Conclusions from the Algebraic Poincaré Lemma

Norbert Dragon

Maximilian Kreuzer
Memorial Conference
Wien
26.06.2011

Variational Derivatives and Noether Currents

Gravitational Energy Momentum Complex

Crucial (Kreuzer) Lemma

Variational Derivatives and Currents

$$\frac{\delta W_{\text{matter}}}{\delta A_m} = j^m{}_{\text{electromagnetic}} = j^m_{\text{Noether, U(1)}}$$

$$\frac{\delta W_{\text{matter}}}{\delta g_{mn}} = -\frac{1}{2}\mathcal{T}^{mn}{}_{EnergyMomentum} = j^{mn}{}_{\text{Noether, Translation}}$$

Why?

Why does a Chern-Simons mass in three dimensions not contribute to the energy momentum tensor?

Jet-Space

Fields $\psi \in \mathcal{F}$ are differentiable maps

$$\psi : \begin{cases} \mathbb{R}^4 & \to & \mathbb{R}^n \\ x & \mapsto & \psi(x) \end{cases}$$

and define a corresponding map $\hat{\psi}$, the prolongation to the jet-space $\mathcal{J}_1 = \mathbb{R}^{4+n+4n}$

$$\hat{\psi} : \begin{cases} \mathbb{R}^4 & \to & \mathbb{R}^{4+n+4n} \\ x & \mapsto & \hat{\psi}(x) = (x, \psi(x), \partial\psi(x)) \end{cases}$$

Higher Jet-space $\mathcal{J}_k = \{\text{Base, Target, derivatives up to } k^{\text{th}} order\} = \pi_k \mathcal{J}_\infty$

Local Action

The Lagrangian density $\mathcal{L}$ is a function of jet-space

$$\mathcal{L} : \begin{cases} \mathbb{R}^{4+n+4n} & \to \mathbb{R} \\ (x, \psi, \partial\psi) & \mapsto \mathcal{L}(x, \psi, \partial\psi) \end{cases}$$

which gives rise to the local action S

$$S : \begin{cases} \mathcal{F} & \to \mathbb{R} \\ \psi & \mapsto S[\psi] = \int d^4x \, (\mathcal{L} \circ \hat{\psi})(x) \end{cases} .$$

The equations of motion state that the Euler-derivative of the Lagrangian vanishes for physical fields

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_m \frac{\partial\mathcal{L}}{\partial\partial_m\psi} , \quad \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} \circ \hat{\psi}_{\text{phys}} = 0 .$$

Noether Theorem

Infinitesimal transformations $\delta\psi$ are maps of the jet-space to $\mathbb{R}^n$

$$\delta\psi : \begin{cases} \mathbb{R}^{4+n+4n} & \to \quad \mathbb{R}^n \\ (x, \psi, \partial\psi) & \mapsto \quad \delta\psi(x, \psi, \partial\psi) \end{cases}$$

They are infinitesimal symmetries of the action $S \Leftrightarrow \exists K^m :$

$$\delta\psi \frac{\partial\mathcal{L}}{\partial\psi} + (\partial_m \delta\psi) \frac{\partial\mathcal{L}}{\partial\partial_m\psi} + \partial_m K^m = 0 \quad \Leftrightarrow$$

$$\delta\psi \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} + \partial_m j^m = 0$$

$$j^m = \delta\psi \frac{\partial\mathcal{L}}{\partial\partial_m\psi} + K^m + \partial_n B^{mn} \; , \; B^{mn} = -B^{nm} \; .$$

To each infinitesimal symmetry of the action there corresponds a conserved current and *vice versa*.

Second Noether Theorem

Infinitesimal gauge symmetry, if $\delta\psi$ is linear in an arbitrary function ξ

$$\delta\psi = \xi R + (\partial_m \xi) R^m \, .$$

To each infinitesimal gauge symmetry of the action there corresponds an identity among the Euler derivatives of the Lagrangian and *vice versa*.

$$\frac{\hat{\partial}}{\hat{\partial}\xi}\left(\delta\psi \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi}\right) = R\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} - \partial_m\big(R^m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi}\big) = 0 \, ,$$

e.g. electromagnetism $\partial_m(\partial_n F^{mn}) = 0$, gravity $D_m G^{mn} = 0$.

$$\big(\xi R + (\partial_m \xi) R^m\big)\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} = \partial_m\big(\xi R^m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi}\big)$$

Frozen Gauge Theories

$$\partial_m(j^m + \xi R^m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi}) = 0$$

$$\delta\phi_i \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_m \phi_i} + \delta A_k \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_m A_k} + K^m + \partial_n B^{nm}$$
$$= -\xi (R^m_{\phi_i} \frac{\hat{\partial}\mathcal{L}_{\text{matter}}}{\hat{\partial}\phi_i} + R^m_{A_k} \frac{\hat{\partial}\mathcal{L}_{\text{matter}}}{\hat{\partial} A_k}) \ .$$

If the matter fields satisfy their equations of motion $\frac{\hat{\partial}\mathcal{L}_{\text{matter}}}{\hat{\partial}\phi} \circ \hat{\phi}_{\text{phys}} = 0$ and for background gauge fields with infinitesimal symmetries, $\delta A_k = 0$,

$$j^m{}_{\text{rigid}} = \delta\phi_i \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_m \phi_i} + K^m + \partial_n B^{mn} = -\xi R^m_{A_k} \frac{\hat{\partial}\mathcal{L}_{\text{matter}}}{\hat{\partial} A_k} \ .$$

In frozen gauge theories the Noether current is, up to improvement terms, the variational derivative with respect to the gauge field.

Example

In gravitational theories, the gauge field is the metric g_{mn} and transforms as

$$\delta g_{kl} = \xi^n \partial_n g_{kl} + \partial_k \xi^n g_{nl} + \partial_l \xi^n g_{kn} = \xi^n \partial_n g_{kl} + R_{n\, g_{kl}}^{m} \partial_m \xi^n \ ,$$

$$R_{n\, g_{kl}}^{m} = \delta_k^m g_{nl} + \delta_l^m g_{nk} \ .$$

The current, corresponding to infinitesimal isometries ξ of $g_{mn} = \eta_{mn}$, is the energy-momentum tensor contracted with the Killing field

$$j^m_{\big|_{\phi_{\text{phys}},\, g_{mn}=\eta_{mn}}} = \mathcal{T}^{mn} \xi_n \ , \quad \mathcal{T}^{mn} = -\frac{1}{2} \frac{\hat{\partial} \mathcal{L}_{\text{matter}}}{\hat{\partial} g_{mn}} \ .$$

Charges from Boundary Values

$$\partial_m (j^m + \xi R^m \frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi}) = 0 \ ,$$

$$Q = \int_{x^0=\text{const}} d^{D-1}x \, j^0 = \int_{x^0=\text{const}} d^{D-1}x \, \partial_i B^{0i} = \int_{\partial V} d^{D-2}x \, n_i B^{0i} \ .$$

In gauge theories, charge is determined by boundary values.

B^{0i} unique up to $\partial_k C^{0ik}$.

Gravitational Energy Momentum Complex

Gravitational energy density cannot be covariant, because gravitational effects can be made to vanish along each chosen geodesic.

$$D_m \mathfrak{T}^{mn} = 0$$

Violation of the conservation of energy and momentum of matter by gravitational effects (by exchange?).

Do the Einstein equations contain exact conservation laws?

$$C^{mn} = f(\mathfrak{g})\, G^{mn} - \kappa\, t^{mn}\,,\ \partial_m C^{mn} = 0\ ,\ \mathfrak{g} = |\det g_{..}|\ .$$

$$C^{mn} = C^{nm} \text{ and } \partial_m C^{mn} = 0 \Leftrightarrow C^{mn} = \partial_k \partial_l X^{kmln}\ ,$$

$$X^{klmn} = -X^{lkmn} = -X^{klnm}\,,\ X^{klmn} + X^{lmkn} + X^{mkln} = 0\,.$$

Noncovariant identity which holds in all coordinate systems.

Proof of the Symmetric Identity

$C^{mn} = \partial_k B^{kmn}$ for all n, with $B^{kmn} = -B^{mkn}$. $C^{mn} = C^{nm}$ implies

$$\partial_k (B^{kmn} - B^{knm}) = 0 \ ,$$

so for each pair $m\,n$

$$B^{kmn} - B^{knm} = -2\partial_l A^{lkmn} \ , \ A^{lkmn} = -A^{klmn} = -A^{lknm} \ .$$

Solve for B^{kmn},

$$B^{kmn} = -\partial_l (A^{lkmn} + A^{lmnk} - A^{lnkm}) \ .$$

For $C^{mn} = \partial_k B^{kmn}$ this means

$$C^{mn} = \partial_k \partial_l (A^{lmkn} + A^{knlm}) = \partial_k \partial_l X^{lmkn} \ .$$

Determination of X^{rmsn}

$$f(\mathfrak{g})\, G^{mn} - \kappa\, t^{mn} = \partial_r \partial_s X^{rmsn}\,.$$

l.h.s.: two derivatives $\to$ X^{rmsn} no derivatives

$$X^{rmsn} = h(\mathfrak{g}) \left(g^{rs} g^{mn} - g^{rn} g^{ms}\right)\,.$$

Comparison of $\partial\partial g$-terms: $h = -f/2$ and $\mathfrak{g}\frac{dh}{d\mathfrak{g}} = h$,
i.e. f and h homogeneous of degree 1. Using $\gamma^{mn} = \sqrt{\mathfrak{g}}\, g^{mn}$, one has

$$\mathfrak{g}\, G^{mn} = -\frac{1}{2} \partial_r \partial_s \left(\gamma^{rs}\gamma^{mn} - \gamma^{rn}\gamma^{ms}\right) + \kappa\, t^{mn}\ ,$$

$$\partial_r \partial_s (\gamma^{rs}\gamma^{mn} - \gamma^{rn}\gamma^{ms}) = 2\kappa \left(\mathfrak{g}\, T^{mn} + t^{mn}\right)\ .$$

Landau Lifschitz Energy Momentum Complex

$$
\begin{aligned}
2\kappa\, t^{mn} &= 2\,\mathfrak{g}\, G^{mn} + \partial_r\partial_s\big(\gamma^{rs}\gamma^{mn} - \gamma^{rn}\gamma^{ms}\big) = \\
&= \partial_k\gamma^{mn}\partial_l\gamma^{kl} - \partial_k\gamma^{km}\partial_l\gamma^{ln} + \frac{1}{2}g^{mn}g_{kl}\partial_r\gamma^{ks}\partial_s\gamma^{lr} - \\
&\; - g_{kl}g^{mr}\partial_r\gamma^{ks}\partial_s\gamma^{ln} - g_{kl}g^{nr}\partial_r\gamma^{ks}\partial_s\gamma^{lm} + g_{kl}g^{rs}\partial_r\gamma^{mk}\partial_s\gamma^{nl} + \\
&\; + \frac{1}{8}\big(2g^{ml}g^{nk} - g^{mn}g^{kl}\big)\big(2g_{pr}g_{qs} - g_{pq}g_{rs}\big)\,\partial_k\gamma^{pq}\partial_l\gamma^{rs}\,.
\end{aligned}
$$

t^{00} not definite. Static gravitational energy density is negative (Newton).
Gravitational Waves in flat space-time to be investigated.

Transformation of the Energy Momentum Complex

$$
\begin{aligned}
&2\kappa\, t'^{\,mn}(x'(x)) = \\
&\quad \left(\det\frac{\partial x}{\partial x'}\right)^2 \frac{\partial x'^{\,m}}{\partial x^t}\frac{\partial x'^{\,n}}{\partial x^u}\, 2\kappa\, t^{tu}(x) + \\
&\quad + \partial_r\partial_s\left(\left(\det\frac{\partial x}{\partial x'}\right)^2 \frac{\partial x'^{\,m}}{\partial x^t}\frac{\partial x'^{\,n}}{\partial x^u}\right)\left(\gamma^{rs}\gamma^{tu} - \gamma^{ru}\gamma^{ts}\right) + \\
&\quad + \partial_s\left(\left(\det\frac{\partial x}{\partial x'}\right)^2 \frac{\partial x'^{\,m}}{\partial x^t}\frac{\partial x'^{\,n}}{\partial x^u}\right)\partial_r\left(2\gamma^{rs}\gamma^{tu} - \gamma^{ru}\gamma^{ts} - \gamma^{su}\gamma^{tr}\right) + \\
&\quad + \partial_s\left(\left(\det\frac{\partial x}{\partial x'}\right)^2 \frac{\partial x'^{\,m}}{\partial x^t}\frac{\partial x'^{\,n}}{\partial x^u}\right)\frac{\partial x^l}{\partial x'^{\,k}}\frac{\partial^2 x'^{\,k}}{\partial x^r \partial x^l}\left(2\gamma^{rs}\gamma^{tu} - \gamma^{ru}\gamma^{ts} - \gamma^{su}\gamma^{tr}\right) + \\
&\quad + \frac{\partial x^k}{\partial x'^{\,l}}\partial_k\left(\left(\det\frac{\partial x}{\partial x'}\right)^2 \frac{\partial x'^{\,m}}{\partial x^t}\frac{\partial x'^{\,n}}{\partial x^u}\right)\frac{\partial^2 x'^{\,l}}{\partial x^s \partial x^r}\left(\gamma^{rs}\gamma^{tu} - \gamma^{ru}\gamma^{ts}\right) \ .
\end{aligned}
$$

Conservation Laws

$$\partial_r \partial_s (\gamma^{rs}\gamma^{mn} - \gamma^{rn}\gamma^{ms}) = 2\kappa \left(\mathfrak{g}\, T^{mn} + t^{mn}\right) \ ,$$

$$\partial_m \left(\mathfrak{g}\, T^{mn} + t^{mn}\right) = 0 \ .$$

Conserved charges (ADM-mass)

$$P^m = \int d^3x \left(\mathfrak{g}\, T^{0m} + t^{0m}\right)$$

are determined by boundary values

$$P^m(V) = \frac{1}{2\kappa}\int_V d^3x\, \partial_r \partial_s \left(\gamma^{rs}\gamma^{m0} - \gamma^{r0}\gamma^{ms}\right) = \frac{1}{2\kappa}\int_{\partial V} d^2 f_i\, \partial_r \left(\gamma^{ri}\gamma^{m0} - \gamma^{r0}\gamma^{mi}\right) \ .$$

Corresponding symmetries: $\delta_n g_{kl} = \mathcal{L}_{v^n} g_{kl}$, $v^n = \gamma^{nl}\partial_l$.

Crucial (Kreuzer) Lemma

Let Ω and η denote invariant differential forms in jet-spaces with star-shaped base and target manifold, which depend on tensors only

$$d\Omega = 0 \Leftrightarrow \Omega = \mathcal{L} d^D x + P(F) + d\eta \ .$$

$$\text{P invariant polynomial in } F = \tfrac{1}{2} dx^m dx^n (\partial_m A_n - \partial_n A_m - [A_m, A_n]) \ ,$$

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\psi} = 0 \Leftrightarrow \mathcal{L} = P(F) + d\eta \ ,$$

$$P(F) + d\eta = 0 \Leftrightarrow P(F) = 0 = d\eta \ .$$

Proof to long to be given here: Part of a series of lectures (available).

Enumerates *all* topological densities, e.g. $D = 4$, gravity,

$$\sqrt{g} R_{klmn} R_{rstu} \epsilon^{mnrs} \epsilon^{klst} \ , \quad R_{mnr}{}^{s} R_{kls}{}^{r} \epsilon^{mnkl} \ .$$

Conerstone to determination of all anomaly candidates.