Chapter 4: The Renormalization Group

We start with the partition function for the Gaussian theory denoted by

$$Z = \int D\phi \exp\left\{-\beta \int dx H[\phi(x)]\right\}, \tag{1}$$

where the Ginzburg-Landau-Wilson Hamiltonian is given by

$$H = \frac{v_2}{2}(\nabla\phi)^2 + \frac{u_2}{2}\phi^2. \tag{2}$$

Going to a momentum space representation of the Hamiltonian, we obtain

$$H = \frac{1}{2}(v_2 k^2 + u_2)|\phi(k)|^2. \tag{3}$$

The upper limit of the momentum space integration is set by a momentum scale Λ, which is initially given by a physical parameter such as some fraction of the inverse lattice spacing.

I. RENORMALIZATION OF THE GAUSSIAN THEORY

The key idea of the renormalization group (RG) is to split the momentum integral into two parts. The low momentum part going from $q = 0$ to $q = \Lambda/b$ and the high momentum part between $q = \Lambda/b$ and $q = \Lambda$. Then, the partition function can be written as

$$Z = Z_{>} \int D\phi \exp\left\{-\beta \int_0^{\lambda/b} dq H[\phi_{<}(q)]\right\}, \tag{4}$$

where the $>$ ($<$) index refers to momenta above (below) the cutoff. As the next step, we rescale the momentum variable as $q' = qk$. Then, the renormalization step is performed by introducing a renormalized order parameter field according to $\phi'(q') = \phi_{<}(q')/z$. Using the rescaled and renormalized expressions, the Hamiltonian is given by

$$H = \int_0^{\Lambda} dq \frac{b^{-d}z^2}{2}(b^{-2}v_2 q^2 + u_2)|\phi'(q')|^2. \tag{5}$$

If we compare the renormalized form of the Hamiltonian to the unrenormalized on in Eq. (3), we find that we can express the renormalization step in terms of a change of the coupling constants according to

$$v_2' = v_2 b^{-d-2} z^2 \tag{6}$$

$$u_2' = u_2 b^- d z^2. \tag{7}$$

In the following, we choose the renormalization factor z such that the v_2 coefficient remains invariant under the RG step. Physically, this implies that the scale of fluctuations remain constant while all other quantities are rescaled. This choice results in $z = b^{1+d/2}$, which in turn leads to $u_2' = b^2 u_2$.

The RG transformation for u_2 has a fixpoint of $u_2^* = 0$, which is unstable. For finite u_2, the RG step implies that thue u_2 gets larger and larger, i.e., it is dominating of the fluctuation term v_2, which remains invariant by construction. This means that the correlations in the order parameter field become weaker and weaker under the RG transformation, i.e., the Gaussian theory describes a disorded phase.

Similarly, we can also calculate the RG step for an external field h. Since the RG step will only lead to a renormalization of the order parameter field in this case, we obtain the relation $h' = zh = b^{1+d/2}h$. Interestingly, we can use the RG transformations for u_2 and h to calculate the critical exponent ν. For this, we first observe that any length gets rescaled by a factor of $1/b$, which implies for the correlation length $\xi' = \xi/b$. Using this, we can write the correlation length as a function of u_2 and h in terms of the renormalized coupling constants,

$$\xi = \xi(u_2, h) = b\xi'(u_2', h') = b\xi(b^2 u_2, b^{1+d/2}h). \tag{8}$$

Let us now assume that the rescaling factor b is chosen such that $b^2 u_2 = 1$, yielding

$$\xi = u_2^{-1/2} \xi(1, h/u_2^{1/2+d/4}) \equiv u_2^{-1/2} \chi(h/u_2^{1/2+d/4}), \tag{9}$$

where we have introduced the scaling function χ. From this expression, we can recover the critical exponent $\nu = 1/2$. Additionally, since the nonanalytic part of the correlation length is already captured in the diverging prefactor, the scaling function χ is an analytic function.

II. PERTURBATIVE RENORMALIZATION GROUP

We now extend our analysis beyond the Gaussian model and include the quartic term of the ϕ^4 theory. Then, we can write the Ginzburg-Landau-Wilson Hamiltonian as

$$H = H_0 + u_4 \int dx \phi(x)^4, \tag{10}$$

where H_0 is the Gaussian part. Following a Fourier transform, we obtain

$$H = H_0 + u_4 \int dq_1 dq_2 dq_3 \phi(q_1)\phi(q_2)\phi(q_3)\phi(-q_1 - q_2 - q_3) \equiv H_0 + U_4. \tag{11}$$

Again, let us now separate the low momentum from the high momentum part. Then, the partition function is given by

$$Z = \int D\phi_< e^{-\beta H_0[\phi_<]} \int D\phi_> e^{-\beta H_0[\phi_>] - U_4[\phi_<, \phi_>]}. \tag{12}$$

Importantly, the quartic term U_4 couples the low momentum part to the high nomentum part, i.e., it behaves like an interaction term for the order parameter field. Multiplying by $Z_0^>$ (and dividing by it), we can see that the last part can be written as an ensemble average over the large momentum part since $\langle A \rangle = Z^{-1} \int D\phi A \exp(-\beta H)$. Then, we obtain

$$Z = Z_0^> \int D\phi_< e^{-\beta H_0[\phi_<]} \langle e^{-U_4[\phi_<, \phi_>]} \rangle_> \tag{13}$$

$$= Z_0^> \int D\phi_< e^{-\beta H_0[\phi_<] + \ln \langle e^{-U_4[\phi_<, \phi_>]} \rangle_>}, \tag{14}$$

where the index for the average value denotes that the averaging is done with respect to the high momentum part only.

The perturbative character from our RG treatment now refers to the cumulant expansion of the logarithm according to

$$\ln \langle e^{-U_4} \rangle = -\langle U_4 \rangle + \frac{1}{2} \left(\langle U^2 \rangle - \langle U \rangle^2 \right) + \ldots. \tag{15}$$

In the following, we will just retain the leading order of the expansion. If we want to evalute $\langle U_4 \rangle$, we have to introduce different cases for the momentum integral in Eq. (11) since the momenta q_1, q_2, q_3, $q_4 = -q_1 - q_2 - q_3$ can lie either above the rescaled cutoff (i.e., contributing to $\phi_>$) or below (contributing to $\phi_<$. Fortunately, all combinations with an odd number of powers in $\phi_>$ vanish due to Wick's theorem [1]. This leaves us with three terms of the form

$$A_1 = \langle \phi_<(q_1)\phi_<(q_2)\phi_<(q_3)\phi_<(q_4)\rangle_> \tag{16}$$

$$A_2 = \langle \phi_>(q_1)\phi_>(q_2)\phi_>(q_3)\phi_>(q_4)\rangle_> \tag{17}$$

$$A_3 = \langle \phi_>(q_1)\phi_>(q_2)\phi_<(q_3)\phi_<(q_4)\rangle_> = \phi_<(q_3)\phi_<(q_4)\langle \phi_>(q_1)\phi_>(q_2)\rangle_>. \tag{18}$$

A_1 is still a fourth order term, but acting completely below the rescaled cutoff. We retain it as the new U_4 of the renormalized theory. Similarly, A_2 acts only above the cutoff and hence will just give rise to an irrelevant constant, as does the $Z_0^>$. We can therefore safely ignore both of them.

A_3 is the interesting term, as it couples the low momentum part to the high momentum part. Since we see that there is a quadratic term remaining below the cutoff, it will give rise to a renormalization of the u_2 term of the theory. Diagramatically, this process can be represented as a one-loop Feynman diagram. Fortunately, we have already calculated the expectation value $\langle \phi_>(q_1)\phi_>(q_2)\rangle$ in the previous chapter,

$$\langle \phi_>(q_1)\phi_>(q_2)\rangle = (2\pi)^{-d}\delta(q_1+q_2)G_0(q_1), \tag{19}$$

where G_0 is the propagator (i.e., the Green's function) of the Gaussian theory, which is given by

$$G_0(q) = \frac{1}{v_2 + u_2 q^2}. \tag{20}$$

Note that the choice of q_1 and q_2 as the momenta above the cutoff was arbitrary in Eq. (18). Since the U_4 term is invariant under exchanging the momentum variables, we also need to consider the cases where q_3 and/or q_4 lie above the cutoff. As long as there are exactly two momenta above the cutoff, these combinations will also contribute to A_3. Since we have to select two momenta out of four possibilites, there are in total $\binom{4}{2} = 6$ combinations. Therefore, we find that the u_2 becomes

$$\tilde{u}_2 = u_2 + 12u_4 \int_{\Lambda_b}^{\Lambda} \frac{d_q}{(2\pi)^d} G_0(q). \tag{21}$$

Therefore, we have the first example where performing the RG step of one coupling constant leads to a modification of another coupling constant. This can even happen in the case where the other coupling constant was not even part of the initial field theory (e.g., for $u_2 = 0$ in the ϕ^4 theory).

After rescaling and renormalization, the Ginzburg-Landau-Wilson Hamiltonian finally becomes

$$H = \int_0^\Lambda dq' \frac{b^{-d} z^2}{2} (b^{-2} v_2 q^2 + \tilde{u}_2) |\phi'(q')|^2 + b^{-3d} z^4 u_4 \int dq_1 dq_2 dq_3 \phi(q_1)\phi(q_2)\phi(q_3)\phi(-q_1 - q_2 - q_3). \tag{22}$$

Requiring that the v_2 coefficient remains invariant under the RG step as before, we obtain the renormalized coupling constants according to

$$u_2' = b^2 \tilde{u}_2 = b^2 \left[u_2 + 12 u_4 \int_{\Lambda_b}^{\Lambda} \frac{d_q}{(2\pi)^d} G_0(q) \right] \tag{23}$$

$$u_4' = b^{4-d} u_4. \tag{24}$$

In the following, we move from discrete RG steps to a continuous process. For this, we set $b = \exp(l)$ and take the limit of infinitesimally small changes in l, which we denote by δl. Then, we may perform a Taylor expansion of the RG equations according to

$$u_2(l) = u_2 + \delta l \frac{du_2}{dl} + O(\delta l^2) \tag{25}$$

$$u_4(l) = u_4 + \delta l \frac{du_4}{dl} + O(\delta l^2). \tag{26}$$

For the u_2 term, we now need to evaluate the momentum integral. As the lower boundary is given by $\Lambda/b = \Lambda e^{-l} = \Lambda(1 - \delta l) + O(\delta l^2)$, we can invoke the fundamental theorem of calculus to obtain

$$\frac{u_2}{dl} = 2u_2 + \frac{12 u_4 S_d \Lambda^d}{(2\pi)^d (u_2 + v_2 \Lambda^2)}, \tag{27}$$

where S_d is the surface area of the d-dimensional unit sphere. For the u_4 term, the continuous form is simply given by

$$\frac{u_4}{dl} = (4 - d) u_4. \tag{28}$$

The right hand sides of Eqs. (27–28) are often refered to as the β functions $\beta(u_2)$ and $\beta(u_4)$, especially in high-energy physics [2]. The fixpoints of the RG flow are given by the roots of the β functions.

We see that the set of nonlinear differential equations has a trivial fixpoint at $(u_2^*, u_4^*) = (0, 0)$. We can analyze the stability of this fixpoint by expanding the differential equation up to first order in $u_{2,4}$, obtaining

$$\frac{d}{dl}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix} = \begin{pmatrix} 2 & 12\frac{S_d \Lambda^{d-2}}{(2\pi)^d v_2} \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} u_2 \\ u_4 \end{pmatrix}. \tag{29}$$

For a 2×2 matrix with a zero on the off-diagonal, the eigenvalues are simply given by the diagonal elements. Again, we obtain an unstable direction associated with the eigenvalue $\lambda_{u_2} = 2$ as in the Gaussian model. Its eigenvalue is given by $(1\ 0)^t$, which means it lies along the $u_4 = 0$ axis. The second eigenvalue $\lambda_{u_4} = 4 - d$ is negative for $d > 4$, i.e., the fixpoint is stable with respect to this direction. This means that depending on where we start in the $u_2 - u_4$ plane, we will always go to a stationary solution corresponding to either $u_2 = -\infty$ or $u_2 = +\infty$. Since the former describes the ordered phase of the ϕ^4 theory while the latter describes the disordered phase, the unstable direction correctly describes the phase transition. Additionally, as the fixpoint is the same as in the Gaussian theory, the critical exponents are the same (i.e., mean-field).

However, this picture is correct only for $d > 4$. For $d < 4$, the second direction is also unstable, meaning that the Gaussian fixpoint cannot describe the phase transition in this case. This is just another manifestation of the fact that $d = 4$ is the upper critical dimension of the ϕ^4 theory, as we have seen earlier in our analysis based on the Ginzburg criterion.

III. THE ϵ EXPANSION

In order to be able to correctly describe the phase transition below four dimensions, we have to go beyond the leading order in the cumulant expansion of Eq. (15). The two-loop expansion for the u_4 term yields [1]

$$\frac{du_4}{dl} = (4-d)u_4 - Bu_4^2, \tag{30}$$

with the coefficient B being given by

$$B = \frac{36 S_d \Lambda^d}{(2\pi)^d (u_2 + v_2\Lambda^2)^2}. \tag{31}$$

Interestingly, this expression results in an additional fixpoint of the RG flow equations for $u_4^* = (4-d)/B$. Close to the new fixpoint, the perturbation theory can now be formulated not in u_4 (which changes under the RG), but in the deviation from the upper critical dimension, i.e., the parameter $\epsilon = 4 - d$ has to be small. Fortunately, this means that if we want to expand our stability analysis to first order in ϵ, we do not have to consider the two-loop expansion for the u_2 term. As a further consequence, we may also calculate all d-dependent terms in the flow equations for $d = 4$, as corrections will correspond to higher order terms in the ϵ expansion. Then, we obtain for the new fixpoint

$$u_2^* = -\frac{3}{18} v_2 \Lambda^2 \epsilon \tag{32}$$

$$u_4^* = \frac{2}{9}\pi^2 v_2^2 \epsilon \tag{33}$$

Linearization close to the fixpoint leads to the diagonal terms $\lambda_{u_2} = 2-(1/3)\epsilon$ and $\lambda_{u_4} = -\epsilon$ [1], i.e, we now have only one unstable direction describing the phase transition. Note also that above the upper critical dimension, this second fixpoint has again two unstable directions since ϵ is negative. In this case, the Gaussian fixpoint is still the only phase transition point in the theory.

Importantly, this second fixpoint is described by a different set of critical exponents than the Gaussian fixpoint, i.e., it corresponds to a new universality class known as the Wilson-Fisher class. One can see that the RG flow of an external field does not change under adding the u_4 term to the Gaussian theory, therefore, the critical exponent ν can still be calculated from the relation $\nu = 1/\lambda_{u_2}$. For $d = 3$, this results in $\nu = 0.6$, which is a significant improvement compared to the mean-field value $\nu_{MF} = 0.5$. Even better estimates can be calculated by systematically going to higher order expansions, e.g., a seven-loop expansion yields $\nu = 0.6304 \pm 0.0013$ [3].

IV. HYPERSCALING RELATIONS

Using a similar strategy as for the correlation length exponent ν, we can also determine the other critical exponents of the theory. For the Wilson-Fisher fixpoint below four dimensions, there are two relevant perturbations, which are increasing under the RG flow: the quadratic term u_2 and the external field h. Importantly, this means that all critical exponents are determined by the associated eigenvalues of the stability analysis, λ_{u_2} and λ_h.

We start by considering the (normalized) free energy density $f = -1/V \ln Z$. Note that we have dropped the explicit temperature dependence as the critical exponents are determined by the singular part. Under the RG flow, f will become [1]

$$f(u_2, h) = -\frac{\ln Z}{V} = -\frac{\ln Z'}{b^d V'} = b^{-d} f(u_2(b), h(b)) = b^{-d} f(b^{\lambda_{u_2}} u_2, b_h^{\lambda} h). \tag{34}$$

As previously, we set the rescaling factor b to $b = u_2^{-1/\lambda_{u_2}}$ such that the first argument becomes constant, which leads to

$$f(u_2, h) = u_2^{d/\lambda_{u_2}} f(1, h/u_2^{\lambda_h/\lambda_{u_2}}) \equiv u_2^{d/\lambda_{u_2}} \chi_f(h/u_2^{\lambda_h/\lambda_{u_2}}), \tag{35}$$

with χ_f being a smooth universal scaling function as before.

The specific heat exponent α is defined according to

$$C = -T \frac{\partial^2 f}{\partial T^2} \sim u_2^{-\alpha}. \tag{36}$$

Taking the derivative $\partial f/\partial T$ produces two singular terms. The first is proportional to $u_2^{1-\alpha}$, while the second is proportional to $u_2^{2-\alpha-\lambda_h/\lambda_{u_2}}$. The second one is subleading compared to the first one since $\lambda_h/\lambda_{u_2} > 1$, so we only retain the latter. Repeating this with the second derivative, we obtian the relation

$$2 - \alpha = \frac{d}{\lambda_{u_2}}. \tag{37}$$

Since we already know that the correlation length exponent $\nu = 1/\lambda_{u_2}$ is also related to λ_{u_2}, we have an equation linking the exponents α and ν,

$$2 - \alpha = d\nu. \tag{38}$$

Such expressions involving the spatial dimension d are called hyperscaling relations [4]. Importantly, the mean-field exponents satisfy this relation only at the upper critical dimension.

The same strategy can also be applied to the order parameter exponent β by inspecting the magnetization

$$m = \frac{1}{V}\frac{\partial \log Z}{\partial h} = u_2^{(d-\lambda_h)/\lambda_{u_2}} \chi_m(h/u_2^{\lambda_h/\lambda_{u_2}}), \tag{39}$$

from which we obtain

$$\beta = \frac{d - \lambda_h}{\lambda_{u_2}}. \tag{40}$$

Taking one further derivative with respect to h leads to the susceptibility exponent γ according to

$$\chi = u_2^{(d-2\lambda_h)/\lambda_{u_2}} \chi_\chi(h/u_2^{\lambda_h/\lambda_{u_2}}), \tag{41}$$

which leads to

$$\gamma = \frac{d - 2\lambda_h}{\lambda_{u_2}}. \tag{42}$$

We can now use Eqs. (37), (40), and (42) to obtain another link between the critical exponents, $\alpha + 2\beta + \gamma = 2$. Similar expression can also be found for the exponents δ and η [1], which completes the set.

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