\phi_2)^\dagger\phi_1. \tag{IX.44d}$$

The term $C_-$ is

$$C_- = i\phi_2^\dagger\mathcal{F}^+_{+12}\phi_2 - i\phi_1^\dagger\mathcal{F}^+_{+12}\phi_1 - \phi_2^\dagger\mathcal{F}^+_{+13}\phi_1 - i\phi_2^\dagger\mathcal{F}^+_{+14}\phi_1 - i\phi_1^\dagger\mathcal{F}_{+14}\phi_2 + \phi_1^\dagger\mathcal{F}_{+13}\phi_2.$$

Therefore, we discover

$$\begin{aligned}&\tfrac{1}{2}|\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 + \tfrac{1}{2}|(\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 + 4|\mathcal{F}^+_{\mathcal{A}_+} + \chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2 \\ &\qquad + 4|\mathcal{F}^+_{\mathcal{A}_-} + \chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2 \\ &= \tfrac{1}{2}|\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 + \tfrac{1}{2}|(\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 + 2|\mathcal{F}_{\mathcal{A}_+}|^2 + 2|\mathcal{F}_{\mathcal{A}_-}|^2 + 4|\chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2 \\ &\quad + 4|\chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2 - \tfrac{1}{2}\{\mathcal{F}_{+\mu\nu}, *\mathcal{F}^{\mu\nu}_+\} - 2\{\mathcal{F}^+_{+\mu\nu}, \chi^{+\mu\nu}_+\} - D_{+\mu}\mathcal{J}^\mu_+ \\ &\quad - \tfrac{1}{2}\{\mathcal{F}_{-\mu\nu}, *\mathcal{F}^{\mu\nu}_-\} - 2\{\mathcal{F}^+_{-\mu\nu}, \chi^{+\mu\nu}_-\} - D_{-\mu}\mathcal{J}^\mu_- \\ &\quad + \tfrac{i}{2}\{\mathcal{F}^+_{+12}, \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger\} - \tfrac{1}{2}\{\mathcal{F}^+_{+13}, \phi_2\phi_1^\dagger - \phi_1\phi_2^\dagger\} + \tfrac{i}{2}\{\mathcal{F}^+_{+14}, \phi_2\phi_1^\dagger + \phi_1\phi_2^\dagger\} + C_+ \\ &\quad - \tfrac{i}{2}\{\mathcal{F}^+_{-12}, \phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\} + \tfrac{1}{2}\{\mathcal{F}^+_{-13}, \phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1\} - \tfrac{i}{2}\{\mathcal{F}^+_{-14}, \phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1\} + C_-.\end{aligned}$$

Now suppose that all operators entering this formula are of proper trace-class, e.g.

$$|\operatorname{Tr}_{\mathcal{H}} \phi_{1,2}| < \infty \qquad \text{and} \qquad |\operatorname{Tr}_{\mathcal{H}} \mathcal{F}_{\pm\mu\nu}| < \infty.$$

Then

$$\begin{aligned}\operatorname{Tr}_{\mathcal{H}}\Big\{ &\tfrac{i}{2}\{\mathcal{F}^+_{+12}, \phi_1\phi_1^\dagger - \phi_2\phi_2^\dagger\} - \tfrac{1}{2}\{\mathcal{F}^+_{+13}, \phi_2\phi_1^\dagger - \phi_1\phi_2^\dagger\} + \tfrac{i}{2}\{\mathcal{F}^+_{+14}, \phi_2\phi_1^\dagger + \phi_1\phi_2^\dagger\} + C_+ \\ &-\tfrac{i}{2}\{\mathcal{F}^+_{-12}, \phi_1^\dagger\phi_1 - \phi_2^\dagger\phi_2\} + \tfrac{1}{2}\{\mathcal{F}^+_{-13}, \phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_1\} - \tfrac{i}{2}\{\mathcal{F}^+_{-14}, \phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1\} + C_- \Big\} = 0,\end{aligned} \tag{IX.46}$$

since we can use the invariance of the trace under cyclic permutations. In fact, one can easily check that for each term in (IX.46) there exists a corresponding term having the opposite sign so that the trace is indeed zero.

Finally we obtain

$$\begin{aligned}\operatorname{Tr}_{\mathcal{H}}\Big\{ &\tfrac{1}{2}|\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 + \tfrac{1}{2}|(\mathcal{D}_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 + 4|\mathcal{F}^+_{\mathcal{A}_+} + \chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2 \\ &\qquad + 4|\mathcal{F}^+_{\mathcal{A}_-} + \chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2\Big\} \\ = \operatorname{Tr}_{\mathcal{H}}\Big\{ &\tfrac{1}{2}|\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi|^2 + \tfrac{1}{2}|(\nabla_{\mathcal{A}_+,\mathcal{A}_-}\Phi)^\dagger|^2 + 2|\mathcal{F}_{\mathcal{A}_+}|^2 + 2|\mathcal{F}_{\mathcal{A}_-}|^2 + 4|\chi^+_+ - \sigma^+(\Phi\otimes\Phi^\dagger)_0|^2 \\ &\qquad + 4|\chi^+_- - \sigma^+(\Phi^*\otimes(\Phi^*)^\dagger)_0|^2\Big\} - \operatorname{Tr}_{\mathcal{H}}\mathcal{T} + \frac{16\pi^2}{|\mathrm{Pf}(2\pi\theta)|}K_\chi,\end{aligned}$$

which is the desired result.

Note that similarly to the commutative case the functionals $E_\chi + 16\pi^2 K_\chi$ and $SW_\chi$ yield the same equations of motion. The integration of (IX.35a) gives the topological charge

$$Q = -\frac{1}{8\pi^2}|\mathrm{Pf}(2\pi\theta)|\operatorname{Tr}_{\mathcal{H}}\mathcal{T} \tag{IX.47}$$

for the considered field configuration $(\mathcal{A}_+, \mathcal{A}_-, \Phi)$ on $\mathbb{R}^4_\theta$.

**Generalized coupled vortex equations.** Let us put one component of $\Phi$ to zero, e.g. consider the case ${}^t\Phi = (\phi_1, \phi_2) =: (\phi, 0)$. Since $\mathbb{R}^4$ is Kähler the field $\phi$ can be regarded as a scalar; cf. section § V.3. Moreover, we choose

$$\chi_{\pm z^1 z^2} = 0 \qquad \text{and} \qquad \chi_{\pm z^1\bar z^{\bar 1}} + \chi_{\pm z^2 \bar z^{\bar 2}} = \mp\tfrac{1}{8}\,v_\pm, \tag{IX.48}$$

where $v_\pm$ are some Hermitian operators acting on $\mathcal{H}$. Then the energy functional (IX.33) turns into

$$\begin{aligned}E_\chi(\mathcal{A}_+, \mathcal{A}_-, \Phi; \theta) = &|\mathrm{Pf}(2\pi\theta)|\operatorname{Tr}_{\mathcal{H}}\Big\{\mathcal{F}_{+\mu\nu}\mathcal{F}_+^{\mu\nu\dagger} + \mathcal{F}_{-\mu\nu}\mathcal{F}_-^{\mu\nu\dagger} + \tfrac{1}{2}\nabla_\mu\phi(\nabla^\mu\phi)^\dagger \\ &+ \tfrac{1}{2}(\nabla_\mu\phi)^\dagger\nabla^\mu\phi + \tfrac{1}{8}(v_+ - \phi\phi^\dagger)^2 + \tfrac{1}{8}(v_- - \phi^\dagger\phi)^2\Big\}\end{aligned} \tag{IX.49a}$$

and $K_\chi$ is given by

$$K_\chi = -\frac{i}{32\pi^2}|\mathrm{Pf}(2\pi\theta)|\operatorname{Tr}_{\mathcal{H}}\left\{\{\mathcal{F}_{+12} + \mathcal{F}_{+34}, v_+\} - \{\mathcal{F}_{-12} + \mathcal{F}_{-34}, v_-\}\right\}. \tag{IX.49b}$$

Also the currents $\mathcal{J}_{\pm\mu}$ have a fairly simple form

$$\mathcal{J}_{+\mu} = \tfrac{i}{4}(\epsilon_{\mu\nu12}+\epsilon_{\mu\nu34})(\phi(\nabla^\nu\phi)^\dagger-(\nabla^\nu\phi)\phi^\dagger), \tag{IX.50a}$$
$$\mathcal{J}_{-\mu} = -\tfrac{i}{4}(\epsilon_{\mu\nu12}+\epsilon_{\mu\nu34})(\phi^\dagger(\nabla^\nu\phi)-(\nabla^\nu\phi)^\dagger\phi), \tag{IX.50b}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the Levi-Civita symbol with $\epsilon_{1234}=1$.

For our choices of $\Phi$ and $\chi_\pm$ the perturbed $U_+(1)\times U_-(1)$ Seiberg-Witten equations in complex coordinates read

$$\mathcal{F}_{+z^1\bar{z}^{\bar{1}}}+\mathcal{F}_{+z^2\bar{z}^{\bar{2}}}=\tfrac{1}{8}(v_+-\phi\phi^\dagger) \quad\text{and}\quad \mathcal{F}_{+z^1z^2}=0, \tag{IX.51a}$$
$$\mathcal{F}_{-z^1\bar{z}^{\bar{1}}}+\mathcal{F}_{-z^2\bar{z}^{\bar{2}}}=-\tfrac{1}{8}(v_--\phi^\dagger\phi) \quad\text{and}\quad \mathcal{F}_{-z^1z^2}=0, \tag{IX.51b}$$
$$\nabla_{\bar{z}^{\bar{1}}}\phi=0 \quad\text{and}\quad \nabla_{\bar{z}^{\bar{2}}}\phi=0. \tag{IX.51c}$$

In the commutative case these equations were considered e.g. in [27].

$U_\pm(1)$ **Seiberg-Witten action functionals.** Having introduced the $U_+(1)\times U_-(1)$ Seiberg-Witten functionals, we are now interested in proper functionals for the $U_\pm(1)$ cases (IX.25) and (IX.26). Let us first discuss the perturbed $U_+(1)$ Seiberg-Witten equations. In this case the Seiberg-Witten action functional takes the following form:

$$SW_\chi(\mathcal{A},\Phi;\theta)=|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\{|\mathcal{D}_{\mathcal{A}}\Phi|^2+2|\mathcal{F}^+_{\mathcal{A}}+\chi^+-\sigma^+(\Phi\otimes\Phi^\dagger)_0|^2\}. \tag{IX.52}$$

Note that now the Dirac operator depends only on $\mathcal{A}$ and the covariant derivatives are given by (IX.25c). Also here the prefactors have been chosen such that the correct commutative limit will be obtained. As before, the functional $SW_\chi$ may be rewritten as

$$SW_\chi(\mathcal{A},\Phi;\theta)=E_\chi(\mathcal{A},\Phi;\theta)+16\pi^2K_\chi-|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\,\mathcal{T}, \tag{IX.53}$$

where $E_\chi$ turns out to be

$$E_\chi(\mathcal{A},\Phi;\theta)=|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\left\{|\mathcal{F}_{\mathcal{A}}|^2+|\nabla_{\mathcal{A}}\Phi|^2+2|\chi^+-\sigma^+(\Phi\otimes\Phi^\dagger)_0|^2\right\}. \tag{IX.54}$$

The Chern-Simons term $K_\chi$ reads

$$K_\chi=-\frac{1}{16\pi^2}|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\,\{\mathcal{F}^+_{\mu\nu},\chi^{+\mu\nu}\} \tag{IX.55}$$

and the topological term $\mathcal{T}$ is given by

$$\mathcal{T}=\tfrac{1}{4}\{\mathcal{F}_{\mu\nu},*\mathcal{F}^{\mu\nu}\}+D_\mu\mathcal{J}^\mu \quad\text{with}\quad D_\mu\mathcal{J}^\mu=\partial_\mu\mathcal{J}^\mu+[\mathcal{A}_\mu,\mathcal{J}^\mu] \tag{IX.56}$$

implying the charge

$$Q=-\frac{1}{8\pi^2}|\mathrm{Pf}(2\pi\theta)|\,\mathrm{Tr}_{\mathcal{H}}\,\mathcal{T}. \tag{IX.57}$$

Again all equations simplify essentially if one chooses ${}^t\Phi = (\phi_1, \phi_2) = (\phi, 0)$, $\chi_{z^1z^2} = 0$ and $\chi_{z^1\bar{z}^{\bar{1}}} + \chi_{z^1\bar{z}^{\bar{1}}} = -\frac{1}{4}v$, where $v$ is some Hermitian operator. Then $E_\chi$ and $K_\chi$ are given by

$$E_\chi(\mathcal{A}, \Phi; \theta) = |\mathrm{Pf}(2\pi\theta)|\, \mathrm{Tr}_{\mathcal{H}} \left\{ \tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu\dagger} + \nabla_\mu\phi(\nabla^\mu\phi)^\dagger + \tfrac{1}{4}(v - \phi\phi^\dagger)^2 \right\} \tag{IX.58a}$$

and

$$K_\chi = -\frac{i}{32\pi^2}|\mathrm{Pf}(2\pi\theta)|\, \mathrm{Tr}_{\mathcal{H}} \left\{ \mathcal{F}_{12} + \mathcal{F}_{34}, v \right\} \tag{IX.58b}$$

The current $\mathcal{J}_\mu$ reduces in this case to

$$\mathcal{J}_\mu = \tfrac{i}{2}(\epsilon_{\mu\nu12} + \epsilon_{\mu\nu34})(\phi(\nabla^\nu\phi)^\dagger - (\nabla^\nu\phi)\phi^\dagger). \tag{IX.59}$$

Finally, the perturbed $U_+(1)$ Seiberg-Witten equations read

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} = \tfrac{1}{4}(v - \phi\phi^\dagger) \qquad \text{and} \qquad \mathcal{F}_{z^1z^2} = 0, \tag{IX.60a}$$

$$\nabla_{\bar{z}^{\bar{1}}}\phi = 0 \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{2}}}\phi = 0. \tag{IX.60b}$$

In the commutative case these $v$-vortex equations in four dimensions were considered e.g. in [27]. Note that in a similar manner one can write down the functionals corresponding to the $U_-(1)$ case (IX.26). Since they look essentially the same as the above introduced $U_+(1)$ functionals we refrain from writing down their explicit form.

## § IX.2 Examples of Solutions

For later convenience we introduce the operators

$$X_{\pm z^a} := \mathcal{A}_{\pm z^a} + \theta_{a\bar{a}}\bar{z}^{\bar{a}} \qquad \text{and} \qquad X_{\pm\bar{z}^{\bar{a}}} := \mathcal{A}_{\pm\bar{z}^{\bar{a}}} + \theta_{\bar{a}a}z^a. \tag{IX.61}$$

Then a short calculation of the Yang-Mills curvature yields

$$\mathcal{F}_{\pm z^1z^2} = [X_{\pm z^1}, X_{\pm z^2}], \qquad \mathcal{F}_{\pm z^1\bar{z}^{\bar{2}}} = [X_{\pm z^1}, X_{\pm\bar{z}^{\bar{2}}}], \qquad \mathcal{F}_{\pm z^2\bar{z}^{\bar{1}}} = [X_{\pm z^2}, X_{\pm\bar{z}^{\bar{1}}}], \tag{IX.62a}$$

$$\mathcal{F}_{\pm z^1\bar{z}^{\bar{1}}} = [X_{\pm z^1}, X_{\pm\bar{z}^{\bar{1}}}] + \theta_{1\bar{1}}, \qquad \mathcal{F}_{\pm z^2\bar{z}^{\bar{2}}} = [X_{\pm z^2}, X_{\pm\bar{z}^{\bar{2}}}] + \theta_{2\bar{2}} \tag{IX.62b}$$

and the covariant derivatives become

$$\nabla_{z^a}\phi = X_{+z^a}\phi - \phi X_{-z^a} \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{a}}}\phi = X_{+\bar{z}^{\bar{a}}}\phi - \phi X_{-\bar{z}^{\bar{a}}}. \tag{IX.63}$$

### § IX.2.1 Solutions to the Perturbed Equations

$U_+(1)\times U_-(1)$ **Seiberg-Witten monopole equations.** Let us consider the equations (IX.51) rewritten in operator form. For the operators (IX.61) we take (cf. e.g. [73, 81])

$$X_{\pm z^a} = \theta_{a\bar{a}}S^N\bar{z}^{\bar{a}}(S^\dagger)^N + \sum_{n=0}^{N-1}\lambda_{a,n}|n\rangle\langle n| \otimes P_0^{(2)}, \tag{IX.64}$$

where the shift operator $S$ is given by (VI.89), $\lambda_{a,n} \in \mathbb{C}$ and $N \in \mathbb{N}$. Then the commutator $[X_{\pm z^a}, X_{\pm \bar{z}^{\bar{a}}}]$ is readily computed to be

$$[X_{\pm z^a}, X_{\pm \bar{z}^{\bar{a}}}] = -\theta_{a\bar{a}}(1 - \mathcal{P}_N), \tag{IX.65}$$

where $\mathcal{P}_N$ is given by

$$\mathcal{P}_N := \sum_{n=0}^{N-1} |n\rangle\langle n| \otimes P_0^{(2)}. \tag{IX.66}$$

It easy to see that the second equations of (IX.51a) and (IX.51b) are trivially satisfied. Choosing $v_- \equiv 0$ and

$$\phi = \sqrt{8(\theta_{1\bar{1}} + \theta_{2\bar{2}})}\, \mathcal{P}_N, \tag{IX.67}$$

we can solve (IX.51c) and the first equation of (IX.51b) consistently, while the first equation of (IX.51a) implies that

$$v_+ = 16(\theta_{1\bar{1}} + \theta_{2\bar{2}})\mathcal{P}_N. \tag{IX.68}$$

Note that the moduli $\lambda_{a,n}$ in (IX.64) can be interpreted as position parameters (see e.g. [73, 85, 78]).

Obviously, the components (IX.62) of the curvature and the field $\phi$ are of proper trace-class. Moreover, it can be easily checked that $\phi$ is covariantly constant, i.e. along with $\nabla_{\bar{z}^{\bar{a}}}\phi = 0$, required by the Seiberg-Witten equations, we also have $\nabla_{z^a}\phi = 0$. This means that the currents $\mathcal{J}_{\pm\mu}$ (IX.50) vanish identically. Thus, it is straightforward to evaluate the topological charge (IX.47) for this configuration. What we find for the topological term (IX.35a) is

$$\mathcal{T} = -8\frac{1}{\theta^{12}\theta^{34}}\mathcal{P}_N. \tag{IX.69}$$

Using $|\mathrm{Pf}(2\pi\theta)| = 4\pi^2|\theta^{12}\theta^{34}|$, we get immediately a charge[2]

$$Q = 4\epsilon_1\epsilon_2 N \qquad \text{with} \qquad \epsilon_1 := \frac{|\theta^{12}|}{\theta^{12}} \qquad \text{and} \qquad \epsilon_2 := \frac{|\theta^{34}|}{\theta^{34}}. \tag{IX.70}$$

Note that $K_\chi$ given by (IX.49b) is also finite. Therefore, the considered field configuration has finite energy

$$E_\chi = 32\pi^2 f(\theta)N \quad \text{with} \quad f(\theta) := |\theta^{12}\theta^{34}| \left[\left(\frac{1}{\theta^{12}}\right)^2 + \frac{1}{\theta^{12}\theta^{34}} + \left(\frac{1}{\theta^{34}}\right)^2\right]. \tag{IX.71}$$

Let us now consider a slight generalization of the ansatz (IX.64). We take first

$$X_{\pm z^a} = \theta_{a\bar{a}} S^{N_\pm} \bar{z}^{\bar{a}} (S^\dagger)^{N_\pm} + \sum_{n=0}^{N_\pm - 1} \lambda^\pm_{a,n} |n\rangle\langle n| \otimes P_0^{(2)}, \tag{IX.72a}$$

and find that $[X_{\pm z^a}, X_{\pm \bar{z}^{\bar{a}}}] = -\theta_{a\bar{a}}(1 - \mathcal{P}_{N_\pm})$. Secondly, choosing

$$\phi = \sqrt{8(\theta_{1\bar{1}} + \theta_{2\bar{2}})}\, S^{N_+}(S^\dagger)^{N_-} \tag{IX.72b}$$

[2] Note that the definition of the charge (IX.47) differs by a factor of 2 in comparison with the standard one.

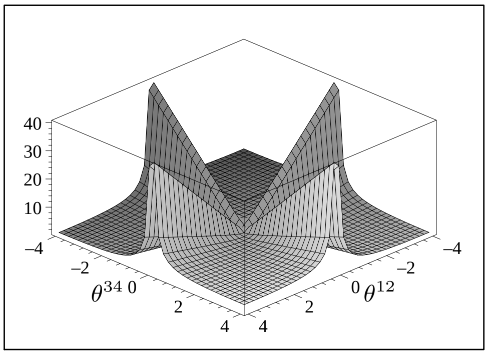


Figure IX.1. $f(\theta)$.

and the perturbations $v_\pm$ such that

$$v_+ = 8(\theta_{1\bar{1}} + \theta_{2\bar{2}}) \qquad \text{and} \qquad v_- = 8(\theta_{1\bar{1}} + \theta_{2\bar{2}})(1 - 2\mathcal{P}_{N_-}), \tag{IX.73}$$

one can easily show that our equations are solved consistently. Again, all our operators are of proper trace-class for $N_+ \neq N_-$. In the case of $N_+ = N_-$ one encounters a slight subtlety since the field $\phi$ is not of trace-class contrary to our assumption. However, potentially dangerous terms, like $\phi F_{-12}\phi^\dagger$ for instance, which occured in (IX.33), are obviously zero when $N_+ = N_-$. Moreover, the field $\phi$ is covariantly constant, as one can readily check. Therefore, the currents $\mathcal{J}_{\pm\mu}$ (IX.50) are identically zero. The topological term (IX.35a) for this configuration thus reads

$$\mathcal{T} = -4\frac{1}{\theta^{12}\theta^{34}}(\mathcal{P}_{N_+} + \mathcal{P}_{N_-}), \tag{IX.74}$$

which produces a topological charge $Q = 2\epsilon_1\epsilon_2(N_+ + N_-)$. The functional $E_\chi$ for this solutions computes to

$$E_\chi = 16\pi^2 f(\theta)(N_+ + N_-), \tag{IX.75}$$

where $f(\theta)$ is given by (IX.71).

$U_+(1)$ **Seiberg-Witten monopole equations.** Consider now the equations (IX.60) and choose the ansatz

$$X_{z^a} = \theta_{a\bar{a}} S^N \bar{z}^{\bar{a}} (S^\dagger)^N + \sum_{n=0}^{N-1} \lambda_{a,n} |n\rangle\langle n| \otimes P_0^{(2)} \qquad \text{and} \qquad \phi = \gamma S^N \tag{IX.76}$$

with $\lambda_{a,n} \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. Then the first equation of (IX.60a) can be solved by $\theta_{1\bar{1}} + \theta_{2\bar{2}} = \frac{1}{4}\gamma^2 = \frac{1}{4}v$, while the other two equations are trivially satisfied. The only nonvanishing components of the curvature are $\mathcal{F}_{z^a\bar{z}^{\bar{a}}} = \theta_{a\bar{a}}\mathcal{P}_N$. Note that the expression for $\phi$ is covariantly constant which implies the vanishing of the current (IX.59). The charge (IX.57) is equal to $Q = \epsilon_1\epsilon_2 N$ and the energy $E_\chi$ is $8\pi^2 f(\theta)N$, where $f(\theta)$ is given by (IX.71).

Consider again the equations (IX.60). Besides the shift type solution for $\phi$ we can also find a projector type solution. Namely, we choose $\phi = \gamma P_0$ with $\gamma \in \mathbb{R}$,

$$v = (4(\theta_{1\bar{1}} + \theta_{2\bar{2}}) + \gamma^2)P_0 \tag{IX.77}$$

and the $X_{z^a}$'s as previously. With this choice the first two equations of (IX.60a) are trivially satisfied, while (IX.60b) yields the condition $P_0 z^a = 0$. Hence, we have to set $\theta^{a\bar{a}} < 0$. Again, the only nonvanishing components of the curvature are $\mathcal{F}_{z^a\bar{z}^{\bar{a}}} = \theta_{a\bar{a}}P_0$. Moreover, we have

$$\nabla_1\phi = -i\nabla_2\phi = -\sqrt{\theta_{1\bar{1}}}\gamma|0,0\rangle\langle 1,0| \qquad \text{and} \qquad \nabla_3\phi = -i\nabla_4\phi = -\sqrt{\theta_{2\bar{2}}}\gamma|0,0\rangle\langle 0,1|, \tag{IX.78}$$

which imply the vanishing of current the $\mathcal{J}_\mu$ (IX.59). The topological charge for this configuration is $Q = \epsilon_1\epsilon_2$.

## § IX.2.2 Solutions to the Unperturbed Equations

In this section we shall consider solutions to the unperturbed Seiberg-Witten equations. We concentrate on the case of the unperturbed $U_+(1)$ equations, i.e. (IX.60) with $v \equiv 0$, as an illustrative example.

$U_+(1)$ **Seiberg-Witten monopole equations with $\Phi \equiv 0$.** In this case we obtain the noncommutative Abelian ASDYM equations. Solutions to these equations, i.e. noncommutative Abelian instantons, have been known for quite some time [147]. However, for the sake of completeness we briefly review their construction. Note that in the case $\Phi \equiv 0$ the current $\mathcal{J}_\mu$ is identically zero.

In terms of the operators (IX.61) the Abelian ASDYM equations read

$$[X_{z^1}, X_{\bar{z}^{\bar{1}}}] + [X_{z^2}, X_{\bar{z}^{\bar{2}}}] + \theta_{1\bar{1}} + \theta_{2\bar{2}} = 0 \qquad \text{and} \qquad [X_{z^1}, X_{z^2}] = 0. \tag{IX.79}$$

Again we consider an ansatz for $X_{z^a}$ of the form (IX.76). This ansatz yields solutions to the equations (IX.79) if the deformation tensor $\theta^{\mu\nu}$ is anti-self-dual what trivially follows from (IX.76). The only nonvanishing components of the curvature are $\mathcal{F}_{z^a\bar{z}^{\bar{a}}} = \theta_{a\bar{a}}\mathcal{P}_N$ implying that the charge (IX.57) is equal to $Q = -N$. The moduli $\lambda_{a,n}$ entering the solution are position parameters showing the location of the noncommutative instantons [63] (see also [78] for a recent review).

It is also possible to consider the case of ASDYM equations on a self-dual background for which $\theta^{1\bar{1}} = \theta^{2\bar{2}} =: \theta$. Let us assume that $\theta > 0$, which leads to the definitions $c_1 = z^1/\sqrt{\theta}$ and $c_2 = z^2/\sqrt{\theta}$. We choose the ansatz [120, 150]

$$X_{z^1} = -\tfrac{1}{\theta}S^\dagger \bar{z}^{\bar{1}} f(N) S = -\tfrac{1}{\sqrt{\theta}}S^\dagger c_1^\dagger f(N) S, \tag{IX.80a}$$
$$X_{z^2} = -\tfrac{1}{\theta}S^\dagger \bar{z}^{\bar{2}} f(N) S = -\tfrac{1}{\sqrt{\theta}}S^\dagger c_2^\dagger f(N) S \tag{IX.80b}$$

and assume that $f(N)|0\rangle = f(0)|0\rangle = 0$, where $N := N_1 + N_2 := c_1^\dagger c_1 + c_2^\dagger c_2$. Then the equation $[X_{z^1}, X_{z^2}] = 0$ is trivially satisfied. A short calculation yields

$$[X_{z^a}, X_{\bar{z}^{\bar{a}}}] = -\tfrac{1}{\theta} S^\dagger [(N_a+1) f^2(N) - N_a f^2(N-1)] S. \tag{IX.81}$$

Using these expressions we obtain from the first equation of (IX.79)

$$S^\dagger [(N+2) f^2(N) - N f^2(N-1) - 2] S = 0, \tag{IX.82}$$

which is equivalent to

$$(n+2) f^2(n) - n f^2(n-1) = 2 \qquad \text{for} \qquad n \geq 1. \tag{IX.83}$$

This recursion relation has the solution [120, 150]

$$f^2(n) = \frac{n(n+3)}{(n+1)(n+2)}. \tag{IX.84}$$

The nonvanishing components of the curvature in this case are

$$\begin{aligned} \mathcal{F}_{z^a \bar{z}^{\bar{b}}} &= \tfrac{1}{\theta} S^\dagger c_a^\dagger (f^2(N+1) - f^2(N)) c_b S, && \text{(IX.85a)} \\ \mathcal{F}_{z^a \bar{z}^{\bar{a}}} &= \tfrac{1}{\theta} S^\dagger [(N_a+1) f^2(N) - N_a f^2(N-1) - 1] S. && \text{(IX.85b)} \end{aligned}$$

The calculation of the topological charge needs the anti-commutators

$$\begin{aligned} \{\mathcal{F}_{z^1 \bar{z}^{\bar{1}}}, \mathcal{F}_{z^2 \bar{z}^{\bar{2}}}\} &= \tfrac{2}{\theta^2} S^\dagger g_1(N_1, N_2) g_2(N_1, N_2) S, && \text{(IX.86a)} \\ \{\mathcal{F}_{z^1 \bar{z}^{\bar{2}}}, \mathcal{F}_{z^2 \bar{z}^{\bar{1}}}\} &= \tfrac{1}{\theta^2} S^\dagger g_3(N_1, N_2) S, && \text{(IX.86b)} \end{aligned}$$

with the abbreviations

$$\begin{aligned} g_{1,2}(N_1, N_2) &:= (N_{1,2}+1) f^2(N) - N_{1,2} f^2(N-1) - 1, && \text{(IX.86c)} \\ g_3(N_1, N_2) &:= (N + 2N_1 N_2)(f^2(N) - f^2(N-1))^2. && \text{(IX.86d)} \end{aligned}$$

Using these expressions, we compute the topological charge (IX.57) to be $-1$. A straightforward extension of the above ansatz allows one to construct multi-instanton configurations, as well [120, 150].

**Fock spaces with indefinite norm.** Let us now discuss the case when $\Phi$ does not vanish identically. For that we relax the condition of positivity of the norm of the Fock spaces $\mathcal{H}_{1,2}$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and assume instead that at least one of them has an indefinite norm (cf. [23]). For instance, we can introduce a collection of the creation and annihilation operators $\{c_a, c_a^\dagger\}_{a=1,2}$, satisfying $[c_1, c_1^\dagger] = 1$, $[c_2, c_2^\dagger] = -1$, and defined by the relations

$$\begin{aligned} c_1 |n_1, n_2\rangle &= \sqrt{n_1}\, |n_1 - 1, n_2\rangle \quad \text{and} \quad c_1^\dagger |n_1, n_2\rangle = \sqrt{n_1+1}\, |n_1+1, n_2\rangle, && \text{(IX.87a)} \\ c_2 |n_1, n_2\rangle &= -\sqrt{n_2}\, |n_1, n_2 - 1\rangle \quad \text{and} \quad c_2^\dagger |n_1, n_2\rangle = \sqrt{n_2+1}\, |n_1, n_2+1\rangle, && \text{(IX.87b)} \end{aligned}$$

The normalization condition is modified to

$$\langle n_1, n_2 | m_1, m_2\rangle = (-1)^{n_2}\delta_{n_1 m_1}\delta_{n_2 m_2}$$

and the identity operator is given by

$$1 = \sum_{n_1,n_2\in\mathbb{N}_0} (-1)^{n_2}|n_1,n_2\rangle\langle n_1,n_2|. \tag{IX.88}$$

Moreover, we have to redefine the relations between $\{c_a, c_a^\dagger\}_{a=1,2}$ and $\{z^a, \bar{z}^{\bar{a}}\}_{a=1,2}$ so that

$$c_1 := \frac{\hat{z}^1}{\sqrt{\theta^{1\bar{1}}}}\frac{1+\mathrm{sgn}(\theta^{1\bar{1}})}{2} + \frac{\hat{\bar{z}}^{\bar{1}}}{\sqrt{\theta^{\bar{1}1}}}\frac{1-\mathrm{sgn}(\theta^{1\bar{1}})}{2} \quad \text{and} \quad c_1^\dagger := \frac{\hat{\bar{z}}^{\bar{1}}}{\sqrt{\theta^{1\bar{1}}}}\frac{1+\mathrm{sgn}(\theta^{1\bar{1}})}{2} + \frac{\hat{z}^1}{\sqrt{\theta^{\bar{1}1}}}\frac{1-\mathrm{sgn}(\theta^{1\bar{1}})}{2}, \tag{IX.89a}$$

$$c_2 := \frac{\hat{\bar{z}}^{\bar{2}}}{\sqrt{\theta^{2\bar{2}}}}\frac{1+\mathrm{sgn}(\theta^{2\bar{2}})}{2} + \frac{\hat{z}^2}{\sqrt{\theta^{\bar{2}2}}}\frac{1-\mathrm{sgn}(\theta^{2\bar{2}})}{2} \quad \text{and} \quad c_2^\dagger := \frac{\hat{z}^2}{\sqrt{\theta^{2\bar{2}}}}\frac{1+\mathrm{sgn}(\theta^{2\bar{2}})}{2} + \frac{\hat{\bar{z}}^{\bar{2}}}{\sqrt{\theta^{\bar{2}2}}}\frac{1-\mathrm{sgn}(\theta^{2\bar{2}})}{2}. \tag{IX.89b}$$

The definition of the shift operator $S^{(1)}$ remains the same, while for $S^{(2)}$ we obtain

$$S^{(2)} : \mathcal{H}_2 \to (1^{(2)} - P_0^{(2)})\mathcal{H}_2 \quad \text{with} \quad S^{(2)\dagger}S^{(2)} = -1^{(2)} \quad \text{and} \quad S^{(2)}S^{(2)\dagger} = -(1^{(2)} - P_0^{(2)}), \tag{IX.90}$$

with the representation

$$S^{(2)} = \sum_{n\in\mathbb{N}_0} (-1)^n |n+1\rangle_2\langle n|_2. \tag{IX.91}$$

For an operator $S : \mathcal{H} \to (1 - P_0)\mathcal{H}$ on $\mathcal{H}$ with the properties $SS^\dagger = 1 - P_0$ and $S^\dagger S = 1$, we can use the old expression (VI.89), since the indefiniteness of the norm does not affect the verification of these properties. Indeed, $S$ contains only the identity, given by (IX.88), and the projector $P_0^{(2)}$ on $\mathcal{H}_2$.

**$U_+(1)$ Seiberg-Witten monopole equations with ${}^t\Phi = (\phi, 0)$.** We start from (IX.60) with $v \equiv 0$. In this case the Seiberg-Witten equations read

$$\mathcal{F}_{z^1\bar{z}^{\bar{1}}} + \mathcal{F}_{z^2\bar{z}^{\bar{2}}} = -\tfrac{1}{4}\phi\phi^\dagger, \qquad \mathcal{F}_{z^1 z^2} = 0 \qquad \text{and} \qquad \nabla_{\bar{z}^{\bar{a}}}\phi = 0, \tag{IX.92}$$

where the covariant derivative $\nabla_{z^a}$ acts as $\nabla_{z^a}\phi = -\theta_{a\bar{a}}\phi\bar{z}^{\bar{a}} + X_{z^a}\phi$. Suppose that $\mathcal{H}_1$ is positive normed while $\mathcal{H}_2$ is endowed with an indefinite norm. Using the definition (VI.89) and the ansatz $X_{z^a} = \theta_{a\bar{a}}S\bar{z}^{\bar{a}}S^\dagger$, we find again (IX.65). Assuming that $\phi = \gamma P_0$ with $\gamma \in \mathbb{R}$, we can solve the last equation of (IX.92) if $P_0 z^a = 0$, i.e. $z^a \sim c_a^\dagger$. For $a = 2$ this is satisfied because of our choice $\theta^{2\bar{2}} > 0$ and for $a = 1$ it implies $\theta^{1\bar{1}} < 0$. The second equation of (IX.92) is again identically satisfied, while the first one yields the condition $\theta_{1\bar{1}} + \theta_{2\bar{2}} = -\frac{1}{4}\gamma^2$, which makes sense due to the different signs of $\theta_{1\bar{1}}$ and $\theta_{2\bar{2}}$. The nonvanishing components of the curvature are $\mathcal{F}_{z^a\bar{z}^{\bar{a}}} = \theta_{a\bar{a}}P_0$ and one can readily verify that the current $\mathcal{J}_\mu$ from (IX.59) has no contributions to the topological charge (IX.57). Moreover, the computation of the topological charge gives $-1$. Note that the introduction of an indefinite norm on $\mathcal{H}_2$ was needed for having solutions to the equation $\theta_{1\bar{1}} + \theta_{2\bar{2}} = -\frac{1}{4}\gamma^2$ on the components of $\theta_{\mu\nu}$.

$U_+(1)$ **Seiberg-Witten monopole equations and vortices.** Finally, we discuss a case which is slightly different from the cases described above, in the sense that we fix the explicit form of the solutions on a two-dimensional subspace of $\mathbb{R}^4_\theta$ from the very beginning. This eventually results in the $U_+(1)$ vortex equations on the complementary two-dimensional subspace.

Again, let $\mathcal{H}_1$ be positive normed and $\mathcal{H}_2$ endowed with an indefinite norm, such that $[z^2, \bar{z}^{\bar{2}}] = \theta^{2\bar{2}} > 0$ and $[c_2, c_2^\dagger] = -1$. Furthermore, we choose an ansatz for $X_{z^2}$ of the form $X_{z^2} = \theta_{2\bar{2}} 1^{(1)} \otimes S^{(2)} \bar{z}^{\bar{2}} S^{(2)\dagger}$. A short calculation yields the result $\mathcal{F}_{z^2\bar{z}^{\bar{2}}} = \theta_{2\bar{2}} 1^{(1)} \otimes P_0^{(2)}$. Assuming that $\mathcal{A}_{z^1} = A_{z^1} \otimes P_0^{(2)}$, we obtain $\mathcal{F}_{z^1\bar{z}^{\bar{1}}} = F_{z^1\bar{z}^{\bar{1}}} \otimes P_0^{(2)}$. Similarly, we take $\phi = \psi \otimes P_0^{(2)}$. Then the equation $\nabla_{\bar{z}^{\bar{2}}}\phi = 0$ implies the condition $P_0^{(2)} z^2 = 0$, which is identically satisfied due to the relation $z^2 \sim c_2^\dagger$. Moreover, the equation $\mathcal{F}_{z^1 z^2} = 0$ is trivially satisfied. Using $\theta_{2\bar{2}} = -1/\theta^{2\bar{2}}$, we finally arrive at the equations

$$\left\{ F_{z^1\bar{z}^{\bar{1}}} - \left( \frac{1}{\theta^{2\bar{2}}} - \frac{1}{4}\psi\psi^\dagger \right) \right\} \otimes P_0^{(2)} = 0, \tag{IX.93a}$$

$$(\partial_{\bar{z}^{\bar{1}}} + A_{\bar{z}^{\bar{1}}})\psi \otimes P_0^{(2)} = 0. \tag{IX.93b}$$

Note that these equations can also be obtained in the context of the perturbed Seiberg-Witten equations if one chooses the perturbation proportional to $P_0^{(2)}$. Then there is no necessity to introduce an indefinite normed space. Rescaling $\psi \mapsto 2\,\tilde{\psi}/\sqrt{\theta^{2\bar{2}}}$ and introducing $\beta := 4/\theta^{2\bar{2}}$, we obtain the equations

$$F_{z^1\bar{z}^{\bar{1}}} = \tfrac{\beta}{4}(1 - \tilde{\psi}\tilde{\psi}^\dagger) \qquad \text{and} \qquad (\partial_{\bar{z}^{\bar{1}}} + A_{\bar{z}^{\bar{1}}})\tilde{\psi} = 0 \tag{IX.94}$$

on $\mathbb{R}^2_\theta \subset \mathbb{R}^4_\theta$. For $\beta = 1$ they coincide with the standard $U_+(1)$ vortex equations. The equations (IX.94) with $\theta^{1\bar{1}} = 2\theta^{12} \neq 0$ and their explicit solutions were extensively discussed in the literature (see e.g. [107, 8, 81]). Note that the choice of the projector $P_0^{(2)}$ on $\mathcal{H}_2$ ensures a finite charge for this solution. To exemplify this case let us rewrite $A_{z^1}$ via $\mathcal{X}_{z^1} = A_{z^1} + \theta_{1\bar{1}}\bar{z}^{\bar{a}}$ with $\mathcal{X}_{z^1} = \theta_{1\bar{1}}(S^{(1)})^N \bar{z}^{\bar{1}} (S^{(1)\dagger})^N$. Moreover, suppose that $\tilde{\psi} = (S^{(1)})^N$. Then one readily verifies that $\theta_{1\bar{1}} = \beta/4 = 1/\theta^{2\bar{2}}$. Therefore, we end up with $\mathcal{F}_{z^1\bar{z}^{\bar{1}}} = \theta_{1\bar{1}}\mathcal{P}_N$ and $\phi = \frac{2}{\sqrt{\theta^{2\bar{2}}}}(S^{(1)})^N \otimes P_0^{(2)}$. Note that $\theta^{1\bar{1}} < 0$. The topological charge (IX.47) for this configuration is $Q = -N$.

To summarize, we have described solutions on $\mathbb{R}^2_{\theta^{12}} \times \mathbb{R}^2_{\theta^{34}}$. It is also allowed to put $\theta^{12}$ to zero in order to have solutions on $\mathbb{R}^2 \times \mathbb{R}^2_{\theta^{34}}$. Then the second equation of (IX.94) reduces to $A_{\bar{z}^{\bar{1}}} = -\partial_{\bar{z}^{\bar{1}}} \log(\tilde{\psi})$. Assuming that $\tilde{\psi} = e^{(u+iv)/2}$ has zeros at points $z^1_n$ in the complex plane, we obtain from the first equation of (IX.94) with $\beta = 1$ (see e.g. [106, 179])

$$\Delta u = e^u - 1 + 4\pi \sum_{n=1}^{N} \delta^{(2)}(z^1 - z^1_n, \bar{z}^{\bar{1}} - \bar{z}^{\bar{1}}_n), \tag{IX.95}$$

i.e. the standard Liouville type equation on $\mathbb{R}^2 \cong \mathbb{C}$. The moduli $z^1_n$ are position parameters of the $N$-vortex solution on the $z^1$-plane. It is well known that this equation exhibits regular $N$-vortex solutions [106].

### § IX.2.3 Noncommutative Solitons and $D$-branes

Without refering to string theory the $U_+(1) \times U_-(1)$ and $U_\pm(1)$ Seiberg-Witten monopole equations are simply understood as noncommutative generalizations of the Abelian Seiberg-Witten equations. In this subsection we shall discuss how one can interpret solutions to the noncommutative Seiberg-Witten equations as $D$-brane configurations in a stringy context.

**Brane-antibrane effective action.** In the simplest case of type II superstrings living in the target space $\mathbb{R}^{9,1}$ a $Dp$-brane with a world volume $\mathbb{R}^{p,1} \hookrightarrow \mathbb{R}^{9,1}$ may be defined via a relative map

$$\varphi : (\Sigma_2, \partial\Sigma_2) \to (\mathbb{R}^{9,1}, \mathbb{R}^{p,1}),$$

where $\Sigma_2$ is a string world sheet with boundary $\partial\Sigma_2$. One may also consider $\overline{Dp}$-branes (= anti-$Dp$-branes) which are $Dp$-branes with opposite orientation and Ramond-Ramond (RR) charge.

It is well known that there are stable BPS $Dp$-branes in type IIA (even $p$) and type IIB (odd $p$) superstring theory. Besides that, it is also well known that a system consisting of coincident $Dp$-brane and $\overline{Dp}$-brane is unstable since open strings ending on these branes have a tachyonic mode ($M^2 < 0$) in the spectrum (see e.g. [208, 176, 154, 152, 187] and references therein). This instability can be seen in the low energy effective action for the brane-antibrane pair. Namely, the effective field theory describing light excitations of this system contains two Abelian gauge potentials $\mathcal{A}_\pm$ and a complex scalar $\phi$ (tachyon). The latter one is associated with modes of the open string stretched between brane and antibrane, and is believed to be subject to a "Mexican hat" potential. The tachyon carries charge one under the diagonal $U(1)$ subgroup of the group $U_+(1) \times U_-(1)$ with the generator $\mathrm{diag}(i, -i)$ corresponding to the gauge potential $\mathcal{A}_+ - \mathcal{A}_-$. After turning on a constant $B$-field (generating a noncommutativity tensor $\theta$ on $\mathbb{R}^{p,1}$ [175]) the theory becomes noncommutative and the tachyon field transforms in the bi-fundamental representation of $U_+(1) \times U_-(1)$ [80].

In perturbative string theory the first computations of the brane-antibrane effective action were performed in [159]. The resulting effective Lagrangian reads

$$\mathcal{L}^{(2)} = \mathcal{F}_{+\hat\alpha\hat\beta}\overline{\mathcal{F}_+^{\hat\alpha\hat\beta}} + \mathcal{F}_{-\hat\alpha\hat\beta}\overline{\mathcal{F}_-^{\hat\alpha\hat\beta}} + (\nabla^{\hat\alpha}\phi)\overline{\nabla_{\hat\alpha}\phi} + \tfrac{1}{4}(\tau^2 - \phi\bar\phi)^2, \tag{IX.96}$$

where the covariant derivative is given by $\nabla_{\hat\alpha}\phi = \partial_{\hat\alpha}\phi + (\mathcal{A}_{+\hat\alpha} - \mathcal{A}_{-\hat\alpha})\phi$, bar denotes complex conjugation, $\tau^2$ is a constant and $\hat\alpha, \hat\beta, \ldots = 0, 1, \ldots, p$. This is not the full effective Lagrangian since the coupling between closed string RR fields and open strings gives an additional term $\mathcal{L}_{\mathrm{CS}}$ called the Chern-Simons term [111]. For constant components of the RR fields, which we now consider, this term does not contribute to the equations of motion, and therefore we will not discuss it here. Note that the terms given by (V.23) and (IX.34) are of such kind.

The tachyon potential given by (IX.96) has the quartic form

$$V(\phi, \bar\phi) = \tfrac{1}{4}(\tau^2 - \phi\bar\phi)^2. \tag{IX.97}$$

Computations in level truncated superstring field theory yield the same result [17]. Any minimum $\phi\bar{\phi} = \tau^2$ of the tachyon potential describes the closed string vacuum (tachyon condensate). The perturbative spectrum around this vacuum is conjectured not to contain any open string excitations. As it was discussed in [80, 81, 83], in the presence of a $B$-field background a noncommutative generalization of the effective theory (IX.96) might be of the form

$$\mathcal{L}^{(2)}_{\rm nc} = \mathcal{F}_{+\hat\alpha\hat\beta}\mathcal{F}_{+}^{\dagger\hat\alpha\hat\beta} + \mathcal{F}_{-\hat\alpha\hat\beta}\mathcal{F}_{-}^{\dagger\hat\alpha\hat\beta} + \tfrac{1}{2}\{\nabla^{\hat\alpha}\phi, (\nabla_{\hat\alpha}\phi)^\dagger\} + \tfrac{1}{8}(\tau^2 - \phi\phi^\dagger)^2 + \tfrac{1}{8}(\tau^2 - \phi^\dagger\phi)^2, \qquad \text{(IX.98)}$$

where $\nabla_{\hat\alpha}\phi = \partial_{\hat\alpha}\phi + \mathcal{A}_{+\hat\alpha}\phi - \phi\mathcal{A}_{-\hat\alpha}$. Higher order corrections to the quartic tachyon potential are known (see e.g. [18, 152]). The result is qualitatively similar ("Mexican hat" form with minima at $\phi\bar\phi < \infty$), and (IX.97) is the leading order term. Note that a quartic form of the potential was also obtained in [65, 175, 43, 41, 2] for bound states in $D(p-2)$-$Dp$ and $D(p-4)$-$Dp$ systems by using scattering calculations in string theory, level truncated superstring field theory and by analyzing the fluctuation spectrum around (noncommutative) vortex and instanton solutions.

Quite different results have been obtained in boundary string field theory (BSFT) (see e.g. [119, 185] and references therein). There (as in [177]) the Lagrangian density is proportional to the tachyon potential itself, and the potential has the form $V \sim \exp(-\phi\bar\phi)$ with a ring of minima at $\phi\bar\phi \to \infty$. It is believed that the level truncation scheme and the BSFT approach can be related by a field redefinition involving all components of the string field [119]. Various 'improved versions' of the effective action of the brane-antibrane system have been introduced (see e.g. [108, 178]), and in the literature there is no final agreement on its form. Thus, the action based on the Lagrangian (IX.96) (or (IX.98) in the presence of a $B$-field) might be considered as an approximation of the low energy effective action of the brane-antibrane system. In any case the discussed Yang-Mills-Higgs theory provides a simple field theoretic model of the more complicated string field theory description of brane-antibrane systems.

**Noncommutative vortices and ABS solitons.** Recall that branes and antibranes have opposite RR charge and therefore they can annihilate into the closed string vacuum state with $\mathcal{A}_\pm = 0$ and $\phi\phi^\dagger = \phi^\dagger\phi = \tau^2$. However, instead of taking the vacuum solution, one can choose as the ground state a (tachyon) soliton solution to the equations of motion for the Lagrangian (IX.98). Such kind of solutions are interpreted as bound states of $Dp$-branes and $\overline{Dp}$-branes, equivalent to $D$-branes of lower dimensionality. Here we discuss the main example of such solutions obtained via the Atiyah-Bott-Shapiro (ABS) construction [4, 208, 90, 82]. These ABS (noncommutative) solitons live in $2n \le p$ dimensions, and for $n = 1$ they coincide with noncommutative vortices on $\mathbb{R}^2_\theta$. For $n = 2$ the ABS solitons are non-Abelian and therefore differ from those (Abelian) Seiberg-Witten solutions which also solve the field equations following from (IX.98).

To describe the ABS solitons on $\mathbb{R}^{2n}_\theta$ we consider a non-Abelian generalization of the Lagrangian (IX.98) with $U_+(q) \times U_-(q)$ gauge group for $q = 2^{n-1}$ and the tachyon field $\phi$ in the bi-fundamental representation $(q, \bar q)$ of this group. This model describes two Hermitian rank $q$

vector bundles $E_\pm \to \mathbb{R}^{p,1}$ with connection one-forms $\mathcal{A}_\pm$ and $\phi \in \mathrm{Hom}(E_-, E_+)$ and is associated with a system of $q$ $Dp$-branes and $q$ $\overline{Dp}$-branes with common world volume $\mathbb{R}^{p,1}$. We assume that $2n \leq p$ and introduce an ansatz for ABS solitons related to Clifford algebras.

Consider the Clifford algebra of the Euclidean space $(\mathbb{R}^{2n}, \delta_{\alpha\beta})$, generated by unity and elements $\Gamma_\alpha$ such that

$$\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = -2\delta_{\alpha\beta} \tag{IX.99}$$

with $\alpha, \beta, \ldots = 1, \ldots, 2n$. Remember that in the $2^n \times 2^n$ matrix representation of this algebra the generators $\Gamma_\alpha$ can be chosen of the form

$$\Gamma_\alpha = \begin{pmatrix} 0 & \gamma_\alpha^\dagger \\ -\gamma_\alpha & 0 \end{pmatrix}, \tag{IX.100}$$

where the $\gamma_\alpha$'s are $q \times q$ gamma matrices with $q = 2^{n-1}$. In this representation the spinor space $S \cong \mathbb{C}^{2q}$ can be decomposed into a direct sum $S \cong S^+ \oplus S^-$ of semi-spinor spaces $S^\pm$ (the spaces of Weyl spinors). Note that for $n = 1$ we have $\gamma_1 = 1$ and $\gamma_2 = i$.

Considering the noncommutative deformation $\mathbb{R}^{2n}_\theta$ of $\mathbb{R}^{2n}$, we introduce the operators

$$T = \gamma_\alpha^\dagger x^\alpha \frac{1}{\sqrt{(\gamma x)(\gamma x)^\dagger}} \qquad \text{and} \qquad T^\dagger = \frac{1}{\sqrt{(\gamma x)(\gamma x)^\dagger}} \gamma_\alpha x^\alpha, \tag{IX.101}$$

where $\gamma x$ is a short hand notation for $\gamma_\alpha x^\alpha$. The operator $T$ defines a map

$$\hat{T} : \mathcal{H} \otimes S^- \to \mathcal{H} \otimes S^+. \tag{IX.102}$$

Here $\mathcal{H}$ is the Hilbert space realized as a representation space of $n$ oscillators defined by formulas similiar to (VI.66) and (VI.67), and $S^\pm$ are the semi-spinor spaces introduced above. In matrix realization $\hat{T}$ looks as

$$\hat{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \tag{IX.103}$$

and hence following the authors of [208, 209, 82, 83], we will not distinguish $T$ and $\hat{T}$ in the sequel.

It is not difficult to see that

$$T^\dagger T = \mathbb{1}_q \qquad \text{and} \qquad TT^\dagger = \mathbb{1}_q - \mathcal{P}_0, \tag{IX.104}$$

where $\mathcal{P}_0$ is the projector onto the kernel of $T^\dagger$. This state is the tensor product of the oscillator ground state with itself and the lowest weight spinor of $SO(2n)$ (the fermion ground state). Also, by introducing $T_N := T^N$ with $T_1 \equiv T$, we have

$$T_N^\dagger T_N = \mathbb{1}_q \qquad \text{and} \qquad T_N T_N^\dagger = \mathbb{1}_q - \mathcal{P}_{N-1}, \tag{IX.105}$$

where $\mathcal{P}_{N-1}$ is the projector onto the kernel of $T_N^\dagger$, an $N$-dimensional subspace in $\mathcal{H} \otimes S$. One can show that

$$\dim \ker T_N = 0 \qquad \text{and} \qquad \dim \operatorname{coker} T_N = \dim \ker T_N^\dagger = N, \tag{IX.106}$$

and therefore the index of $T_N$ is given by

$$\operatorname{ind} T_N = \dim \ker T_N - \dim \operatorname{coker} T_N = -N. \tag{IX.107}$$

The operators $T_N$ and $T_N^\dagger$ are Toeplitz operators.

Now we reduce the equations of motion for the Lagrangian (IX.98) to the space $\mathbb{R}^{2n}_\theta \hookrightarrow \mathbb{R}^{p+1}_\theta$ by assuming that all fields depend only on $x^\alpha$ and by taking $\mathcal{A}_{\pm\alpha}$ as the only nonvanishing components of the gauge potentials $\mathcal{A}_\pm$. To solve the reduced field equations we consider $T_N$ as $q \times q$ matrices with operator entries and introduce the ansatz

$$\mathcal{A}_{+z^a} = -(\mathcal{A}_{+\bar{z}^{\bar{a}}})^\dagger = \tau\theta_{a\bar{a}}(T_{N_+}\bar{z}^{\bar{a}}T_{N_+}^\dagger - \bar{z}^{\bar{a}}), \quad \mathcal{A}_{-z^a} = 0 = \mathcal{A}_{-\bar{z}^{\bar{a}}} \quad \text{and} \quad \phi = \tau T_{N_+}, \tag{IX.108}$$

which solves the equation of motion for (IX.98) [81, 83]. More general solutions can also be constructed. For $n = 1$ the solution (IX.108) describes $N_+$ vortices on $\mathbb{R}^2_\theta$. In the case $n = 2$ it gives a solution for the $U_+(2) \times U_-(2)$ Yang-Mills-Higgs model on $\mathbb{R}^4_\theta$. Some noncommutative Seiberg-Witten solutions (e.g. (IX.72a), (IX.72b) with $N_- = 0$) solve the above equations as well but for the gauge group $U_+(1) \times U_-(1)$. These solutions can therefore be regarded as a new kind of tachyon solitons.

**Noncommutative Seiberg-Witten solitons.** Note that the Seiberg-Witten equations (IX.51) with $v_+ = v_- = \tau^2 = \text{const}$ coincide with the first order BPS equations for the Lagrangian (IX.98), and therefore their solutions also satisfy the equations of motion for (IX.98). In particular, the solution given by (IX.72) with $N_+ \geq 1$ and $N_- = 0$ is such a solution. Following the general logic of Sen's proposal, it is natural to interpret this solution as a configuration of $N_+$ stable $D(p-4)$-branes, corresponding to the topologically stable codimension four Seiberg-Witten soliton on a $Dp - \overline{Dp}$ brane pair.

The more general configuration described by (IX.72) with $N_\pm \geq 1$ can be interpreted in two different ways. Firstly, we may again choose $v_+ = v_- = \tau^2 = \text{const}$. Then one can show that this configuration satisfies the second order equations of motion for the Lagrangian (IX.98) but does not satisfy the first order equations (IX.51) with constant $v_\pm$. This is natural since this solution corresponds to a system of $N_+$ $D(p-4)$-branes and $N_-$ $\overline{D(p-4)}$-branes which is not a BPS configuration from the point of view of the Lagrangian (IX.98). Secondly, one may consider the situation where $v_+$ depends on $\phi\phi^\dagger$ and $v_-$ on $\phi^\dagger\phi$ and take as the low energy effective action the sum of the functional (IX.49a) and the Chern-Simons term (IX.49b), where the latter contributes to the equations of motion for nonconstant $v_\pm$. Then for proper choices of $v_\pm$ one can obtain configurations which satisfy the noncommutative Seiberg-Witten equations. For instance, the choice

$$v_+ = \tau^2 \qquad \text{and} \qquad v_- = -\tau^2 + 2\phi^\dagger\phi \tag{IX.109}$$

corresponds to the same tachyon potential $V \sim (\tau^2 - \phi\phi^\dagger)^2 + (\tau^2 - \phi^\dagger\phi)^2$ as in (IX.98). However, for the above choice of $v_\pm$ the Chern-Simons term (IX.49b) becomes nontrivial and contributes to the equations (IX.51) which are the BPS equations for the action $E_\chi + 16\pi^2 K_\chi$ with $E_\chi$ and $K_\chi$ determined by (IX.49). Therefore, for $v_\pm$ given by (IX.109) the configuration (IX.72) with $N_\pm \geq 1$ is a solution to the Seiberg-Witten equations (IX.51). So, in both cases the configuration (IX.72) may be interpreted as $N_+$ $D(p-4)$-branes and $N_-$ $\overline{D(p-4)}$-branes. Other solutions to the noncommutative Seiberg-Witten equations can be analyzed similarly.

# Concluding Remarks

In this thesis we have considered three different noncommutative field theories and investigated the construction of nonperturbative solutions to the corresponding equations of motion.

In the part on noncommutative instantons we have extended the self-dual BPST instanton solution to Yang-Mills theory defined on a noncommutative Euclidean space. We have chosen the noncommutative deformation matrix $\theta^{\mu\nu}$ to be self-dual as well, because self-dual instantons on an anti-self-dual background cannot be captured with our noncommutative extension of the Belavin-Zakharov ansatz. Our calculations demonstrate that noncommutativity with a self-dual $\theta^{\mu\nu}$ causes no difficulties in constructing solutions. Moreover, in the framework of the dressing and splitting approaches described in this part we were able to solve the reality problem of the gauge field which was encountered by the authors of [35] in generalizing the BPST ansatz [11]. It is tempting to recycle the constructed one-instanton solution as the seed solution in the dressing method, in order to generate multi-instantons. Perhaps a combination of the dressing and splitting method will do the job. One may hope that computing the two-instanton configuration in terms of a matrix $\Upsilon_{N=2}$ will point towards a recursive procedure for the construction of $N$-instantons. However, further work in this direction needs to be done.

In the second part of the thesis we have applied the method of dressing transformations to a noncommutative modified $U(n)$ sigma model in order to generate non-Abelian multi-soliton and soliton-antisoliton solutions. However, a natural question arises concerning the construction of Abelian soliton-antisoliton configurations. We have explained, that Abelian multi-soliton configurations do exist and actually we constructed them. Hence, pure Abelian antisolitions exist as well, since we have an ambiguity in what we are calling soliton and what antisoliton, i.e. the sign of the definition of the topological charge is rather a matter of taste. However, no Abelian scattering solutions of this model have been found so far, rather ring-like structures with solitons (or antisolitons) sitting at the origin and changing their shape with time (cf. [128]). Considering the Abelian soliton-antisoliton case, we were led to even more negative results in such a way, that using the techniques discussed above we did not obtain any stable configuration. It might be possible that Abelian solutions do exist considering higher order poles in the dressing ansatz.

The splitting and dressing approaches presented here have recently been lifted from gauge field theory to string field theory [130, 115]. Since $10d$ superstrings in Berkovits' nonpolynomial formulation [14] as well as $4d$ self-dual strings à la Berkovits and Siegel [16] are classically integrable [125], their field equations can be linearized and classical backgrounds can be constructed using these methods. Moreover, string field theory may be viewed as an infinite-dimensional noncommutative field theory, so that the techniques of the present thesis are directly applicable. A program in this direction has been initiated [130, 115].

In the last part of this thesis we have discussed different noncommutative deformations of the (perturbed) Seiberg-Witten monopole equations on Euclidean four-dimensional space. Namely, starting from properly deformed $U(2)$ self-duality type Yang-Mills equations in eight dimensions, we performed a reduction to $U(2)$ noncommutative Seiberg-Witten equations on $\mathbb{R}^4_\theta$ with the matter field living in the adjoint representation of the gauge group. We then concentrated on the $U_+(1) \times U_-(1) \subset U(2)$ noncommutative Seiberg-Witten equations with the matter in the bi-fundamental representation of $U_+(1) \times U_-(1)$. Perturbed versions of these equations have also been discussed. Then, by considering the matter field $\Phi$ as an element of a right or left $\mathbb{R}^4_\theta$-module, we have introduced the (perturbed) $U_+(1)$ and $U_-(1)$ Seiberg-Witten equations. The commutative limits of all these three Abelian cases are identical to the standard (perturbed) Abelian Seiberg-Witten equations on $\mathbb{R}^4$. It has been shown that $\mathbb{R}^4_\theta$ supports regular finite-action solutions to the Seiberg-Witten equations even if there are no such solutions in the commutative case. This is a well known phenomenon related to the fact that, due to the noncommutativity tensor $\theta$, an additional length scale enters the theory. We have constructed explicit solutions to the $U_+(1) \times U_-(1)$ and $U_\pm(1)$ noncommutative Seiberg-Witten equations and interpreted them as $D$-brane configurations in type II superstring theory. It would be illuminating to generalize the present results to non-Abelian noncommutative Seiberg-Witten theory and discuss the latter's relation to superstring theory.

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# Index

**Declaration**

Hereby I declare that I have composed this thesis without illegitimate help of third parties and that I have marked all sources as such. This work has not yet been submitted to this or any other university in order to earn an acedemic degree.

Martin Wolf, Dresden - Hannover, July 4th 2003

**Erklärung**

Hiermit erkläre ich, daß ich diese Arbeit ohne unzulässige Hilfe Dritter verfaßt und alle Quellen als solche gekennzeichnet habe. Diese Arbeit wurde bisher weder an dieser noch an irgendeiner anderen Universität zur Erlangung eines akademischen Grades eingereicht.

Martin Wolf, Dresden - Hannover, 04. Juli 2003