Quantum Physics with Python, Problems set

Dr. Hendrik Weimer, Summer term 2016, 13 June 2016

General rules:

- Choose one of the problems you want to solve.
- You are encouraged to work in small groups, as long as you clearly document the individual contributions to the solution.
- Your solution must consist of a single Python script containing the code and all output files (PDF) generated by the script.
- Send your solutions to Meghana Raghunandan (`Meghana.Raghunandan@itp.uni-hannover.de`) by **4 July 2016** at the latest.
- One student from your group will have to present the solution on the last day of the course (11 July 2016).

1. Ground state of the Heisenberg model

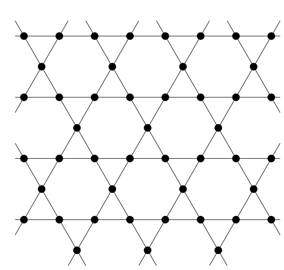
The Heisenberg Hamiltonian has the form

$$H = J\sum_{\langle ij\rangle}\left[\sigma_x^{(i)}\sigma_x^{(j)} + \sigma_y^{(i)}\sigma_y^{(j)} + \sigma_z^{(i)}\sigma_z^{(j)}\right], \tag{1}$$

where $\langle ij\rangle$ denotes that the interaction is restricted to nearest neighbors.

(a) Compute the ground state energy and the energy of the first excited state for a one-dimensional chain with periodic boundary conditions and $J = 1$ (antiferromagnetic interaction). How does the energy difference between these states behave when you increase the number of sites in the problem? What is the energy difference in the case of $J = -1$ (ferromagnetic interactions)?

(b) Consider the ground state energy antiferromagnetic Heisenberg model on the two-dimensional Kagome lattice (depicted on the right). How does the energy per site change when you add more sites to the lattice? How does your result compare to the solution for periodic boundary conditions [Phys. Rev. B **83**, 212401 (2011), arXiv:1103.1159]?



2. Landau-Zener sweeps

The Landau-Zener model is a time-dependent Hamiltonian for two-level systems, given by

$$H = \begin{pmatrix} x(t) & g \\ g & -x(t) \end{pmatrix}. \tag{2}$$

The initial state is the ground state of the problem in the limit of $t \to -\infty$.

(a) Consider a linear function $x(t) = vt$ and compute the instantaneous eigenenergies of the problem as a function of time.

(b) Compute the state of the system for large positive times for different values of v and calculate the probability p_e to find the system in its excited state. Compare with the Landau-Zener formula $p_e = \exp(-\pi/v)$.

(c) Change the time-dependent terms to $x(t) = vt^3$ and $x(t) = \mathrm{sgn}(t)v\sqrt{|t|}$. How does this affect the excited state probability p_e?

(d) Revisit the Ising model of Lecture 5. Starting in one of the ferromagnetic ground states at $g_0 = 0.2$ at $t = 0$, perform a linear ramp $g = vt + g_0$ until the value $g_1 = 5$ is reached. How does the overlap of the final state with the ground state for $g = g_1$ depend on v?

3. The Jaynes-Cummings model

The Jaynes-Cummings model describes the interaction of a two-level atom with a single mode of the radiation field inside a cavity. Its Hamiltonian is given by

$$H = \Delta a^\dagger a + g(\sigma_+ a + \sigma_- a^\dagger), \tag{3}$$

where $a^\dagger$ and a are bosonic creation and annihilation operators and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are spin flip operators. Initially, the atom is in the ground state $|0\rangle$ with $\sigma_-|0\rangle = 0$ and the radiation field is in a coherent state $|\alpha\rangle$.

(a) First, consider the resonant case ($\delta = 0$) and calculate the time evolution of the system, where you should see collapse and revivals of the Rabi oscillations between atom and cavity. How do these collapse and revivals depend on α?

(b) Calculate the entanglement between the atom and the cavity as a function of time. Use the entanglement entropy $S = -\mathrm{Tr}\{\rho_a \log \rho_a\}$ as a measure for entanglement, where ρ_a is the reduced density matrix of the atom.

(c) How do the Rabi oscillations change when δ, the detuning between atom and cavity, is nonzero? What happens to the entanglement in the system?

(d) Consider the case where the cavity is lossy, i.e., there are incoherent processes described by the jump operator $c = \sqrt{\kappa}a$, where κ is the cavity loss rate. Calculate the time evolution by solving the quantum master equation. How does the steady state change when you add an additional driving term $H' = \Omega\sigma_x$ to the Hamiltonian?

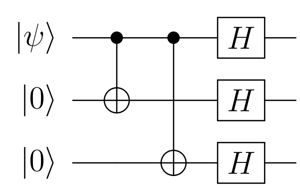
4. Quantum error correction

Consider a qubit initially prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \exp(i\phi)|1\rangle). \tag{4}$$

(a) Unavoidable couplings to the environment generically lead to decoherence and therefore lead to a corruption of the qubit state. You can model this decoherence by arandom gate performing a rotation around the z axis. The angle of the random rotation can be chosen as a Gaussian random number with zero mean and a variance of λ. Gaussian random numbers can be created using the `random.normal` function provided by NumPy. Study the overlap of the final state (after applying the random gate) with the initial state as a function of the decoherence parameter λ. Since you are studying a random process, you have to average over several realizations of the process. Do the results depend on the value of ϕ?

(b) Quantum error correction is a way to preserve quantum information even in the presence of decoherence. A simple quantum error correction encoding scheme is shown below:



Here, $|0\rangle$ means that the two auxiliary qubits are initialized in $|0\rangle$. The decoding scheme is the inverse of the encoding scheme. Finally, the actual error correction is performed by a single Toffoli gate with the two auxiliary qubits as the control qubits and the original qubit as the target.

First, assume that the quantum gates for encoding, decoding, and error correction are perfect. How does the overlap between the final (decoded and corrected) state and the initial state change compared to the case without error correction?

(c) Now, consider the case where the quantum gates for encoding and decoding are subject to decoherence as well. You can model this process by applying random z rotations on all of the qubits after each quantum gate. Now have a look at the overlap between the initial state and the final state again. Is there a critical value for λ, where you gain more from the quantum error correction than you lose from the additional overhead in implementing the additional quantum gates?