Irrelevance of $D = 26$
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Stone-von Neumann Theorem

Covariant String

Timelike String

Lightcone String

Representations of Lie-Algebras

Blessing in Disguise
Unitary Representations

Quantum theories give probability distributions for results of measurements

\[ w(i, A, \Psi) = \frac{|\langle \Lambda_i|\Psi \rangle|^2}{\langle \Lambda_i|\Lambda_i \rangle \langle \Psi|\Psi \rangle} , \Psi, \Lambda_i \in \mathcal{H} , w(i, A, \Lambda_j) = \delta_{ij} \]

Wigner: Symmetries are realized by unitary or antiunitary transformations.

Relativistic is a quantum theory, if a unitary representation (of the connected part of the cover) of the Poincaré group acts on its Hilbert space.

String theory is no relativistic quantum theory, because it lacks a Hilbert space with a unitary representation of the Poincaré group

Recognize problems not as fans but with scientific impartiality and curiosity
Stone-von Neummann Theorem

All unitary, irreducible representations of

\[ e^{itP} e^{isX} = e^{it \cdot s} e^{isX} e^{itP}, \quad [X_m, P^n] = i\hbar \delta_m^n, \quad t, s \in \mathbb{R}^D \]

are unitarily equivalent to the multiplicative and differential operations

\[ (P^n \Psi)(p) = p^n \Psi(p), \quad (X_m \Psi)(p) = -i \frac{\partial \Psi}{\partial p^m}(p) \]

Algebra of \( X^m, P_n \) acts on the Schwartz space \( S(\mathbb{R}^D) \) with scalar product

\[ \langle \Phi | \Psi \rangle = \int d^Dp \ \Phi^*(p) \Psi(p) \]

representation is irreducible

no compatible restriction to functions which vanish in some fixed domain
The Covariant String

The covariant string (flag ship) employs

\[ [X_m, P^n] = i \delta_m^n, \ m, n \in \{0, 1, \ldots D - 1\} \]

and restricts physical states to mass shells (D dimensional volume = 0)

\[ \left( ((P^0)^2 - \vec{P}^2) - 2(N - 1) \right) \Psi_{\text{phys}} = 0 \]

where \( N \) has a discrete spectrum.

\[ \Rightarrow \langle \Phi | \Psi_{\text{phys}} \rangle = 0 \ \forall \Phi \Rightarrow \Psi_{\text{phys}} = 0 \]

Theorem: Covariant String has no physical state.

To compare: acceptable Quantum theories contain \( \leq (D - 1) \) Heisenberg pairs
The Timelike Gauge

employs excitation operators $\alpha^i_{-n}$ which commute with $\vec{P}$

Incompatible with a unitary representation of the Lorentz group
The Lightcone String

related problem: continuous helicity \( \hbar \)

\[
D = \begin{pmatrix}
\frac{\partial p_x}{\partial p_y} \\
\frac{\partial p_y}{\partial p_z} \\
\frac{\partial p_z}{\partial p_x}
\end{pmatrix} - \frac{i\hbar}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix}
p_y \\
-p_x \\
0
\end{pmatrix}
\]

noncommutative (monopolian) geometry

\[
[D_i, D_j] = F_{ij} = i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|^3}
\]

\[-iM_{ij} = -(p^i D_j - p^j D_i) - i \hbar \epsilon_{ijk} \frac{p^k}{|\vec{p}|}, \quad -iM_{0i} = |\vec{p}| D_i
\]

satisfy the Lorentz algebra for all \( \hbar \in \mathbb{R} \)? Possible?
Representation of a Lie-Algebra $\text{Lie}(G)$

$D \subset H$ dense subspace

Representation $\pi : \text{Lie}(G) \to L(D) : \pi(x)D \subset D$

$$\pi(\alpha x + \beta y) = \alpha \pi(x) + \beta \pi(y)$$

$$\pi([x, y]) = \pi(x)\pi(y) − \pi(y)\pi(x)$$

$$\langle \pi(x)\Phi|\Psi\rangle = −\langle \Phi|\pi(x)\Psi\rangle$$

$\forall x, y \in \text{Lie}(G), \forall \alpha, \beta \in \mathbb{R}, \forall \Phi, \Psi \in D$

Schm"udgen: $U_g : H \to H, \langle \Phi|U_g\Psi\rangle$ measurable

$D = \text{Gårding space } \{ \Psi_f : \Psi_f = \int_G d\mu_g f(g) U_g\Psi \}$

$f$ smooth, compact support, $U_g\Psi_f = \Psi_{f \circ g^{-1}}, U_g\Psi$ smooth function of $g$

$U_{e^x} = e^{\pi(x)}, \pi(x)\Psi = \lim_{t \to 0}(e^{t\pi(x)} − 1)/t \Psi$ defined $\forall \Psi \in D$,

$D \sim S(\mathbb{R}^{D-1})$ (massive) or $D \sim S(S^{D-2} \times \mathbb{R})$ (massless, tachyonic)
The Lightcone String

The Lightcone string employs operators \((p_+)^{-1}\) and \(\alpha_i\)

which do not map \(D\) to \(D\)

whatever Lie algebra they formally satisfy

they cannot generate a unitary representation of \(G=\text{Poincaré group}\)

\((p_+)^{-n}\) diverges on tachyonic and massless states

excitations \(\alpha_m^i\) cannot relate smoothly massive and massless shells

\(\alpha_m^i(P + k(P)) = P\alpha_m^i, \ q + k(q) = p(q)\)

\((\alpha\Psi)(p(q)) = U(q) \left( \det \frac{\partial p}{\partial q} \frac{N(q)}{N(p)} \right)^{\frac{1}{2}} \Psi(q)\)

\(p\) massive, \(q\) massless \(\exists q : \frac{\partial p}{\partial q} = 0\) or \(\infty\)

\(\alpha_m^i\) generate negative energy shells and states with diverging \(\langle \Psi | (P)^n | \Psi \rangle\)
Blessing in Disguise

If the canonical generators of string theory generated the $D = 26$ Poincaré group:
25 continuous, unbounded spatial momenta would be predicted irrevocably!

Canonical generators of the lightcone gauged string do not generate Poincaré

Does not rule out successful other constructions (unknown however)

Compactification to $D = 4$ has to occur before the quantization

and requires a check of the $D = 4$ Poincaré representation afterwards