Surprises in Relativistic Quantum Physics

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Domains matter

Points matter

Pictures matter

Language matters

Stringtheory inconsistent

Massive very different from Massless

Interaction $P'^n = M'u^n$, $P^n = Mu^n$, $M'_{kl} = M_{kl}$

Axiomatic Quantum Field Theory? Canonical Quantization?

The Wigner Rotation $W(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_p$

Area δ of an hyperbolic triangle or segment

Calculation a Herculean effort, tedious manipulations

Back of the envelope calculation in $\mathrm{SL}(2, \mathbb{C})$

$$\tan\frac{\delta}{2} = \frac{\sin\varphi}{\left(\coth\frac{a}{2}\,\coth\frac{b}{2}\right) + \cos\varphi}$$

Determines generators of Poincaré group $\mathfrak{P}$ for massive states

$$(P^m\Psi)(p) = p^m\Psi(p)\ ,\ P^2 = m^2 > 0\ ,$$

$$\left(-iM_{ij}\Psi\right)(p) = -\left(p^i\partial_{p^j} - p^j\partial_{p^i}\right)\Psi(p) + \Gamma_{ij}\Psi(p)\ ,$$

$$\left(-iM_{0i}\Psi\right)(p) = p^0\,\partial_{p^i}\Psi(p) + \Gamma_{ij}\frac{p^j}{p^0 + m}\Psi(p)$$

Language and Notation

$\Psi(p)$ rather than $|p_1, s_1 \dots p_n, s_n\rangle$ allows to check the domain of the generators

Gårding space of smoothened states (smooth f of compact support)

$$\Psi_f = \int_{\mathfrak{P}} d\mu_g \, f(g) \, U_g \Psi \ , \ U_g \Psi_f = \Psi_{f \circ g^{-1}}$$

requires $g \mapsto U_g \Psi$ measurable $\forall \Psi$ i.e. U_g strongly measurable (a very weak req.)

Introduces differential geometry into Hilbert space

The Gårding space of any strongly measurable unitary representation of a Lie group is dense and invariant under all U_g and the algebra of its generators.

Poincaré algebra acts on Schwartz space $\mathcal{S}_{\mathcal{M}} \subset \mathfrak{L}^2(\mathcal{M})$, $|p^\alpha \partial_\beta \Psi(p)| < C_{\alpha,\beta,\Psi}$

Sweet poisoned gift

Death of the Covariant String

$$[P^m, P^n] = 0 = [X_m, X_n]\ ,\ [P^n, X_m] = i\delta^n{}_m\ ,\ e^{iP\cdot a}X_m e^{-iP\cdot a} = X_m - a_m$$

Stone-von Neumann Theorem (1931)

$$\big(P^m\Psi\big)(p) = p^m\Psi(p)\ ,\ \big(X_n\Psi\big)(p) = -i\frac{\partial}{\partial p^n}\Psi(p)\ ,\ \langle\Psi\,|\,\Phi\rangle = \int d^D p\,\Psi^*(p)\Phi(p)$$

Algebra acts on $\mathcal{S}^D$, spectrum of $P^m P^n \eta_{mn}$ continuous

$$\text{Constraint:}\qquad \big(P^m P^n \eta_{mn} - M^2(N)\big)\Psi_{\text{phys}} = 0$$

No nonvanishing smooth solution, not even a measurable solution,

support of Ψ_{phys} has D-dimensional measure $0\ \Rightarrow \Psi_{\text{phys}} = 0$

The covariant string has no physical states.

$$0 = \langle\Psi|f^n(P)X_n(P^2 - M^2)\Psi\rangle - \langle X_n f^n(P)(P^2 - M^2)\Psi|\Psi\rangle = -2i\,\langle\Psi|f^n(P)P_n\Psi\rangle ?$$

Death of the Lightcone String

$$[P^+, X^-] = i\ ,\ M^2 = P^+P^- - \vec{P}_T^2\ ,\ P^- = \frac{m^2 + \vec{P}_T^2}{P^+}$$

P^- not smooth in the spectrum of P^+

$$V_b = e^{iX^-b}\ , \quad \big(V_b\Psi\big)(p_T, p^+) = \Psi(p_T, p^+ + b)\ ,$$

$$\Psi \neq 0 \rightarrow |(m^2 + p_T^2)\Psi(p_T, -b)| > \epsilon \text{ in some } \mathcal{U}_{-b}$$

$$\big\langle P^-V_b\Psi \big| P^-V_b\Psi \big\rangle > \int_{\mathcal{U}} dp^+ d^{D-2}p\, \frac{\epsilon^2}{(p^+)^2} = \infty$$

There is no common, invariant domain of P^- and V_b. They do not generate an algebra. The calculation of $D = 26$ is meaningless.

The algebra of the lightcone string has no domain.

Though the points with $P^+ = 0$ have measure 0, the singularity matters – hard to see in terms of $|p_1, s_1, \ldots p_n, s_n\rangle$.

Contradiction

If the lightcone string represented rotations unitarily (generators M_{ij}), then $\vec{X} = (X_1, \ldots X_{D-1})$ existed together with $\vec{P} = (P^1, \ldots P^{D-1})$.

They are Heisenberg pairs and vectors under M_{ij} and $L_{ij} = X_i P_j - X_j P_i$.

So $S_{ij} = M_{ij} - L_{ij}$ commutes with X_i and P_j and therefore with L_{ij}

L_{ij} and $M_{ij} = L_{ij} + S_{ij}$ satisfy the angular momentum algebra of SO$(D-1)$

$\Rightarrow$ S_{ij} satisfies the angular momentum algebra of SO$(D-1)$.

? Massless states of the lightcone string are only helicity multiplets of SO$(D-2)$!

Massless Representation $p^0 = |\vec{p}| > 0$

$\mathcal{M}_0 = S^2 \times \mathbb{R} \neq \mathbb{R}^3$, stabilizer E(2) $\neq$SO(3), no exercise for students

States: sections rather than functions, coordinate patches, not defined at $p = 0$,

Wigner rotation calculated in $\mathrm{SL}(2,\mathbb{C})$, h helicity

$$\big(-iM_{12}\Psi\big)_N(p) = -\big(p_x\partial_{p_y} - p_y\partial_{p_x}\big)\Psi_N(p) - i\,h\,\Psi_N(p)\ ,$$

$$\big(-iM_{31}\Psi\big)_N(p) = -\big(p_z\partial_{p_x} - p_x\partial_{p_z}\big)\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,$$

$$\big(-iM_{32}\Psi\big)_N(p) = -\big(p_z\partial_{p_y} - p_y\partial_{p_z}\big)\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,$$

$$\big(-iM_{01}\Psi\big)_N(p) = |\vec{p}|\partial_{p_x}\Psi_N(p) - i\,h\,\frac{p_y}{|\vec{p}| + p_z}\Psi_N(p)\ ,$$

$$\big(-iM_{02}\Psi\big)_N(p) = |\vec{p}|\partial_{p_y}\Psi_N(p) + i\,h\,\frac{p_x}{|\vec{p}| + p_z}\Psi_N(p)\ ,$$

$$\big(-iM_{03}\Psi\big)_N(p) = |\vec{p}|\partial_{p_z}\Psi_N(p)$$

Singularity at $\mathcal{A}_- = \{\ |\vec{p}|(1,0,0,-1) : |\vec{p}| > 0\}$

Outside $\mathcal{A}_-$ the generators M_{kl} satisfy the Lorentz algebra for each value of h

Smooth Ψ_N with $\Psi_{N|_{\mathcal{A}_-}} \neq 0$ not in the domain, e.g. $\langle M_{13}\Psi_N | M_{13}\Psi_N \rangle = \infty$

$$\Psi_S(p) = e^{2i\,h\varphi(p)}\,\Psi_N(p)\ ,\ e^{2i\,h\varphi(p)} = \Big((p_x + ip_y)/\sqrt{p_x^2 + p_y^2}\Big)^{2h}\ ,\ 2h \in \mathbb{Z}$$

$$\begin{aligned}
\big(-iM_{12}\Psi\big)_S(p) &= -\big(p_x\partial_{p_y} - p_y\partial_{p_x}\big)\Psi_S(p) + i\,h\,\Psi_S(p)\ ,\\
\big(-iM_{31}\Psi\big)_S(p) &= -\big(p_z\partial_{p_x} - p_x\partial_{p_z}\big)\Psi_S(p) - i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\big(-iM_{32}\Psi\big)_S(p) &= -\big(p_z\partial_{p_y} - p_y\partial_{p_z}\big)\Psi_S(p) + i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\big(-iM_{01}\Psi\big)_S(p) &= |\vec{p}|\partial_{p_x}\Psi_S(p) + i\,h\,\frac{p_y}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\big(-iM_{02}\Psi\big)_S(p) &= |\vec{p}|\partial_{p_y}\Psi_S(p) - i\,h\,\frac{p_x}{|\vec{p}| - p_z}\Psi_S(p)\ ,\\
\big(-iM_{03}\Psi\big)_S(p) &= |\vec{p}|\partial_{p_z}\Psi_S(p)
\end{aligned}$$

Noncommutative Geometry

$$D_i = \partial_{p^i} + A_i - \frac{p^i}{2|\vec{p}|^2}$$

$$\vec{A}_N(p) = \frac{i\,h}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix} \ , \ \vec{A}_S(p) = \frac{i\,h}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix}$$

$$[P^i, P^j] = 0 \ , \ [P^i, D_j] = -\delta^i{}_j \ , \ [D_i, D_j] = F_{ij} = \partial_{p^i} A_j - \partial_{p^j} A_i = i\,h\,\epsilon_{ijk} \frac{P^k}{|\vec{P}|^3}$$

The geometry of massless particles with nonvanishing helicity is noncommutative.

$$-iM_{ij} = -(P^i D_j - P^j D_i) - i\,h\,\epsilon_{ijk} \frac{P^k}{|\vec{P}|} \ , \ -iM_{0i} = |\vec{P}|^{1/2} D_i |\vec{P}|^{1/2}$$

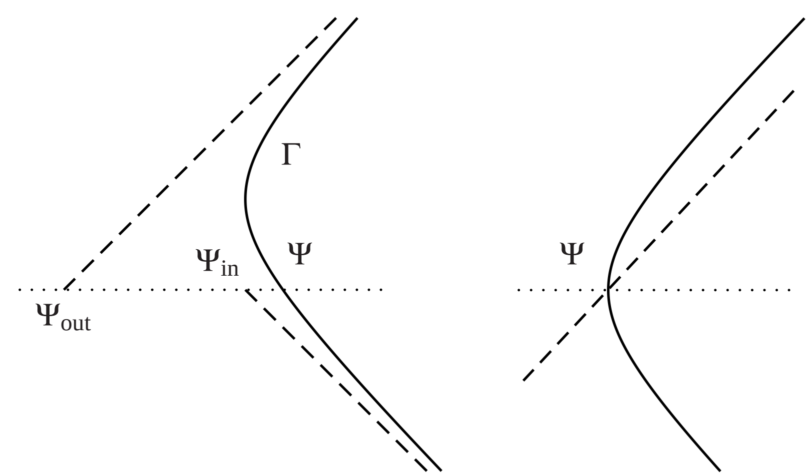
Momentum Space Monopole: $\displaystyle \frac{1}{4\pi} \int_{\mathcal{S}} \mathbb{F} = i\,h$

Scattering

Product rule ⇔ free time evolution: $(P^m\Psi)^{ij}(p_1,p_2) = (p_1^m + p_2^m)\,\Psi^{ij}(p_1,p_2)$

Nontrivial group of unitary motion has no limit!

$$\Psi(s+t) = U'(s)\Psi(t)\ ,\ \Psi_\pm = U'(s)\Psi_\pm\ ,\ \Psi(t)-\Psi_\pm = U'(t)(\Psi(0)-\Psi_\pm)$$



$$\Omega_\pm = \underset{t\to\mp\infty}{\text{s-lim}}\, e^{iHt}\, e^{-iH't}\ ,\ S = \Omega_+\Omega_-^{-1} = \underset{t,\,t'\to\infty}{\text{s-lim}}\ e^{-iHt}e^{iH'(t+t')}e^{-iHt'}\ ,\ [H,H'] \neq 0$$

$H\Omega_+ = \Omega_+ H'$ not reconstructible from Wightman distributions

Schrödinger Picture

Interaction concerns time evolution of many particle states.

Language and pictures matter, e.g. Ptolemaic system.

Not the states but their time evolution is interacting. In, out and interacting are relations not properties. Ψ_{out} is the *initial* state of a future asymptote, not a state with all momenta outwards as opposed to Ψ_{in} with all momenta inwards.

States are just states, they determine probabilities of results.

Interacting fields, whatever they are, are not needed.

Lorentz transformations transform states and consequently paths irrespective of their time evolution: $M'_{kl} = M_{kl}$.

Claim: $P'^{\,m} = u^m M'$, $P^m = u^m M$, $M = \sqrt{P^2}$, $[u^m, M'] = 0$, $[M', M] \neq 0$.

? Canonical Quantization, Haag's Theorem

Center Variables

$$(p_1, \ldots p_n) \leftrightarrow (u, q_1, \ldots q_n) : \sum_i q_i = 0, q_i \in \mathbb{R}^3$$

$$u = \frac{\sum_i p_i}{m}\ ,\ m^2 = (\sum_i p_i)^2\ ,\ u^2 = 1\ ,$$

$$p_{i\,\|} = (p_i \cdot u)\, u\ ,\ p_{i\,\perp} = p_i - p_{i\,\|}\ ,\ q_i = (L_u)^{-1} p_{i\,\perp}$$

$$p_i(u, q) = \sqrt{m_i^2 + \vec{q}_i^{\,2}}\, u + L_u\, q_i = \sqrt{m_i^2 + \vec{q}_i^{\,2}} \begin{pmatrix} \sqrt{1 + \vec{u}^2} \\ \vec{u} \end{pmatrix} + \begin{pmatrix} \vec{u} \cdot \vec{q}_i \\ \vec{q}_i + \frac{(\vec{u} \cdot \vec{q}_i)\, \vec{u}}{1 + \sqrt{1 + \vec{u}^2}} \end{pmatrix}$$

$$M = \sum_i \sqrt{m_i^2 + (\vec{q}_i)^2}$$

$$H_2 = \sqrt{1 + \vec{u}^2} \left(\sqrt{m_1^2 + \vec{q}^{\,2}} + \sqrt{m_2^2 + \vec{q}^{\,2}} \right) = m_1 + m_2 + \frac{(m_1 + m_2)\vec{u}^2}{2} + \frac{\vec{q}^{\,2}}{2\mu} + \ldots$$

Generalized Wave Operators

$$\lim_{t\to\infty}\left(e^{i\sqrt{1+\vec{u}^2}M\,t}e^{-iH't}\Psi\right)(u,q) = \lim_{t'\to\infty}\left(e^{iM\,t'}e^{-iH't'/\sqrt{1+\vec{u}^2}}\Psi\right)(u,q)$$

$H'/\sqrt{1+\vec{u}^2}$ hermitian, if $[H', u^0] = 0 \Rightarrow H' = u^0 M' \Rightarrow P'^{\,m} = u^m M'$

$$\Omega_\pm(H,H') = \operatorname*{s-lim}_{t\to\mp\infty} e^{i\sqrt{1+\vec{u}^2}M\,t}e^{-i\sqrt{1+\vec{u}^2}M't} = \operatorname*{s-lim}_{t\to\mp\infty} e^{iM\,t}e^{-iM't} = \Omega_\pm(M,M')$$

$M' = M + V$, $[V, M] \neq 0$, V translational and rotational invariant ($\sum q_i = 0$),

Invariance Principle:
f monotonously increasing $\Rightarrow \Omega_\pm(M,M') = \Omega_\pm(f(M), f(M'))$

Relativistic and Nonrelativistic scattering and binding analogous

List of Further Results

Particles not localized in the forward and backward cone of a compact domain.

Boosts transform states ($\neq$ fields) nonlocally by convolution.

The state $\int_{\mathcal{U}} d^4x\, f(x)\, \Phi(x)\Omega$, $\Phi(x)$ a local field, is not localized within $\mathcal{U}$.

Hilbert space does not allow for a massless vector field. Time order of a massive vector field does not yield a covariant propagator

$$\langle \mathrm{T}\, \Phi^m(x)\, \Phi^n(0)\rangle \stackrel{?}{=} \int \tilde{d}p \left(\theta(x^0)\, e^{-ipx} + \theta(-x^0)\, e^{ipx}\right)\left(-\eta^{mn} + p^m p^n/m^2\right)$$

$$\begin{aligned}\langle \mathrm{T}\, \Phi^m(x)\, \Phi^n(0)\rangle &= \lim_{\epsilon\to 0+} i \int \frac{d^4p}{(2\pi)^4} \frac{-\eta^{mn} + p^m p^n/m^2}{p^2 - m^2 + i\epsilon}\, e^{ipx} \\ &= -\Big(\eta^{mn} + \frac{\partial^m\partial^n}{m^2}\Big) \lim_{\epsilon\to 0+} i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2 - m^2 + i\epsilon}\end{aligned}$$

The $1/m^2$-term is not cancelled by adding to Φ^m a gradient $\partial^m\phi$.

Continued List

Massless particles have no position operator.

Massless fields have algebraically special Fouriertransformation of rank 1

$$
\begin{aligned}
\Phi_{\alpha_1\dots\alpha_{2h}}(x) &= \int \tilde{d}p\, n_{\alpha_1}(p)\dots n_{\alpha_{2h}}(p)\left(e^{i\,p\,x}\, b^\star_N(p) + e^{-i\,p\,x}\, a_N(p)\right)\\
&= \int \tilde{d}p\, s_{\alpha_1}(p)\dots s_{\alpha_{2h}}(p)\left(e^{i\,p\,x}\, b^\star_S(p) + e^{-i\,p\,x}\, a_S(p)\right)
\end{aligned}
$$

Antiparticles not because of CPT but if real representations $\hat{D}$ of internal symmetries decompose over complex numbers, $\hat{D} = D + D^*$.

Though Wigner's fundamental paper celebrates its 80th birthday basic questions are still unresolved, e.g. the relation of M' to $\mathcal{L}_{\text{int}}$.