Chapter 3: Landau Theory for Phase Transitions

In our treatment of mean-fielf theory of phase transitions, we have seen that the central point is the behavior of the free energy close to the phase transiton point. In the context of the appearance of spontaneous symmetry breaking in the mexican hat potential $F(m) = am^2 + bm^4$, the value of the coffecients a and b were not particularly interesting, except for the fact that a can become negative. Landau theory for phase transitions captures this notion on very general grounds, being a phenomenological approach that does not require to solve any microscopic statistical mechanics problem. Importantly, Landau theory will serve as a starting point, upon which many further systematic extensions can be built.

I. THE LANDAU FREE ENERGY FUNCTIONAL

The central idea of Landau theory is to introduce an order parameter field $\phi(x)$ as the degree of freedom that is being analyzed. Then, one considers an energy density functional of the form

$$f[\phi(x)] = u_0 + u_1\phi(x) + u_2\phi(x)^2 + u_3\phi(x)^3 + u_4\phi(x)^4 + \dots \tag{1}$$

Close to a continuous phase transiton, we are interested in the regime where the order parameter is small, meaning that we can truncate the series at a relatively low order. Next, we require that the free energy has the same symmetries as the microscopic model we want to describe [1]. In the case of the Ising model, the $\mathbb{Z}_2$ symmetry translates to the relation $f(\phi(x)) = f(-\phi(x)$, causing the odd terms of the expansion to vanish.

The parition function corresponding to such an energy density can be written as a functional integral over $\phi(x)$, according to

$$Z = \int D\phi \exp\left\{-\int dx f[\phi(x)]\right\}. \tag{2}$$

As the simplest approximation to the functional integral, we first take the saddlepoint approximation, i.e., we replace the exponential in the integral by the largest value it can possibly take, which in turn is where the integral over $f(x)$ becomes minimal [2]. This minimum is reached by solutions $\phi(x)$ which are constant, i.e., $\phi(x) = \Phi$. The value of Φ can then be found by solving the equation

$$\frac{\partial f(\Phi)}{\partial \Phi} = 0. \tag{3}$$

Remarkably, for the Ising model this gives rise to the same critical exponents that we found within mean-field theory. Assuming $2u_2 = at \equiv a(T - T_c)/T_c$, we find $\Phi \sim t^{1/2}$ for $T < T_c$. We can therefore interpret Eq. (1) as a free energy functional. However, here we have not made any particular assumption about the microscopic model, other than imposing a symmmetry constraint. Hence, it is more appropriate to refer to the class of all microscopic theories which give rise to the same Landau theory, which is called a ϕ^4 theory in this case, as we have expanded the order parameter up to the fourth order.

II. GRADIENT TERMS: GINZBURG-LANDAU THEORY

The first systematic extension we are adding to the Landau free energy is a spatial gradient of the order parameter field. The motivation behind this is to include low-momentum variations of the order parameter, which will become important to capture effects beyond a mean-field theory treatment. Then, the free energy functional (sometimes referred to as a Ginzburg-Landau theory) becomes for the ϕ^4 theory in an external field $H(x)$

$$f[\phi(x)] = \frac{1}{2}[\nabla\phi(x)]^2 - H(x)\phi(x) + u_2\phi(x)^2 + u_4\phi(x)^4. \tag{4}$$

Here, the linear gradiant term again vanishes due to symmetry (there is no prefered direction) and we have chosen units such that the prefactor of the spatial gradient term becomes $1/2$.

Using the gradient term, we can now calculate the correlation length of the system. For this, we take an external field of the form $H(x) = -\lambda\delta(x)$. Before turning to the saddlepoint approximation, we perform a partial integration of the spatial gradient term, leading to

$$F = \int dx f(x) = \int dx - \frac{1}{2}\phi(x)\nabla^2\phi(x) + \lambda\delta(x)\phi(x) + u_2\phi(x)^2 + u_4\phi(x)^4, \tag{5}$$

where we have discarded the boundary term because it is irrelevant in the thermodynamic limit. The saddlepoint approximation then results in the differential equation

$$-\left(\nabla^2 + 2u_2\right)\phi(x) - 4u_4\phi(x)^3 - \lambda\delta(x) = 0. \tag{6}$$

In the disordered phase ($t > 0$), we can safely neglect the cubic term as $\phi(x)$ can be expected to be on the order of λ, which we consider to be infinitesimally small. Then, we can Fourier transform the differential equation and solve for $\phi(k)$,

$$\phi(k) = -\frac{\lambda}{k^2 + 2u_2}. \tag{7}$$

The inverse Fourier transform results in a Bessel function that can be approximated for large values of x by

$$\phi(x) \sim x^{2-d} \exp(-x/\xi), \tag{8}$$

where we have introduced the correlation length $\xi = (at)^{-1/2}$. This equation can be understood as a correlation function for the order parameter field, as it describes the response of the system to having a nonzero order parameter at the origin. It corresponds to an Ornstein-Zernike form, which is in general given by

$$C(0,x) \sim x^{2-d-\eta} \exp(-ct^{\nu}), \tag{9}$$

where η and ν are another set of critical exponents, which we find to be $\eta = 0$ and $\nu = 1/2$ for the ϕ^4 theory.

In the ordered phase at $t < 0$, the cubic term in Eq. (6) can no longer be neglected. However, we can expand the order parameter field as $\phi(x) = \Phi + \delta\phi(x)$ up to first order in $\delta\phi(x)$, which merely results in a correlation length ξ which is twice as large as in the disordered phase [2].

[1] L. Landau, Ukr. J. Phys. **53**, 25 (2008), originally published in Zh. Eksp. Teor. Fiz. **7**, 19–32 (1937).

[2] K. Huang, *Statistical Mechanics* (John Wiley and Sons, New York, 1987).