Irrelevance of $\mathrm{D} = 26$
Lessons from Unitary Representations of the Poincaré Group

Norbert Dragon
Hannover
June 1st, 2018

Florian Oppermann, Olaf Lechtenfeld, Domenico Giulini, Elmar Schrohe, ITP LUH

Irrelevance of $\mathrm{D} = 26$
Lessons from Unitary Representations of the Poincaré Group

Stone-von Neumann Theorem

Representations of Lie-Algebras

Covariant String

Static Gauge

Lightcone String

The Blessing of Failure

Sine Ira et Studio

Quantum Theory, String Theory, 2-dimensional Conformal Field Theory:

Mathematical Statements, $2+2=4$, are independent of the pedigree of a theory.

A logical flaw is not corrected by repeating the flaw elsewhere.

Relativistic Quantum Theory

Quantum theories give probability distributions for results of measurements

$$w(i, A, \Psi) = \frac{|\langle \Lambda_i | \Psi \rangle|^2}{\langle \Lambda_i | \Lambda_i \rangle \langle \Psi | \Psi \rangle} \ , \Psi, \Lambda_i \in \mathcal{H} \ , w(i, A, \Lambda_j) = \delta_{ij}$$

Wigner: Symmetries are realized by unitary or antiunitary transformations.

Relativistic is a quantum theory, if a unitary representation (of the connected part of the cover) of the Poincaré group acts on its Hilbert space.

Preview of my conclusion as of now:

In no dimension is string theory a relativistic quantum theory.

It lacks a Hilbert space with a unitary representation of the Poincaré group.

Reminder: Stone-von Neumann Theorem

All unitary, irreducible representations of

$$e^{itP}e^{isX} = e^{-it\cdot s}e^{isX}e^{itP}\ ,\ [X^m, P^n] = -i\,\eta^{mn}\ ,\ t, s \in \mathbb{R}^D$$

are unitarily equivalent to the multiplicative and differential operations

$$(P^n\Psi)(p) = p^n\Psi(p)\ ,\ (X^m\Psi)(p) = -i\eta^{mn}\frac{\partial\Psi}{\partial p^n}(p)$$

Algebra of X^m, P^n acts on the Schwartz space $\mathcal{S}(\mathbb{R}^D) \subset \mathfrak{L}^2(\mathbb{R}^D, d^Dp)$

$$\langle\Phi|\Psi\rangle = \int d^Dp\,\Phi^*(p)\,\Psi(p)$$

unitary representation is irreducible, though $\mathcal{S}$ is a genuine invariant subspace

no compatible restriction to functions which vanish in some fixed domain

Reminder: Schwartz Space

$$\mathcal{S}(\mathbb{R}^D) = \left\{ f : \mathbb{R}^D \to \mathbb{C} \,\middle|\, |x^\alpha \partial^\beta f| < C_{\alpha,\beta} \right\}$$

$$x^\alpha = (x^1)^{\alpha_1}(x^2)^{\alpha_2} \ldots (x^D)^{\alpha_D} \,, \;\; \alpha \in \mathbb{N}^D$$

f deserves the denomination physical state

$$\left\langle f \middle| X^\alpha P^\beta f \right\rangle < \infty$$

all expectation values of position and momentum exist

Unitary Representation of a Lie-Group G and its Algebra

$U_g : \mathcal{H} \to \mathcal{H}$, $U_{g_2} U_{g_1} = U_{g_2 g_1}$, $g \mapsto \langle \Phi | U_g \Psi \rangle$ measurable

Gårding space $\mathcal{D}$ spanned by group averaged states $\Psi_f = \int_G \mathrm{d}\mu_g \, f(g) \, U_g \Psi$

f smooth, compact support (neighbourhood of e)

$U_g \Psi_f = \Psi_{f \circ g^{-1}}$, $U_g \Psi_f$ smooth function of g: differential geometry

$\mathcal{D} \subset \mathcal{H}$ core of the representation $\pi : \mathrm{Lie}(G) \to L(\mathcal{D}) : \pi(x)\Psi = \frac{\mathrm{d}}{\mathrm{d}\alpha}\big|_{\alpha=0} U_{e^{\alpha x}} \Psi$

$\pi(x)\mathcal{D} \subset \mathcal{D}$ invariant, $\forall x, y \in \mathrm{Lie}(G)$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall \Phi, \Psi \in \mathcal{D}$

$$
\begin{aligned}
\pi(\alpha x + \beta y) &= \alpha \pi(x) + \beta \pi(y) \\
\pi([x, y]) &= \pi(x)\pi(y) - \pi(y)\pi(x) \\
\langle \pi(x)\Phi | \Psi \rangle &= - \langle \Phi | \pi(x) \Psi \rangle
\end{aligned}
$$

Unitary Representation of a Lie-Group G and its Algebra

Note well:

the core of $\pi(\mathrm{Lie}(\mathrm{G}))$ is invariant: $\pi(x)\mathcal{D} \subset \mathcal{D}$

all $\pi(x)$ can be applied repeatedly

Necessary for $\sum_{n=0}^{N} \frac{1}{n!}(\pi(x))^n$ to exist for all N

Domain of π(Lie(Poincaré)), space of physical states:

single particle: $\mathcal{D} = \mathcal{S}(\mathbb{R}^{D-1})$ (massive) or $\mathcal{D} = \mathcal{S}(S^{D-2} \times \mathbb{R})$ (massless, tachyon)

Poisoned mathematical gift: smooth generators.

The Covariant String $\mathcal{H}_1 = \mathcal{H} \times \mathcal{H}_N$

$\mathcal{H}_N$: denumerable sum of finite dimensional eigenspaces of a level operator N

D Heisenberg pairs of momentum and position operators: $\mathcal{H} \to \mathcal{H}$

$$[X^m, P^n] = -i\,\eta^{mn}\ ,\ \ m, n \in \{0, 1, \ldots D - 1\}$$

Physical states constrained to discrete mass shells (D dimensional volume = 0)

$$\Big(\big((P^0)^2 - \vec{P}^2\big) - M^2(N)\Big)\Psi_{\text{phys}} = 0$$

$$\forall \Phi :\ \langle \Phi \,|\, \Psi_{\text{phys}}\rangle_{\mathcal{H}_1} = \int d^D p\ \langle \Phi^*(p) | \Psi_{\text{phys}}(p)\rangle_{\mathcal{H}_N} = 0\ \Rightarrow \Psi_{\text{phys}} = 0.$$

The Absence of Physical States of the Covariant String

$$\Psi_{\text{phys}} = 0$$

The covariant string predicts the absence of physical states!

Conclusion not dared:

Bahns, Rejzner, Zahn (2017) Grundling, Hurst (1993) Dimock (2002)

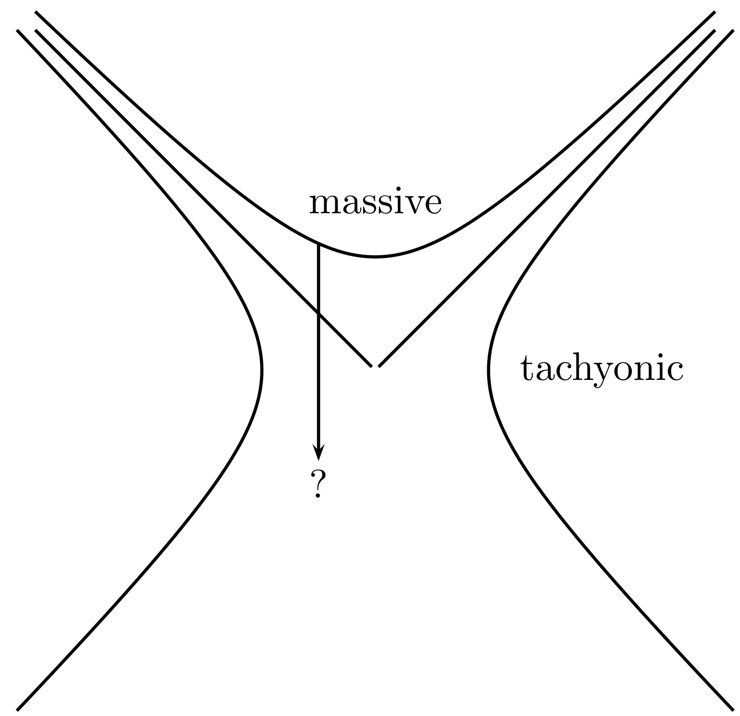
To compare: acceptable Quantum theories contain $\leq (\mathrm{D}-1)$ Heisenberg pairs

Klein-Gordon in D dimensions has no solution in $\mathfrak{L}^2(\mathbb{R}^{\mathrm{D}}, \mathrm{d}^{\mathrm{D}}x)$

but determines the time evolution of states in $\mathfrak{L}^2(\mathbb{R}^{\mathrm{D}-1}, \mathrm{d}^{\mathrm{D}-1}x)$.

The Static Gauge

employs invertible excitation operators α_n^i of a tachyon which commute with $\vec{P}$



Visibly incompatible with a unitary representation of the Lorentz group

The Lightcone String

$D - 1$ Heisenberg pairs: $\vec{P}_T = (P^1, \ldots P^{D-2})$, $\vec{X}_T = (X^1, \ldots X^{D-2})$, X^-, P^+,

Spectrum of $M^2 = P^+P^- - \vec{P}_T^2$ discrete, so $P^- = (m^2 + \vec{P}_T^2)/P^+$ on $\mathcal{H}_m$

$\vec{P}_T, P^+, P^-$ multiplicative on $\mathcal{L}^2(\mathbb{R}^{D-1}, \mathrm{d}^{D-2}p\, \mathrm{d}p^+)$ but $P^-\mathcal{S}(\mathbb{R}^{D-1}) \not\subseteq \mathcal{S}(\mathbb{R}^{D-1})$

X^- generates p^+ translations $U_a = e^{-iX^-a}$

$$\big(U_a\Psi\big)(p_T, p^+) = \Psi(p_T, p^+ - a)\ ,\ a \in \mathbb{R}\ ,\ (p_T, p^+) \in \mathbb{R}^{D-1}$$

The Core of the Lightcone String vanishes

$\Psi \neq 0$: $\exists \mathcal{U}_{(\underline{p},a)} : |(m^2 + \vec{p}_T^{\,2})\, \Psi(p)| > \epsilon > 0$ almost everywhere in $\mathcal{U}_{(\underline{p},a)}$

So $\Phi = U_a \Psi$ is not in the domain of P^-

$$\left\langle P^- \Phi \,\middle|\, P^- \Phi \right\rangle \geq \int_{\mathcal{U}_{(\underline{p},0)}} dp^+ d^{D-2}p \, \frac{\epsilon^2}{(p^+)^2} = \infty$$

no common, invariant, dense domain of X^-, P^+, P^-: inconsistent!

Blessing of Failure

If the canonical generators of string theory generated the $\mathrm{D} = 26$ Poincaré group:

25 continuous, unbounded spatial momenta would be predicted irrevocably!

Compactification to $\mathrm{D} = 4$ has to occur *before* the quantization

requires a subsequent check of the $\mathrm{D} = 4$ Poincaré representation.

Calculation of $\mathrm{D} = 26$ faulty (no domain) and irrelevant for compactified string.

General Problems for Theories with Several Mass Shells

Massless particles and tachyons do not allow translators of momentum

$$\mathcal{S}(\mathbb{R}^{D-1}) \neq \mathcal{S}(S^{D-2} \times \mathbb{R})$$

Massless and tachyonic representations cannot be related to massive ones

by momentum local, invertible excitation operators α^i_n

$$\mathcal{S}(\mathbb{R}^{D-1}) \neq \mathcal{S}(S^{D-2} \times \mathbb{R})$$

Comparison to quantum fields: operator $\neq$ operator valued distribution

Continuous Helicity h

$$D_{N,S} = \begin{pmatrix} \partial_{p_x} \\ \partial_{p_y} \\ \partial_{p_z} \end{pmatrix} - \frac{ih}{|\vec{p}|(|\vec{p}| \pm p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix} \ , \ D_S = e^{2i\,h\varphi(p)}\, D_N\, e^{-2i\,h\varphi(p)}$$

noncommutative (monopolian) geometry on $\mathcal{S}(S^2 \times \mathbb{R}) \neq \mathcal{S}(\mathbb{R}^3)$

$$[D_i, D_j] = F_{ij} = i\,h\,\epsilon_{ijk}\,\frac{p^k}{|\vec{p}|}$$

$$-iM_{ij} = -(p^i\,D_j - p^j\,D_i) - i\,h\,\epsilon_{ijk}\,\frac{p^k}{|\vec{p}|}\ ,\ -iM_{0i} = |\vec{p}|\,D_i$$

$\forall h \in \mathbb{R}$: Lorentz algebra ✓, $\vec{P}\vec{J} = h|\vec{P}|$ ✓, skew hermitean only if $2h \in \mathbb{Z}$

Conclusion

String theory has to reconsider its foundations

as of now it has

no physical states

with a unitary representation of the Poincaré group.