Insights into Relativistic Quantum Physics

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13.10.2020

Insights imply Corrections

The algebra of the Covariant String disallows physical states.

The algebra of the Lightcone String has no domain. It does not exist.

Concrete Contradictions: $0 = \mathrm{i}$, massless $\mathrm{SO}(\mathrm{D}-1)$ multiplets.

Interaction leaves all Lorentz generators unchanged.

Interaction changes the invariant mass: $M' = \Omega_\pm M \Omega_\pm^{-1}$, $[M', M] \neq 0$

Wightman distributions are insensitive to interactions.

Hilbert space does not allow for a massless vector field.

Antiparticles not by $\mathcal{CPT}$ but by internal symmetries and local fields.

The Covariant String has no Physical States

$$[P^m, P^n] = 0 = [X_m, X_n]\ ,\ [P^n, X_m] = i\delta^n{}_m\ ,\ e^{iP\cdot a} X_m e^{-iP\cdot a} = X_m - a_m$$

Stone-von Neumann Theorem (1931)

$$\big(P^m \Psi\big)(p) = p^m \Psi(p)\ ,\ \big(X_n \Psi\big)(p) = -i\frac{\partial}{\partial p^n}\Psi(p)\ ,$$

$$\langle \Psi | \Phi \rangle = \int d^D p\, \Psi^*(p) \Phi(p)\ .$$

Algebra acts on Schwartz space $\mathcal{S}(\mathbb{R}^D) \subset \mathfrak{L}^2(\mathbb{R}^D)$, $|p^\alpha \partial_\beta \Psi(p)| < C_{\alpha,\beta,\Psi}$.

Spectrum of $P^m P^n \eta_{mn}$ purely continuous, no eigenvalues.

Constraints for Physical One-Particle States: $\big(P^m P^n \eta_{mn} - M^2(N)\big)\Psi_{\text{phys}} \overset{?}{=} 0$

Restricts Support of Ψ_{phys} to have D-dimensional measure zero: $\Psi_{\text{phys}} = 0$

The covariant string has no physical states.

Schrödinger eq. determines not states but paths in $\mathcal{H}$, ∂_t not an operator in $\mathcal{H}$.

Devastating Contradiction

$$\Psi \in \mathcal{H}_{\text{phys}}: \ 0 - 0 = \langle \Psi | P^m X_n (P^2 - M^2) \Psi \rangle - \langle (P^2 - M^2) \Psi | P^m X_n \Psi \rangle$$

$$= -2i \, \langle \Psi | P^m P_n \Psi \rangle = -2i \, \langle P^m \Psi | P_n \Psi \rangle$$

$$\langle P^m \Psi | P^n \Psi \rangle = 0 \ \Rightarrow |P^m \Psi| = 0 \ , \ \text{Hilbert space} \ \Rightarrow P^m \Psi = 0$$

$$0 - 0 = \langle \Psi | X_n P^m \Psi \rangle - \langle P^m \Psi | X_n \Psi \rangle$$

$$= -i \delta^m{}_n \, \langle \Psi | \Psi \rangle \, , \ \text{Hilbert space}$$

$$0 = i$$

$$\mathcal{H}_{\text{phys}} \subset \mathcal{F}_{\text{string}} = \mathcal{H}_{\text{Heisenberg}} \otimes \mathcal{V}_{\text{indefinite}} \ \text{no help.}$$

Culprit X^0, $\vec{X}$ known to exist for massive states.

Can a community of theorists err for a long time?

Acceptance does not Prove Consistency or Usefulness

Historic Example: Ptolemaic System

Shared by the whole scientific community

Blocked progress for 2000 years by misleading intuition

Not even wrong in General Relativity

The Heisenberg Picture gives no intuitive understanding of scattering.

Does the Lightcone Gauge show the Consistency of String Theory?

The Hare and the Hedgehog

Gårding Space

Gårding space of group-smoothened states (f smooth, of compact support)

$$\Psi_f = \int_G d\mu_g \, f(g) \, U_g \Psi \ , \quad U_g \Psi_f = \Psi_{f \circ g^{-1}} \ , \text{ smooth}$$

requires $U : g \mapsto U_g \Psi$ measurable $\forall \Psi$ i.e. U_g strongly measurable (very weak)

Gårding states Ψ_f are smooth objects of differential geometry.

Smoothness is a property of Ψ, not encoded in its arguments $|p_1, s_1 \ldots p_n, s_n\rangle$

The Gårding space of a unitary representation of a Lie group is dense.

It is the domain of the algebra of the generators and invariant under all U_g.

Sweet poisoned gift: justifies algebraic calculations but disallows poles.

The Algebra of the Lightcone String has no Domain

$$[P^+, X^-] = i \ , \ M^2 = P^+ P^- - \vec{P}_T^2 \ , \ P^- = \frac{m^2 + \vec{P}_T^2}{P^+}$$

Dubious: P^- not smooth, $P^- \mathcal{S}(\mathbb{R}^{D-1}) \not\subseteq \mathcal{S}(\mathbb{R}^{D-1})$

$$V_b = e^{-iX^- b} \ , \quad V_b \Psi = \Psi \circ T_{-b} \ , \ T_b(p_T, p^+) = (p_T, p^+ + b) \ ,$$

$$\Psi \neq 0 \to |(m^2 + p_T^2)\Psi(p_T, b)| > \epsilon \ \text{ a.e. in some } \mathcal{U}' = T_b \mathcal{U}$$

$$\left\langle P^- V_b \Psi \middle| P^- V_b \Psi \right\rangle > \int_{\mathcal{U}} dp^+ d^{D-2} p \, \frac{\epsilon^2}{(p^+)^2} = \infty$$

There is no common, invariant domain of P^- and the generator X^- of V_b.

As the algebra of the lightcone string has no domain, it does not exist.

The calculation of $D = 26$ is as meaningless as continuous helicity.

Though the points with $P^+ = 0$ have measure 0, the singularity matters.

Concrete Contradiction: Massless SO$(D-1)$ Multiplets

If the lightcone string represented rotations with generators M_{ij},
then $\vec{X} = (X_1, \dots X_{D-1})$ existed together with $\vec{P} = (P^1, \dots P^{D-1})$.

They are Heisenberg pairs and vectors under both M_{ij} and $L_{ij} = X_i P_j - X_j P_i$.

So $S_{ij} = M_{ij} - L_{ij}$ commutes with X_i and P_j and therefore with L_{ij}

L_{ij} and $M_{ij} = L_{ij} + S_{ij}$ satisfy the angular momentum algebra of SO$(D-1)$

$\Rightarrow S_{ij}$ satisfies the angular momentum algebra of SO$(D-1)$.

Massless states of the lightcone string are only helicity multiplets of SO$(D-2)$!

The massive and the massless shells are not homeomorphic.
No excitations $(\alpha\Psi)(p) = \Psi'(f(p))$, $f : \mathcal{M}_0 \stackrel{?}{\leftrightarrow} \mathcal{M}_m$

Wigner's Conjecture: Massless spin multiplets have no position operator $\vec{X}$.
Proof: Lorentz generators not smooth at $\vec{p} = 0$, breaks translational invariance.

The Lorentz Generators

Generators of Poincaré group $\mathfrak{P}$ on wave functions Ψ of $u = p/m$, $m > 0$,

$$(P^n\Psi)(u) = u^n\, M\Psi(u)\ ,\ M\Psi = m\Psi\ ,\ u^2 = 1\ ,$$
$$\big(-iM_{ij}\Psi\big)(u) = -\big(u^i\partial_{u^j} - u^j\partial_{u^i}\big)\Psi(u) + \Gamma_{ij}\Psi(u)\ ,$$
$$\big(-iM_{0i}\Psi\big)(u) = \sqrt{1+\vec{u}^2}\,\partial_{u^i}\Psi(u) + \Gamma_{ij}\frac{u^j}{1+\sqrt{1+\vec{u}^2}}\Psi(u)\ .$$

Irreducible if m, s discrete, purely continuous M-spectrum for scattering states

Interacting representation of Poincaré transformations: dispensable

$$M_{mn} \neq X_m P_n - X_n P_m + \Gamma_{mn}(\alpha)\ \text{(Covariant String)}$$

Algebra acts on Schwartz space $\mathcal{S}(H^3) \subset \mathfrak{L}^2(H^3)$, Γ_{ij} spin matrices $\mathfrak{so}(D-1)$

Massless Representation $p^0 = |\vec{p}| > 0$

$\mathcal{M}_0 = S^2 \times \mathbb{R} = \mathbb{R}^3 - \{0\}$, stabilizer E(2) $\neq$SO(3), no exercise for students.

States: not functions but sections of a complex line bundle over $\mathcal{M}_0$,h helicity,

northern patch $\mathcal{U}_N = \{ p : p^0 = |\,\vec{p}|\,,\, p_z > -p^0 \}$, $\mathcal{A}_- = \{ p = |\,\vec{p}|(1, 0, 0, -1) \}$

$$\begin{aligned}
\big(-iM_{12}\Psi\big)_N(p) &= -\big(p_x\partial_{p_y} - p_y\partial_{p_x}\big)\Psi_N(p) - i\,h\,\Psi_N(p)\ , \\
\big(-iM_{31}\Psi\big)_N(p) &= -\big(p_z\partial_{p_x} - p_x\partial_{p_z}\big)\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ , \\
\big(-iM_{32}\Psi\big)_N(p) &= -\big(p_z\partial_{p_y} - p_y\partial_{p_z}\big)\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ , \\
\big(-iM_{01}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_x}\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ , \\
\big(-iM_{02}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_y}\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ , \\
\big(-iM_{03}\Psi\big)_N(p) &= |\vec{p}|\partial_{p_z}\Psi_N(p)\ .
\end{aligned}$$

Southern Section

Outside $\mathcal{A}_-$ the generators M_{kl} satisfy the Lorentz algebra for each value of h.

Ψ_N not smooth and nonvanishing at $\mathcal{A}_-$ else $\langle M_{13}\Psi_N | M_{13}\Psi_N\rangle = \infty$.

$$\Psi_S(p) = e^{2ih\varphi(p)}\Psi_N(p)\ ,\ e^{i\varphi(p)} = (p_x + ip_y)/\sqrt{p_x^2 + p_y^2}\ ,\ 2h \in \mathbb{Z}$$

$$\begin{aligned}
\bigl(-iM_{12}\Psi\bigr)_S(p) &= -\bigl(p_x\partial_{p_y} - p_y\partial_{p_x}\bigr)\Psi_S(p) + i\,h\,\Psi_S(p)\ ,\\
\bigl(-iM_{31}\Psi\bigr)_S(p) &= -\bigl(p_z\partial_{p_x} - p_x\partial_{p_z}\bigr)\Psi_S(p) - i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\bigl(-iM_{32}\Psi\bigr)_S(p) &= -\bigl(p_z\partial_{p_y} - p_y\partial_{p_z}\bigr)\Psi_S(p) + i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\bigl(-iM_{01}\Psi\bigr)_S(p) &= |\vec{p}|\partial_{p_x}\Psi_S(p) + i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\bigl(-iM_{02}\Psi\bigr)_S(p) &= |\vec{p}|\partial_{p_y}\Psi_S(p) - i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\bigl(-iM_{03}\Psi\bigr)_S(p) &= |\vec{p}|\partial_{p_z}\Psi_S(p)\ .
\end{aligned}$$

Noncommutative Geometry

$$D_i = \partial_{p^i} + A_i - \frac{p^i}{2|\vec{p}|^2}$$

$$\vec{A}_N(p) = \frac{i\,h}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix} \ , \ \vec{A}_S(p) = \frac{i\,h}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix}$$

$$[P^i, P^j] = 0 \ , \ [P^i, D_j] = -\delta^i{}_j \ , \ \ [D_i, D_j] = F_{ij} = \partial_{p^i} A_j - \partial_{p^j} A_i = i\,h\,\epsilon_{ijk} \frac{P^k}{|\vec{P}|^3}$$

Momentum space geometry of massless particles with $h \neq 0$ is noncommutative.

$$-iM_{ij} = -(P^i D_j - P^j D_i) - i\,h\,\epsilon_{ijk} \frac{P^k}{|\vec{P}|} \ , \ -iM_{0i} = |\vec{P}|^{1/2} D_i |\vec{P}|^{1/2}$$

Lorentz algebra $\forall h$, skew hermitian only if $h \in 2\mathbb{Z}$, $|p^\alpha \partial_\beta \Psi(p)|/|\vec{p}|^n < C_{\alpha,\beta,n,\Psi}$

Momentum Space Monopole: $\dfrac{1}{4\pi} \displaystyle\int_{S^2} \mathbb{F} = i\,h$

Goethe

Wer das erste Knopfloch verfehlt, kommt mit dem Zuknöpfen nicht zu Rande.

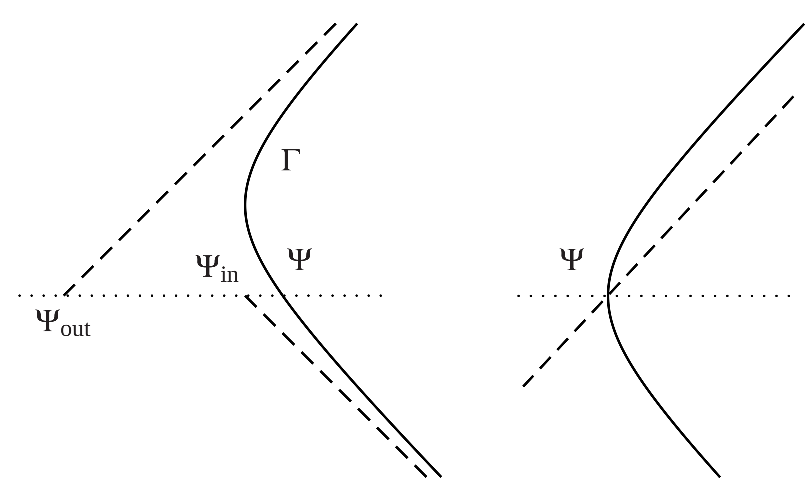
(If you start to button your shirt wrong you will not end correct.)

Naive (Reed Simon) Scattering: N-Particle States, $N \geq 2$

Product rule $\Leftrightarrow$ free time evolution: $(P^m \Psi)^{ij}(p_1, p_2) = (p_1^m + p_2^m)\,\Psi^{ij}(p_1, p_2)$

Unitary group of motion has no limit! At best: $\lim_{t\to\infty}\big(U(t)\Psi_{out} - U'(t)\Psi\big) = 0\,.$

$$\Psi_{out} = \lim_{t\to\infty} U(-t)U'(t)\Psi = \Omega_- \Psi\ ,\ \ \Omega_\pm(H, H') = \underset{t\to\mp\infty}{\text{s-lim}}\, e^{iHt}\, e^{-iH't}$$



$$S = \Omega_+ \Omega_-^{-1} = \underset{t,t'\to\infty}{\text{s-lim}}\, e^{-iHt} e^{iH'(t+t')} e^{-iHt'}\text{, in product basis} \sim \text{cross section}$$

$H\Omega_\pm = \Omega_\pm H'$, $[H, H'] \neq 0$, H' not reconstructible from Wightman distributions

Clarifying Language

Not the states but their time evolution is interacting (Schrödinger picture).

World lines of states do not constitute a Hilbert space.

In, out and interacting are relations not properties of states. Ψ_{out} is the *initial* state of the future asymptote of Γ_Ψ not a state with all momenta outwards.

$\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}_{N\geq 2}$

States are just states, they determine probabilities of results.

Interacting fields, interacting representation, . . . dispensable concepts.

Transformations U_g transform states and consequently all paths *irrespective of their time evolution*: Generators M_{mn}, P^n are independent of the coupling.

Claim: $H' = u^0 M'$, $P^n = u^n M$, $M = \sqrt{P^2}$, $[u^n, M'] = 0$, $[M', U_\Lambda] = 0$

rather than $P'^{\,i} = P^i$, $P'^{\,0} \neq P^0$, $M'_{ij} = M_{ij}$, $M'_{0i} \neq M_{0i}$.

Axiomatic Quantum Field Theory (Haag's Theorem)? Canonical Quantisation?

Center Variables

$$(p_1, \ldots p_n) \leftrightarrow (u, q_1, \ldots q_n) : \sum_i q_i = 0$$

$$u = \frac{\sum_i p_i}{m} \ , \ m^2 = (\sum_i p_i)^2 \ , \ u^2 = 1 \ ,$$

$$p_{i\,\|} = (p_i \cdot u)\, u \ , \ p_{i\,\perp} = p_i - p_{i\,\|} \ , \ q_i = (L_u)^{-1} p_{i\,\perp} \in \mathbb{R}^3 \ ,$$

$$p_i = \sqrt{m_i^2 + \vec{q}_i^{\,2}}\, u + L_u\, q_i = \sqrt{m_i^2 + \vec{q}_i^{\,2}} \begin{pmatrix} \sqrt{1 + \vec{u}^2} \\ \vec{u} \end{pmatrix} + \begin{pmatrix} \vec{u} \cdot \vec{q}_i \\ \vec{q}_i + \frac{(\vec{u} \cdot \vec{q}_i)\, \vec{u}}{1 + \sqrt{1 + \vec{u}^2}} \end{pmatrix}$$

$$P^m = u^m M \ , \ M = \sum_i \sqrt{m_i^2 + (\vec{q}_i)^2}$$

$$\sqrt{1 + \vec{u}^2}\Big(\sqrt{m_1^2 + \vec{q}^{\,2}} + \sqrt{m_2^2 + \vec{q}^{\,2}}\Big) = m_1 + m_2 + \frac{(m_1 + m_2)\vec{u}^2}{2} + \frac{\vec{q}^{\,2}}{2\mu} + \ldots$$

$\Lambda : (u, q) \rightarrow (\Lambda u, W(\Lambda, u) q)$, Wigner rotation $W(\Lambda, u) = L_{\Lambda u}^{-1} \Lambda L_u \in \mathrm{SO}(3)$

Factorisation of Momentum

$$\lim_{t\to\infty}\left(e^{i\sqrt{1+\vec{u}^2}M\,t}e^{-iH't}\Psi\right)(u,q)=\lim_{t'\to\infty}\left(e^{iM\,t'}e^{-iH't'/\sqrt{1+\vec{u}^2}}\Psi\right)(u,q)$$

$H'/\sqrt{1+\vec{u}^2}$ hermitian, if $[H',u^0]=0 \Rightarrow H'=u^0M' \Rightarrow P'^m=u^mM'$

$$\Omega_\pm(H,H')=\underset{t\to\mp\infty}{\text{s-lim}}\,e^{i\sqrt{1+\vec{u}^2}M\,t}e^{-i\sqrt{1+\vec{u}^2}M't}=\underset{t\to\mp\infty}{\text{s-lim}}\,e^{iM\,t}e^{-iM't}=\Omega_\pm(M,M')$$

$M'=M+V$, $[V,M]\neq 0$, V translational and rotational invariant ($\sum q_i=0$),

Invariance Principle: $f\ \nearrow\ \Rightarrow \Omega_\pm(M,M')=\Omega_\pm(f(M),f(M'))$.

Relativistic and Nonrelativistic scattering and binding analogous.

Eigenvalues and eigenstates of M and M' have to coincide (adapt M to coupling).

Scattering states belong to the continuous spectrum of H and H'.

Each hermitian H is completely determined by its spectrum (up to a basis).

Suitable pairs H,H' with the same spectra define $\Omega_\pm(H,H')$.

Further Results

Particles not localisable to the forward and backward cone of a compact domain.

Boosts transform states ($\neq$ fields) nonlocally by convolution.

Dirac field is not a position wave function.

$\Psi = \int \mathrm{d}^4x\, f(x)\, \Phi(x)\Omega$, $\Phi(x)$ a local field, are not localised to support of f.

Hilbert space does not allow for a massless vector field:

$$\langle \mathrm{T}\, \Phi^m(x)\, \Phi^n(0)\rangle = -\Big(\eta^{mn} + \frac{\partial^m\partial^n}{m^2}\Big) \lim_{\epsilon\to 0+} \mathrm{i} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{\mathrm{i}px}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

The $1/m^2$-term is not canceled by adding a gradient, $\Phi'^{\,m} = \Phi^m + a\partial^m\phi$,

$$\langle \mathrm{T}\, \Phi'^{\,m}(x)\, \Phi'^{\,n}(0)\rangle = -\Big(\eta^{mn} + (1+a^2)\frac{\partial^m\partial^n}{m^2}\Big) \lim_{\epsilon\to 0+} \mathrm{i} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{\mathrm{i}px}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

BRST-Symmetry no luxury of a massless vector field but a necessity

Continued List

Antiparticles coincide in mass not because of $\mathcal{CPT}$ (in Axiomatic QFT proven only for massive theories) but because in field theory representations $\hat{D}$ of internal symmetries must be real,

$U_g \Phi U_g^{-1} = \hat{D}_g^{-1} \Phi$ for fields Φ, which are mapped to each other by conjugation.

$\hat{D} = D + D^*$ or $\hat{D} = \hat{D}^*$.

So each field theoretic model contains with each charged particle (eigenstate of a chosen Cartan subalgebra) the oppositely charged particle of equal mass.

Conclusion

Though Wigner's fundamental paper on unitary representations of the Poincaré group celebrates its 80^{th} birthday basic questions are still unresolved, e.g. the relation of $\mathcal{M}'$ to $\mathcal{L}_{\text{int}}$ and the calculation of bound states (hadron, hydrogen) from first principles.