Lecture 7: Quantum master equations

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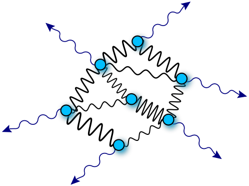
Outline of the course

1. Introduction to Python
2. SciPy/NumPy packages
3. Plotting and fitting
4. QuTiP: states and operators
5. Ground state problems
6. Time evolution and quantum quenches
7. **Quantum master equations**
8. Generation of squeezed states
9. Quantum computing
10. Grover's algorithm and quantum machine learning
11. Student presentations

Open quantum systems

- Quantum systems are never perfectly isolated from the outside world
- Description in terms of pure state vectors no longer holds
- Dynamics no longer described by the unitary Schrödinger equation



The density operator

- Statistical mixture of pure quantum states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- Proability p_i to find the system in the pure state $|\psi_i\rangle$
- Conservation of probability: $\mathrm{Tr}\{\rho\} = 1$, positivity: $\rho \geq 0$.
- Unitary part of the Dynamics: Liouville-von Neumann equation:

$$\frac{d}{dt}\rho = \frac{d}{dt}\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \left[\left(\frac{d}{dt}|\psi_i\rangle\right)\langle\psi_i| + |\psi_i\rangle\left(\frac{d}{dt}\langle\psi_i|\right)\right]$$

$$= -\sum_i p_i \frac{i}{\hbar}\left(H|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|H\right) = -\frac{i}{\hbar}\left[H, \rho\right]$$

Reduced density matrices

- Biparite system (parts A and B)

$$|\psi\rangle = \sum_{ij} c_{ij}|i\rangle_A|j\rangle_B \equiv \sum_{ij} c_{ij}|ij\rangle$$

- Reduced density matrices: partial trace

$$\rho_A = \mathrm{Tr}_B\{\rho\} = \sum_{i,i'}\sum_j \langle ij|\rho|i'j\rangle|i\rangle\langle i'|$$

- Pure states:

$$\rho_A = \sum_{i,i'}\sum_j c_{ij}^* c_{i'j}|i\rangle\langle i'|$$

Example: Bell states

`Qobj.ptrace(list)`
`list` contains the parts that should be *kept*

```
import numpy as np
from qutip import basis, tensor
psi = (tensor(basis(2, 0), basis(2, 0)) +\
    tensor(basis(2, 1), basis(2, 1)))/np.sqrt(2)
print(psi.ptrace([0]))
```

Output:

```
Quantum object: dims = [[2], [2]], shape = [2, 2], type = oper,
Qobj data =
[[ 0.5  0. ]
 [ 0.   0.5]]
```

Entanglement entropy

For pure states, we can use the von Neumann entropy of the reduced density matrices as a measure of entanglement between A and B.

$$S = -\mathsf{Tr}\{\rho_A \log \rho_A\}$$

```
from qutip import entropy_vn
print(entropy_vn(psi))
print(entropy_vn(psi.ptrace([0])))
```

Output:

```
2.220446049250313e-16
0.6931471805599454
```

Quantum master equations

- Interaction with the environment leads to incoherent (i.e., non-unitary) dynamics
- Assumption: The environment has a short memory time (Markovian)
- Well justified in quantum optics
- $\Rightarrow$ Lindblad form of the quantum master equation

$$\frac{d}{dt}\rho = -i[H,\rho] + \sum_{i=1}^{d^2-1} c_i \rho c_i^\dagger - \frac{1}{2}\left\{c_i^\dagger c_i, \rho\right\}$$

- Jump operators c_i can be non-Hermitian (dimension: square root of a rate)

[Breuer, Petruccione: The Theory of Open Quantum Systems, OUP (2007)]

Steady state solution

Steady state:

$$\frac{d}{dt}\rho = 0$$

$$H = \sigma_x \qquad c = \sigma_- = |0\rangle\langle 1|$$

```
from qutip import steadystate, sigmax, destroy
H = sigmax()
c = np.sqrt(2)*destroy(2)
print(steadystate(H, [c]))
```

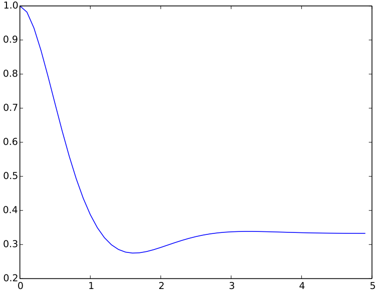
Output:

```
Quantum object: dims = [[2], [2]], shape = [2, 2], type = oper,
Qobj data =
[[ 0.66666667+0.j          0.00000000+0.33333333j]
 [ 0.00000000-0.33333333j  0.33333333+0.j        ]]
```

```
mesolve(H, rho0, tlist, c_opes, e_ops, ...)
```

```
from qutip import mesolve, ket2dm, sigmaz
rho0 = ket2dm(basis(2,0))
tlist = np.arange(0, 5, 0.1)
res = mesolve(H, rho0, tlist, [c], [sigmaz()])
plt.plot(tlist, res.expect[0])
plt.show()
```

Output:



Example: Single atom laser

- Single two-level atom coupled to a single cavity mode

$$H = g(\sigma_- a^\dagger + \sigma_+ a)$$

- Pump term for the atom:

$$c_1 = \sqrt{\gamma_P}\sigma_+$$

- Cavity loss:

$$c_2 = \sqrt{\kappa} a$$