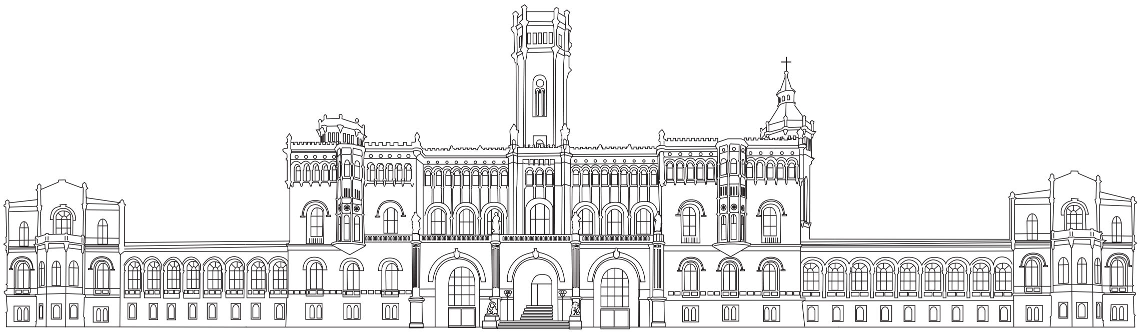

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Abstract

The contextuality of an empirical model is of interest because we can see it as a resource for quantum computation. We can understand this in the setting of measurement based quantum computation. This is an alternative to computing with quantum circuit models in which information is processed by a sequence of single-qubit measurements on an entangled multi-qubit resource state. Contextuality describes the dependence of a measurement outcome on a particular measurement context and therefore is the power of measurement based quantum computing. In this work we use the sheaf-theoretic structure of contextuality, the method of bundle diagrams and the powerful tool of Čech cohomology to study many examples of empirical models.

Contents

1 Introduction

Contextuality is one of the key attributes of quantum mechanics [4] and distinguishes quantum mechanics from classical physics. We can understand it, by considering whether it is possible to assign "pre-existing" outcomes to measurements of quantum observables. If this is possible, we can describe the situation with classical mechanics. We assume a model with two different sets C_1 and C_2 of compatible, i.e. commuting, observables containing a given observable x. It is reasonable to require that the outcome value o attached to the observable x is a property of x alone and thus agrees in C_1 and C_2. This constraint is called "context independence" [32]. But it turns out not to be the case for all settings in quantum mechanics [25, 8]. This fact is often referred to as contextuality. Analogously non-contextuality is the property that any measurement has an outcome independent of other compatible measurements being carried out at the same time and can be explained by classical models with hidden variables [11]. There have been a number of experimental verifications that there are contextual behaviors in nature [5, 7, 13, 11, 23, 26, 29, 38, 40, 41].

The main concept from Abramsky and Brandenburger [1] is to understand contextuality as an effect arising where there is a data family which is locally consistent but globally inconsistent. We can visualize this graphically in so called "bundle diagrams". Therefore we have a base space of sets of variables which can be jointly measured and a space of observations fibered over this space. A family of local sections depict the values of the variables in these fibers. Contextuality means that the data is consistent locally, but not globally, i.e. there is no global section defined on all the variables which reconciles all the local data [4]. Sheaf theory [27] is responsible for the passage from local to global and therefore useful for discussing contextuality. Cohomology describes a sequence of abelian groups defined from a cochain complex and is one of the major tools of modern mathematics [4]. In particular Čech cohomology approaches the intersection properties of open covers of a topological space. Hence it is the natural mathematical setting for analyzing contextuality. This was studied by Abramsky and coworkers in [1, 3].

We are interested in contextuality because we can see it as a resource for quantum computation. We can understand this in the setting of measurement-based quantum computation (MBQC). This is an alternative to computing with quantum circuit models in which the computation is driven by using only local measurements as computational steps as opposed to unitary gates [32]. Information is processed by a sequence of single-qubit measurements on an entangled multi-qubit resource state, such as a cluster state or graph state [9, 19, 20]. For example the logic NAND gate can be produced by three measurements using the Grenberger-Horne-Zeilinger state as a resource [6]. In addition to [10], [34] shows that any quantum logic circuit can be implemented on a cluster state and that any quantum circuit can be implemented on a sufficiently large cluster. Furthermore [18] presents many universal resource states and computational schemes for MBQC.

Each measurement based quantum computer consists of two components, a correlated multipartite resource state and a classical control computer [6]. The state is consumed, i.e. the entanglement is destroyed, by the process of computation [34]. It consists of a number of parties, for example Alice and Bob, which exchange classical information with the control computer. Between the parties no direct communication is allowed. The outputs are correlated, but solely due to their joint history. Each party receives an input from the control computer. This input is an element of a set of measurement settings. These settings are sets of compatible measurements and command definitely which measurement operation the chosen party implements. The party returns an outcome after the measurement.

The classical computer can store classical information, exchange it with the parties and compute certain functions. It has rules for the classical side-processing of measurement outcomes. The only part of the model where active computation takes place is here. Furthermore the control computer processes and feeds forward measurement outcomes and directs future adaptive measurements. The correlated measurement outcomes enable it to compute problems beyond its own power [6].

Measurement based models are especially interesting because they have no classical analogues. Additionally they offer a new perspective on the feature of entanglement in quantum computation and may also be interesting for experimental considerations. They suggest a different kind of computer architecture and offer interesting possibilities for further issues [22]. For reviews of MBQC, see [35, 15, 28].

Contextuality describes the dependence of a measurement outcome on a particular measurement context. In MBQC no quantum speedup can occur without contextuality [16, 21, 36]. For this reason it is the source of the power of MBQC. A cohomological framework based on group cohomology developed by Abramsky and coworkers has been recently proposed by Raussendorf and collaborators to understand contextuality as a resource in MBQC [33, 32, 37, 39].

In this work we study different empirical models. We depict them in bundle diagrams and thereby conclude whether the models are contextual or not. We also build the Čech cohomologies to compute sets of global sections.

In Chapter 2 we approach the sheaf-theoretic structure of contextuality and therefore recall the most important definitions from [1]. Introducing the method of bundle diagrams helps to understand the sheaf structure. Furthermore we explain how one can deduce contextuality from the non-existence of a global section. In Chapter 3 we depict a range of empirical models by bundle diagrams. We also explain how one can read out of these diagrams whether a model is contextual. Chapter 4 explains the powerful tool of Čech cohomology [3] which we use to study the sets of global sections. Many examples follow in Chapter 5. The last chapter includes a conclusion and open questions.

2 The sheaf-theoretic structure of contextuality

The aim of this chapter is to understand how a structure of sheafs can describe the contextuality of an empirical model. We restrict ourselves to use Pauli measurement operations as observables in this work.

In our basic setting there are several agents, who can each select from a set of measurements and observe outcomes. We name running a prepared system, i.e. each agent performs a measurement on the system and observes an outcome, an event. A probability distribution on these events summarizes repeated runs.

Definition 2.1. *An empirical model is a family of probability distributions on events, one for each choice of measurements. A set of jointly performable measurements is named measurement context.*

For the beginning we study a bipartite qubit model, where each of the parties can apply a measurement operation on the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

We choose exemplary the operations $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and index the measurement operations with the initials of the executers Alice or Bob. One can imagine a measurement setting where Alice measures $X_A = X \otimes \mathbb{1}$ and Bob $X_B = \mathbb{1} \otimes X$. These two measurements commute and hence build a context. There are four sets of those jointly performable measurements

$$\begin{aligned} C_1 &= \{X_A, X_B\} \\ C_2 &= \{X_A, Z_B\} \\ C_3 &= \{Z_A, X_B\} \\ C_4 &= \{Z_A, Z_B\}. \end{aligned}$$

Using the state $|\Psi\rangle$ we can calculate the possibilities that $|\Psi\rangle$ passes into $|\nu_i\mu_j\rangle$, i.e. we get the outcome ij, after the measurement. The eigenstates of Z are $|\nu_i\rangle = |i\rangle$

for $i \in \{0, 1\}$. X has eigenstates

$$|\mu_0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|\mu_1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

For instance the probability for the event $X_A = +, Z_B = 0$, i.e. to get the outcome 00 measuring in the context C_2, is

$$\begin{aligned}\text{Prob}(X_A = +, Z_B = 0) &= \langle\Psi|\,|{+0}\rangle\,\langle{+0}|\,|\Psi\rangle \\ &= |\,\langle\Psi| + 0\rangle\,|^2 \\ &= 1/4.\end{aligned}$$

Calculating all possibilities this way we get an empirical model for the chosen setting.

A	B	00	10	01	11
X_A	X_B	1/2	0	0	1/2
X_A	Z_B	1/4	1/4	1/4	1/4
Z_A	X_B	1/4	1/4	1/4	1/4
Z_A	Z_B	1/2	0	0	1/2

Figure 2.1: Empirical model for measuring on $|\Psi\rangle$

2.1 Bundle diagrams

Carù [14] presents a kind of diagram to depict empirical models by representing measurements as vertices. A set of vertices forms a face whenever the corresponding measurements can be performed jointly, hence contexts correspond to faces of the complex. It follows, that the base of the bundle diagram suitable to the above described model is a square, where the four edges represent the contexts.

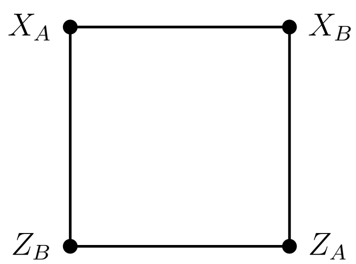


Figure 2.2: Base of the bundle diagram

Above each vertex is a fiber of the outcome values that can be assigned to each measurement (for example 0 or 1).

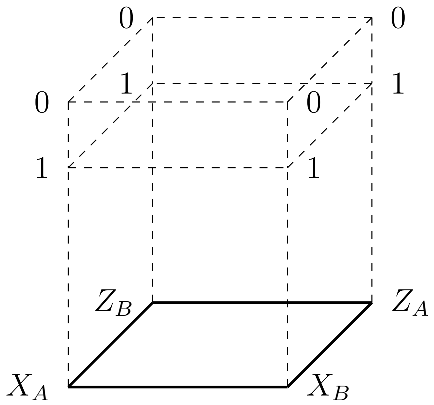


Figure 2.3: Fibers of the bundle diagram

Concerning the above empirical model, for each measurement context there exist outcome combinations with a probability larger than 0. Such a possible outcome pair is represented by an edge connecting the outcomes involved above the corresponding context.

A	B	00	10	01	11
X_A	X_B	1/2	0	0	1/2
X_A	Z_B	1/4	1/4	1/4	1/4
Z_A	X_B	1/4	1/4	1/4	1/4
Z_A	Z_B	1/2	0	0	1/2

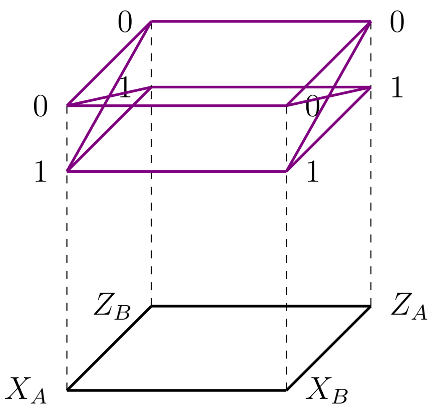


Figure 2.4: Bundle diagram for measuring on $|\Psi\rangle$

At this point it becomes clear, why this method only describes possibilistic models which contain only the information whether the possibilities are 0 or not, thoroughly. Nevertheless, we will see in Chapter 3 that the bundle diagrams are useful if we are interested in the contextuality of an empirical model.

2.2 Sheaf structure

After describing empirical models graphically we want to use the mathematical language of sheaf structure to describe the possible outcomes on the different measurement contexts. Therefor we remind the most important definitions from [1] in this section.

We assume that the set of all measurement operations M of a special model and the set of outcomes O are finite. For each set of measurements $U \subset M$, for example

$U = C_3 = \{Z_A, X_B\}$, a section over U is a function $s : U \to O$. Such a section describes the event in which a measurement x was performed and the outcome $s(x)$ was observed for each $x \in U$. We assume only events with probabilities larger then 0 and write $\mathcal{E} : U \to O^U$ for the assignment of the set of sections over U to each set of measurements U. O^U denotes the set of functions from U to O. For example each of the purple lines right above the set of measurements C_3 in Figure 2.4 depicts a section in the set $\mathcal{E}(C_3)$:

$$\begin{aligned}\mathcal{E}(C_3) = \{&s_{C_300}(Z_A) = 0, s_{C_300}(X_B) = 0;\\ &s_{C_301}(Z_A) = 0, s_{C_301}(X_B) = 1;\\ &s_{C_310}(Z_A) = 1, s_{C_310}(X_B) = 0;\\ &s_{C_311}(Z_A) = 1, s_{C_311}(X_B) = 1; \}\end{aligned}$$

Definition 2.2. *If $s : U' \to O$ is a function and $U \subseteq U'$, we write $s|U : U \to O$ for the restriction of s to U. It exists a restriction map*

$$res_U^{U'} : \mathcal{E}(U') \to \mathcal{E}(U) :: s \to s|U.$$

The map $res_U^{U'}$ makes $\mathcal{E}$ a presheaf if $res_U^{U'} \circ res_{U'}^{U''} = res_U^{U''}$ for $U \subset U' \subset U''$ and $res_U^U = id_U$ hold.

But $\mathcal{E}$ has an additional important property. In the bundle diagram we can recognize that some sections touch each other at a vertex.

Definition 2.3. *Suppose we have a family of sets which covers U, that means $\{U_i\}_{i \in I}$ with $\bigcup_{i \in I} U_i = U$. A family of sections $\{s_i \in \mathcal{E}(U_i)\}_{i \in I}$ is compatible, if for all $i, j, \in I$*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j.$$

For each compatible family there is a unique section $s \in \mathcal{E}(U)$ such that $s|U_i = s_i$ for all $i \in I$. This property is known as the sheaf condition. We name $\mathcal{E}$ a sheaf of events.

The condition says that we can glue together local data which is compatible in the sense of agreeing on overlaps and the glued section is uniquely determined. It is trivial that it holds for $\mathcal{E}$, because we assume functions on a discrete space in this case and can glue together partial functions which agree on their overlaps by taking the union of their graphs.

Next up we study the distributions on events mentioned in Definition 2.1 in detail. Given a set U and a semiring R, the support of a function $\Phi : U \to R$ is the set of $x \in U$ such that $\Phi(x) \neq 0$.

Definition 2.4. *An R-distribution on U is a function $d : U \to R$ which has finite support and such that $\sum_{x \in U} d(x) = 1$. We write $\mathcal{D}_R(U)$ for the set of R-distributions on U.*

Choosing the non-negative reals for R, $\mathcal{D}_R(U)$ is the set of probability distributions with finite support on U. Given a function $s : U \to O$, we define the probability distributions on s

$$\mathcal{D}_R(s) : \mathcal{D}_R(U) \to \mathcal{D}_R(O) :: d \to \left[y \to \sum_{f(x)=y} d(x) \right].$$

Remember that the event sheaf $\mathcal{E}(U)$ is the set of sections over U. Hence $\mathcal{D}_R(\mathcal{E}(U))$ is the set of distributions on these sections. In other words we can form a presheaf $\mathcal{D}_R\mathcal{E} : \mathcal{P}(U)^{op} \to \text{Set}$, which assigns to each set of measurements U the set $\mathcal{D}_R(\mathcal{E}(U))$ of distributions on U-sections [1].

We discussed the sheaf-theoretic structure considering an arbitrary set $U \subset M$. In general it is not possible to perform all measurements together. How one can see in our beginning example the idea is to build contexts C with measurements that can be performed jointly instead of arbitrary sets U. In the case of spatially separated agents measurements of different agents can be performed jointly, but one can imagine more complex situations, where compatible sets of measurements may overlap in complicated ways [1].

Definition 2.5. *A measurement cover $\mathcal{M}$ on the set M is a family of subsets of M such that $\bigcup \mathcal{M} = M$ and $\mathcal{M}$ is an anti-chain, i.e. for $C, C' \in \mathcal{M}$ and $C \subseteq C'$ implies $C = C'$.*

We define a no-signaling [2] empirical model for $\mathcal{M}$ to be a compatible family for the cover $\mathcal{M}$ with respect to the presheaf $\mathcal{D}_R\mathcal{E}$. This means that for each measurement context $C \in \mathcal{M}$ there is a distribution $e(C) \in \mathcal{D}_R\mathcal{E}$. In the sense of the sheaf condition this family of distributions is compatible, that means for all $C, C' \in \mathcal{M}$

$$e_C|C \cap C' = e_{C'}|C \cap C'.$$

After discussing the sheaf-theoretical description we can describe the chosen qubit model not only graphically with bundle diagrams but also mathematically by writing out the presheaf and distributions of the four contexts.

$$\begin{aligned}
\mathcal{E}(C_1) = \{ & s_{C_1 00}(X_A) = 0, s_{C_1 00}(X_B) = 0; \\
& s_{C_1 11}(X_A) = 1, s_{C_1 11}(X_B) = 1; \} \\
\mathcal{E}(C_2) = \{ & s_{C_2 00}(X_A) = 0, s_{C_2 00}(Z_B) = 0; \\
& s_{C_2 01}(X_A) = 0, s_{C_2 01}(Z_B) = 1; \\
& s_{C_2 10}(X_A) = 1, s_{C_2 10}(Z_B) = 0; \\
& s_{C_2 11}(X_A) = 1, s_{C_2 11}(Z_B) = 1; \} \\
\mathcal{E}(C_3) = \{ & s_{C_3 00}(Z_A) = 0, s_{C_2 00}(X_B) = 0; \\
& s_{C_3 01}(Z_A) = 0, s_{C_3 01}(X_B) = 1;
\end{aligned}$$

$$
\begin{aligned}
& s_{C_3 10}(Z_A) = 1, s_{C_3 10}(X_B) = 0; \\
& s_{C_3 11}(Z_A) = 1, s_{C_3 11}(X_B) = 1; \} \\
\mathcal{E}(C_4) = \{ & s_{C_4 00}(Z_A) = 0, s_{C_4 00}(Z_B) = 0; \\
& s_{C_4 11}(Z_A) = 1, s_{C_4 11}(Z_B) = 1; \} \\
e(C_1) = \{ & s_{C_1 00} \to 1/2, s_{C_1 11} \to 1/2\} \\
e(C_2) = \{ & s_{C_2 00} \to 1/4, s_{C_2 01} \to 1/4, s_{C_2 10} \to 1/4, s_{C_2 11} \to 1/4\} \\
e(C_3) = \{ & s_{C_3 00} \to 1/4, s_{C_3 01} \to 1/4, s_{C_3 10} \to 1/4, s_{C_3 11} \to 1/4\} \\
e(C_4) = \{ & s_{C_4 00} \to 1/2, s_{C_4 11} \to 1/2\}
\end{aligned}
$$

2.3 Global sections

In the following part we want to introduce and interpret global sections [1].

Definition 2.6. *A global section $s \in \mathcal{E}(U)$ is a section with*

$$s|C_i = s_i \quad \forall i \in I$$

where I is the set of numbers of the contexts.

This global assignment glues together a certain compatible family on a presheaf and allocates a definite outcome to each measurement. A model, where the measurement outcomes only depend on the measurement operator and not on the operators measured simultaneously with it, is called a noncontextual hidden variable model. In particular a deterministic hidden-variable model means that the model is deterministic for each fixed value of the hidden variable. Abramsky and his coworkers [1] determined locality and non-contextuality in terms of the existence of global sections:

Proposition 2.7. *The existence of a global section for an empirical model implies the existence of a non-contextual deterministic hidden-variable model which realizes it.*

Thus given an empirical model we have to check whether we can extend every section to a global one, i.e. we can realize every possible outcome combination for a context in a local hidden variable model. If this is the case the model is non-contextual. We name a model strongly contextual, if there is no global section at all.

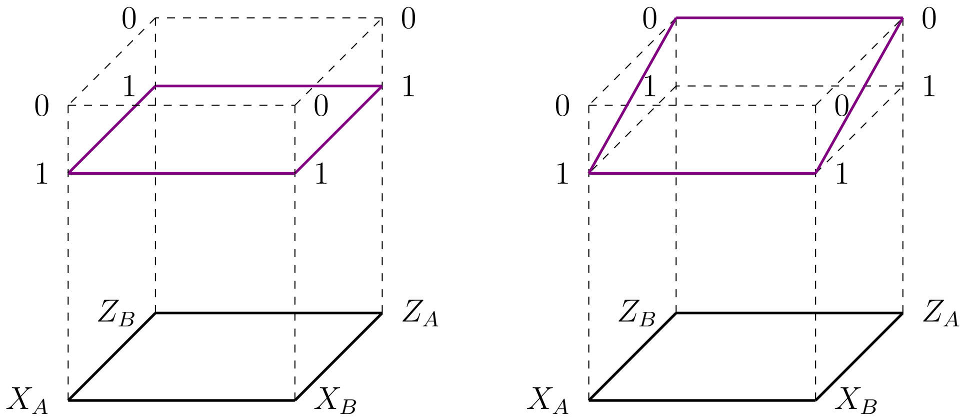


Figure 2.5: Examples for global sections measuring on $|\Psi\rangle$

A global section in the bundle diagram is represented by a closed path traversing all the fibers exactly once. Using this representation, we can have a direct feedback on the contextuality of empirical models. The above discussed model is non-contextual, because every section is expendable to a global one. Two examples of global sections are depicted in Figure 2.5.

In the last section we saw that the distributions $\mathcal{D}_R\mathcal{E}(C)$ on C form presheaf. We assume now that the sheaf condition holds for a specific model and $\mathcal{D}_R\mathcal{E}(C)$ is a sheaf. It follows that there exists a global section $d \in \mathcal{D}_R\mathcal{E}$ defined on the entire set of measurements M. Such a global section determines a distribution on the set $\mathcal{E}(M) = O^M$, which specifies assignments of outcomes to all measurements. This distribution, restricted to the probabilities specified by the empirical model, must maintain on all the measurement contexts in $\mathcal{M}$, that means

$$d|C = e_C \quad \forall\, C \in \mathcal{M}.$$

The idea of extendability of probability distributions is studied by Fine [17] in a general mathematical form. In this work we restrict ourselves to global sections of the sheaf $\mathcal{E}(M)$, because we showed that a possibilistic approach is sufficient to find out whether a model is contextual or not.

3 Examples of empirical models and bundle diagrams

3.1 Qutrit example

In Chapter 2 we considered a qubit model. Analogous we can build the same bipartite scenario with qutrits using the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle).$$

The eigenstates of Z are analogously defined by $|\nu_i\rangle = |i\rangle$ for $i \in \{0, 1, 3\}$. The eigenstates belonging to X change to

$$|\mu_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$$
$$|\mu_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \mu\,|1\rangle + \mu^2\,|2\rangle)$$
$$|\mu_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \mu^2\,|1\rangle + \mu\,|2\rangle)$$

with $\mu = \exp^{\frac{i2\pi}{3}}$. We built the empirical model by calculating the possibilities for different outcomes. Using a bundle diagram we can depict the model.

A	B	00	01	02	10	11	12	20	21	22
X_A	X_B	1/3	0	0	0	0	1/3	0	1/3	0
X_A	Z_B	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9
Z_B	X_B	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9	1/9
Z_B	Z_B	1/3	0	0	0	0	1/3	0	1/3	0

Table 3.1: Empirical model for the qutrit model

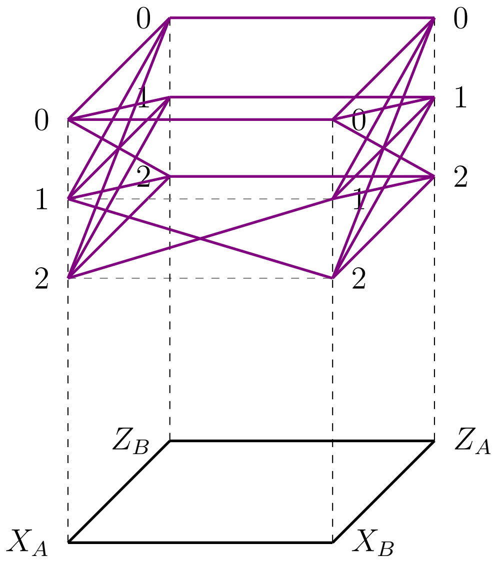


Figure 3.1: Bundle diagram for the qutrit model

Note that every section is expandable to a global one. It results that the model is non-contextual. Figure 3.2 shows two examples for global sections.

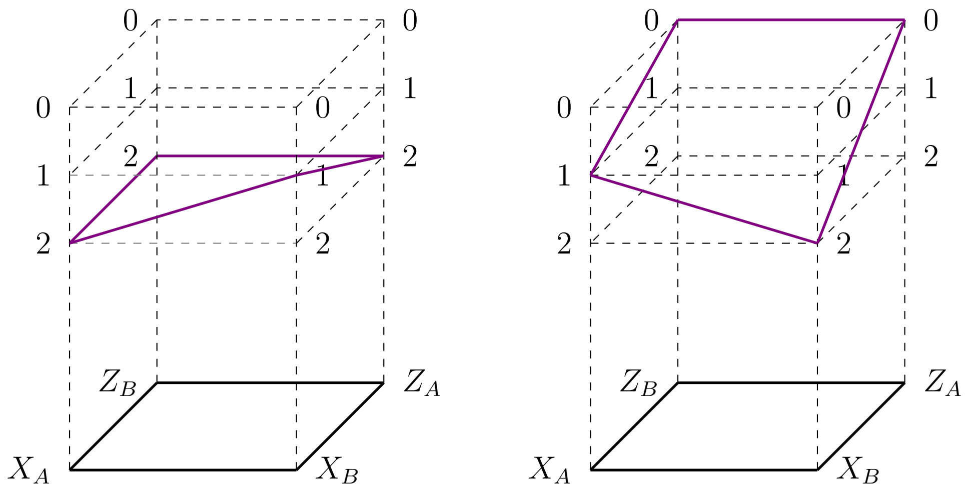


Figure 3.2: Examples for global sections for qutrit case

We also write out the presheaf and distributions of the contexts for this qutrit model example.

$$
\begin{aligned}
\mathcal{E}(C_1) = \{&s_{C_100}(X_A) = 0, s_{C_100}(X_B) = 0;\\
&s_{C_112}(X_A) = 1, s_{C_112}(X_B) = 2;\\
&s_{C_121}(X_A) = 2, s_{C_121}(X_B) = 1;\}\\
\mathcal{E}(C_2) = \{&s_{C_200}(X_A) = 0, s_{C_200}(Z_B) = 0;\\
&s_{C_201}(X_A) = 0, s_{C_201}(Z_B) = 1;
\end{aligned}
$$

$$
\begin{aligned}
& s_{C_202}(X_A) = 0, s_{C_202}(Z_B) = 2; \\
& s_{C_210}(X_A) = 1, s_{C_210}(Z_B) = 0; \\
& s_{C_211}(X_A) = 1, s_{C_211}(Z_B) = 1; \\
& s_{C_212}(X_A) = 1, s_{C_212}(Z_B) = 2; \\
& s_{C_220}(X_A) = 2, s_{C_220}(Z_B) = 0; \\
& s_{C_221}(X_A) = 2, s_{C_221}(Z_B) = 1; \\
& s_{C_222}(X_A) = 2, s_{C_222}(Z_B) = 2; \} \\
\mathcal{E}(C_3) = \{ & s_{C_300}(Z_A) = 0, s_{C_300}(X_B) = 0; \\
& s_{C_301}(Z_A) = 0, s_{C_301}(X_B) = 1; \\
& s_{C_302}(Z_A) = 0, s_{C_302}(X_B) = 2; \\
& s_{C_310}(Z_A) = 1, s_{C_310}(X_B) = 0; \\
& s_{C_311}(Z_A) = 1, s_{C_311}(X_B) = 1; \\
& s_{C_312}(Z_A) = 1, s_{C_312}(X_B) = 2; \\
& s_{C_320}(Z_A) = 2, s_{C_320}(X_B) = 0; \\
& s_{C_321}(Z_A) = 2, s_{C_321}(X_B) = 1; \\
& s_{C_322}(Z_A) = 2, s_{C_322}(X_B) = 2; \} \\
\mathcal{E}(C_4) = \{ & s_{C_400}(Z_A) = 0, s_{C_400}(Z_B) = 0; \\
& s_{C_411}(Z_A) = 1, s_{C_411}(Z_B) = 1; \} \\
& s_{C_422}(Z_A) = 2, s_{C_422}(Z_B) = 2; \} \\
e(C_1) = \{ & s_{C_100} \to 1/3, s_{C_112} \to 1/3, s_{C_121} \to 1/3\} \\
e(C_2) = \{ & s_{C_200} \to 1/9, s_{C_201} \to 1/9, s_{C_202} \to 1/9, s_{C_210} \to 1/9, s_{C_211} \to 1/9, \\
& s_{C_212} \to 1/9, s_{C_220} \to 1/9, s_{C_221} \to 1/9, s_{C_222} \to 1/9\} \\
e(C_3) = \{ & s_{C_300} \to 1/9, s_{C_301} \to 1/9, s_{C_302} \to 1/9, s_{C_310} \to 1/9, s_{C_311} \to 1/9, \\
& s_{C_312} \to 1/9, s_{C_320} \to 1/9, s_{C_3221} \to 1/9, s_{C_322} \to 1/9\} \\
e(C_4) = \{ & s_{C_400} \to 1/3, s_{C_411} \to 1/3, s_{C_422} \to 1/3\}
\end{aligned}
$$

3.2 Greenberger-Horne-Zeilinger state

We introduced measurement based quantum computing in Chapter 1. Browne and Briegel [10] present this method using the Greenberger-Horne-Zeilinger state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle)$ as a resource. In this example the classical control computer, as explained in Chapter 1, is a device containing only NOT$= |1\rangle\langle 0| + |0\rangle\langle 1|$ and CNOT$= |1\rangle\langle 1| \otimes X + |0\rangle\langle 0| \otimes \mathbb{1}$ operations. Adding any deterministic two-bit gate which is not itself a product of NOT and CNOT operations constitutes a classical universal gate set [10]. An example for this is the NAND-gate, which is described within Table 3.2 using 0 as false and 1 as true. By performing three measurements on the GHZ state it is possible to implement this gate [10]. We want to build the bundle diagram for these measurements and explore the contextuality of this tripartite model.

a	b	NAND
0	0	1
0	1	1
1	0	1
1	1	0

Table 3.2: Truth table for NAND-gate

In this example we have the set of measurements $X = \{X_A, X_B, X_C, Y_A, Y_B, Y_C\}$ with $Y = -i\left|0\right\rangle\left\langle 1\right| + i\left|1\right\rangle\left\langle 0\right|$. The choices of the measurements of Alice and Bob depend on the inputs a and b of the NAND-gate. Charlie measures addicted to a third supplemented input $a \oplus b$. If the input is 0 an X-measurement is accomplished. The input 1 suggests a Y-measurement. Due to this the contexts

$$\begin{aligned} C_1 &= \{X_A, X_B, X_C\} \\ C_2 &= \{X_A, Y_B, Y_C\} \\ C_3 &= \{Y_A, X_B, Y_C\} \\ C_4 &= \{Y_A, Y_B, X_C\} \end{aligned}$$

are allowed. We use the basis

$$\left|+\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) \qquad \left|-\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle)$$

for the X-measurements and

$$\left|\circlearrowleft\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + i\left|1\right\rangle) \qquad \left|\circlearrowright\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - i\left|1\right\rangle)$$

for the Z-measurements. We get the following empirical model.

	A	B	C	000	001	010	011	100	101	110	111
C_1	X_A	X_B	X_C	0	1/4	1/4	0	1/4	0	0	1/4
C_2	X_A	Y_B	Y_C	0	1/4	1/4	0	1/4	0	0	1/4
C_3	Y_A	X_B	Y_C	0	1/4	1/4	0	1/4	0	0	1/4
C_4	Y_A	Y_B	X_C	1/4	0	0	1/4	0	1/4	1/4	0

Table 3.3: Empirical model for MBQC

The bottom of the appropriate bundle diagram is depicted in Figure 3.3.

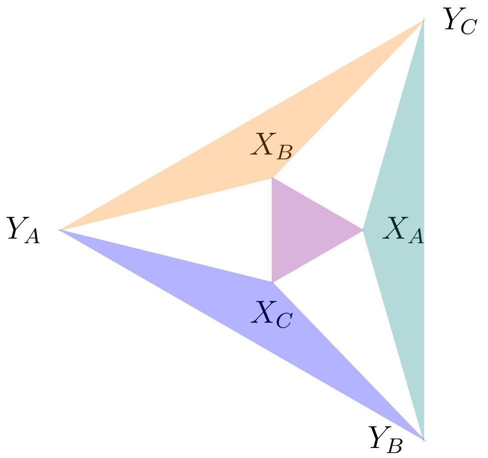


Figure 3.3: Bottom of bundle diagram for MBQC

For a better overview we consider the different contexts individually first and afterwards all together. Every context is assigned to a color. The violet triangles represent the sections above the context C_1. The context C_2 is marked with teal triangles. The sections related to C_3 are depicted in orange, the ones belonging to C_4 in blue.

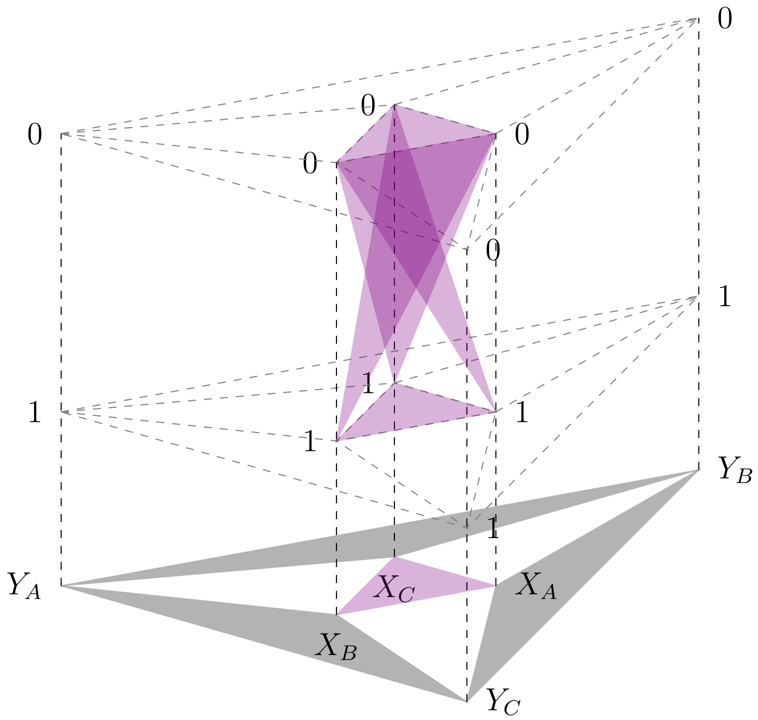


Figure 3.4: Bundle diagram for C_1

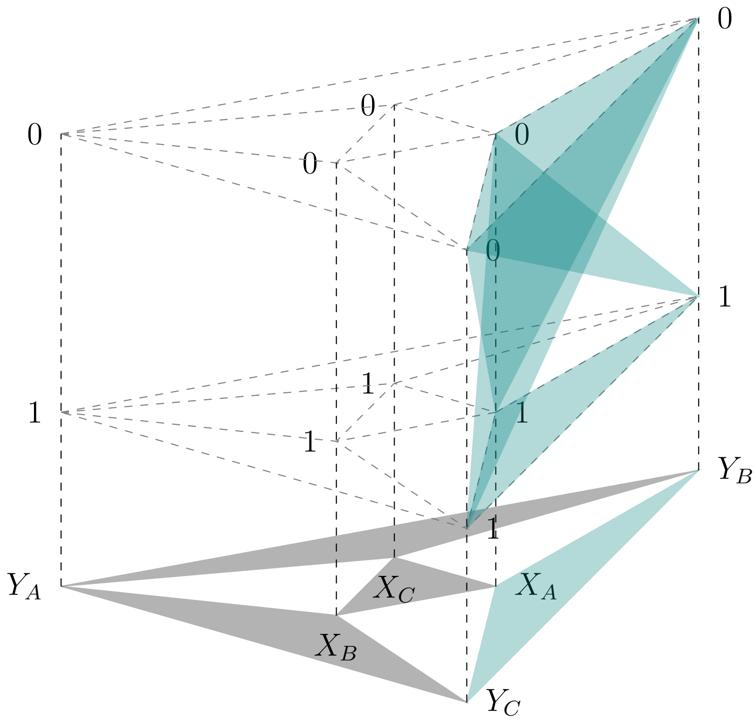


Figure 3.5: Bundle diagram for C_2

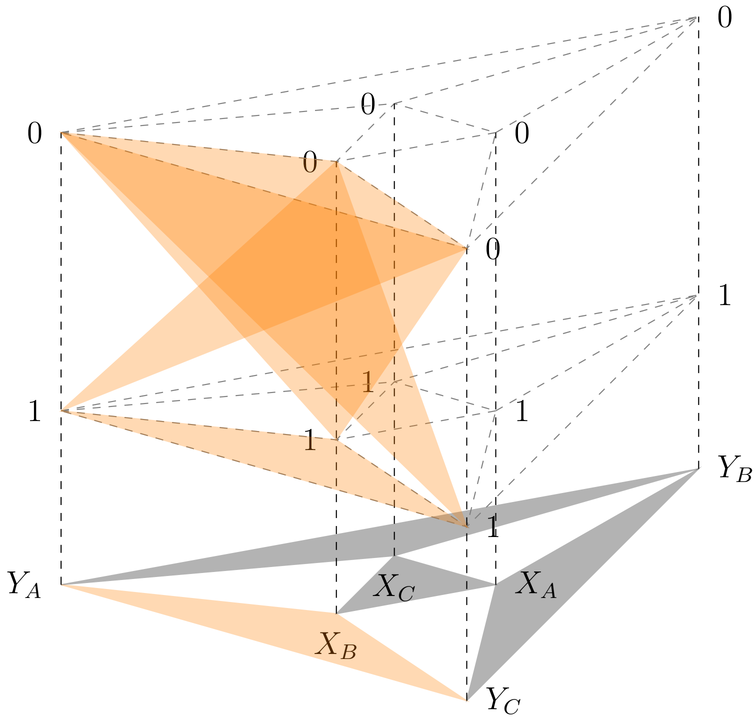


Figure 3.6: Bundle diagram for C_3

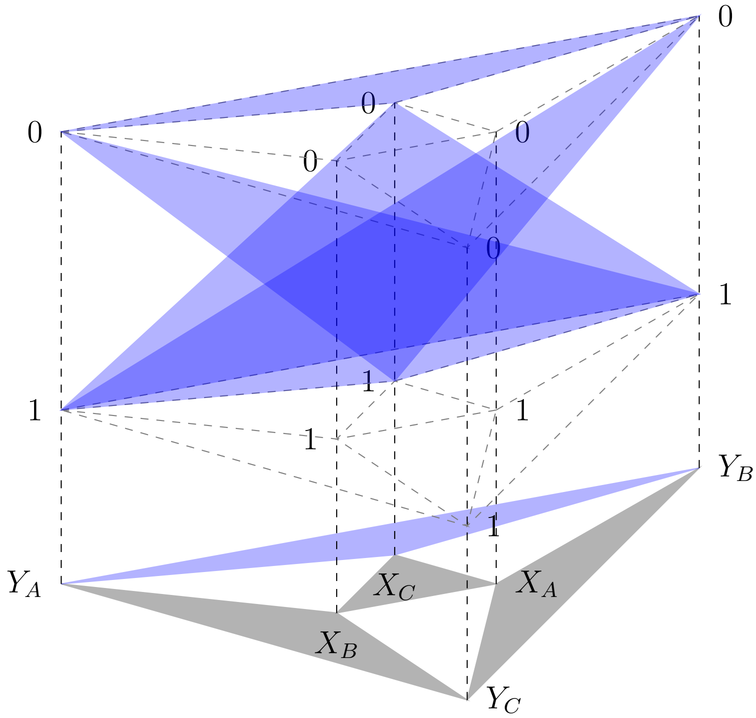


Figure 3.7: Bundle diagram for C_4

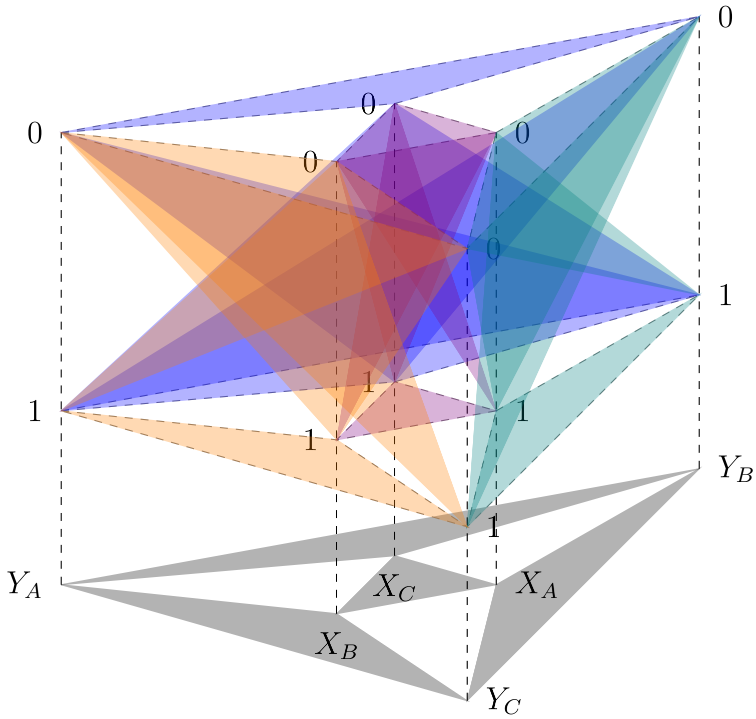


Figure 3.8: Bundle diagram with all contexts for MBQC

If there exists a triangle which cannot be continued to a global section, the model is contextual. We want to consider the violet triangle $X_A1 - X_B1 - X_C1$ and depict all ways to find a global section in a tree diagram.

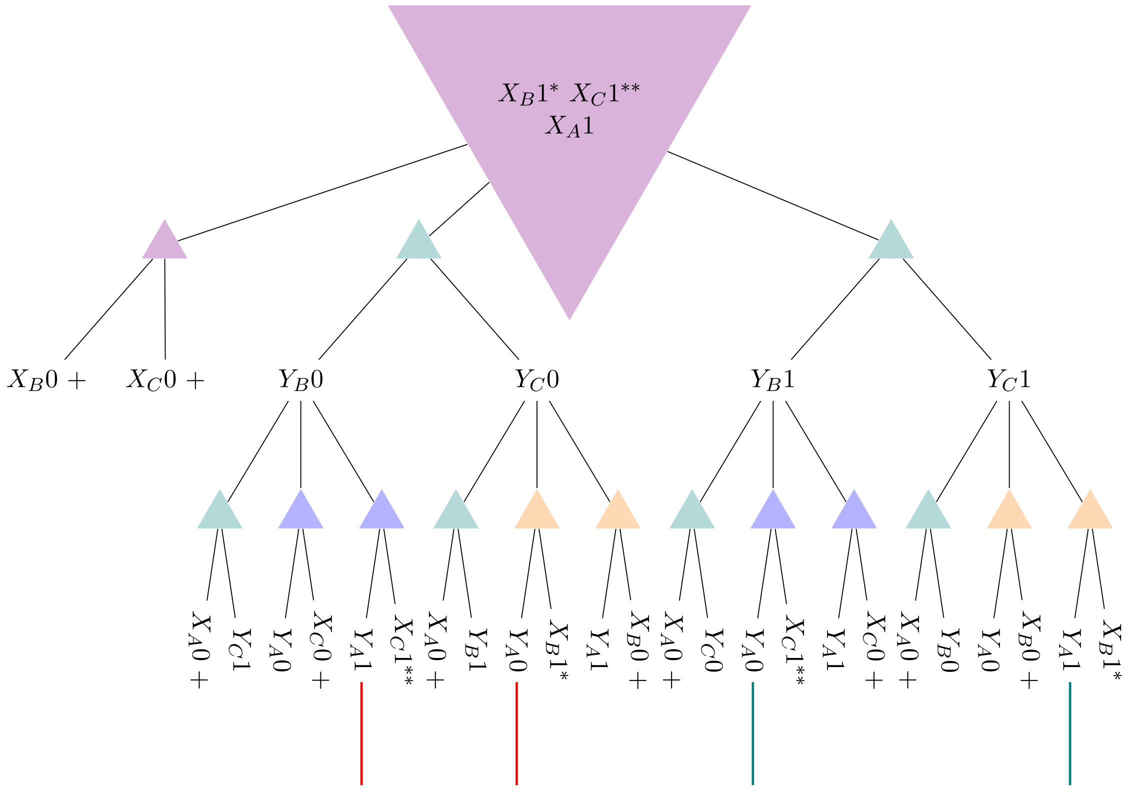


Figure 3.9: Trying to built a global section with the triangle $X_A1 - X_B1 - X_C1$.

We start with the chosen triangle and draw all triangles connected to the edge X_A1. In a global section one outcome per measurement is allowed only. We tag leafs of the tree where we can stop building the tree because this is not the case anymore with a + symbol. For example our start triangle contains X_A1, so we can stop when we meet the point X_A0.

Of course we also have to connect the edges X_B1 and X_C1 of the starting triangle correctly. When we meet them again we mark the nodes with $*$ or $**$. We can see that there is a possibility to connect the edges X_B1 and X_C1 of the start triangle and touch all measurements X_A, X_B, Y_A and Y_B ones, unless $Y_A1 \neq Y_A0$. This situation is tagged with red lines. A similar case applied on the teal lines. We can see graphically how building a global section fails in these two cases in Figure 3.10.

Using the tree diagram we see that it is not possible to build a complete global section with the triangle $X_A1 - X_B0 - X_C0$. Hence the model is contextual.

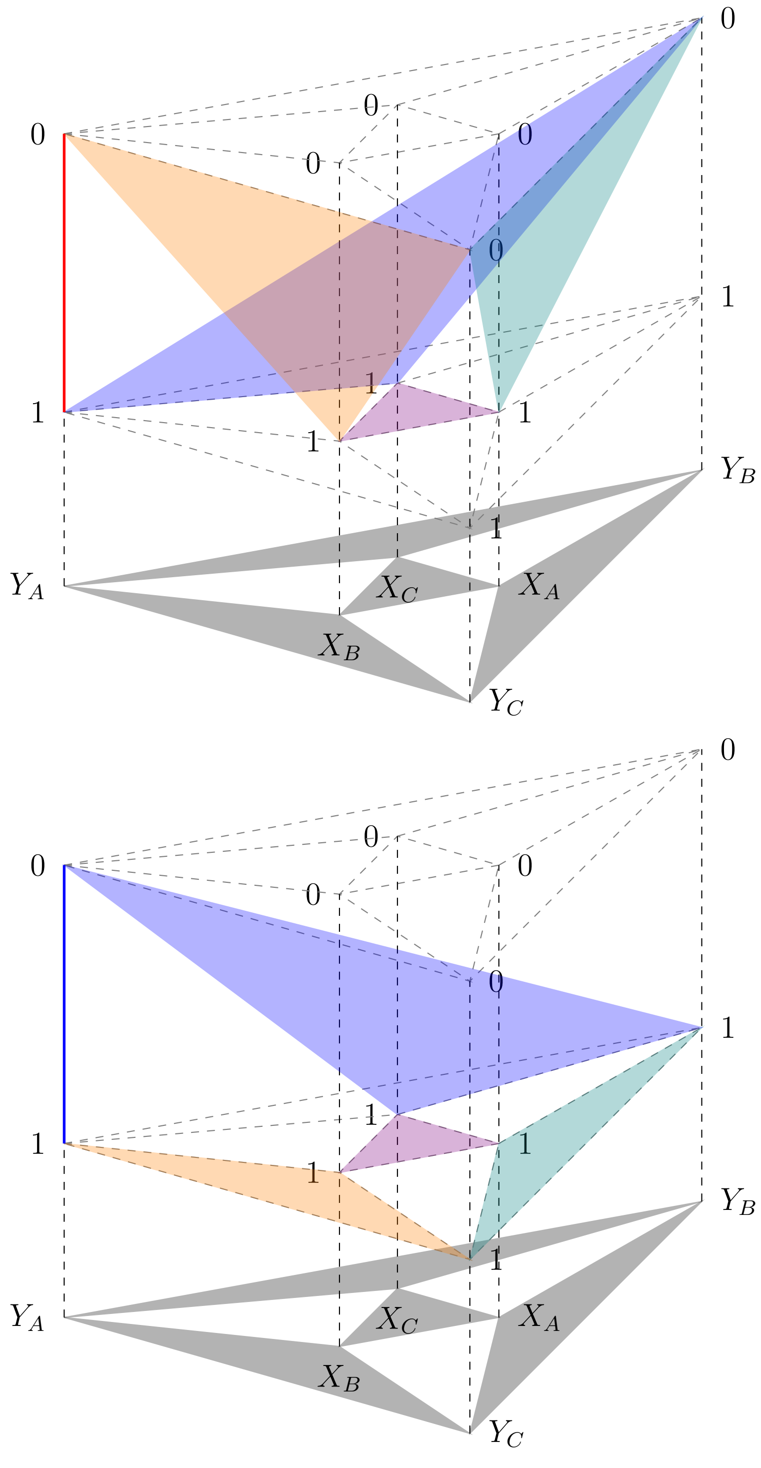


Figure 3.10: Examples for failed global sections for MBQC

We can show also that the model is strongly contextual, i.e. there is no global section at all. Therefore we define a function f : Support$[P(O)] \to \{\pm 1\}$ for each context, where $P(O)$ is the possibility to get the outcome $O = (a, b, c)$ measuring in the context. Considering C_1, the function is defined by

$$f_{C_1}(X_A = a, X_B = b, X_C = c) = (-1)^a(-1)^b(-1)^c$$

and produces the following codomain

$$\begin{aligned} f_{C_1}(0,0,1) &= -1 \\ f_{C_1}(0,1,0) &= -1 \\ f_{C_1}(1,0,0) &= -1 \\ f_{C_1}(1,1,1) &= -1. \end{aligned}$$

Defining the other functions analogously and computing the codomains we can observe that the codomain for one particular context is either $\{-1\}$ or $\{1\}$. So we can summarize the functions f_{C_i} for $i \in \{1, \dots, 4\}$ with an overall parity function $F : C_i \to \{\pm 1\}$ defined on $\{C_i | i \in \{1, \dots, 4\}\}$. We depict the values of the parity function for each context in Figure 3.11.

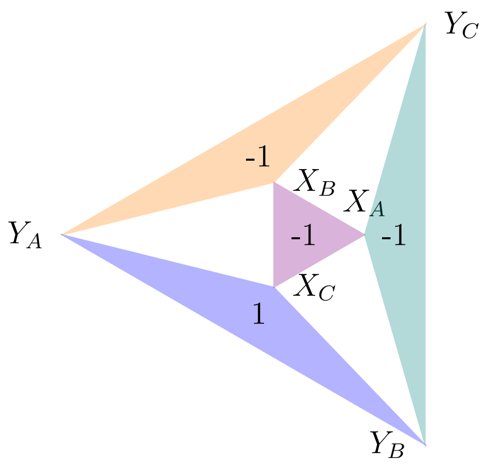


Figure 3.11: Values of parity function

It follows that all triangles have an odd number of edges on the 1-plain, except the orange ones. So it is not possible to built any global section. Also the structure of the NAND gate becomes clear in this approach, i.e.

$$\begin{aligned} X_A X_B X_C |\Psi\rangle &= -|\Psi\rangle \\ X_A Y_B Y_C |\Psi\rangle &= -|\Psi\rangle \\ Y_A X_B Y_C |\Psi\rangle &= -|\Psi\rangle \\ Y_A Y_B X_C |\Psi\rangle &= |\Psi\rangle. \end{aligned}$$

We were not able to find a global section using the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle)$. As a next step we want to figure out whether there exists a state which produces a model with a global section using the same contexts. Thus we search for a model which fulfills $F(C_i) = -1$ for an even number of contexts C_i. An appropriate set of conditions on $|\Psi\rangle$ is depicted in Figure 3.12 and requires

$$\begin{aligned}
X_A X_B X_C |\Psi\rangle &= - |\Psi\rangle \\
X_A Y_B Y_C |\Psi\rangle &= |\Psi\rangle \\
Y_A X_B Y_C |\Psi\rangle &= - |\Psi\rangle \\
Y_A Y_B X_C |\Psi\rangle &= |\Psi\rangle .
\end{aligned}$$

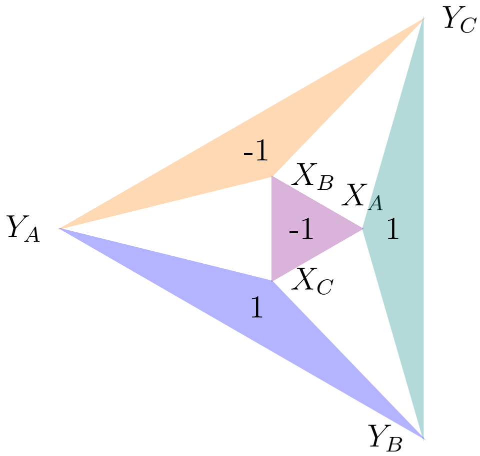


Figure 3.12: Values of parity function

We search for a state $|\Psi\rangle$, which fulfill the conditions. Therefore we build projectors

$$\begin{aligned}
P_1 &= \frac{\mathbb{1} - X_A X_B X_C}{2} \\
P_2 &= \frac{\mathbb{1} + X_A Y_B Y_C}{2} \\
P_3 &= \frac{\mathbb{1} - Y_A Y_B X_C}{2} \\
P_4 &= \frac{\mathbb{1} + X_A X_B Y_C}{2}.
\end{aligned}$$

If $P_1 |\Psi\rangle = P_2 |\Psi\rangle = P_3 |\Psi\rangle = P_4 |\Psi\rangle = |\Psi\rangle$ yields than $\mathrm{tr}(P_1 P_2 P_3 P_4) = 1$ and therefore $P_1 P_2 P_3 P_4 = |\Psi\rangle \langle\Psi|$ [31]. In the case depicted in Figure 3.12 we get $P_1 P_2 P_3 P_4 = 0$. Hence the set of conditions does not lead us to a state $|\Psi\rangle \neq 0$.

Calculating the states resulting of all sets of conditions satisfying $F(C_i) = -1$ for an even number of the four contexts C_i analogously, we find an interesting fact.

Lemma 3.1. *The states with an even number of minus signs in the projectors vanish* *using the contexts*

$$\begin{aligned} C_1 &= \{X_A, X_B, X_C\} \\ C_2 &= \{X_A, Y_B, Y_C\} \\ C_3 &= \{Y_A, X_B, Y_C\} \\ C_4 &= \{Y_A, Y_B, X_C\}. \end{aligned}$$

It is only possible to find a state $|\Psi\rangle \neq 0$ *for an odd number of minus signs in the projectors.*

Proof. See appendix. □

On the one hand we reasoned that for a state with an odd number of minus signs in the projection there is no global face. On the other hand we proved that there is no state $|\Psi\rangle \neq 0$ satisfying $F(C_i) = -1$ for an even number of C_i. Thus there is no state producing an empirical model with a global section using the contexts defined at the beginning of the section at all.

3.3 Cluster state ring

In this chapter we consider a tripartite model made up of a ring of n qubits. The measurement contexts are defined as $C_j = \{Z_{(j-1) \bmod n}, X_j, Z_{(j+1) \bmod n}\}$ for $j \in \{1, ..., n\}$. The context C_1 is depicted in Figure 3.13. With this setting measurement based quantum computing is feasible. The resource state is a cluster state [30] which satisfies $Z_{(j-1) \bmod n} X_j Z_{(j+1) \bmod n} |\psi\rangle = |\psi\rangle$ for all j.

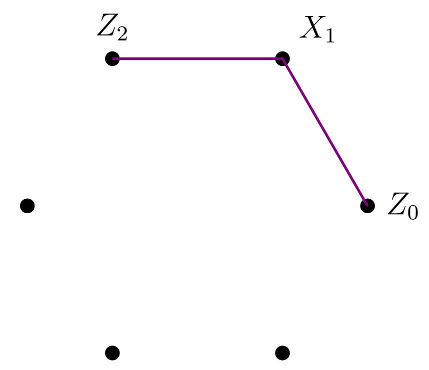


Figure 3.13: Context C_1 for a cluster state model n=6

3.3.1 Three qubits

We consider a cluster state model with $n = 3$ first, hence the model possesses three contexts. A stabilizer state for this is

$$|\Psi\rangle = \frac{1}{\sqrt{8}}(-|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

with stabilizer equalities

$$XZZ\,|\Psi\rangle = |\Psi\rangle$$
$$ZXZ\,|\Psi\rangle = |\Psi\rangle$$
$$ZZX\,|\Psi\rangle = |\Psi\rangle$$

where XZZ is the shortcut for $X \otimes Z \otimes Z$.

	1	2	3	000	001	010	011	100	101	110	111
C_1	X_1	Z_2	Z_3	1/4	0	0	1/4	0	1/4	1/4	0
C_2	Z_1	X_2	Z_3	1/4	0	0	1/4	0	1/4	1/4	0
C_3	Z_1	Z_2	X_3	1/4	0	0	1/4	0	1/4	1/4	0

Table 3.4: Empirical model for cluster state $n = 3$

The resulting empirical model is presented in Table 3.4. Using the same colors for C_1 to C_3 as before we depict it in a bundle diagram.

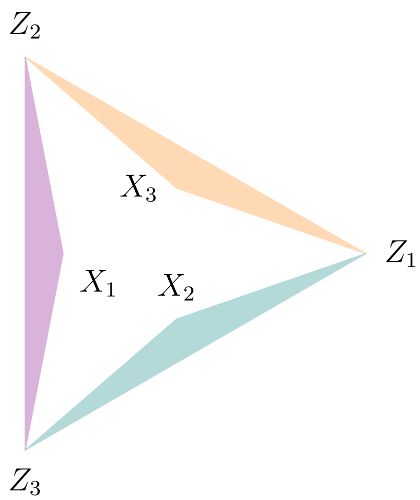


Figure 3.14: Base of bundle diagram for cluster state n=3

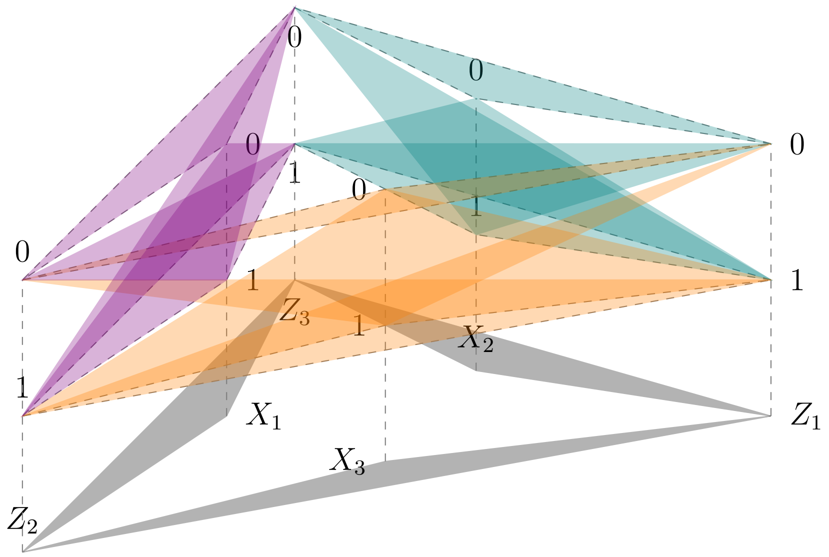


Figure 3.15: Bundle diagram for cluster state $n = 3$

Again we want to work out whether the model is contextual. Because the sections behave the same over all contexts we choose context $C_2 = \{Z_1, X_2, Z_3\}$ and consider each of the teal triangles.

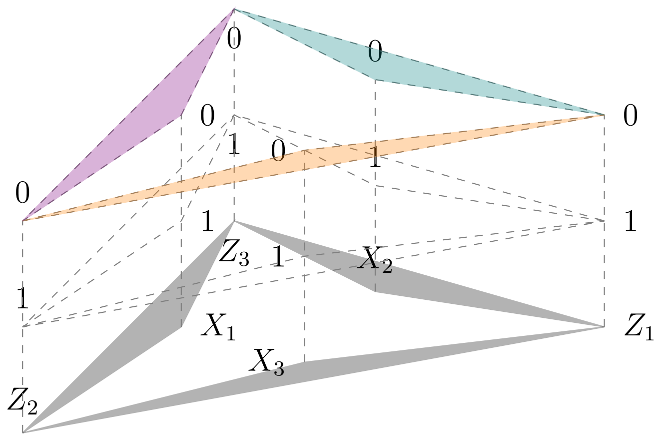


Figure 3.16: Global section for outcome (000) in context C_2

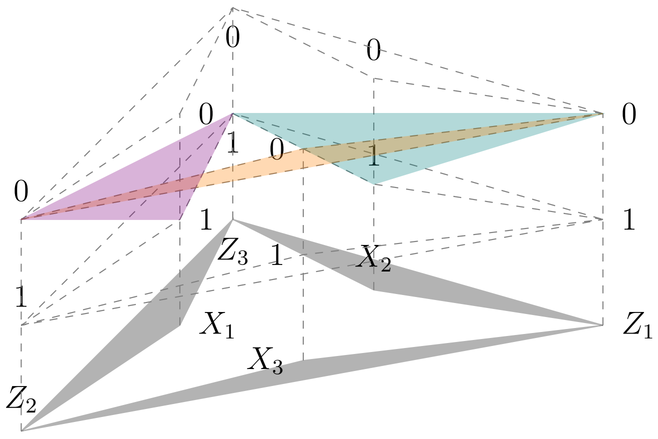


Figure 3.17: Global section for outcome (011) in context C_2

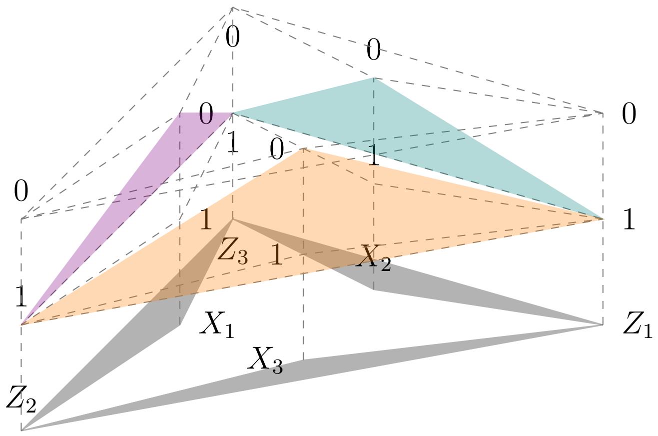


Figure 3.18: Global section for outcome (101) in context C_2

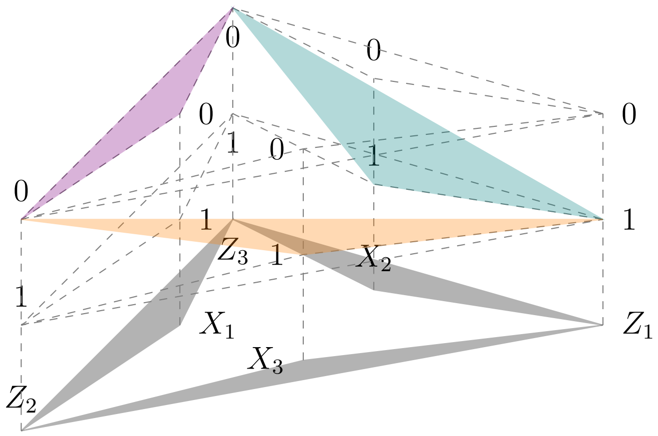


Figure 3.19: Global section for outcome (110) in context C_2

We find a global section for every section over C_2 and analogously for every section over C_1 and C_3. The model is non-contextual. In the appendix the cases of cluster state rings with four and five qubits are listed.

3.3.2 Six qubits

Lastly we set $n = 6$ and take a look at the resulting model. The state

$$
\begin{aligned}
|\Psi\rangle = \frac{1}{8}(&- |000000\rangle - |000001\rangle - |000010\rangle + |000011\rangle - |000100\rangle - |000101\rangle \\
&+ |000110\rangle - |000111\rangle - |000111\rangle - |001001\rangle - |001010\rangle + |001011\rangle \\
&+ |001100\rangle + |001101\rangle - |001110\rangle + |001111\rangle - |010000\rangle - |010001\rangle \\
&- |010010\rangle + |010011\rangle - |010100\rangle - |010101\rangle + |010110\rangle - |010111\rangle \\
&+ |010111\rangle + |011001\rangle + |011010\rangle - |011011\rangle - |011100\rangle - |011101\rangle \\
&+ |011110\rangle - |011111\rangle - |100000\rangle + |100001\rangle - |100010\rangle - |100011\rangle \\
&- |100100\rangle + |100101\rangle + |100110\rangle + |100111\rangle - |100111\rangle + |101001\rangle \\
&- |101010\rangle - |101011\rangle + |101100\rangle - |101101\rangle - |101110\rangle - |101111\rangle \\
&+ |110000\rangle - |110001\rangle + |110010\rangle + |110011\rangle + |110100\rangle - |110101\rangle \\
&- |110110\rangle - |110111\rangle - |110111\rangle + |111001\rangle - |111010\rangle - |111011\rangle \\
&+ |111100\rangle - |111101\rangle - |111110\rangle - |111111\rangle)
\end{aligned}
$$

is a stabilizer.

$$
\begin{aligned}
XZ\mathbb{1}\mathbb{1}\mathbb{1}Z\,|\Psi\rangle &= |\Psi\rangle \\
ZXZ\mathbb{1}\mathbb{1}\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\
\mathbb{1}ZXZ\mathbb{1}\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\
\mathbb{1}\mathbb{1}ZXZ\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\
\mathbb{1}\mathbb{1}\mathbb{1}ZXZ\,|\Psi\rangle &= |\Psi\rangle \\
Z\mathbb{1}\mathbb{1}\mathbb{1}ZX\,|\Psi\rangle &= |\Psi\rangle
\end{aligned}
$$

	1	2	3	4	5	6	000	001	010	011	100	101	110	111
C_1	X_1	Z_2	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	Z_6	1/4	0	0	1/4	0	1/4	1/4	0
C_2	Z_1	X_2	Z_3	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_3	$\mathbb{1}$	Z_2	X_3	Z_4	$\mathbb{1}$	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_4	$\mathbb{1}$	$\mathbb{1}$	Z_3	X_4	Z_5	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_5	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	Z_4	X_5	Z_6	1/4	0	0	1/4	0	1/4	1/4	0
C_6	Z_1	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	Z_5	X_6	1/4	0	0	1/4	0	1/4	1/4	0

Table 3.5: Empirical model for cluster state $n = 6$

Depicting the resulting empirical model in a bundle diagram we can make a statement concerning the contextuality of the model. Analogously to the bundle models built for the example before we use the same colors. The sections belonging to the context C_4 and C_5 are depicted in blue and pink. The ones above C_6 are drawn in red.

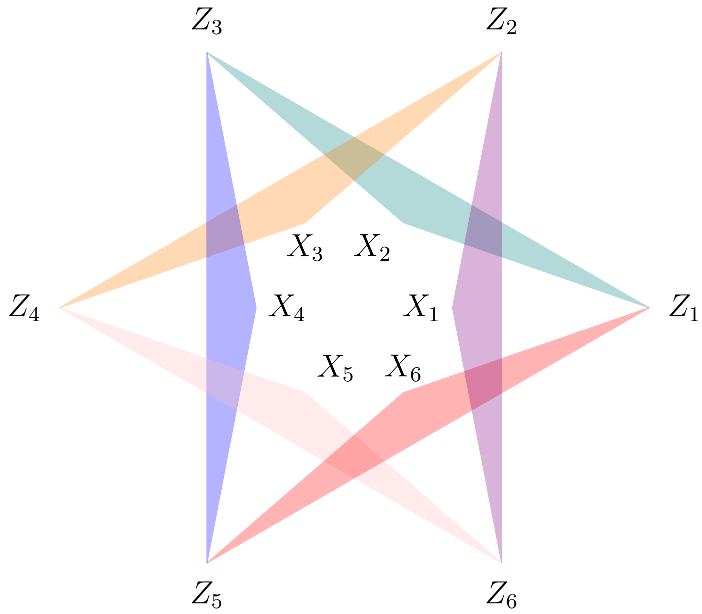


Figure 3.20: Base of bundle diagram for cluster state n=6

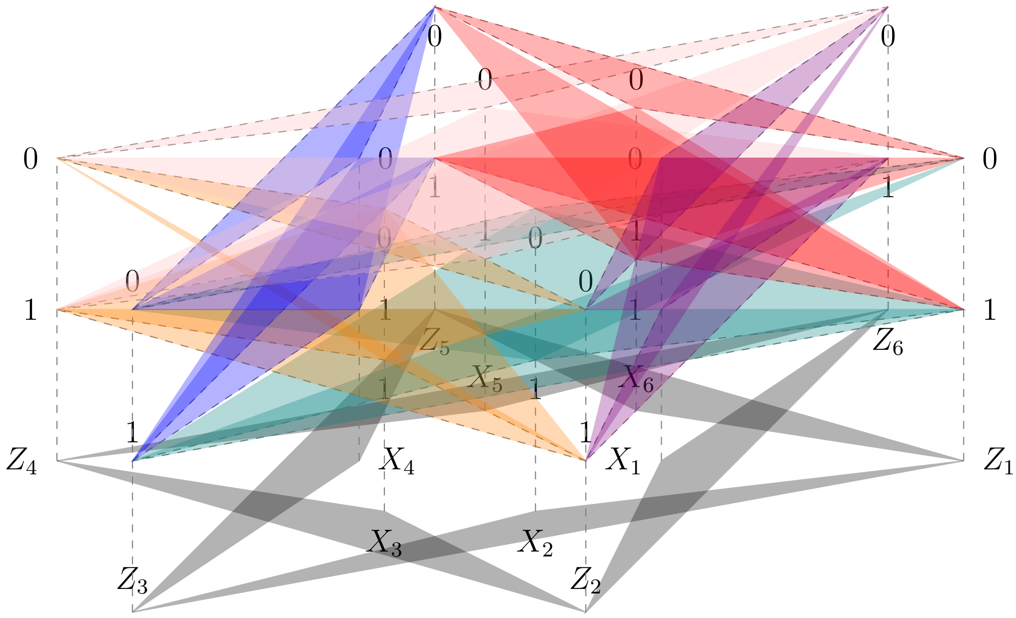


Figure 3.21: Bundle diagram for cluster state n=6

As in the case $n = 4$, depicted in the appendix, one can see that every global section is divided in two rings. Figure 3.3.2 is an example for a global section. Building analogously the bundle diagrams it results that this is the fact for all cluster state ring models with even numbers of qubits.

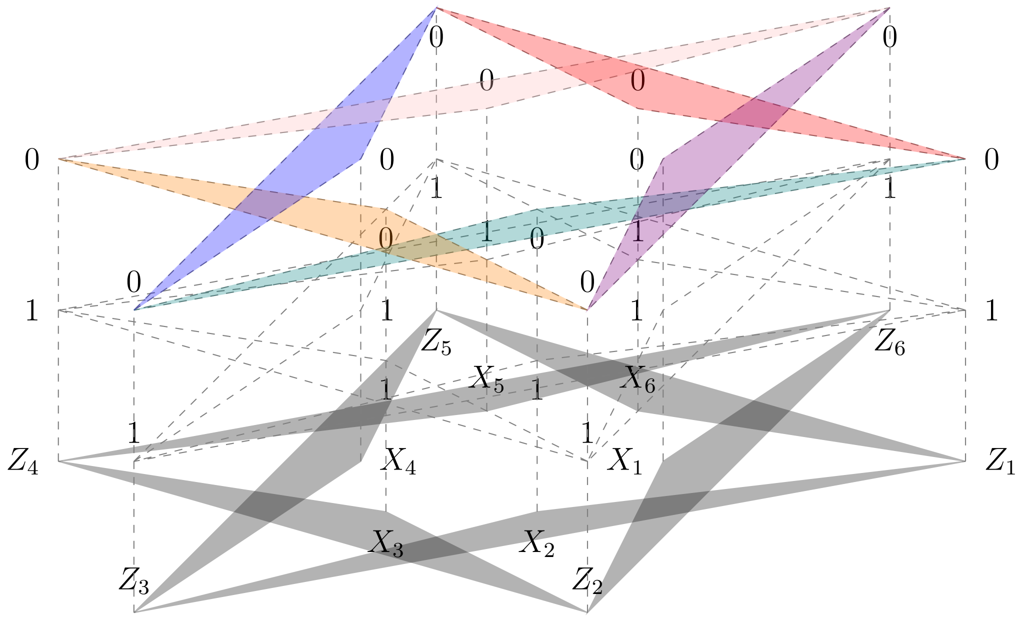


Figure 3.22: Example for global section

Likewise this model is non-contextual, because we can find a global section for each triangle. Looking at the empirical models for $n = 3$ to $n = 6$ there is no reason, why the pattern should not hold for all $n \in \mathbb{N}$. It follows that the cluster state models are non-contextual for all $n \in \mathbb{N}$ using the stabilizer state Ψ determined at the beginning of this section. In the following we study this more generally for all stabilizer states using the stabilizer properties.

3.4 State-independent cluster model

Using the properties of the stabilizer states, i.e. $|\Psi\rangle S = |\Psi\rangle$, measuring becomes a lot simpler. When we measure σ_k^α with $\alpha \in \{X, Z\}$ and $k \in \{1, \ldots, n\}$ we have to build the projector $P_\pm = \frac{\mathbb{1}+\sigma_k^\alpha}{2}$ to get the probabilities

$$p_\pm = \langle\Psi| P_\pm |\Psi\rangle = \frac{1}{2} \pm \frac{1}{2} \langle\Psi| \sigma_k^\alpha |\Psi\rangle .$$

It is not necessary to write out the state. Instead we can use the information whether the measurement operator commutes with all stabilizers S_j.
Suppose $\sigma_k^\alpha |\Psi\rangle = \sigma_k^\alpha S |\Psi\rangle = -S\sigma_k^\alpha |\Psi\rangle$ then

$$\begin{aligned} \langle\Psi| \sigma_k^\alpha |\Psi\rangle &= \langle\Psi| \sigma_k^\alpha S |\Psi\rangle \\ &= -\langle\Psi| S\sigma_k^\alpha |\Psi\rangle \\ &= -\langle\Psi| \sigma_k^\alpha |\Psi\rangle \end{aligned}$$

because $\langle\Psi| S = \langle\Psi|$ and consequently the term vanishes. We find the relation

$$\{S_j, \sigma_k^\alpha\} = 0 \Rightarrow \langle\Psi| \sigma_k^\alpha |\Psi\rangle = 0.$$

In the considered case $+\sigma_k^\alpha$ or $-\sigma_k^\alpha$ is a member of the stabilizer group, that means

$$\langle\Psi| \sigma_k^\alpha |\Psi\rangle = \mathrm{tr}\left(\sigma_k^\alpha |\Psi\rangle\langle\Psi|\right) = \pm 1.$$

So it remains to evaluate what occurs when $[S_j, \sigma_k^\alpha] = 0\ \forall j$.

3.4.1 In one dimension

With this state-independent description we can use the rules from last section to easily recalculate the probabilities for measurement outcomes. First we want to consider a one dimensional cluster state with $n = 4$. The code can be found in the appendix. With o_i for the measurement outcome on the ith qubit the following probabilities result:

$$\begin{aligned}
\mathrm{Prob}[X, Z, \mathbb{1}, Z] &= \frac{1}{4}(1 + o_1 o_2 o_4) \\
\mathrm{Prob}[Z, X, Z, \mathbb{1}] &= \frac{1}{4}(1 + o_1 o_2 o_3) \\
\mathrm{Prob}[\mathbb{1}, Z, X, Z] &= \frac{1}{4}(1 + o_2 o_3 o_4) \\
\mathrm{Prob}[Z, \mathbb{1}, Z, X] &= \frac{1}{4}(1 + o_1 o_3 o_4).
\end{aligned}$$

This corresponds to our results of Table A.1 in the appendix using the X-base instead of the Z-base, i.e. the first outcome value is named -1 instead of 0 here. Doing the same for $n = 5$, we also get the model in Table A.2:

$$\begin{aligned}
\mathrm{Prob}[X, Z, \mathbb{1}, \mathbb{1}, Z] &= \frac{1}{4}(1 + o_1 o_2 o_5) \\
\mathrm{Prob}[Z, X, Z, \mathbb{1}, \mathbb{1}] &= \frac{1}{4}(1 + o_1 o_2 o_3) \\
\mathrm{Prob}[\mathbb{1}, Z, X, Z, \mathbb{1}] &= \frac{1}{4}(1 + o_2 o_3 o_4) \\
\mathrm{Prob}[\mathbb{1}, \mathbb{1}, Z, X, Z] &= \frac{1}{4}(1 + o_3 o_4 o_5) \\
\mathrm{Prob}[Z, \mathbb{1}, \mathbb{1}, Z, X] &= \frac{1}{4}(1 + o_1 o_4 o_5).
\end{aligned}$$

So we proved that the possibilities are independent of the choice of the stabilizer state.

3.4.2 In two dimensions

Preparing Chapter 5, we consider a two dimensional cluster state model. The stabilizers in this model appear as stars out of X- and Z- measurement operators in a lattice of qubits. The shape of the operator and the names of the outcomes are defined in Figure 3.23.

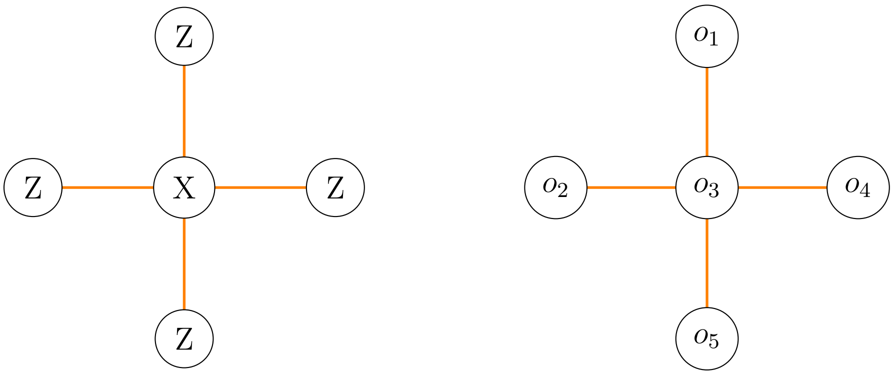


Figure 3.23: Measurements on stars

There are three different cases which can appear. We can measure a star inside of the cluster state, a star at one of the four sides or the stars in the four corners. Depending on the position of the measured star more or less commutators vanish.

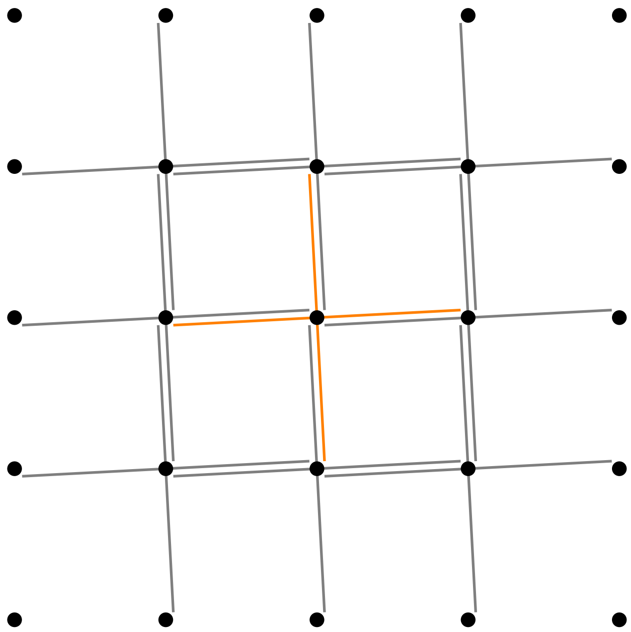


Figure 3.24: Star in the middle of a cluster

First we want to calculate a measurement in the middle of the cluster. The code is situated in the appendix. We can reveal that the probability is

$$\text{Prob}[Z, Z, X, Z, Z] = \frac{1}{32}(1 + o_1 o_2 o_3 o_4 o_5).$$

You can see that the outcome possibilities for a star in the middle of a cluster are rotational symmetric. For a star at one frame side of the cluster there are less stabilizers and one can not see the symmetry anymore.

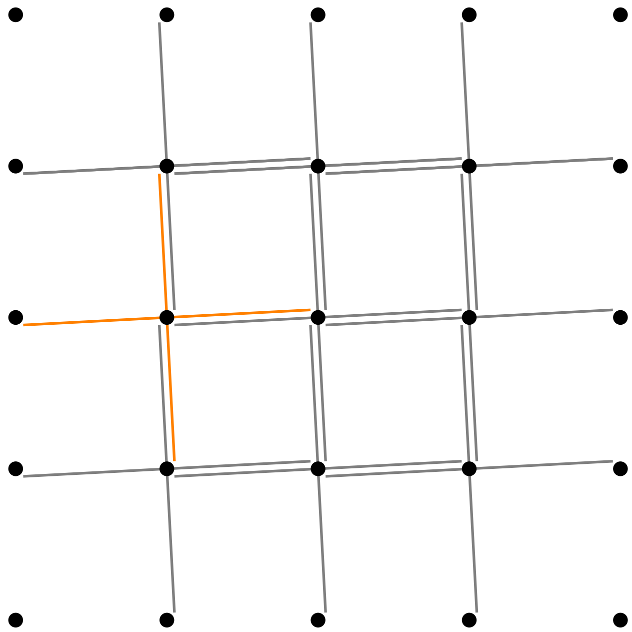


Figure 3.25: Star in the left side of a cluster

The four stabilizers for a side star are

$$\begin{aligned} S_1 &= X\mathbb{1}Z\mathbb{1}\mathbb{1} \\ S_2 &= ZZXZZ \\ S_3 &= \mathbb{1}\mathbb{1}ZX\mathbb{1} \\ S_4 &= \mathbb{1}\mathbb{1}Z\mathbb{1}X. \end{aligned}$$

For the star on the left side we get the probability

$$\text{Prob}[Z, Z, X, Z, Z] = \frac{1}{32}(1 + o_2 + o_1 o_3 o_4 o_5 + o_1 o_2 o_3 o_4 o_5).$$

It is easy to work out analogously the probability on one of the other three sides of the cluster now.

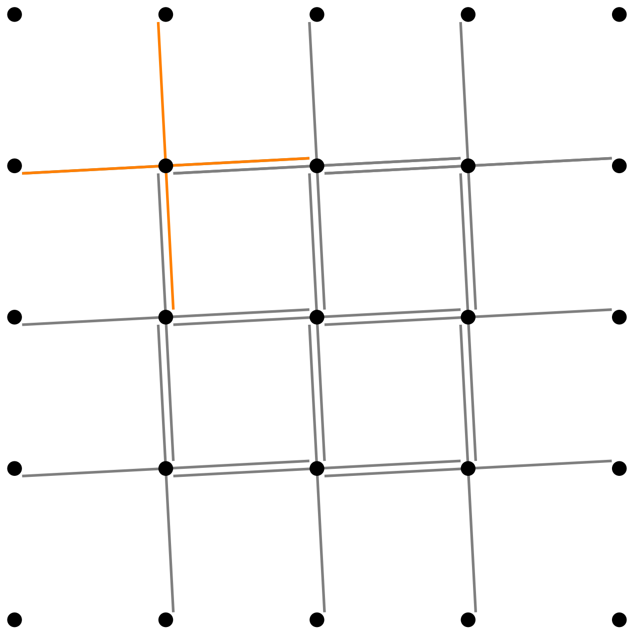


Figure 3.26: Star in the left upper corner of a cluster

For a measurement in one corner of the cluster the stabilizers are different again.

$$S_1 = ZZXZZ$$
$$S_2 = \mathbb{1}\mathbb{1}ZX\mathbb{1}$$
$$S_3 = \mathbb{1}\mathbb{1}Z\mathbb{1}X$$

$$\text{Prob}[Z,Z,X,Z,Z] = \frac{1}{32}(1 + o_1 + o_2 + o_1o_2 + o_3o_4o_5 + o_1o_3o_4o_5 + o_2o_3o_4o_5 + o_1o_2o_3o_4o_5).$$

Also in that instance it is not difficult to generalize the result to the other corners of the cluster. We can now summarize our results. In Table 3.7 one can see the possibilities for special outcomes $(o_1, o_2, o_3, o_4, o_5)$ considering the three different cases.

Outcomes	Middle star	Left Side star	Left Upper Corner star
$(-1,-1,-1,-1,-1)$	0	0	0
$(-1,-1,-1,-1,1)$	1/16	0	0
$(-1,-1,-1,1,-1)$	1/16	0	0
$(-1,-1,-1,1,1)$	0	0	0
$(-1,-1,1,-1,-1)$	1/16	0	0
$(-1,-1,1,-1,1)$	0	0	0
$(-1,-1,1,1,-1)$	0	0	0
$(-1,-1,1,1,1)$	1/16	0	0
$(-1,1,-1,-1,-1)$	1/16	1/8	0
$(-1,1,-1,-1,1)$	0	0	0
$(-1,1,-1,1,-1)$	0	0	0
$(-1,1,-1,1,1)$	1/16	1/8	0
$(-1,1,1,-1,-1)$	0	0	0
$(-1,1,1,-1,1)$	1/16	1/8	0
$(-1,1,1,1,-1)$	1/16	1/8	0
$(-1,1,1,1,1)$	0	0	0
$(1,-1,-1,-1,-1)$	1/16	0	0
$(1,-1,-1,-1,1)$	0	0	0
$(1,-1,-1,1,-1)$	0	0	0
$(1,-1,-1,1,1)$	1/16	0	0
$(1,-1,1,-1,-1)$	0	0	0
$(1,-1,1,-1,1)$	1/16	0	0
$(1,-1,1,1,-1)$	1/16	0	0
$(1,-1,1,1,1)$	0	0	0
$(1,1,-1,-1,-1)$	0	0	0
$(1,1,-1,-1,1)$	1/16	1/8	1/4
$(1,1,-1,1,-1)$	1/16	1/8	1/4
$(1,1,-1,1,1)$	0	0	0
$(1,1,1,-1,-1)$	1/16	1/8	1/4
$(1,1,1,-1,1)$	0	0	0
$(1,1,1,1,-1)$	0	0	0
$(1,1,1,1,1)$	1/16	1/8	1/4

Table 3.6: Empirical model of a 2D cluster state

3.5 Toric code

Another example of a stabilizer code is the toric code, which we will consider in Chapter 5. It is defined on a 2-dimensional lattice with stabilizer operators around each vertex and each plaquette. Gross and coworkers [18] discussed variations of Kitaev's toric code states [24] as resource states for MBQC.

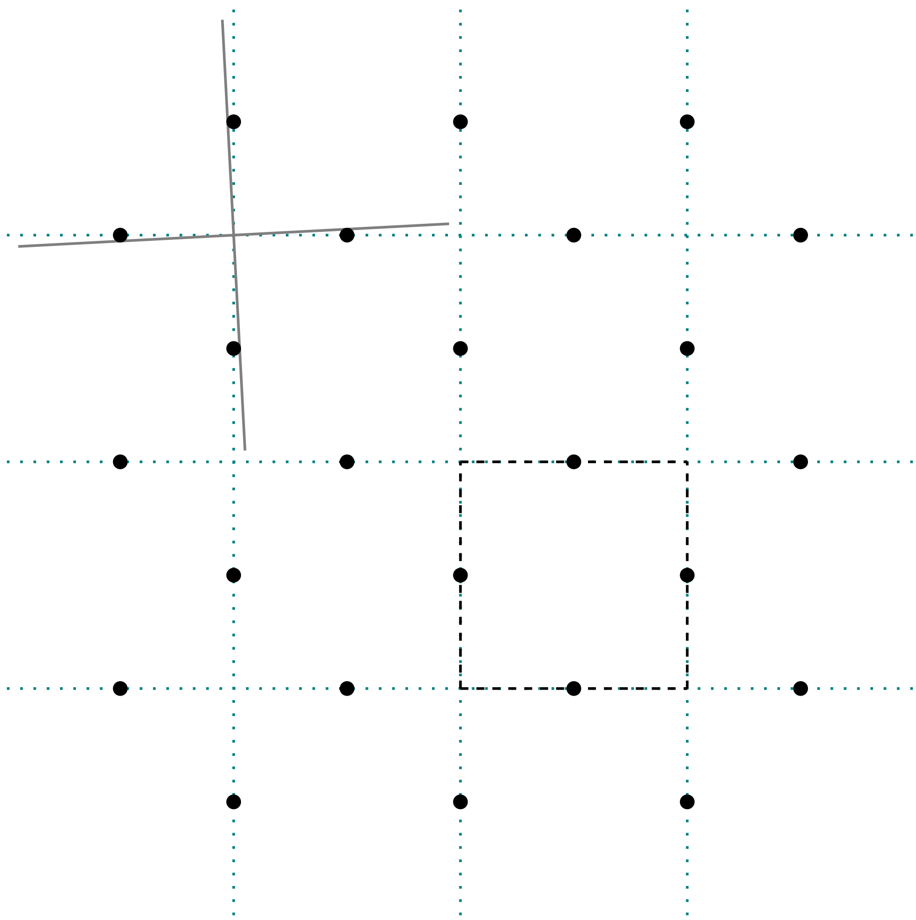


Figure 3.27: Star in the middle of a toric code

The vertex stabilizer operator is defined by $A_v = \prod_{i \in v} X_i$ on the qubits i around each vertex v. Analogously the plaquette operator is $B_p = \prod_{i \in p} X_i$ around each plaquette. We name the outcomes of the measurement operators o_1, o_2, o_3, o_4 in the following order.

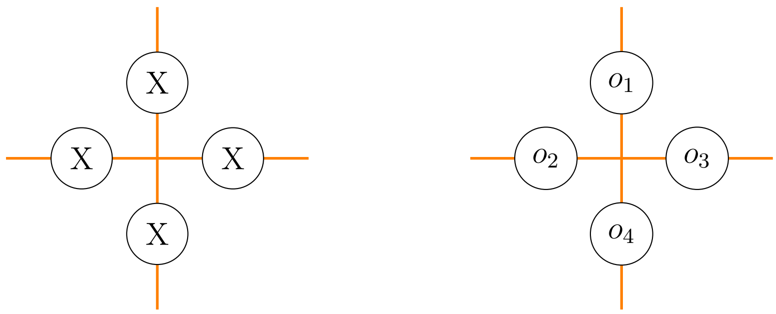


Figure 3.28: Measurements A_v and appropriate outcomes on vertices

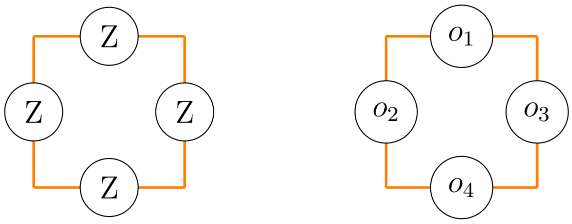


Figure 3.29: Measurements B_p and appropriate outcomes on plaquettes

To begin we calculate the probabilities for measuring an A_V operator in the middle of the toric code.

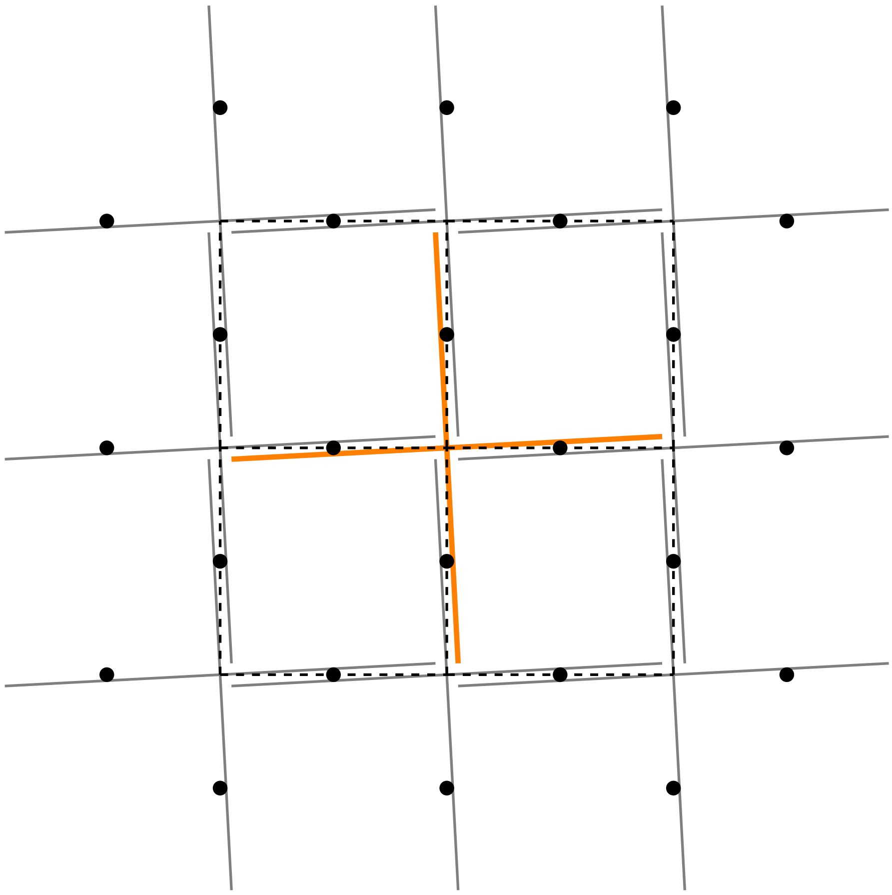


Figure 3.30: Star in the middle of a toric code

The stabilizers for this case are defined the following way.

$$
\begin{aligned}
S_1 &= ZZ\mathbb{1}\mathbb{1}\\
S_2 &= Z\mathbb{1}Z\mathbb{1}\\
S_3 &= XXXX\\
S_4 &= \mathbb{1}Z\mathbb{1}Z\\
S_5 &= \mathbb{1}\mathbb{1}ZZ\\
S_6 &= X\mathbb{1}\mathbb{1}\mathbb{1}\\
S_7 &= \mathbb{1}X\mathbb{1}\mathbb{1}\\
S_8 &= \mathbb{1}\mathbb{1}X\mathbb{1}\\
S_9 &= \mathbb{1}\mathbb{1}\mathbb{1}X
\end{aligned}
$$

The probability to get outcome (o_1, o_2, o_3, o_4) is

$$\text{Prob}[X, X, X, X] = \frac{1}{16}(1 + o_1 o_2 o_3 o_4).$$

We take also a view on the probabilities for measurement operator B_p.

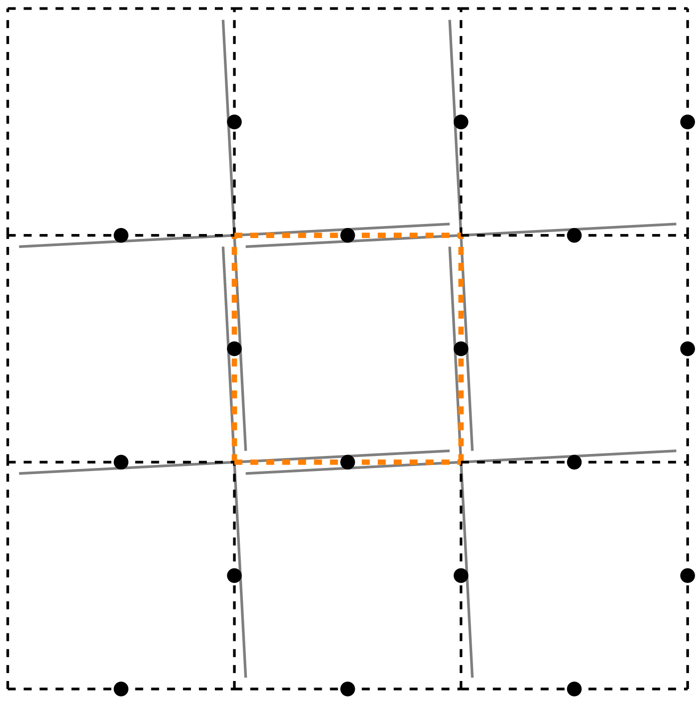


Figure 3.31: Plaquette in the middle of a toric code

The stabilizers change to

$$\begin{aligned}
S_1 &= XX\mathbb{1}\mathbb{1} \\
S_2 &= X\mathbb{1}, X\mathbb{1} \\
S_3 &= ZZZZ \\
S_4 &= \mathbb{1}X\mathbb{1}X \\
S_5 &= \mathbb{1}\mathbb{1}XX \\
S_6 &= Z\mathbb{1}\mathbb{1}\mathbb{1} \\
S_7 &= \mathbb{1}Z\mathbb{1}\mathbb{1} \\
S_8 &= \mathbb{1}\mathbb{1}Z\mathbb{1} \\
S_9 &= \mathbb{1}\mathbb{1}\mathbb{1}Z.
\end{aligned}$$

The probability for a particular outcome is again

$$\text{Prob}[X, X, X, X] = \frac{1}{16}(1 + o_1 o_2 o_3 o_4).$$

We end up with the following empirical model:

Outcomes	A_v	B_q
$(-1,-1,-1,-1)$	1/8	1/8
$(-1,-1,-1,1)$	0	0
$(-1,-1,1,-1)$	0	0
$(-1,-1,1,1)$	1/8	1/8
$(-1,1,-1,-1)$	0	0
$(-1,1,-1,1)$	1/8	1/8
$(-1,1,1,-1)$	1/8	1/8
$(-1,1,1,1)$	0	0
$(1,-1,-1,-1)$	0	0
$(1,-1,-1,1)$	1/8	1/8
$(1,-1,1,-1)$	1/8	1/8
$(1,-1,1,1)$	0	0
$(1,1,-1,-1)$	1/8	1/8
$(1,1,-1,1)$	0	0
$(1,1,1,-1)$	0	0
$(1,1,1,1)$	1/8	1/8

Table 3.7: Empirical model of a toric code

4 The cohomology of contextuality

In the last two chapters we used sheaf theory to analyze the structure of contextuality and showed that this phenomenon can be characterized in therms of obstructions to the existence of global sections. The aim is now to use the powerful tools of Čech cohomology [3] on an abelian presheaf to study the structure of contextuality. It can be proven that the zeroth cohomology group describes the compatible families and thus the global sections.

4.1 Coboundary maps on Čech cochain complexes

We use the same notations as in Chapter 2 and consider presheafs $\mathcal{E} : U \to O^U$, a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U and restriction maps $\mathrm{res}_U^{U'} : \mathcal{E}(U') \to \mathcal{E}(U)$.

Definition 4.1. *A q-simplex is a list $\sigma = (U_0, ..., U_q)$ of elements of $\mathcal{U}$ with*

$$|\sigma| := \bigcap_{i=0}^{q} U_i \neq \emptyset.$$

We define the nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ as an abstract simplicial complex, i.e. if $\sigma' \subseteq \sigma$ and $\sigma \in N(\mathcal{U})$ then $\sigma' \in N(\mathcal{U})$.

Notice that a single element of the cover is a 0-simplex. We write $N(\mathcal{U})^q$ to name the set of q-simplices. For a given $q+1$-simplex $\sigma = (U_0, ..., U_{q+1})$ we can built q-simplices by omitting one of the elements. We write

$$\partial_j(\sigma) := (U_0, ..., \hat{U}_j, ..., U_{q+1})$$

for $0 \leq j \leq q$ and know that $|\sigma| \subseteq |\partial_j(\sigma)|$.

As an example we consider a cluster ring model out of n qudits. The measurement contexts $C_k = \{Z_{k-1}, X_k, Z_{k+1}\}$ are the elements in the cover $\mathcal{U}$. We showed in section 3.3 that the presheaf over a context C_k includes four possible sections with an even number of 1's.

$$\mathcal{E}(C_k) = \{000, 011, 101, 110\}$$

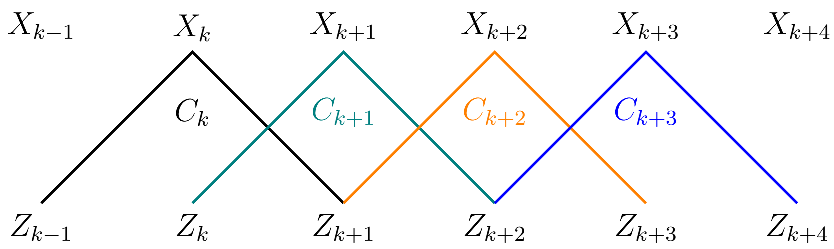


Figure 4.1: Segment out of three contexts of a cluster state ring

The 0-simplices $\sigma_k^{(0)} = \langle C_k \rangle$ correspond to the contexts and the 1-simplices $\sigma_k^{(1)} = \langle C_k, C_{k+2} \rangle$ occupy two contexts. In Figure 4.1 it becomes clear that there no sets of contexts with two or more common elements exist, i.e. there are no 2-simplices. The nerves are defined the following way.

$$\mathcal{N}(\mathcal{U})^0 = \{\sigma_k^{(0)} |\, k \in \mathbb{N}\}$$
$$\mathcal{N}(\mathcal{U})^1 = \{\sigma_k^{(1)} |\, k \in \mathbb{N}\}$$

Definition 4.2. *For $q \geq 0$ we define the Čech cochain complex*

$$C^q(\mathcal{U}, \mathcal{E}) := \prod_{\sigma \in N(\mathcal{U})^q} \mathcal{E}(|\sigma|).$$

In the framework of our cluster ring example we get

$$C^0(\mathcal{U}, \mathcal{E}) = \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{n} \{000, 011, 101, 110\} = \{000, 011, 101, 110\}^{\times n}$$
$$C^1(\mathcal{U}, \mathcal{E}) = \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{n} \{Z_{k+1}\} = \prod_{k=0}^{n} \{0, 1\} = \{0, 1\}^{\times n}$$

with $\mathcal{E}(C_k \cap C_{k+2}) = \mathcal{E}(\{Z_{k+1}\}) = \{0, 1\}$. $C^0(\mathcal{U}, \mathcal{E})$ comprises the possible sets of sections which can be realized over the n contexts not considering any compatibility at the overlaps. One can see in Figure 4.1 that a section on C_k is compatible with a section over C_{k+2} if the measurement outcomes of Z_{k+1} coincides in the two contexts. $C^1(\mathcal{U}, \mathcal{E})$ contains strings representing possibilities for these n outcomes of the Z-measurements.

The aim is now to find a coboundary map $\delta^q : C^q(\mathcal{U}, \mathcal{E}) \to C^{q+1}(\mathcal{U}, \mathcal{E})$ which vanishes if the measurement outcomes agree on the overlaps.

Definition 4.3. *With $\omega = (\omega(\tau))_{\tau \in N(\mathcal{U})^q} \in C^q(\mathcal{U}, \mathcal{E})$ we define the group homomorphism*

$$\delta^q(\omega)(\sigma) := \sum_{j=0}^{q+1} (-1)^j res_{|\sigma|}^{|\partial_j(\sigma)|} \omega(\partial_j \sigma)$$

for a q-simplex $\tau \in N(\mathcal{U})^q$ and a $q+1$-simplex $\sigma \in N(\mathcal{U})^{q+1}$.

These functions fulfill the following identity [3].

Theorem 4.4. *For each q it is $\delta^{q+1} \circ \delta^q = 0$.*

To calculate an example for a group homomorphism we consider

$$\omega = (\dots, 000, \underbrace{011}_{C_k}, \underbrace{101}_{C_{k+1}}, \underbrace{101}_{C_{k+2}}, \dots) \in C^0(\mathcal{U}, \mathcal{E})$$

and $q = 0$. Using

$$\partial_j \sigma_k^{(1)} = \partial_j \langle C_k, C_{k+2} \rangle = \begin{cases} \langle C_{k+2} \rangle \text{ if } j = 0 \\ \langle C_k \rangle \text{ if } j = 1 \end{cases}$$

we compute first the two particular summands and than $\delta^0(\omega)(\sigma_k)$.

$$\begin{aligned} j = 0\text{: } &\mathrm{res}^{|\partial_0 \sigma_k^{(1)}|}_{|\sigma_k^{(1)}|} \omega(\partial_0 \sigma_k^{(1)}) = \mathrm{res}^{C_{k+2}}_{C_k \cap C_{k+2}} \omega(C_{k+2}) = \mathrm{res}^{C_{k+2}}_{\{Z_{k+1}\}} 101 = 1 \\ j = 1\text{: } &\mathrm{res}^{|\partial_1 \sigma_k^{(1)}|}_{|\sigma_k^{(1)}|} \omega(\partial_1 \sigma_k^{(1)}) = \mathrm{res}^{C_k}_{C_k \cap C_{k+2}} \omega(C_k) = \mathrm{res}^{C_k}_{\{Z_{k+1}\}} 011 = 1 \\ \delta^0(\omega)(\sigma_k) &= \sum_{j=0}^{1} (-1)^j \mathrm{res}^{|\partial_j \sigma_k^{(1)}|}_{|\sigma_k^{(1)}|} \omega(\partial_j \sigma_k^{(1)}) = 1 - 1 = 0 \end{aligned}$$

Based on the computed example one can see that

$$\begin{aligned} \mathrm{Im}\, \delta^0(\omega)(\sigma_k^{(1)}) =& \{0, 1\}^{\times n} \\ \ker \delta^0(\omega)(\sigma_k^{(1)}) =& \{S \in C^0(\mathcal{U}, \mathcal{E}) \mid S_k[[3]] \oplus S_{k+2}[[1]] = 0\} \end{aligned}$$

holds for arbitrary ω concerning the cluster state ring model.

4.2 Čech cohomology of a presheaf

The complex $0 \to C^0(\mathcal{U}, \mathcal{E}) \to ...$ is named augmented complex.

Definition 4.5. *We name the kernel of δ^q the q-cocycles $Z^q(\mathcal{U}, \mathcal{E})$ and the image of δ^{q-1} the q-coboundaries $B^q(\mathcal{U}, \mathcal{E})$.*

Both are subgroups of $C^q(\mathcal{U}, \mathcal{E})$ and because of Theorem 4.1 it yields $B^q(\mathcal{U}, \mathcal{E}) \subseteq Z^q(\mathcal{U}, \mathcal{E})$.

Definition 4.6. *We define the q-th Čech cohomology group $\check{H}^q(\mathcal{U}, \mathcal{E})$ as the quotient group*

$$\check{H}^q(\mathcal{U}, \mathcal{E}) := Z^q(\mathcal{U}, \mathcal{E}) / B^q(\mathcal{U}, \mathcal{E}).$$

Since $B^0(\mathcal{U},\mathcal{E}) = 0$ we define $\check{H}^0(\mathcal{U},\mathcal{E}) \widehat{=} Z^0(\mathcal{U},\mathcal{E})$. The cohomology class $[z]$ for a given cocycle $z \in Z^q(\mathcal{U},\mathcal{E})$ is the image of z under the canonical map $Z^q(\mathcal{U},\mathcal{E}) \to \check{H}^q(\mathcal{U},\mathcal{E})$.

For the cluster state model the augmented complex is

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0.$$

We create the cocycles, coboundaries and the cohomology group for this case.

$$\begin{aligned} Z^0(\mathcal{U},\mathcal{E}) &= \text{0-cocycles } = \ker\delta^0 = \{S \in C^0(\mathcal{U},\mathcal{E}) \mid S_k[[3]] \oplus S_{k+2}[[1]] = 0, k \in \mathbb{N}\} \\ B^0(\mathcal{U},\mathcal{E}) &= \text{0-coboundaries } = \text{Im }\delta^{-1} = \{000\}^{\times n} \\ \check{H}^0(\mathcal{U},\mathcal{E}) &= \text{0th cohomology group} = Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) \\ &= \{S \in C^0(\mathcal{U},\mathcal{E}) \mid S_k[[3]] \oplus S_{k+2}[[1]] = 0, k \in \mathbb{N}\} \end{aligned}$$

$$\begin{aligned} Z^1(\mathcal{U},\mathcal{E}) &= \text{1-cocycles } = \ker\delta^1 = \{0,1\}^{\times n} \\ B^1(\mathcal{U},\mathcal{E}) &= \text{1-coboundaries } = \text{Im }\delta^0 = \{0,1\}^{\times n} \\ \check{H}^1(\mathcal{U},\mathcal{E}) &= \text{1th cohomology group} = Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times n} \end{aligned}$$

One can see that the elements in $\ker\delta^0(\omega)(\sigma_k^{(1)})$ are outcome strings referring to the contexts C_k and C_{k+2} for certain k regarding the overlapping. Hence $\check{H}^0(\mathcal{U},\mathcal{E}) = Z^0(\mathcal{U},\mathcal{E})$ includes the strings, which are compatible in every C_k for $k \in \mathbb{N}$. This can be proven generally [3]:

Theorem 4.7. *There is a bijection between compatible families and elements of the zeroth cohomology group $\check{H}^0(\mathcal{U},\mathcal{E})$.*

Proof. The cochains $c = (r_i)_{U_i \in U}$ in $C^0(\mathcal{U},\mathcal{E}) = \prod_{\sigma=U_i\in\mathcal{U}} \mathcal{E}(\sigma)$ correspond to families $\{r_i \in \mathcal{E}(U_i)\}$, i.e. sets that include one section for each context. We use $\check{H}^0(\mathcal{U},\mathcal{E}) \widehat{=} Z^0(\mathcal{U},\mathcal{E}) = \ker\delta^0$. For each 1-simplex $\sigma = (C_i, C_j)$ the map

$$\delta^0(c)(\sigma) = r_i|C_i \cap C_j - r_j|C_i \cap C_j$$

exists. If and only if the corresponding family is compatible, $\delta^0(c) = 0$. □

The compatible families correspond to the global sections we discussed and depicted in Chapter 3.

5 Examples of Čech cohomologies

After introducing the definitions of the cohomology of a presheaf and giving a intuition for the interpretation we consider a few examples.

5.1 Cluster rings for odd and even numbers of qubits

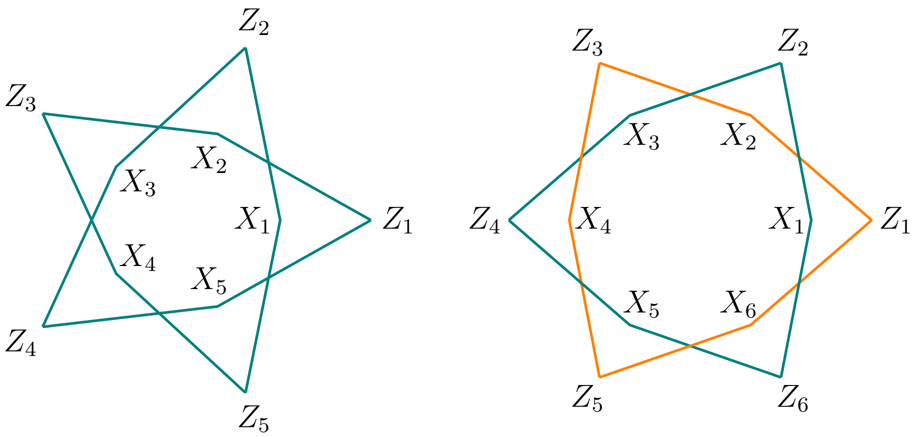


Figure 5.1: Contexts for a cluster ring for $n = 5$ and $n = 6$

In Section 3.3 we considered the bundle diagrams of cluster ring models for number of qubits $n \in \{3, 4, 5, 6\}$. In the following part we approach the difference between cluster ring models out of n qubits for odd and even n by means of Čech cohomologies. In Figure 5.1 one can see that for an odd number of qubits the potential global section is divided in two independent parts whereas for the other cases the global section is consistence. We want to figure out, whether this makes differences in the cohomologies.

Therefor we start with $n = 5$ and compute the cohomology group. In Figure 5.1 the context C_1 is depicted by a violet line. The second context is painted in teal, C_4 in orange and the forth in blue. Context C_5 is represented with a red curve.

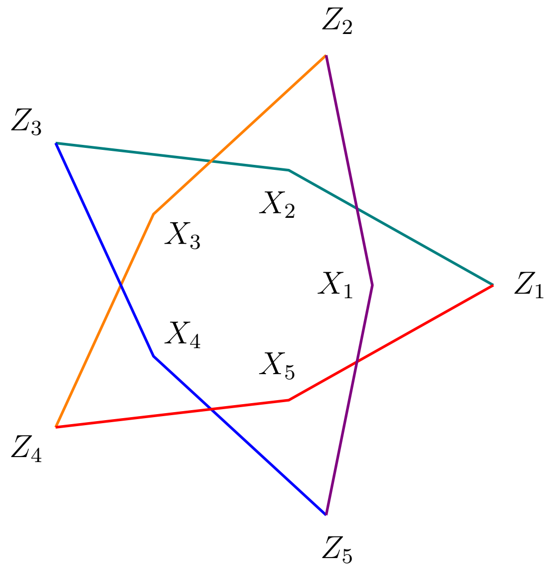


Figure 5.2: Contexts for cluster state n=5

Using the measurement context we built the simplices, nerves, the sheaf and the cochains.

$$
\begin{array}{lrcl}
\text{Contexts:} & C_1 & = & \{Z_5, X_1, Z_2\} \\
& C_2 & = & \{Z_1, X_2, Z_3\} \\
& C_3 & = & \{Z_2, X_3, Z_4\} \\
& C_4 & = & \{Z_3, X_4, Z_5\} \\
& C_5 & = & \{Z_4, X_5, Z_1\} \\
& \partial_j \sigma_k^{(1)} & = & \partial_j \langle C_k, C_{k+2} \rangle = \begin{cases} \langle C_{k+2} \rangle & \text{if } j = 0 \\ \langle C_k \rangle & \text{if } j = 1 \end{cases} \\
\text{Simplices:} & \sigma_k^{(0)} & = & \langle C_k | \, k = 1 \dots 5 \rangle \\
& \sigma_k^{(1)} & = & \langle C_k, C_{k+2} | \, k = 1 \dots 5 \rangle \\
\text{Nerves:} & \mathcal{N}(\mathcal{U})^0 & = & \{\sigma_k^{(0)} | \, k = 1 \dots 5\} \\
& \mathcal{N}(\mathcal{U})^1 & = & \{\{C_1, C_3\}, \{C_2, C_4\}, \{C_3, C_5\}, \{C_4, C_1\}, \{C_5, C_2\}\} \\
\text{Sheaf:} & \mathcal{E}(C_k) & = & \text{empirical model} = \text{even numbers of 1s} \\
& & = & \{000, 011, 101, 110\} \\
& \mathcal{E}(C_k \cap C_{k+2}) & = & \mathcal{E}(\{Z_{k+1}\}) = \{0, 1\} \\
\text{Cochains:} & C^0(\mathcal{U}, \mathcal{E}) & = & \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{5} \{000, 011, 101, 110\} \\
& & = & \{000, 011, 101, 110\}^{\times 5} \\
& C^1(\mathcal{U}, \mathcal{E}) & = & \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{5} \{Z_{k+1}\} = \prod_{k=0}^{5} \{0, 1\} \\
& & = & \{0, 1\}^{\times 5}
\end{array}
$$

Further we compute analogously to the Chapter 4 the Čech cohomology group for the augmented complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0.$$

$$
\begin{aligned}
Z^0(\mathcal{U},\mathcal{E}) &= \ker\delta^0 = \{S \in C^0(\mathcal{U},\mathcal{E}) \,|\, S_k[[3]] \oplus S_{k+2}[[1]] = 0,\ k = 1\dots 5\} \\
B^0(\mathcal{U},\mathcal{E}) &= \mathrm{Im}\ \delta^{-1} = \{000\}^{\times 5} \\
\check{H}^0(\mathcal{U},\mathcal{E}) &= Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) \\
&= \{S \in C^0(\mathcal{U},\mathcal{E}) \,|\, S_k[[3]] \oplus S_{k+2}[[1]] = 0,\ k = 1\dots 5\}
\end{aligned}
$$

$$
\begin{aligned}
Z^1(\mathcal{U},\mathcal{E}) &= \ker\delta^1 = \{0,1\}^{\times 5} \\
B^1(\mathcal{U},\mathcal{E}) &= \mathrm{Im}\ \delta^0 = \{0,1\}^{\times 5} \\
\check{H}^1(\mathcal{U},\mathcal{E}) &= Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times 5}
\end{aligned}
$$

$\check{H}^0(\mathcal{U},\mathcal{E})$ is the set of strings which represents the global sections. To compute these specifically we choose another notation representing measurement outcomes X_i and Z_i with the string v.

$$
v = \{Z_1, X_2, Z_3, X_4, Z_5, X_1, Z_2, X_3, Z_4, X_5\}
$$

$$
M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

By solving $Mv = 0$ we get $2^n = 2^5 = 32$ strings out of $\check{H}^0(\mathcal{U},\mathcal{E})$, which fulfill the conditions needed to be a global section.

$$
\begin{aligned}
\check{H}^0(\mathcal{U},\mathcal{E}) = \{v \,|\, & Z_3 = -X_2 - Z_1, \\
& Z_5 = X_2 - X_4 + Z_1, \\
& Z_2 = -X_1 - X_2 + X_4 - Z_1, \\
& Z_4 = X_1 + X_2 - X_3 - X_4 + Z_1, \\
& X_5 = -X_1 - X_2 + X_3 + X_4 - 2Z_1\}. \\
= \{& \{0,0,0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,1,1,1\}, \{0,0,0,0,0,1,1,0,1,1\}, \\
& \{0,0,0,0,0,1,1,1,0,0\}, \{0,0,0,1,1,0,1,0,1,1\}, \{0,0,0,1,1,0,1,1,0,0\}, \\
& \{0,0,0,1,1,1,0,0,0,0\}, \{0,0,0,1,1,1,0,1,1,1\}, \{0,1,1,0,1,0,1,0,1,1\}, \\
& \{0,1,1,0,1,0,1,1,0,0\}, \{0,1,1,0,1,1,0,0,0,0\}, \{0,1,1,0,1,1,0,1,1,1\}, \\
& \{0,1,1,1,0,0,0,0,0,0\}, \{0,1,1,1,0,0,0,1,1,1\}, \{0,1,1,1,0,1,1,0,1,1\}, \\
& \{0,1,1,1,0,1,1,1,0,0\}, \{1,0,1,0,1,0,1,0,1,0\}, \{1,0,1,0,1,0,1,1,0,1\}, \\
& \{1,0,1,0,1,1,0,0,0,1\}, \{1,0,1,0,1,1,0,1,1,0\}, \{1,0,1,1,0,0,0,0,0,1\}, \\
& \{1,0,1,1,0,0,0,1,1,0\}, \{1,0,1,1,0,1,1,0,1,0\}, \{1,0,1,1,0,1,1,1,0,1\}, \\
& \{1,1,0,0,0,0,0,0,0,1\}, \{1,1,0,0,0,0,0,1,1,0\}, \{1,1,0,0,0,1,1,0,1,0\}, \\
& \{1,1,0,0,0,1,1,1,0,1\}, \{1,1,0,1,1,0,1,0,1,0\}, \{1,1,0,1,1,0,1,1,0,1\}, \\
& \{1,1,0,1,1,1,0,0,0,1\}, \{1,1,0,1,1,1,0,1,1,0\}\}
\end{aligned}
$$

Now we compare this above result with a model out of six qubits to find out whether differences arise from the division of the global section depicted in Figure 5.1. Analogously to the $n = 5$ case the context are mapped in Figure 5.3 with color black related to the context C_6.

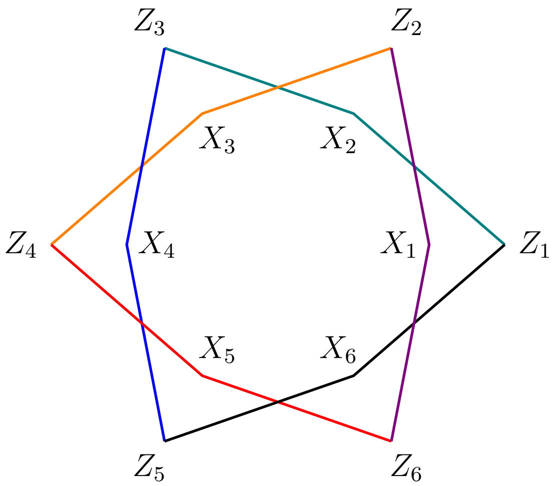


Figure 5.3: Contexts for cluster state n=6

Contexts:
$$\begin{aligned}
C_1 &= \{Z_6, X_1, Z_2\}\\
C_2 &= \{Z_1, X_2, Z_3\}\\
C_3 &= \{Z_2, X_3, Z_4\}\\
C_4 &= \{Z_3, X_4, Z_5\}\\
C_5 &= \{Z_4, X_5, Z_6\}\\
C_6 &= \{Z_5, X_6, Z_1\}
\end{aligned}$$

Simplices:
$$\begin{aligned}
\sigma_k^{(0)} &= \langle C_k |\, k = 1 \ldots 6\rangle\\
\sigma_k^{(1)} &= \langle C_k, C_{k+2} |\, k = 1 \ldots 6\rangle\\
\partial_j \sigma_k^{(1)} &= \partial_j \langle C_k, C_{k+2}\rangle = \begin{cases} \langle C_{k+2}\rangle & \text{if } j = 0\\ \langle C_k\rangle & \text{if } j = 1\end{cases}
\end{aligned}$$

Nerves:
$$\begin{aligned}
\mathcal{N}(\mathcal{U})^0 &= \{\sigma_k^{(0)} |\, k = 1 \ldots 6\}\\
\mathcal{N}(\mathcal{U})^1 &= \{\{C_1, C_3\}, \{C_2, C_4\}, \{C_3, C_5\}, \{C_4, C_6\}, \{C_5, C_0\},\\
&\quad \{C_6, C_1\}\}
\end{aligned}$$

Sheaf:
$$\begin{aligned}
\mathcal{E}(C_k) &= \text{empirical model} = \text{even numbers of 1s}\\
&\quad \{000, 011, 101, 110\}\\
\mathcal{E}(C_k \cap C_{k+2}) &= \mathcal{E}(\{Z_{k+1}\}) = \{0, 1\}
\end{aligned}$$

Cochains:
$$\begin{aligned}
C^0(\mathcal{U}, \mathcal{E}) &= \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{6} \{000, 011, 101, 110\}\\
&= \{000, 011, 101, 110\}^{\times 6}\\
C^1(\mathcal{U}, \mathcal{E}) &= \prod_{\sigma_k \in \mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_k|) = \prod_{k=0}^{6} \{Z_{k+1}\} = \prod_{k=0}^{6} \{0, 1\}\\
&= \{0, 1\}^{\times 6}
\end{aligned}$$

We are interested in Čech cohomology group for the complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$$

and compute the strings belonging to global sections the same way as in the odd case.

$$\begin{aligned}
Z^0(\mathcal{U},\mathcal{E}) &= \ker \delta^0 = \{S \in C^0(\mathcal{U},\mathcal{E}) \,|\, S_k[[3]] \oplus S_{k+2}[[1]] = 0,\ k = 1 \dots 7\} \\
B^0(\mathcal{U},\mathcal{E}) &= \text{Im } \delta^{-1} = \{000\}^{\times 6} \\
\check{H}^0(\mathcal{U},\mathcal{E}) &= Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) \\
&= \{S \in C^0(\mathcal{U},\mathcal{E}) \,|\, S_k[[3]] \oplus S_{k+2}[[1]] = 0,\ k = 1 \dots 7\}
\end{aligned}$$

$$\begin{aligned}
Z^1(\mathcal{U},\mathcal{E}) &= \ker \delta^1 = \{0,1\}^{\times 6} \\
B^1(\mathcal{U},\mathcal{E}) &= \text{Im } \delta^0 = \{0,1\}^{\times 6} \\
\check{H}^1(\mathcal{U},\mathcal{E}) &= Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times 6}
\end{aligned}$$

The first entry in the set of $\check{H}^0(\mathcal{U},\mathcal{E})$ describes the global section depicted in figure 3.3.2. We use the notation $\{Z_1, X_2, Z_3, X_4, Z_5, X_6, X_1, Z_2, X_3, Z_4, X_5, Z_6\}$.

$$\begin{aligned}
\check{H}^0(\mathcal{U},\mathcal{E}) = \{&\{0,0,0,0,0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,0,0,1,1,0\}, \\
&\{0,0,0,0,0,0,0,1,1,0,1,1\}\{0,0,0,0,0,0,0,1,1,1,0,1\}, \\
&\{0,0,0,0,0,0,1,0,1,0,1,1\}, \{0,0,0,0,0,0,1,0,1,1,0,1\}, \\
&\{0,0,0,0,0,0,1,1,0,0,0,0\}, \{0,0,0,0,0,0,1,1,0,1,1,0\}, \\
&\{0,0,0,1,1,1,0,0,0,0,0,0\}, \{0,0,0,1,1,1,0,0,0,1,1,0\}, \\
&\{0,0,0,1,1,1,0,1,1,0,1,1\}, \{0,0,0,1,1,1,0,1,1,1,0,1\}, \\
&\{0,0,0,1,1,1,1,0,1,0,1,1\}, \{0,0,0,1,1,1,1,0,1,1,0,1\}, \\
&\{0,0,0,1,1,1,1,1,0,0,0,0\}, \{0,0,0,1,1,1,1,1,0,1,1,0\}, \\
&\{0,1,1,0,1,1,0,0,0,0,0,0\}, \{0,1,1,0,1,1,0,0,0,1,1,0\}, \\
&\{0,1,1,0,1,1,0,1,1,0,1,1\}, \{0,1,1,0,1,1,0,1,1,1,0,1\}, \\
&\{0,1,1,0,1,1,1,0,1,0,1,1\}, \{0,1,1,0,1,1,1,0,1,1,0,1\}, \\
&\{0,1,1,0,1,1,1,1,0,0,0,0\}, \{0,1,1,0,1,1,1,1,0,1,1,0\}, \\
&\{0,1,1,1,0,0,0,0,0,0,0,0\}, \{0,1,1,1,0,0,0,0,0,1,1,0\}, \\
&\{0,1,1,1,0,0,0,1,1,0,1,1\}, \{0,1,1,1,0,0,0,1,1,1,0,1\}, \\
&\{0,1,1,1,0,0,1,0,1,0,1,1\}, \{0,1,1,1,0,0,1,0,1,1,0,1\}, \\
&\{0,1,1,1,0,0,1,1,0,0,0,0\}, \{0,1,1,1,0,0,1,1,0,1,1,0\}, \\
&\{1,0,1,0,1,0,0,0,0,0,0,0\}, \{1,0,1,0,1,0,0,0,0,1,1,0\}, \\
&\{1,0,1,0,1,0,0,1,1,0,1,1\}, \{1,0,1,0,1,0,0,1,1,1,0,1\}, \\
&\{1,0,1,0,1,0,1,0,1,0,1,1\}, \{1,0,1,0,1,0,1,0,1,1,0,1\},
\end{aligned}$$

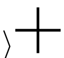
$$\{1,0,1,0,1,0,1,1,0,0,0,0\},\{1,0,1,0,1,0,1,1,0,1,1,0\},$$
$$\{1,0,1,1,0,1,0,0,0,0,0,0\},\{1,0,1,1,0,1,0,0,0,1,1,0\},$$
$$\{1,0,1,1,0,1,0,1,1,0,1,1\},\{1,0,1,1,0,1,0,1,1,1,0,1\},$$
$$\{1,0,1,1,0,1,1,0,1,0,1,1\},\{1,0,1,1,0,1,1,0,1,1,0,1\},$$
$$\{1,0,1,1,0,1,1,1,0,0,0,0\},\{1,0,1,1,0,1,1,1,0,1,1,0\},$$
$$\{1,1,0,0,0,1,0,0,0,0,0,0\},\{1,1,0,0,0,1,0,0,0,1,1,0\},$$
$$\{1,1,0,0,0,1,0,1,1,0,1,1\},\{1,1,0,0,0,1,0,1,1,1,0,1\},$$
$$\{1,1,0,0,0,1,1,0,1,0,1,1\},\{1,1,0,0,0,1,1,0,1,1,0,1\},$$
$$\{1,1,0,0,0,1,1,1,0,0,0,0\},\{1,1,0,0,0,1,1,1,0,1,1,0\},$$
$$\{1,1,0,1,1,0,0,0,0,0,0,0\},\{1,1,0,1,1,0,0,0,0,1,1,0\},$$
$$\{1,1,0,1,1,0,0,1,1,0,1,1\},\{1,1,0,1,1,0,0,1,1,1,0,1\},$$
$$\{1,1,0,1,1,0,1,0,1,0,1,1\},\{1,1,0,1,1,0,1,0,1,1,0,1\},$$
$$\{1,1,0,1,1,0,1,1,0,0,0,0\},\{1,1,0,1,1,0,1,1,0,1,1,0\}\}$$

It results, that $\check{H}^0(\mathcal{U},\mathcal{E})$ includes $2^n = 2^5 = 64$ possible strings. Continuing this method to larger values of n it results that there exist 2^n global sections in a cluster ring model out of n qubits. Finally we can observe no difference between the odd and the even case in the cohomology groups $\check{H}^0(\mathcal{U},\mathcal{E})$ and $\check{H}^1(\mathcal{U},\mathcal{E})$.

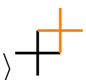
5.2 Cluster state in two dimensions

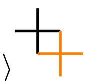
Carrying on Section 3.4.2 we consider cluster states on latices. Because of dimension increase and the star structure of the measurement context, there exist four different types of 1-simplices instead of one type as in the above example. Additionally there are sets out of three and four contexts containing one common element, i.e. nontrivial 2- and 3-simplices. The 2-simplices exist in four different patterns as well as the 1-simplices. The pictures beside the simplices depict their compositions.

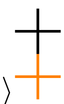
Contexts: $C_{k,l} = \{Z_{k,l+1}, Z_{k-1,l}, X_{k,l}, Z_{k+1,l}, Z_{k,l-1}\}$

Simplices: $\sigma^{(0)}_{k,l} = \langle C_{k,l}\rangle$ 

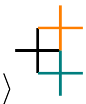
$\sigma^{(1)}_{k,l} = \langle C_{k,l}, C_{k+2,l}\rangle$ 

$\sigma^{(1)\prime}_{k,l} = \langle C_{k,l}, C_{k+1,l+1}\rangle$ 

$\sigma^{(1)\prime\prime}_{k,l} = \langle C_{k,l}, C_{k+1,l-1}\rangle$ 

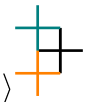
$\sigma^{(1)\prime\prime\prime}_{k,l} = \langle C_{k,l}, C_{k,l-2}\rangle$ 

$$\sigma_{k,l}^{(2)} = \langle C_{k,l}, C_{k+1,l+1}, C_{k+1,l-1} \rangle$$

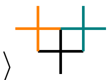


$$\sigma_{k,l}^{(2)\prime} = \langle C_{k,l}, C_{k-1,l-1}, C_{k+1,l-1} \rangle$$

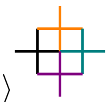
$$\sigma_{k,l}^{(2)\prime\prime} = \langle C_{k,l}, C_{k-1,l-1}, C_{k-1,l+1} \rangle$$



$$\sigma_{k,l}^{(2)\prime\prime\prime} = \langle C_{k,l}, C_{k-1,l+1}, C_{k+1,l+1} \rangle$$



$$\sigma_{k,l}^{(3)} = \langle C_{k,l}, C_{k+1,l+1}, C_{k+2,l}, C_{k+1,l-1} \rangle$$



Nerves:

$$\begin{aligned}
\mathcal{N}(\mathcal{U})^0 &= \{\sigma_{k,l}^{(0)} \,|\, k,l \in \mathbb{N}\} \\
\mathcal{N}(\mathcal{U})^1 &= \{\sigma_{k,l}^{(1)}, \sigma_{k,l}^{(1)\prime}, \sigma_{k,l}^{(1)\prime\prime}, \sigma_{k,l}^{(1)\prime\prime\prime} \,|\, k,l \in \mathbb{N}\} \\
\mathcal{N}(\mathcal{U})^2 &= \{\sigma_{k,l}^{(2)}, \sigma_{k,l}^{(2)\prime}, \sigma_{k,l}^{(2)\prime\prime}, \sigma_{k,l}^{(2)\prime\prime\prime} \,|\, k,l \in \mathbb{N}\} \\
\mathcal{N}(\mathcal{U})^3 &= \{\sigma_{k,l}^{(3)} \,|\, k,l \in \mathbb{N}\}
\end{aligned}$$

Sheaf:

$$\begin{aligned}
\mathcal{E}(C_{k,l}) &= \text{odd numbers of 1s} \\
&= \{00001, 00010, 00100, 00111, 01000, 01011, 01101, 01110, \\
&\quad 10000, 10011, 10101, 10110, 11001, 11010, 11100, 11111\} \\
\mathcal{E}(|\sigma_{k,l}^{(1)}|) &= \mathcal{E}(\{Z_{k+1,l}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(1)\prime}|) &= \mathcal{E}(\{Z_{k,l+1}, Z_{k+1,l}\}) = \{0,1\}^{\times 2} = \{00, 01, 10, 11\} \\
\mathcal{E}(|\sigma_{k,l}^{(1)\prime\prime}|) &= \mathcal{E}(\{Z_{k+1,l}, Z_{k,l-1}\}) = \{0,1\}^{\times 2} = \{00, 01, 10, 11\} \\
\mathcal{E}(|\sigma_{k,l}^{(1)\prime\prime\prime}|) &= \mathcal{E}(\{Z_{k,l-1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(2)}|) &= \mathcal{E}(\{Z_{k+1,l}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(2)\prime}|) &= \mathcal{E}(\{Z_{k,l-1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(2)\prime\prime}|) &= \mathcal{E}(\{Z_{k-1,l}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(2)\prime\prime\prime}|) &= \mathcal{E}(\{Z_{k,l+1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(3)}|) &= \mathcal{E}(\{Z_{k+1,l}\}) = \{0,1\}
\end{aligned}$$

Cochains:

$$\begin{aligned}
C^0(\mathcal{U},\mathcal{E}) &= \textstyle\prod_{\sigma_{k,l} \in \mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_{k,l}|) = \prod_{\sigma_{k,l} \in \mathcal{N}(\mathcal{U})^0} \mathcal{E}(\sigma_{k,l}) = \mathcal{E}(C_{k,l})^{\times n \times n} \\
C^1(\mathcal{U},\mathcal{E}) &= \textstyle\prod_{\sigma_{k,l} \in \mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_{k,l}|) \\
&= (\{0,1\}^{\times n \times n}, \{00,01,10,11\}^{\times n \times n}, \{00,01,10,11\}^{\times n \times n}, \\
&\quad \{0,1\}^{\times n \times n})
\end{aligned}$$

$$
\begin{aligned}
C^2(\mathcal{U},\mathcal{E}) = &\ \prod_{\sigma_{k,l}\in\mathcal{N}(\mathcal{U})^2} \mathcal{E}(|\sigma_{k,l}|)\\
= &\ (\{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n})\\
C^3(\mathcal{U},\mathcal{E}) = &\ \prod_{\sigma_{k,l}\in\mathcal{N}(\mathcal{U})^3} \mathcal{E}(|\sigma_{k,l}|) = \{0,1\}^{\times n\times n}
\end{aligned}
$$

To calculate examples for the coboundary maps we chose an exemplary segment of $\omega \in C^0(\mathcal{U},\mathcal{E})$, in particular

$$
\omega = \begin{pmatrix}
\underbrace{00001}_{C_{k,l+2}} & 10101 & 00100\\
01110 & \underbrace{11100}_{C_{k+1,l+1}} & 00001\\
\underbrace{00001}_{C_{k,l}} & 10000 & \underbrace{11111}_{C_{k+2,l}}\\
11100 & \underbrace{10000}_{C_{k+1,l-1}} & 11111
\end{pmatrix}.
$$

With this example we can compute $\delta^0(\omega)(\sigma^{(1)}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)\prime}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l})$ and $\delta^0(\omega)(\sigma^{(1)\prime\prime\prime}_{k,l})$.

$$
\begin{aligned}
j = 0\text{: } & \operatorname{res}^{|\partial_0\sigma^{(1)}_{k,l}|}_{|\sigma^{(1)}_{k,l}|}\omega(\partial_0\sigma^{(1)}_{k,l}) = \operatorname{res}^{C_{k+2,l}}_{\{Z_{k+1,l}\}}\omega(C_{k+2,l}) = 1\\
j = 1\text{: } & \operatorname{res}^{|\partial_1\sigma^{(1)}_{k,l}|}_{|\sigma^{(1)}_{k,l}|}\omega(\partial_1\sigma^{(1)}_{k,l}) = \operatorname{res}^{C_{k,l}}_{\{Z_{k+1,l}\}}\omega(C_{k,l}) = 0\\
\delta^0(\omega)(\sigma^{(1)}_{k,l}) = & 1 - 0 = 1
\end{aligned}
$$

$$
\begin{aligned}
j = 0\text{: } & \operatorname{res}^{|\partial_0\sigma^{(1)\prime}_{k,l}|}_{|\sigma^{(1)\prime}_{k,l}|}\omega(\partial_0\sigma^{(1)\prime}_{k,l}) = \operatorname{res}^{C_{k+1,l+1}}_{\{Z_{k+1,l},Z_{k,l+1}\}}\omega(C_{k+1,l+1}) = \{0,1\}\\
j = 1\text{: } & \operatorname{res}^{|\partial_1\sigma^{(1)\prime}_{k,l}|}_{|\sigma^{(1)\prime}_{k,l}|}\omega(\partial_1\sigma^{(1)\prime}_{k,l}) = \operatorname{res}^{C_{k,l}}_{\{Z_{k+1,l},Z_{k,l+1}\}}\omega(C_{k,l}) = \{0,0\}\\
\delta^0(\omega)(\sigma^{(1)\prime}_{k,l}) = & \{0,1\} - \{0,0\} = \{0,1\}
\end{aligned}
$$

$$
\begin{aligned}
j = 0\text{: } & \operatorname{res}^{|\partial_0\sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|}\omega(\partial_0\sigma^{(1)\prime\prime}_{k,l}) = \operatorname{res}^{C_{k+1,l-1}}_{\{Z_{k+1,l},Z_{k,l-1}\}}\omega(C_{k+1,l-1}) = \{1,0\}\\
j = 1\text{: } & \operatorname{res}^{|\partial_1\sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|}\omega(\partial_1\sigma^{(1)\prime\prime}_{k,l}) = \operatorname{res}^{C_{k,l}}_{\{Z_{k+1,l},Z_{k,l-1}\}}\omega(C_{k,l}) = \{0,1\}\\
\delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) = & \{1,0\} - \{0,1\} = \{1,1\}
\end{aligned}
$$

$$j = 0\text{: } \mathrm{res}^{|\partial_0 \sigma^{(1)'''}_{k,l}|}_{|\sigma^{(1)'''}_{k,l}|} \omega(\partial_0 \sigma^{(1)'''}_{k,l}) = \mathrm{res}^{C_{k,l+2}}_{\{Z_{k,l+1}\}} \omega(C_{k,l+2}) = 1$$

$$j = 1\text{: } \mathrm{res}^{|\partial_1 \sigma^{(1)'''}_{k,l}|}_{|\sigma^{(1)'''}_{k,l}|} \omega(\partial_1 \sigma^{(1)'''}_{k,l}) = \mathrm{res}^{C_{k,l}}_{\{Z_{k,l+1}\}} \omega(C_{k,l}) = 0$$

$$\delta^0(\omega)(\sigma^{(1)'''}_{k,l}) = 1 - 0 = 1$$

Calculating the values of the coboundary maps explicitly for the chosen $\omega \in C^0(\mathcal{U}, \mathcal{E})$ helps to postulate the image and kernels of $\delta^0(\omega)(\sigma^{(1)}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)'}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)''}_{k,l})$ and $\delta^0(\omega)(\sigma^{(1)'''}_{k,l})$ for general ω:

$$\begin{aligned} \mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{k,l}) &= \{0,1\}^{\times n \times n} \\ \ker \delta(\omega)^0(\sigma^{(1)}_{k,l}) &= \{M \in C^0(\mathcal{U}, \mathcal{E}) \mid M_{k,l}[[4]] \oplus M_{k,l+2}[[2]] = 0\} \end{aligned}$$

$$\begin{aligned} \mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)'}_{k,l}) &= \{00, 01, 10, 11\}^{\times n \times n} \\ \ker \delta^0(\omega)(\sigma^{(1)'}_{k,l}) &= \{M \in C^0(\mathcal{U}, \mathcal{E}) \mid M_{k,l}[[1]] \oplus M_{k+1,l+1}[[2]] = 0, \\ &\quad M_{k,l}[[4]] \oplus M_{k+1,l+1}[[5]] = 0\} \end{aligned}$$

$$\begin{aligned} \mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)''}_{k,l}) &= \{00, 01, 10, 11\}^{\times n \times n} \\ \ker \delta^0(\omega)(\sigma^{(1)''}_{k,l}) &= \{M \in C^0(\mathcal{U}, \mathcal{E}) \mid M_{k,l}[[4]] \oplus M_{k+1,l-1}[[1]] = 0, \\ &\quad M_{k,l}[[5]] \oplus M_{k+1,l-1}[[2]] = 0\} \end{aligned}$$

$$\begin{aligned} \mathrm{Im}\ \delta(\omega)(\sigma^{(1)'''}_{k,l}) &= \{0,1\}^{\times n \times n} \\ \ker \delta(\omega)(\sigma^{(1)'''}_{k,l}) &= \{M \in C^0(\mathcal{U}, \mathcal{E}) \mid M_{k,l}[[1]] \oplus M_{k,l+2}[[5]] = 0\} \end{aligned}$$

After computing examples of coboundary maps for $q = 0$ we want to compute values of $\delta^1(\omega)(\sigma^{(2)}_{k,l})$, $\delta^1(\omega)(\sigma^{(2)'}_{k,l})$, $\delta^1(\omega)(\sigma^{(2)''}_{k,l})$ and $\delta^1(\omega)(\sigma^{(2)'''}_{k,l})$. Considering the kernels of the four coboundary maps for $q = 0$ and the augmented complex one can see, that $\omega \in C^1(\mathcal{U}, \mathcal{E})$ has two parts with strings length 1 and two parts with strings of two values.

We choose

$$\omega = \left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & \underbrace{0}_{\sigma^{(1)}_{k,l}} & 1 \\ \underbrace{0}_{\sigma^{(1)}_{k-1,l-1}} & 0 & \underbrace{1}_{\sigma^{(1)}_{k-1,l+1}} \end{pmatrix}, \begin{pmatrix} 01 & 11 & 11 \\ 00 & \underbrace{00}_{\sigma^{(1)\prime}_{k,l}} & 11 \\ \underbrace{01}_{\sigma^{(1)\prime}_{k-1,l-1}} & 11 & 10 \end{pmatrix}, \begin{pmatrix} \underbrace{11}_{\sigma^{(1)\prime\prime}_{k-1,l+1}} & 11 & 01 \\ 00 & \underbrace{11}_{\sigma^{(1)\prime\prime}_{k,l}} & 11 \\ 01 & 01 & 10 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 1 & \underbrace{0}_{\sigma^{(1)\prime\prime\prime}_{k+1,l+1}} \\ 1 & 0 & 1 \\ \underbrace{1}_{\sigma^{(1)\prime\prime\prime}_{k-1,l-1}} & 0 & 1 \end{pmatrix} \right).$$

as an example.

$$\begin{aligned}
j = 0\colon\ & \mathrm{res}^{|\partial_0 \sigma^{(2)}_{k,l}|}_{|\sigma^{(2)}_{k,l}|} \omega(\partial_0 \sigma^{(2)}_{k,l}) = \mathrm{res}^{|C_{k+1,l+1}, C_{k+1,l-1}|}_{|C_{k,l}, C_{k+1,l+1}, C_{k+1,l-1}|} \omega(C_{k+1,l+1}, C_{k+1,l-1}) \\
&= \mathrm{res}^{\{Z_{k+1,l}\}}_{\{Z_{k+1,l}\}} \omega(\sigma^{(1)\prime\prime\prime}_{k+1,l+1}) = 0 \\
j = 1\colon\ & \mathrm{res}^{|\partial_1 \sigma^{(2)}_{k,l}|}_{|\sigma^{(2)}_{k,l}|} \omega(\partial_1 \sigma^{(2)}_{k,l}) = \mathrm{res}^{|C_{k,l}, C_{k+1,l-1}|}_{|C_{k,l}, C_{k+1,l+1}, C_{k+1,l-1}|} \omega(C_{k,l}, C_{k+1,l-1}) \\
&= \mathrm{res}^{\{Z_{k,l-1}, Z_{k+1,l}\}}_{\{Z_{k+1,l}\}} \omega(\sigma^{(1)\prime\prime}_{k,l}) = 1 \\
j = 2\colon\ & \mathrm{res}^{|\partial_2 \sigma^{(2)}_{k,l}|}_{|\sigma^{(2)}_{k,l}|} \omega(\partial_2 \sigma^{(2)}_{k,l}) = \mathrm{res}^{|C_{k,l}, C_{k+1,l+1}|}_{|C_{k,l}, C_{k+1,l+1}, C_{k+1,l-1}|} \omega(C_{k,l}, C_{k+1,l+1}) \\
&= \mathrm{res}^{\{Z_{k+1,l}, Z_{k,l+1}\}}_{\{Z_{k+1,l}\}} \omega(\sigma^{(1)\prime}_{k,l}) = 0 \\
\delta^1(\omega)(\sigma^{(2)}_{k,l}) =& 0 - 1 + 0 = 1
\end{aligned}$$

$$\begin{aligned}
j = 0\colon\ & \mathrm{res}^{|\partial_0 \sigma^{(2)\prime}_{k,l}|}_{|\sigma^{(2)\prime}_{k,l}|} \omega(\partial_0 \sigma^{(2)\prime}_{k,l}) = \mathrm{res}^{|C_{k-1,l-1}, C_{k+1,l-1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k+1,l-1}|} \omega(C_{k-1,l-1}, C_{k+1,l-1}) \\
&= \mathrm{res}^{\{Z_{k,l-1}\}}_{\{Z_{k,l-1}\}} \omega(\sigma^{(1)}_{k-1,l-1}) = 0 \\
j = 1\colon\ & \mathrm{res}^{|\partial_1 \sigma^{(2)\prime}_{k,l}|}_{|\sigma^{(2)\prime}_{k,l}|} \omega(\partial_1 \sigma^{(2)\prime}_{k,l}) = \mathrm{res}^{|C_{k,l}, C_{k+1,l-1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k+1,l-1}|} \omega(C_{k,l}, C_{k+1,l-1}) \\
&= \mathrm{res}^{\{Z_{k,l-1}, Z_{k+1,l}\}}_{\{Z_{k,l-1}\}} \omega(\sigma^{(1)\prime\prime}_{k,l}) = 1 \\
j = 2\colon\ & \mathrm{res}^{|\partial_2 \sigma^{(2)\prime}_{k,l}|}_{|\sigma^{(2)\prime}_{k,l}|} \omega(\partial_2 \sigma^{(2)\prime}_{k,l}) = \mathrm{res}^{|C_{k,l}, C_{k-1,l-1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k+1,l-1}|} \omega(C_{k,l}, C_{k-1,l-1}) \\
&= \mathrm{res}^{\{Z_{k,l-1}, Z_{k-1,l}\}}_{\{Z_{k,l-1}\}} \omega(\sigma^{(1)\prime}_{k-1,l-1}) = 0
\end{aligned}$$

$$\delta^1(\omega)(\sigma_{k,l}^{(2)\prime}) = 0 - 1 + 0 = 1$$

$$\begin{aligned}
j = 0\colon\ & \mathrm{res}^{|\partial_0 \sigma_{k,l}^{(2)\prime\prime}|}_{|\sigma_{k,l}^{(2)\prime\prime}|} \omega(\partial_0 \sigma_{k,l}^{(2)\prime\prime}) = \mathrm{res}^{|C_{k-1,l-1}, C_{k-1,l+1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k-1,l+1}|} \omega(C_{k-1,l-1}, C_{k-1,l+1}) \\
& = \mathrm{res}^{\{Z_{k-1,l}\}}_{\{Z_{k-1,l}\}} \omega(\sigma_{k-1,l-1}^{(1)\prime\prime\prime}) = 1 \\
j = 1\colon\ & \mathrm{res}^{|\partial_1 \sigma_{k,l}^{(2)\prime\prime}|}_{|\sigma_{k,l}^{(2)\prime\prime}|} \omega(\partial_1 \sigma_{k,l}^{(2)\prime\prime}) = \mathrm{res}^{|C_{k,l}, C_{k-1,l+1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k-1,l+1}|} \omega(C_{k,l}, C_{k-1,l+1}) \\
& = \mathrm{res}^{\{Z_{k-1,l}, Z_{k,l+1}\}}_{\{Z_{k-1,l}\}} \omega(\sigma_{k-1,l+1}^{(1)\prime\prime}) = 1 \\
j = 2\colon\ & \mathrm{res}^{|\partial_2 \sigma_{k,l}^{(2)\prime\prime}|}_{|\sigma_{k,l}^{(2)\prime\prime}|} \omega(\partial_2 \sigma_{k,l}^{(2)\prime\prime}) = \mathrm{res}^{|C_{k,l}, C_{k-1,l-1}|}_{|C_{k,l}, C_{k-1,l-1}, C_{k-1,l+1}|} \omega(C_{k,l}, C_{k-1,l-1}) \\
& = \mathrm{res}^{\{Z_{k-1,l}, Z_{k,l-1}\}}_{\{Z_{k-1,l}\}} \omega(\sigma_{k-1,l-1}^{(1)\prime}) = 0
\end{aligned}$$

$$\delta^1(\omega)(\sigma_{k,l}^{(2)\prime\prime}) = 1 - 1 + 0 = 0$$

$$\begin{aligned}
j = 0\colon\ & \mathrm{res}^{|\partial_0 \sigma_{k,l}^{(2)\prime\prime\prime}|}_{|\sigma_{k,l}^{(2)\prime\prime\prime}|} \omega(\partial_0 \sigma_{k,l}^{(2)\prime\prime}) = \mathrm{res}^{|C_{k-1,l+1}, C_{k+1,l+1}|}_{|C_{k,l}, C_{k-1,l+1}, C_{k+1,l+1}|} \omega(C_{k-1,l+1}, C_{k+1,l+1}) \\
& = \mathrm{res}^{\{Z_{k,l+1}\}}_{\{Z_{k,l+1}\}} \omega(\sigma_{k-1,l+1}^{(1)}) = 1 \\
j = 1\colon\ & \mathrm{res}^{|\partial_1 \sigma_{k,l}^{(2)\prime\prime\prime}|}_{|\sigma_{k-1,l+1}^{(2)}|} \omega(\partial_1 \sigma_{k,l}^{(2)\prime\prime\prime}) = \mathrm{res}^{|C_{k,l}, C_{k+1,l+1}|}_{|C_{k,l}, C_{k-1,l+1}, C_{k+1,l+1}|} \omega(C_{k,l}, C_{k+1,l+1}) \\
& = \mathrm{res}^{\{Z_{k,l+1}, Z_{k+1,l}\}}_{\{Z_{k,l+1}\}} \omega(\sigma_{k,l}^{(1)\prime}) = 0 \\
j = 2\colon\ & \mathrm{res}^{|\partial_2 \sigma_{k,l}^{(2)\prime\prime\prime}|}_{|\sigma_{k,l}^{(2)\prime\prime\prime}|} \omega(\partial_2 \sigma_{k,l}^{(2)\prime\prime\prime}) = \mathrm{res}^{|C_{k,l}, C_{k-1,l+1}|}_{|C_{k,l}, C_{k-1,l+1}, C_{k+1,l+1}|} \omega(C_{k,l}, C_{k-1,l+1}) \\
& = \mathrm{res}^{\{Z_{k,l+1}, Z_{k-1,l}\}}_{\{Z_{k,l+1}\}} \omega(\sigma_{k-1,l+1}^{(1)\prime\prime}) = 1
\end{aligned}$$

$$\delta^1(\omega)(\sigma_{k,l}^{(2)\prime\prime\prime}) = 1 - 0 + 1 = 0$$

For general $\omega \in C^1(\mathcal{U}, \mathcal{E})$ we get accordingly:

$$\begin{aligned}
\mathrm{Im}\ \delta^1(\omega(\sigma_{k,l}^{(2)})) &= \{0,1\}^{\times n \times n} \\
\ker \delta^1(\omega(\sigma_{k,l}^{(2)})) &= \{M \in C^1(\mathcal{U}, \mathcal{E}) \mid M_{2,k,l}[[2]] \oplus M_{3,k-1,l+1}[[1]] \oplus M_{4,k,l+2} = 0\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\ \delta^1(\omega)(\sigma_{k,l}^{(2)\prime}) &= \{0,1\}^{\times n \times n} \\
\ker \delta^1(\omega)(\sigma_{k,l}^{(2)\prime}) &= \{M \in C^1(\mathcal{U}, \mathcal{E}) \mid M_{1,k-1,l-1} \oplus M_{3,k,l}[[2]] \oplus M_{2,k-1,l-1}[[1]] = 0\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\ \delta^1(\omega)(\sigma_{k,l}^{(2)\prime\prime}) &= \{0,1\}^{\times n \times n} \\
\ker \delta^1(\omega)(\sigma_{k,l}^{(2)\prime\prime}) &= \{M \in C^1(\mathcal{U}, \mathcal{E}) \mid M_{4,k-1,l-1}[[2]] \oplus M_{3,k,l}[[2]] \oplus M_{2,k-1,l-1}[[1]] = 0\}
\end{aligned}$$

$$\begin{aligned}\text{Im } \delta^1(\omega)(\sigma^{(2)'''}_{k,l}) =& \{0,1\}^{\times n \times n}\\ \ker \delta^1(\omega)(\sigma^{(2)'''}_{k,l}) =& \{M \in C^1(\mathcal{U},\mathcal{E}) \mid M_{1,k-1,l+1} \oplus M_{2,k,l}[[1]] \oplus M_{3,k-1,l+1}[[1]] = 0\}\end{aligned}$$

To make the example complete we choose

$$\omega = \left(\begin{pmatrix} 0 & 1 & 1 \\ \underbrace{0}_{\sigma^{(2)}_{k,l}} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \underbrace{0}_{\sigma^{(2)'}_{k+1,l+1}} & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & \underbrace{1}_{\sigma^{(2)''}_{k+2,l}} \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & \underbrace{1}_{\sigma^{(2)'''}_{k+1,l-1}} & 0 \end{pmatrix} \right)$$

to compute $\delta^2(\omega)(\sigma^{(3)}_{k,l})$.

$$\begin{aligned}
j=0\colon\ & \text{res}^{|\partial_0\sigma^{(3)}_{k,l}|}_{|\sigma^{(3)}_{k,l}|}\omega(\partial_0\sigma^{(3)}_{k,l}) = \text{res}^{|C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1}|}_{|C_{k,l},C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1}|}\omega(C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1})\\
&= \text{res}^{\{Z_{k+1,l}\}}_{\{Z_{k+1,l}\}}\omega(\sigma^{(2)''}_{k+2,l}) = 1\\
j=1\colon\ & \text{res}^{|\partial_1\sigma^{(3)}_{k,l}|}_{|\sigma^{(3)}_{k,l}|}\omega(\partial_1\sigma^{(3)}_{k,l}) = \text{res}^{|C_{k,l},C_{k+2,l},C_{k+1,l-1}|}_{|C_{k,l},C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1}|}\omega(C_{k,l},C_{k+2,l},C_{k+1,l-1})\\
&= \text{res}^{\{Z_{k+1,l}\}}_{\{Z_{k+1,l}\}}\omega(\sigma^{(2)'''}_{k+1,l-1}) = 1\\
j=2\colon\ & \text{res}^{|\partial_2\sigma^{(3)}_{k,l}|}_{|\sigma^{(3)}_{k,l}|}\omega(\partial_2\sigma^{(3)}_{k,l}) = \text{res}^{|C_{k,l},C_{k+1,l+1},C_{k+1,l-1}|}_{|C_{k,l},C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1}|}\omega(C_{k,l},C_{k+1,l+1},C_{k+1,l-1})\\
&= \text{res}^{\{Z_{k+1,l}\}}_{\{Z_{k+1,l}\}}\omega(\sigma^{(2)}_{k,l}) = 0\\
j=3\colon\ & \text{res}^{|\partial_3\sigma^{(3)}_{k,l}|}_{|\sigma^{(3)}_{k,l}|}\omega(\partial_3\sigma^{(3)}_{k,l}) = \text{res}^{|C_{k,l},C_{k+1,l+1},C_{k+2,l}|}_{|C_{k,l},C_{k+1,l+1},C_{k+2,l},C_{k+1,l-1}|}\omega(C_{k,l},C_{k+1,l+1},C_{k+2,l})\\
&= \text{res}^{\{Z_{k+1,l}\}}_{\{Z_{k+1,l}\}}\omega(\sigma^{(2)'}_{k+1,l+1}) = 0
\end{aligned}$$

$$\delta^2(\omega)(\sigma^{(3)}_{k,l}) = 1 - 1 + 0 - 0 = 0$$

Generalizing the calculation for $\omega \in C^2(\mathcal{U},\mathcal{E})$ we end up with:

$$\begin{aligned}\text{Im } \delta^2(\omega)(\sigma^{(3)}_{k,l}) =& \{0,1\}^{\times n \times n}\\ \ker \delta^2(\omega)(\sigma^{(3)}_{k,l}) =& \{M \in C^2(\mathcal{U},\mathcal{E}) \mid M_{1,k,l} \oplus M_{2,k+1,l+1} \oplus M_{3,k+2,l} \oplus M_{4,k+1,l-1} = 0\}\end{aligned}$$

After considering the images and kernels of the all functions concerning the augmented complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} C^3 \xrightarrow{\delta^3} 0$$

for special types of simplices we are able to compute $\check{H}^0(\mathcal{U},\mathcal{E})$. The set of 0-cocycles $Z^0(\mathcal{U},\mathcal{E})$ is the intersection of $\ker \delta^0(\omega)(\sigma^{(1)}_k)$, $\ker \delta^0(\omega)(\sigma^{(1)'}_k)$, $\ker \delta^0(\omega)(\sigma^{(1)''}_k)$ and

$\ker \delta^0(\omega)(\sigma_k^{(1)'''})$:

$$\begin{aligned} Z^0(\mathcal{U},\mathcal{E}) = \ker \delta^0 = \{M \in C^0(\mathcal{U},\mathcal{E}) \mid & M_{k,l}[[4]] \oplus M_{k,l+2}[[2]] = 0, \\ & M_{k,l}[[1]] \oplus M_{k+1,l+1}[[2]] = 0, M_{k,l}[[4]] \oplus M_{k+1,l+1}[[5]] = 0, \\ & M_{k,l}[[4]] \oplus M_{k+1,l-1}[[1]] = 0, M_{k,l}[[5]] \oplus M_{k+1,l-1}[[2]] = 0, \\ & M_{k,l}[[1]] \oplus M_{k,l+2}[[5]] = 0, k,l \in \mathbb{N}\} \end{aligned}$$

In these measurement outcome structures every context $C_{k,l}$ with $k,l \in \mathbb{N}$ is compatible with the adjacent contexts $C_{k+2,l}$, $C_{k+1,l+1}$, $C_{k+1,l-1}$ and $C_{k,l-2}$. Because the 0-coboundaries are

$$B^0(\mathcal{U},\mathcal{E}) = \text{Im } \delta^{-1} = \{0000\}^{\times n \times n},$$

it is $Z^0(\mathcal{U},\mathcal{E}) = \check{H}^0(\mathcal{U},\mathcal{E})$. Finally $\check{H}^0(\mathcal{U},\mathcal{E})$ contains global section strings for which all contexts are compatible.

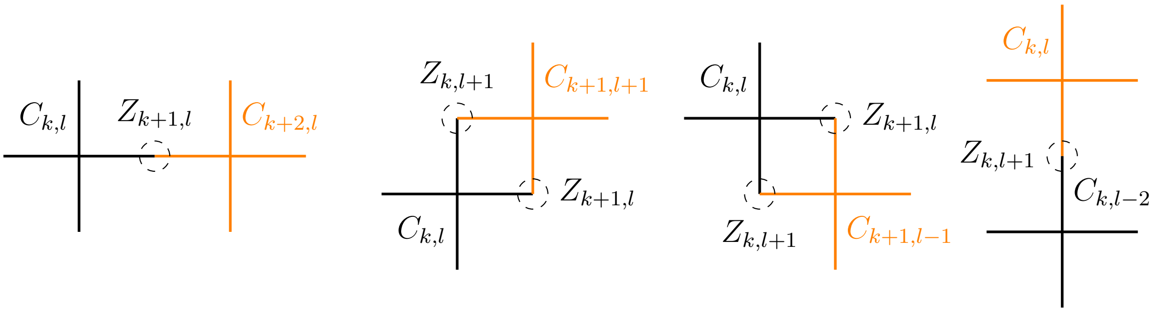


Figure 5.4: The four ways each context $C_{k,l}$ has to be compatible with its neighbor contexts in a global section

One can see no interesting structure in the cohomology groups for $q \in \{2,3,4\}$:

$$\begin{aligned} Z^1(\mathcal{U},\mathcal{E}) = \ker \delta^1 = \{M \in C^1(\mathcal{U},\mathcal{E}) \mid & \\ & M_{2,k,l}[[2]] \oplus M_{3,k-1,l+1}[[1]] \oplus M_{4,k,l+2} = 0, \\ & M_{1,k-1,l-1} \oplus M_{3,k,l}[[2]] \oplus M_{2,k-1,l-1}[[2]] = 0, \\ & M_{4,k-1,l-1}[[2]] \oplus M_{3,k,l}[[2]] \oplus M_{2,k-1,l-1}[[1]] = 0, \\ & M_{1,k-1,l+1} \oplus M_{2,k,l}[[1]] \oplus M_{3,k-1,l+1}[[1]] = 0, k,l \in \mathbb{N}\} \\ B^1(\mathcal{U},\mathcal{E}) = \text{Im } \delta^0 = \big(& \{0,1\}^{\times n \times n}, \{00,01,10,11\}^{\times n \times n}, \\ & \{00,01,10,11\}^{\times n \times n}, \{0,1\}^{\times n \times n}\big) \\ \check{H}^1(\mathcal{U},\mathcal{E}) = & Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) \\ = & \big(\{0,1\}^{\times n \times n}, \{00,01,10,11\}^{\times n \times n}, \{00,01,10,11\}^{\times n \times n}, \{0,1\}^{\times n \times n}\big) \end{aligned}$$

$$\begin{aligned} Z^2(\mathcal{U},\mathcal{E}) = \ker \delta^2 = \{M \in C^2(\mathcal{U},\mathcal{E}) \mid & \\ & M_{1,k,l} \oplus M_{2,k+1,l+1} \oplus M_{3,k+2,l} \oplus M_{4,k+1,l-1} = 0\} \end{aligned}$$

$$
\begin{aligned}
B^2(\mathcal{U},\mathcal{E}) =& \text{Im } \delta^1 = \{0,1\}^{\times n\times n} \times \{0,1\}^{\times n\times n} \times \{0,1\}^{\times n\times n} \\
& \times \{0,1\}^{\times n\times n} \\
\check{H}^2(\mathcal{U},\mathcal{E}) =& Z^2(\mathcal{U},\mathcal{E})/B^2(\mathcal{U},\mathcal{E}) = \big(\{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}, \\
& \{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}\big)
\end{aligned}
$$

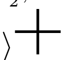
$$
\begin{aligned}
Z^3(\mathcal{U},\mathcal{E}) =& \ker \delta^3 = \{0,1\}^{\times n\times n} \\
B^3(\mathcal{U},\mathcal{E}) =& \text{Im } \delta^2 = \{0,1\}^{\times n\times n} \\
\check{H}^3(\mathcal{U},\mathcal{E}) =& Z^3(\mathcal{U},\mathcal{E})/B^3(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times n\times n}
\end{aligned}
$$

5.3 Toric code

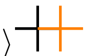
In Section 3.5 we considered toric code on a plane. In the following we want to discuss the belonging cohomology group. After explaining the calculations detailed in the last section we put parts of the next examples in the appendix.

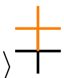
Contexts: $A_{k,l} = \{X_{k,l+\frac{1}{2}}, X_{k-\frac{1}{2},l}, X_{k+\frac{1}{2},l}, X_{k,l-\frac{1}{2}}\}$

$P_{k,l} = \{Z_{k+\frac{1}{2},l}, Z_{k,l-\frac{1}{2}}, Z_{k+\frac{1}{2},l-1}, X_{k+1,l-\frac{1}{2}}\}$

Simplices: $\sigma^{(0)}_{k,l} = \langle A_{k,l}\rangle$ 

$\sigma^{(0)\prime}_{k,l} = \langle P_{k,l}\rangle$ 

$\sigma^{(1)}_{k,l} = \langle A_{k,l}, A_{k+1,l}\rangle$ 

$\sigma^{(1)\prime}_{k,l} = \langle A_{k,l}, A_{k,l+1}\rangle$ 

$\sigma^{(1)\prime\prime}_{k,l} = \langle P_{k,l}, P_{k+1,l}\rangle$ 

$\sigma^{(1)\prime\prime\prime}_{k,l} = \langle P_{k,l}, P_{k,l+1}\rangle$ 

Nerves: $\mathcal{N}(\mathcal{U})^0 = \{\sigma^{(0)}_{k,l}, \sigma^{(0)\prime}_{k,l} \,|\, k,l \in \mathbb{N}\}$

$\mathcal{N}(\mathcal{U})^1 = \{\sigma^{(1)}_{k,l}, \sigma^{(1)\prime}_{k,l}, \sigma^{(1)\prime\prime}_{k,l}, \sigma^{(1)\prime\prime\prime}_{k,l} \,|\, k,l \in \mathbb{N}\}$

Sheaf: $\mathcal{E}(A_{k,l}) =$ even numbers of 1s $= \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$

$\mathcal{E}(P_{k,l}) =$ even numbers of 1s $= \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$

$\mathcal{E}(|\sigma^{(1)}_{k,l}|) = \mathcal{E}(\{X_{k+\frac{1}{2},l}\}) = \{0,1\}$

$\mathcal{E}(|\sigma^{(1)\prime}_{k,l}|) = \mathcal{E}(\{X_{k,l-\frac{1}{2}}\}) = \{0,1\}$

$$\begin{aligned}
\mathcal{E}(|\sigma_{k,l}^{(1)\prime\prime}|) = &\quad \mathcal{E}(\{Z_{k+1,l-\frac{1}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma_{k,l}^{(1)\prime\prime\prime}|) = &\quad \mathcal{E}(\{Z_{k+\frac{1}{2},l}\}) = \{0,1\} \\
\text{Cochains:} \quad C^0(\mathcal{U},\mathcal{E}) = &\quad \textstyle\prod_{\sigma_{k,l}\in\mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_{k,l}|) = (\mathcal{E}(A_{k,l})^{\times n\times n}, \mathcal{E}(P_{k,l})^{\times n\times n}) \\
C^1(\mathcal{U},\mathcal{E}) = &\quad \textstyle\prod_{\sigma_{k,l}\in\mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_{k,l}|) \\
= &\quad (\{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n}, \{0,1\}^{\times n\times n})
\end{aligned}$$

For the augmented complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$$

we computed the cohomology group in the appendix:

$$\begin{aligned}
Z^0(\mathcal{U},\mathcal{E}) =& \ker\delta^0 = \{M \in C^0(\mathcal{U},\mathcal{E}) \mid M_{1,k,l}[[3]] \oplus M_{1,k+1,l}[[2]] = 0, \\
& M_{1,k,l}[[1]] \oplus M_{1,k,l+1}[[4]] = 0, M_{2,k,l}[[4]] \oplus M_{2,k+1,l}[[2]] = 0, \\
& M_{2,k,l}[[1]] \oplus M_{2,k+1,l}[[3]] = 0, k,l \in \mathbb{N}\} \\
B^0(\mathcal{U},\mathcal{E}) =& \text{Im } \delta^{-1} = \{0000\}^{\times n\times n} \\
\check{H}^0(\mathcal{U},\mathcal{E}) =& Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) = \{M \in C^0(\mathcal{U},\mathcal{E}) \mid \\
& M_{1,k,l}[[3]] \oplus M_{1,k+1,l}[[2]] = 0, M_{1,k,l}[[1]] \oplus M_{1,k,l+1}[[4]] = 0, \\
& M_{2,k,l}[[4]] \oplus M_{2,k+1,l}[[2]] = 0, M_{2,k,l}[[1]] \oplus M_{2,k+1,l}[[3]] = 0, k,l \in \mathbb{N}\}
\end{aligned}$$

$$\begin{aligned}
Z^1(\mathcal{U},\mathcal{E}) =& \ker\delta^1 = \{0,1\}^{\times n\times n} \\
B^1(\mathcal{U},\mathcal{E}) =& \text{Im } \delta^0 = \{0,1\}^{\times n\times n} \\
\check{H}^1(\mathcal{U},\mathcal{E}) =& Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \left[\{0,1\}^{\times n\times n}\right] \\
=& \{0,1\}^{\times n\times n} \oplus \{0,1\}^{\times n\times n} = \{0,1\}^{\times n\times n}
\end{aligned}$$

The solutions of $\check{H}^0(\mathcal{U},\mathcal{E})$ correspond to the global sections in the plane. We are not able to express these in specific finite strings because we worked with an infinite plane.

5.3.1 On a disk

We can lay the toric code on a disk as depicted in Figure 5.5. The plaquette operators behave like in a plane, but the star operators differ. In contrast to a plane oric code, the model on a disk includes stars with only two or three tails on the edges of the shape. First we calculate the probabilities for the outcome sets on these stars.

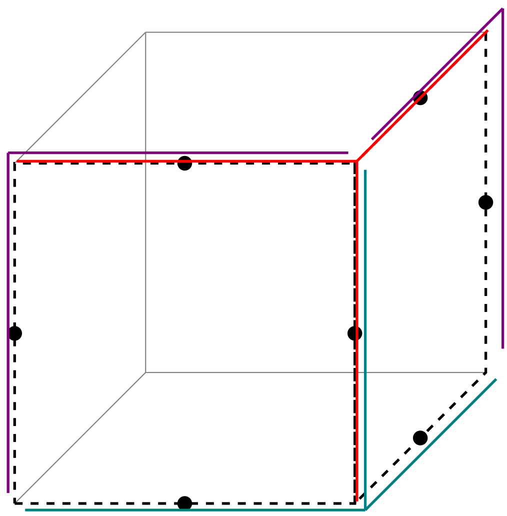


Figure 5.5: Toric code on a disk

We start considering the stars with two tails.

Figure 5.6: Measurements A_v on edges

Therefore we use stabilizers analogously to Section 3.4.1:

$$S_1 = ZZ$$
$$S_2 = XX$$
$$S_3 = \mathbb{1}X$$

The probability to get outcome $(o_1\,o_2)$ with $o_i \in \{-1, 1\}$ is

$$\text{Prob}[X, X] = \frac{1}{4}(1 + o_1 o_2).$$

For the stars with three tails we also use the stabilizer algorithm to calculate the probabilities.

Figure 5.7: Measurements A_v on edges

$$\begin{aligned}
S_1 &= ZZ\mathbb{1}\\
S_2 &= Z\mathbb{1}Z\\
S_3 &= \mathbb{1}ZZ\\
S_4 &= XXX\\
S_5 &= X\mathbb{1}\mathbb{1}\\
S_6 &= \mathbb{1}X\mathbb{1}\\
S_7 &= \mathbb{1}\mathbb{1}X
\end{aligned}$$

The chance to get outcome $(o_1\, o_2\, o_3)$ is

$$\text{Prob}[X, X, X] = \frac{1}{8}(1 + o_1 o_2 o_3).$$

Now it is known which outcome sets are possible, i.e. Prob > 0. With these results we are able to compute the appropriate cohomology group.

$$\begin{array}{lrl}
\text{Contexts:} & A_{1,1,0} = & \{X_{1,\frac{1}{2},0}, X_{1,1,\frac{1}{2}}, X_{\frac{1}{2},1,0}\}\\
& A_{1,0,0} = & \{X_{\frac{1}{2},0,0}, X_{1,0,\frac{1}{2}}, X_{1,\frac{1}{2},0}\}\\
& A_{1,1,1} = & \{X_{1,1,\frac{1}{2}}, X_{1,\frac{1}{2},1}\}\\
& A_{0,1,0} = & \{X_{0,\frac{1}{2},0}, X_{\frac{1}{2},1,0}\}\\
& P_{0,0,0} = & \{Z_{\frac{1}{2},0,0}, Z_{1,\frac{1}{2},0}, Z_{\frac{1}{2},1,0}, Z_{0,\frac{1}{2},0}\}\\
& P_{1,0,0} = & \{Z_{1,0,\frac{1}{2}}, Z_{1,\frac{1}{2},1}, Z_{1,1,\frac{1}{2}}, Z_{1,\frac{1}{2},0}, \}\\
\text{Simplices:} & \sigma^{(0)}_{k,l,j} = & \langle A_{k,l,j}\rangle\,,\ klj \in \{110, 100, 111, 010\}\\
& \sigma^{(0)\prime}_{k,l,j} = & \langle P_{k,l,j}\rangle\,,\ klj \in \{000, 100\}\\
& \sigma^{(1)}_{1,0,0} = & \langle A_{1,0,0}, A_{1,1,0}\rangle\\
& \sigma^{(1)}_{1,1,1} = & \langle A_{1,1,1}, A_{1,1,0}\rangle\\
& \sigma^{(1)}_{0,1,0} = & \langle A_{0,1,0}, A_{1,1,0}\rangle\\
& \sigma^{(1)\prime}_{0,0,0} = & \langle P_{0,0,0}, P_{1,0,0}\rangle\\
\text{Nerves:} & \mathcal{N}(\mathcal{U})^0 = & \left\{\sigma^{(0)}_{k,l,j},\ klj \in \{110, 100, 111, 010\},\ \sigma^{(0)\prime}_{k,l,j},\ klj \in \{000, 100\}\right\}\\
& \mathcal{N}(\mathcal{U})^1 = & \left\{\sigma^{(1)}_{k,l,j},\ klj \in \{100, 111, 010\},\ \sigma^{(1)\prime}_{0,0,0}\right\}\\
\text{Sheaf:} & \mathcal{E}(A_{k,l,j}) = & \{000, 011, 101, 110\},\ klj \in \{110, 100\}\\
& \mathcal{E}(A_{k,l,j}) = & \{00, 11\},\ klj \in \{111, 010\}\\
& \mathcal{E}(P_{k,l,j}) = & \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}
\end{array}$$

$$
\begin{aligned}
& \mathcal{E}(|\sigma^{(1)}_{1,0,0}|) = && \mathcal{E}(\{X_{1,\frac{1}{2},0}\}) = \{0,1\} \\
& \mathcal{E}(|\sigma^{(1)}_{1,1,1}|) = && \mathcal{E}(\{X_{1,1,\frac{1}{2}}\}) = \{0,1\} \\
& \mathcal{E}(|\sigma^{(1)}_{0,1,0}|) = && \mathcal{E}(\{X_{\frac{1}{2},1,0}\}) = \{0,1\} \\
& \mathcal{E}(|\sigma^{(1)\prime}_{0,0,0}|) = && \mathcal{E}(\{Z_{1,\frac{1}{2},0}\}) = \{0,1\} \\
\text{Cochains:}\quad & C^0(\mathcal{U},\mathcal{E}) = && \textstyle\prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_{k,l,j}|) = (\mathcal{E}(A_{k,l,j})^{\times 4},\, \mathcal{E}(P_{k,l,j})^{\times 3}) \\
& C^1(\mathcal{U},\mathcal{E}) = && \textstyle\prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_{k,l,j}|) = (\{0,1\}^{\times 3},\, \{0,1\}^{\times 3})
\end{aligned}
$$

After considering the whole augmented complex in the appendix

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$$

we are able to write down the cocycles, coboundaries and the cohomology group for the model.

$$
\begin{aligned}
Z^0(\mathcal{U},\mathcal{E}) =& \ker\delta^0 = \{S \in C^0(\mathcal{U},\mathcal{E}) \mid A_{100}[[3]] \oplus A_{110}[[1]] = 0, \\
& A_{111}[[1]] \oplus A_{110}[[2]] = 0,\ A_{010}[[2]] \oplus A_{110}[[3]] = 0,\ P_{000}[[2]] \oplus P_{100}[[4]] = 0\} \\
B^0(\mathcal{U},\mathcal{E}) =& \operatorname{Im}\,\delta^{-1} = \{000\}^{\times n} \\
\check{H}^0(\mathcal{U},\mathcal{E}) =& Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) = Z^0(\mathcal{U},\mathcal{E}) \\
Z^1(\mathcal{U},\mathcal{E}) =& \ker\delta^1 = \{0,1\}^{\times n} \\
B^1(\mathcal{U},\mathcal{E}) =& \operatorname{Im}\,\delta^0 = \{0,1\}^{\times n} \\
\check{H}^1(\mathcal{U},\mathcal{E}) =& Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times n}
\end{aligned}
$$

Because there is no common constraint, we can divide the solution in the outcomes of the X- and Z-measurements. Starting with the values belonging to the vertex operators we get the structure

$$
\begin{aligned}
v =& \{X_{1\frac{1}{2}0}, X_{11\frac{1}{2}}, X_{\frac{1}{2}10}, X_{\frac{1}{2}00}, X_{10\frac{1}{2}}, X_{1\frac{1}{2}1}, X_{0\frac{1}{2}0}\} \\
M =& \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Solving $Mv = 0$ we get 8 possible strings.

$$
\begin{aligned}
\check{H}^0(\mathcal{U},\mathcal{E})|_X = & \{v \mid X_{\frac{1}{2}10} = -X_{11\frac{1}{2}} - X_{1\frac{1}{2}0}, X_{10\frac{1}{2}} = -X_{1\frac{1}{2}0} - X_{\frac{1}{2}00}, \\
& X_{1\frac{1}{2}1} = -X_{11\frac{1}{2}}, X_{0\frac{1}{2}0} = X_{11\frac{1}{2}} + X_{1\frac{1}{2}0}\}. \\
= & \{\{0,0,0,0,0,0,0\}, \{0,0,0,1,1,0,0\}, \{0,1,1,0,0,1,1\},
\end{aligned}
$$

$$\{0,1,1,1,1,1,1\},\{1,0,1,0,1,0,1\},\{1,0,1,1,0,0,1\},$$
$$\{1,1,0,0,1,1,0\},\{1,1,0,1,0,1,0\}\}$$

Repeating the procedure to get the global sections considering the Z-measurements we end up with 32 possible strings.

$$v = \{Z_{\frac{1}{2}00}, Z_{1\frac{1}{2}0}, Z_{\frac{1}{2}10}, Z_{0\frac{1}{2}0}, Z_{10\frac{1}{2}}, Z_{1\frac{1}{2}1}, Z_{11\frac{1}{2}}\}$$
$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned}\check{H}^0(\mathcal{U},\mathcal{E})|_Z = \{v \mid Z_{020} = -Z_{120} - Z_{200} - Z_{210}, Z_{112} = -Z_{102} - Z_{120} - Z_{121}\}.\\
= \{\{0,0,0,0,0,0,0\},\{0,0,0,0,0,1,1\},\{0,0,0,0,1,0,1\},\\
\{0,0,0,0,1,1,0\},\{0,0,1,1,0,0,0\},\{0,0,1,1,0,1,1\},\\
\{0,0,1,1,1,0,1\},\{0,0,1,1,1,1,0\},\{0,1,0,1,0,0,1\},\\
\{0,1,0,1,0,1,0\},\{0,1,0,1,1,0,0\},\{0,1,0,1,1,1,1\},\\
\{0,1,1,0,0,0,1\},\{0,1,1,0,0,1,0\},\{0,1,1,0,1,0,0\},\\
\{0,1,1,0,1,1,1\},\{1,0,0,1,0,0,0\},\{1,0,0,1,0,1,1\},\\
\{1,0,0,1,1,0,1\},\{1,0,0,1,1,1,0\},\{1,0,1,0,0,0,0\},\\
\{1,0,1,0,0,1,1\},\{1,0,1,0,1,0,1\},\{1,0,1,0,1,1,0\},\\
\{1,1,0,0,0,0,1\},\{1,1,0,0,0,1,0\},\{1,1,0,0,1,0,0\},\\
\{1,1,0,0,1,1,1\},\{1,1,1,1,0,0,1\},\{1,1,1,1,0,1,0\},\\
\{1,1,1,1,1,0,0\},\{1,1,1,1,1,1,1\}\}\end{aligned}$$

Finally there exist $8 \times 32 = 256$ possible global sections for the toric code depicted on Figure 5.5, because

$$\check{H}^0(\mathcal{U},\mathcal{E})|_X \times \check{H}^0(\mathcal{U},\mathcal{E})|_Z = \check{H}^0(\mathcal{U},\mathcal{E}).$$

5.3.2 On an edge

In this section we consider toric code on an edge. Here we only have stars with three tails.

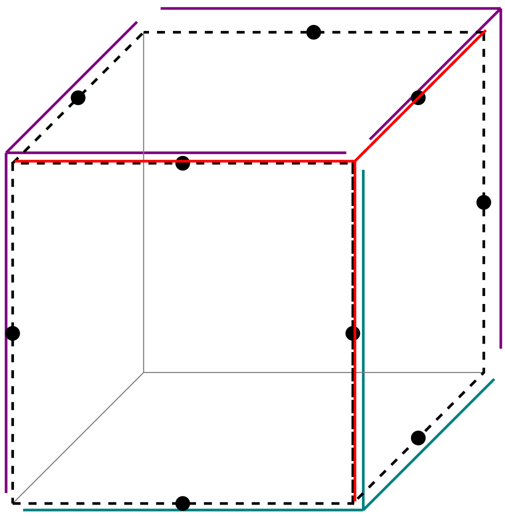


Figure 5.8: Toric code on a edge

$$
\begin{aligned}
\text{Contexts:}\quad & A_{1,1,0} = && \{X_{1,\frac{1}{2},0}, X_{1,1,\frac{1}{2}}, X_{\frac{1}{2},1,0}\}\\
& A_{1,0,0} = && \{X_{\frac{1}{2},0,0}, X_{1,0,\frac{1}{2}}, X_{1,\frac{1}{2},0}\}\\
& A_{1,1,1} = && \{X_{1,1,\frac{1}{2}}, X_{1,\frac{1}{2},1}, X_{\frac{1}{2},1,1}\}\\
& A_{0,1,0} = && \{X_{0,\frac{1}{2},0}, X_{\frac{1}{2},1,0}, X_{0,1,\frac{1}{2}}\}\\
& P_{0,0,0} = && \{Z_{\frac{1}{2},0,0}, Z_{1,\frac{1}{2},0}, Z_{\frac{1}{2},1,0}, Z_{0,\frac{1}{2},0}\}\\
& P_{1,0,0} = && \{Z_{1,0,\frac{1}{2}}, Z_{1,\frac{1}{2},1}, Z_{1,1,\frac{1}{2}}, Z_{1,\frac{1}{2},0},\}\\
& P_{0,1,0} = && \{Z_{\frac{1}{2},1,0}, Z_{1,1,\frac{1}{2}}, Z_{\frac{1}{2},1,1}, Z_{0,1,\frac{1}{2}}\}\\
\text{Simplices:}\quad & \sigma^{(0)}_{k,l,j} = && \langle A_{k,l,j}\rangle\ ,\ klj \in \{110, 100, 111, 010\}\\
& \sigma^{(0)\prime}_{k,l,j} = && \langle P_{k,l,j}\rangle\ ,\ klj \in \{000, 100, 010\}\\
& \sigma^{(1)}_{1,0,0} = && \langle A_{1,0,0}, A_{1,1,0}\rangle\\
& \sigma^{(1)}_{1,1,1} = && \langle A_{1,1,1}, A_{1,1,0}\rangle\\
& \sigma^{(1)}_{0,1,0} = && \langle A_{0,1,0}, A_{1,1,0}\rangle\\
& \sigma^{(1)\prime}_{0,0,0} = && \langle P_{0,0,0}, P_{1,0,0}\rangle\\
& \sigma^{(1)\prime}_{1,0,0} = && \langle P_{1,0,0}, P_{0,1,0}\rangle\\
& \sigma^{(1)\prime}_{0,1,0} = && \langle P_{0,1,0}, P_{0,0,0}\rangle\\
\text{Nerves:}\quad & \mathcal{N}(\mathcal{U})^0 = && \Big\{\sigma^{(0)}_{k,l,j},\ klj \in \{110, 100, 111, 010\},\\
& && \sigma^{(0)\prime}_{k,l,j},\ klj \in \{000, 100, 010\}\Big\}\\
& \mathcal{N}(\mathcal{U})^1 = && \Big\{\sigma^{(1)}_{k,l,j},\ klj \in \{100, 111, 010\},\ \sigma^{(1)\prime}_{k,l,j},\ klj \in \{000, 100, 010\}\Big\}
\end{aligned}
$$

$$\begin{aligned}
\text{Sheaf:} \quad \mathcal{E}(A_{k,l,j}) =& \quad \{000, 011, 101, 110\},\ klj \in \{110, 100, 111, 010\} \\
\mathcal{E}(P_{k,l,j}) =& \quad \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\} \\
\mathcal{E}(|\sigma^{(1)}_{1,0,0}|) =& \quad \mathcal{E}(\{X_{1,\frac{1}{2},0}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)}_{1,1,1}|) =& \quad \mathcal{E}(\{X_{1,1,\frac{1}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)}_{0,1,0}|) =& \quad \mathcal{E}(\{X_{\frac{1}{2},1,0}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{0,0,0}|) =& \quad \mathcal{E}(\{Z_{1,\frac{1}{2},0}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{1,0,0}|) =& \quad \mathcal{E}(\{Z_{1,1,\frac{1}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{0,1,0}|) =& \quad \mathcal{E}(\{Z_{\frac{1}{2},1,0}\}) = \{0,1\} \\
\text{Cochains:} \quad C^0(\mathcal{U},\mathcal{E}) =& \quad \textstyle\prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_{k,l,j}|) = (\mathcal{E}(A_{k,l,j})^{\times 4},\ \mathcal{E}(P_{k,l,j})^{\times 3}) \\
C^1(\mathcal{U},\mathcal{E}) =& \quad \textstyle\prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_{k,l,j}|) = (\{0,1\}^{\times 3},\ \{0,1\}^{\times 3})
\end{aligned}$$

One can see in the appendix that we end up with the following cohomology group.

$$\begin{aligned}
Z^0(\mathcal{U},\mathcal{E}) =& \ker \delta^0 = \{S \in C^0(\mathcal{U},\mathcal{E}) \,|\, A_{100}[[3]] \oplus A_{110}[[1]] = 0, \\
& A_{111}[[1]] \oplus A_{110}[[2]] = 0,\ A_{010}[[2]] \oplus A_{110}[[3]] = 0, \\
& P_{000}[[2]] \oplus P_{100}[[4]] = 0,\ P_{100}[[3]] \oplus P_{100}[[2]] = 0,\ P_{010}[[1]] \oplus P_{000}[[3]] = 0\} \\
B^0(\mathcal{U},\mathcal{E}) =& \mathrm{Im}\ \delta^{-1} = \{000\}^{\times n} \\
\check{H}^0(\mathcal{U},\mathcal{E}) =& Z^0(\mathcal{U},\mathcal{E})/B^0(\mathcal{U},\mathcal{E}) = \{S \in C^0(\mathcal{U},\mathcal{E}) \,| \\
& A_{100}[[3]] \oplus A_{110}[[1]] = 0,\ A_{111}[[1]] \oplus A_{110}[[2]] = 0,\ A_{010}[[2]] \oplus A_{110}[[3]] = 0, \\
& P_{000}[[2]] \oplus P_{100}[[4]] = 0,\ P_{100}[[3]] \oplus P_{100}[[2]] = 0,\ P_{010}[[1]] \oplus P_{000}[[3]] = 0\} \\
Z^1(\mathcal{U},\mathcal{E}) =& \ker \delta^1 = \{0,1\}^{\times n} \\
B^1(\mathcal{U},\mathcal{E}) =& \mathrm{Im}\ \delta^0 = \{0,1\}^{\times n} \\
\check{H}^1(\mathcal{U},\mathcal{E}) =& Z^1(\mathcal{U},\mathcal{E})/B^1(\mathcal{U},\mathcal{E}) = \{0,1\}^{\times n}
\end{aligned}$$

Again we can divide the solution in the part of X- and of Z-measurements.

$$v = \{X_{1\frac{1}{2}0}, X_{11\frac{1}{2}}, X_{\frac{1}{2}10}, X_{\frac{1}{2}00}, X_{10\frac{1}{2}}, X_{1\frac{1}{2}1}, X_{\frac{1}{2}11}, X_{0\frac{1}{2}0}, X_{01\frac{1}{2}}\}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Solving $Mv = 0$ we get 32 possible strings.

$$\check{H}^0(\mathcal{U},\mathcal{E})|_X = \{v \,|\, X_{\frac{1}{2}10} = -X_{11\frac{1}{2}} - X_{1\frac{1}{2}0},\ X_{10\frac{1}{2}} = -X_{1\frac{1}{2}0} - X_{\frac{1}{2}00},$$

$$
\begin{aligned}
&X_{\frac{1}{2}11} = -X_{11\frac{1}{2}} - X_{1\frac{1}{2}1},\ X_{01\frac{1}{2}} = -X_{0\frac{1}{2}0} + X_{11\frac{1}{2}} + X_{1\frac{1}{2}0}\}. \\
= \{&\{0,0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,1,1\}, \{0,0,0,0,0,1,1,0,0\}, \\
&\{0,0,0,0,0,1,1,1,1\}, \{0,0,0,1,1,0,0,0,0\}, \{0,0,0,1,1,0,0,1,1\}, \\
&\{0,0,0,1,1,1,1,0,0\}, \{0,0,0,1,1,1,1,1,1\}, \{0,1,1,0,0,0,1,0,1\}, \\
&\{0,1,1,0,0,0,1,1,0\}, \{0,1,1,0,0,1,0,0,1\}, \{0,1,1,0,0,1,0,1,0\}, \\
&\{0,1,1,1,1,0,1,0,1\}, \{0,1,1,1,1,0,1,1,0\}, \{0,1,1,1,1,1,0,0,1\}, \\
&\{0,1,1,1,1,1,0,1,0\}, \{1,0,1,0,1,0,0,0,1\}, \{1,0,1,0,1,0,0,1,0\}, \\
&\{1,0,1,0,1,1,1,0,1\}, \{1,0,1,0,1,1,1,1,0\}, \{1,0,1,1,0,0,0,0,1\}, \\
&\{1,0,1,1,0,0,0,1,0\}, \{1,0,1,1,0,1,1,0,1\}, \{1,0,1,1,0,1,1,1,0\}, \\
&\{1,1,0,0,1,0,1,0,0\}, \{1,1,0,0,1,0,1,1,1\}, \{1,1,0,0,1,1,0,0,0\}, \\
&\{1,1,0,0,1,1,0,1,1\}, \{1,1,0,1,0,0,1,0,0\}, \{1,1,0,1,0,0,1,1,1\}, \\
&\{1,1,0,1,0,1,0,0,0\}, \{1,1,0,1,0,1,0,1,1\}\}
\end{aligned}
$$

We have to concern also the Z-measurements.

$$
\begin{aligned}
v =& \{Z_{\frac{1}{2}00}, Z_{1\frac{1}{2}0}, Z_{\frac{1}{2}10}, Z_{0\frac{1}{2}0}, Z_{10\frac{1}{2}}, Z_{1\frac{1}{2}1}, Z_{11\frac{1}{2}}, Z_{\frac{1}{2}11}, Z_{01\frac{1}{2}}\} \\
M =& \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

The compatible sections over the Z-measurements are 64 strings:

$$
\begin{aligned}
\check{H}^0(\mathcal{U},\mathcal{E})|_Z = &\{v \mid Z_{020} = -Z_{120} - Z_{200} - Z_{210},\ Z_{112} = -Z_{102} - Z_{120} - Z_{121}, \\
&Z_{012} = Z_{102} + Z_{120} + Z_{121} - Z_{210} - Z_{211}\}. \\
= \{&\{0,0,0,0,0,0,0,0,0\}, \{0,0,0,0,0,0,0,1,1\}, \{0,0,0,0,0,1,1,0,1\}, \\
&\{0,0,0,0,0,1,1,1,0\}, \{0,0,0,0,1,0,1,0,1\}, \{0,0,0,0,1,0,1,1,0\}, \\
&\{0,0,0,0,1,1,0,0,0\}, \{0,0,0,0,1,1,0,1,1\}, \{0,0,1,1,0,0,0,0,1\}, \\
&\{0,0,1,1,0,0,0,1,0\}, \{0,0,1,1,0,1,1,0,0\}, \{0,0,1,1,0,1,1,1,1\}, \\
&\{0,0,1,1,1,0,1,0,0\}, \{0,0,1,1,1,0,1,1,1\}, \{0,0,1,1,1,1,0,0,1\}, \\
&\{0,0,1,1,1,1,0,1,0\}, \{0,1,0,1,0,0,1,0,1\}, \{0,1,0,1,0,0,1,1,0\}, \\
&\{0,1,0,1,0,1,0,0,0\}, \{0,1,0,1,0,1,0,1,1\}, \{0,1,0,1,1,0,0,0,0\}, \\
&\{0,1,0,1,1,0,0,1,1\}, \{0,1,0,1,1,1,1,0,1\}, \{0,1,0,1,1,1,1,1,0\}, \\
&\{0,1,1,0,0,0,1,0,0\}, \{0,1,1,0,0,0,1,1,1\}, \{0,1,1,0,0,1,0,0,1\}, \\
&\{0,1,1,0,0,1,0,1,0\}, \{0,1,1,0,1,0,0,0,1\}, \{0,1,1,0,1,0,0,1,0\}, \\
&\{0,1,1,0,1,1,1,0,0\}, \{0,1,1,0,1,1,1,1,1\}, \{1,0,0,1,0,0,0,0,0\}, \\
&\{1,0,0,1,0,0,0,1,1\}, \{1,0,0,1,0,1,1,0,1\}, \{1,0,0,1,0,1,1,1,0\}, \\
&\{1,0,0,1,1,0,1,0,1\}, \{1,0,0,1,1,0,1,1,0\}, \{1,0,0,1,1,1,0,0,0\},
\end{aligned}
$$

$$\{1,0,0,1,1,1,0,1,1\},\{1,0,1,0,0,0,0,0,1\},\{1,0,1,0,0,0,0,1,0\},$$
$$\{1,0,1,0,0,1,1,0,0\},\{1,0,1,0,0,1,1,1,1\},\{1,0,1,0,1,0,1,0,0\},$$
$$\{1,0,1,0,1,0,1,1,1\},\{1,0,1,0,1,1,0,0,1\},\{1,0,1,0,1,1,0,1,0\},$$
$$\{1,1,0,0,0,0,1,0,1\},\{1,1,0,0,0,0,1,1,0\},\{1,1,0,0,0,1,0,0,0\},$$
$$\{1,1,0,0,0,1,0,1,1\},\{1,1,0,0,1,0,0,0,0\},\{1,1,0,0,1,0,0,1,1\},$$
$$\{1,1,0,0,1,1,1,0,1\},\{1,1,0,0,1,1,1,1,0\},\{1,1,1,1,0,0,1,0,0\},$$
$$\{1,1,1,1,0,0,1,1,1\},\{1,1,1,1,0,1,0,0,1\},\{1,1,1,1,0,1,0,1,0\},$$
$$\{1,1,1,1,1,0,0,0,1\},\{1,1,1,1,1,0,0,1,0\},\{1,1,1,1,1,1,1,0,0\},$$
$$\{1,1,1,1,1,1,1,1,1\}\}$$

Finally we get $32 \times 64 = 4096$ possible strings with the identity

$$\check{H}^0(\mathcal{U},\mathcal{E})|_X \times \check{H}^0(\mathcal{U},\mathcal{E})|_Z = \check{H}^0(\mathcal{U},\mathcal{E}).$$

5.3.3 On a cube

The toric code on an open cube is made up of toric code on a plane, on disks and on edges. We want to calculate how many possible outcome strings exist for the hole cube and begin with the X-measurements. Therefore we have to multiply the numbers of possible outcome strings for each star, but consider how many possibilities disappeared because of the already determined outcomes.

We start with an upper edge star. The outcomes $\{000, 011, 101, 110\}$ are possible. The to the edge connected stars have eight possible outcome strings

$$\{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\},$$

but half of them disappear because the connecting point is determined by the first considered operator. If a star is overlapping an other four-outcome-star at outcome, one outcome is determined, for example $o_1 = 1$, and only four strings are left, here $\{1001, 1010, 1100, 1111\}$. This is the way we go through the whole cube multiplying the numbers of possible sections.

Four sections belong to a red edge star, see Figure 5.9. Further eight outcome sections are possible for each of the three the connecting stars. The connecting point excludes one half of the stars. Finally the first four products are $4 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2}$. Continuing this process we get

$$\begin{aligned}\#_X =& 4 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{2} \times \frac{4}{4} \times \frac{4}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{4}{4} \times \frac{8}{8} \times \frac{4}{2} \\ & \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{8} \\ =& 524288\end{aligned}$$

global sections over the X-measurements. The factor $\frac{4}{8}$ comes from the last edge star with three outcomes which are overall determined by other stars. This gives a constrain to the stars set before.

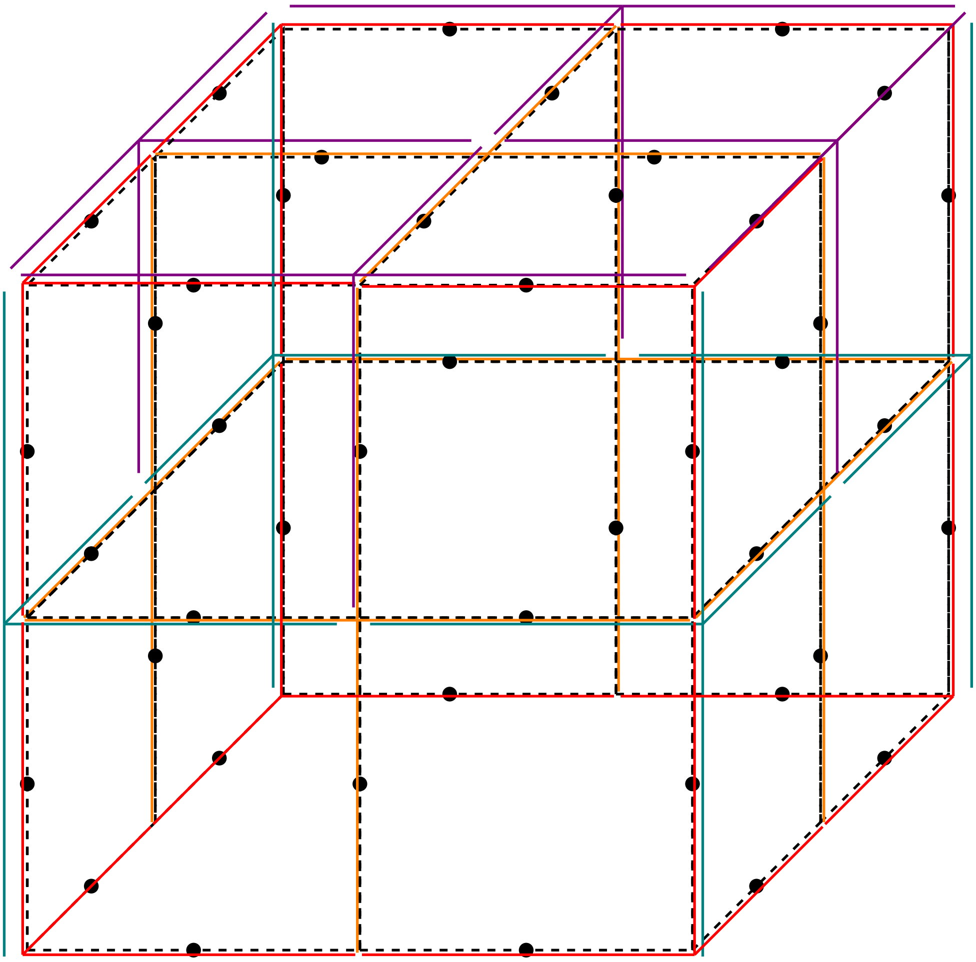


Figure 5.9: Toric code on a cube open on the bottom

We have to do this also for the number of possible strings for Z-outcomes.

$$\begin{aligned}\#_Z =& 4 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \\ & \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \\ =& 16777216\end{aligned}$$

Of course we have to multiply both quantities and end up with a number of

$$\# = \#_X \times \#_Z = 8.796093 \times 10^{12}$$

possible strings, which describe the number of different global sections on the whole cube. In the appendix lists are situated of measurement context, simplices, nerves, the sheaf and cochains on the whole cube.

5.3.4 On a cylinder

It is interesting how the shape of the cube has influence on the number of global sections. The aim is to compute the number for a toric code on a cylinder instead of a cube. We assume the same number of 20 plaquette operators as before and determine the length of the cylinder as four.

$$\begin{aligned}\#_X =& 8 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{4}{2} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \\ & \times \frac{8}{8} \times \frac{4}{2} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{4} \times \frac{4}{8} \\ =& 2097152 \\ \#_Z =& 8 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \\ & \times \frac{8}{4} \times \frac{8}{8} \\ =& 33554432 \\ \# =& \#_X \times \#_Z = 7.0368744 \times 10^{13}\end{aligned}$$

It is not surprising that the numbers of global section increases. The cylinder shape constrains less outcomes to be equal.

5.3.5 On a torus

To extend the constrains we consider a torus out of 20 plaquettes.

$$\begin{aligned}\#_X =& 8 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{8} \\ & \times \frac{8}{8} \times \frac{8}{16} \\ =& 1048576 \\ \#_Z =& 8 \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{2} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{4} \times \frac{8}{8} \times \frac{8}{8} \\ & \times \frac{8}{8} \times \frac{8}{16} \\ =& 1048576\end{aligned}$$

It is interesting, that for the torus case the number of sections over the X-measurements $\#_X$ equals the number of sections belonging to the Z-measurements $\#_Z$. Finally the number of global sections is

$$\# = \#_X \times \#_Z = 1.0995116 \times 10^{12}.$$

The number of global sections in this case is approximately a ninth of the cube case.

6 Conclusion

In this chapter we summarize our results and collect open questions. An aim of this work was to give examples for describing empirical models using their sheaf-theoretic structure. Therefore we explained the definitions following a concrete example in details. We also introduced bundle diagrams to depict the existence or not-existence of global sections and make statements about the contextuality of the model. We considered bipartite qubit and qutrit models as trivial examples. Generalizing bundle diagrams from bipartite models to multipartite models we reconstruct the contextuality of an MBQC model using the Greenberger-Horne-Zeilinger-state. Building empirical models and using the contexts defined in [10] we showed that all states produce models without global sections, i.e. contextual models. Finally we studied cluster ring models for small numbers of qubits and proved the non-contextuality of these models. We proved that the empirical model describing cluster rings and cluster states on a plane is independent of the chosen stabilizer state. At the end of the chapter we worked out the empirical model of the toric code on a plane.

We studied the power of cohomology for understanding the contextuality of empirical models in this work. After introducing the definitions and meanings of Čech cohomology groups we studied the difference between cluster state rings composed of an even or odd numbers of qubits. The result is that the visual difference in the diagrams is not reflected in the cohomology groups. The number of global sections is always 2^n with the number of qubits n. It turned out that computing the cohomology group is more useful for finding the set of global sections than using bundle diagrams.

We also built the cohomology groups for cluster states in two dimensions. Considering the toric code we studied how the number of global sections changes when the code lies at an open cube, a cylinder or a torus. The latter gives the most constraints to a potential global section and hence the number of global sections is the lowest.

An open question remaining of this work is in which models the higher cohomology groups matter, because here we only studied models where just the zeroth cohomology group is not trivial. It would be interesting to find models where this is not the case. Furthermore we only considered models using contexts out of Pauli observables. It could be worthwhile to look at models with measurements in other bases. Moreover in this work we not discussed graph-theoretic approach to contextuality [12]. It is an outstanding issue to discuss this for out chosen examples.

Overall we gave an introduction into contextuality, also in the setting of measurement based quantum computing, discussed the sheaf structure in detail and studied the cohomology of contextuality for many examples.

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Appendix

Cluster state ring with four and five qubits

Four qubits

In this paragraph we consider a cluster state ring model with $n = 4$. The state

$$\begin{aligned}|\Psi\rangle = \frac{1}{\sqrt{4}}(& -|0000\rangle - |0001\rangle - |0010\rangle + |0011\rangle - |0100\rangle - |0101\rangle + |0110\rangle - |0111\rangle \\ & - |0111\rangle + |1001\rangle - |1010\rangle - |1011\rangle + |1100\rangle - |1101\rangle - |1110\rangle - |1111\rangle)\end{aligned}$$

is a stabilizer.

$$\begin{aligned} XZ\mathbb{1}Z\,|\Psi\rangle &= |\Psi\rangle \\ ZXZ\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\ \mathbb{1}ZXZ\,|\Psi\rangle &= |\Psi\rangle \\ Z\mathbb{1}ZX\,|\Psi\rangle &= |\Psi\rangle \end{aligned}$$

Calculating the probabilities for all different measurement outcomes we build the empirical model and the bundle diagram. The sections belonging to the forth context are depicted in blue.

	1	2	3	4	000	001	010	011	100	101	110	111
C_1	X_1	Z_2	$\mathbb{1}$	Z_4	1/4	0	0	1/4	0	1/4	1/4	0
C_2	Z_1	X_2	Z_3	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_3	$\mathbb{1}$	Z_2	X_3	Z_4	1/4	0	0	1/4	0	1/4	1/4	0
C_4	Z_1	$\mathbb{1}$	Z_3	X_4	1/4	0	0	1/4	0	1/4	1/4	0

Table A.1: Empirical model for cluster state $n = 4$

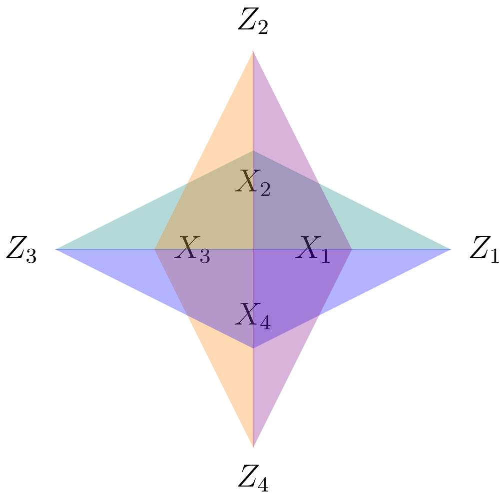


Figure A.1: Base of bundle diagram for cluster state n=4

It becomes clear already in the base of the bundle diagram that the global sections are divided in two parts, violet-orange and teal-blue, not influencing each other.

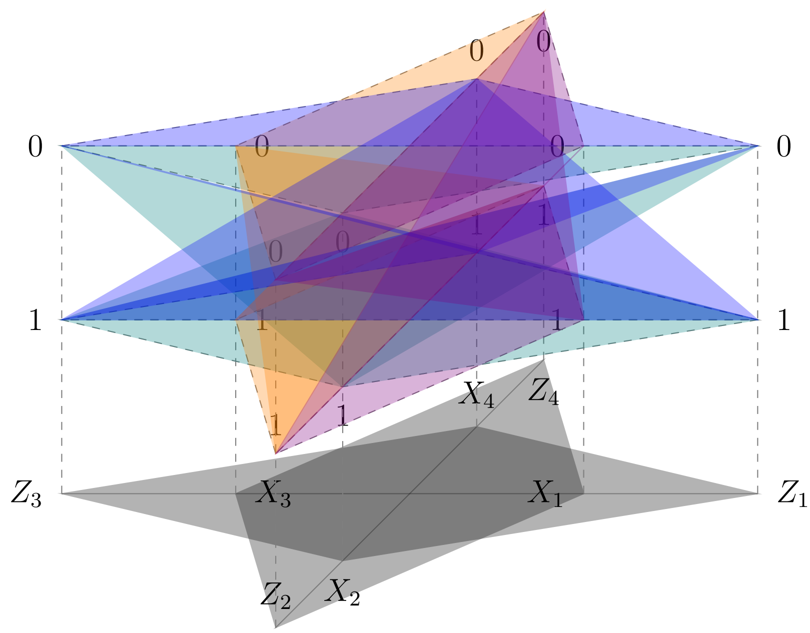


Figure A.2: Bundle diagram for cluster state n=4

Analogous to the case $n = 3$, it is possible to build global sections for each triangle over a chosen context and see with the same argument as before that the model is non-contextual.

Five qubits

Continuing the last subsection we consider a cluster state model with five contexts now. The state

$$\begin{aligned}|\Psi\rangle = \frac{1}{4\sqrt{2}}(&-|00000\rangle - |00001\rangle - |00010\rangle + |00011\rangle - |00100\rangle - |00101\rangle + |00110\rangle \\ &- |00111\rangle - |00111\rangle - |01001\rangle - |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle \\ &- |01110\rangle + |01111\rangle - |10000\rangle + |10001\rangle - |10010\rangle - |10011\rangle - |10100\rangle \\ &+ |10101\rangle + |10110\rangle + |10111\rangle + |10111\rangle - |11001\rangle + |11010\rangle + |11011\rangle \\ &- |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle)\end{aligned}$$

is a stabilizer for these contexts.

$$\begin{aligned}XZ\mathbb{1}\mathbb{1}Z\,|\Psi\rangle &= |\Psi\rangle \\ ZXZ\mathbb{1}\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\ \mathbb{1}ZXZ\mathbb{1}\,|\Psi\rangle &= |\Psi\rangle \\ \mathbb{1}\mathbb{1}ZXZ\,|\Psi\rangle &= |\Psi\rangle \\ Z\mathbb{1}\mathbb{1}ZX\,|\Psi\rangle &= |\Psi\rangle\end{aligned}$$

The probabilities are used for building the bundle diagram. The sections to the new added context C_5 are depicted in pink.

	1	2	3	4	5	000	001	010	011	100	101	110	111
C_1	X_1	Z_2	$\mathbb{1}$	$\mathbb{1}$	Z_5	1/4	0	0	1/4	0	1/4	1/4	0
C_2	Z_1	X_2	Z_3	$\mathbb{1}$	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_3	$\mathbb{1}$	Z_2	X_3	Z_4	$\mathbb{1}$	1/4	0	0	1/4	0	1/4	1/4	0
C_4	$\mathbb{1}$	$\mathbb{1}$	Z_3	X_4	Z_5	1/4	0	0	1/4	0	1/4	1/4	0
C_5	Z_1	$\mathbb{1}$	$\mathbb{1}$	Z_4	X_5	1/4	0	0	1/4	0	1/4	1/4	0

Table A.2: Empirical model for cluster state $n = 5$

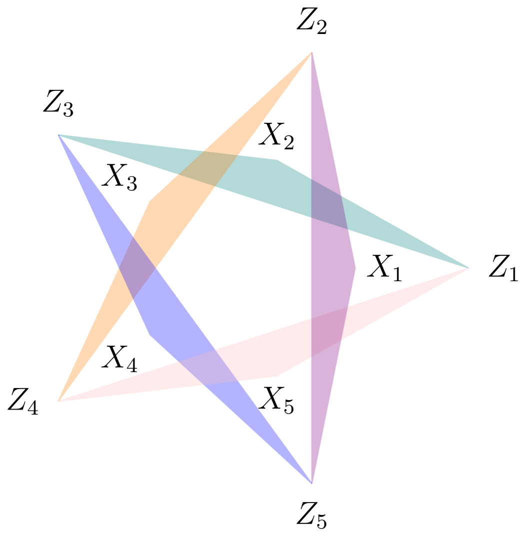


Figure A.3: Base of bundle diagram for cluster state n=5

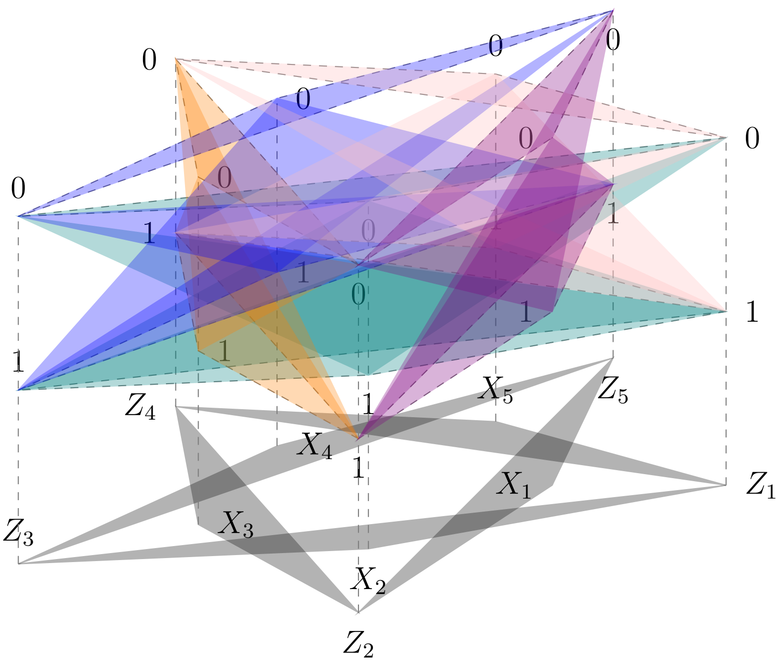


Figure A.4: Bundle diagram for cluster state n=5

This cluster ring model is also non-contextual. This can be shown analogously to the last example.

Proof of Lemma 3.1

Lemma. *The states with an even number of minus signs in the projectors vanish using the contexts*

$$C_1 = \{X_A, X_B, X_C\}$$
$$C_2 = \{X_A, Y_B, Y_C\}$$
$$C_3 = \{Y_A, X_B, Y_C\}$$
$$C_4 = \{Y_A, Y_B, X_C\}.$$

It is only possible to find a state $|\Psi\rangle \neq 0$ *for an odd number of minus signs in the projectors.*

Proof. We can prove this by calculating $P_1 P_2 P_3 P_4 = |\Psi\rangle \langle\Psi|$ for all cases.

```
id := IdentityMatrix[8];
X := {{0, 1}, {1, 0}};
Y := {{0, -I}, {I, 0}};
a := {a0, a1};
b := {b0, b1};
c := {c0, c1};

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{1/2, 0, 0, 0, 0, 0, 0, -(1/2)}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0,
0, 0, 0,0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {-(1/2), 0, 0, 0,
0, 0, 0, 1/2}}

P1 = (id + KroneckerProduct[X, X, X])/2;
```

```
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 1/2, 1/2, 0, 0, 0}, {0, 0, 0, 1/2, 1/2, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1/2, 0,
0, 1/2, 0,0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 1/2, 0, 0, 1/2, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0,
0, 0, 0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
```

```
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id + KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 1/2, 0, 0, 0, 0, -(1/2), 0}, {0, 0, 0,
0, 0, 0,0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, -(1/2), 0, 0, 0, 0, 1/2, 0}, {0, 0, 0,
0, 0, 0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 1/2, 0, 0, 0, 0, 1/2, 0}, {0, 0, 0, 0,
0, 0, 0,0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 1/2, 0, 0, 0, 0, 1/2, 0}, {0, 0, 0, 0,
0, 0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;
```

```
P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id + KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1/2, 0, 0,
-(1/2), 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, -(1/2), 0, 0, 1/2, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0,
0, 0, 0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id + KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 1/2, -(1/2), 0, 0, 0}, {0, 0, 0, -(1/2), 1/2, 0,
0, 0}, {0, 0,0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0,
0, 0, 0, 0, 0}}

P1 = (id + KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
```

```
{{1/2, 0, 0, 0, 0, 0, 0, 1/2}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0,
0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {1/2, 0, 0, 0, 0,
0, 0, 1/2}}

P1 = (id - KroneckerProduct[X, X, X])/2;
P2 = (id - KroneckerProduct[X, Y, Y])/2;
P3 = (id - KroneckerProduct[Y, X, Y])/2;
P4 = (id - KroneckerProduct[Y, Y, X])/2;

P1.P2.P3.P4
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}
```

□

Code for state-independent cluster

In one dimension

In Section 3.4.1 we calculated state-independently the probabilities for measurement outcomes in the cluster setting. The Mathematica code for the case $n = 4$ is written down exemplarily here.

```
k[a_, b_] := a.b + b.a

t[m1_, m2_, m3_, m4_] :=
ArrayFlatten[ArrayFlatten[TensorProduct[m1, m2, m3, m4], 4]];
(*tensor product*)

S1 := t[Z, X, Z, Id];
S2 := t[Id, Z, X, Z];
S3 := t[Z, Id, Z, X];
S4 := t[X, Z, Id, Z];
(*stabilizers*)

Com[a_, b_, c_, d_] :=
k[t[a, b, c, d], S1] != ConstantArray[0, {16, 16}] &&
k[t[a, b, c, d], S2] != ConstantArray[0, {16, 16}] &&
k[t[a, b, c, d], S3] != ConstantArray[0, {16, 16}] &&
k[t[a, b, c, d], S4] != ConstantArray[0, {16, 16}];
(* anticommutator of t[a,b,c,d] with all stabilizers is not 0*)
```

```
T[a_, b_, c_, d_] := If[Com[a, b, c, d] == True, 1, 0]
(* 1 if anticommutator of t[a,b,c,d] with all stabilizers is not 0*)

Prob[a1_, a2_, a3_, a4_] :=
Module[{o1 = o1, o2 = o2, o3 = o3, o4 = o4, a = a1, b = a2, c = a3 ,
d = a4},
If[a == Id, o1 = 0];
If[b == Id, o2 = 0];
If[c == Id, o3 = 0];
If[d == Id, o4 = 0];
1/32 (1 + o1 T[a, Id, Id, Id] + o2 T[Id, b, Id, Id]
+ o3 T[Id, Id, c, Id] + o4 T[Id, Id, Id, d]
+ o1 o2 T[a, b, Id, Id] + o1 o3 T[a, Id, c, Id]
+ o1 o4 T[a, Id, Id, d] + o2 o3 T[Id, b, c, Id]
+ o2 o4 T[Id, b, Id, d] + o3 o4 T[Id, Id, c, d]
+ o1 o2 o3 T[a, b, c, Id] + o2 o3 o4 T[Id, b, c, d]
+ o3 o4 o1 T[a, Id, c, d] + o4 o1 o2 T[a, b, Id, d])]
(*probability of measuring ai on qubit i with outcome oi*)
```

In two dimensions

We also considered a state-independent cluster model in Section 3.4.2. In the following the code for calculating the outcome probabilities is notated regarding a star in the middle of a cluster lattice as depicted in Figure 3.24. Stars on other positions can be treated analogously.

```
k[a_, b_] := a.b - b.a

s[q01_, q10_, q11_, q12_, q21_] := Module[{
q}, L = Table[Id, {j, 9}]; q = L[[1]];
L[[2]] = q01;
L[[4]] = q10;
L[[5]] = q11;
L[[6]] = q12;
L[[8]] = q21;
For[i = 2, i <= 9, i++, q = ArrayFlatten[TensorProduct[q, L[[i]]]]]; q]
(*make states in 2D*)


S1 := s[X, Id, Z, Id, Id]
S2 := s[Id, X, Z, Id, Id]
S3 := s[Z, Z, X, Z, Z]
S4 := s[Id, Id, Z, X, Id]
S5 := s[Id, Id, Z, Id, X]
```

```
(*stabilizers*)
d = Dimensions[S1];

Com[s_] :=
k[s, S1] == ConstantArray[0, d] && k[s, S2] == ConstantArray[0, d] &&
k[s, S3] == ConstantArray[0, d] && k[s, S4] == ConstantArray[0, d] &&
k[s, S5] == ConstantArray[0, d];
(*anticommutator of s with all stabilizers is not 0*)
T[s_] := If[Com[s] == True, 1, 0]
(*1 if anticommutator of s with all stabilizers is not 0*)
G[a1_, a2_, a3_, a4_, a5_] := T[s[a1, a2, a3, a4, a5]]

Prob[a1_, a2_, a3_, a4_, a5_] :=
Module[{o1 = o1, o2 = o2, o3 = o3, o4 = o4, o5 = o5, a = a1, b = a2,
c = a3 , d = a4, e = a5},
If[a == Id, o1 = 0];
If[b == Id, o2 = 0];
If[c == Id, o3 = 0];
If[d == Id, o4 = 0];
If[e == Id, o5 = 0];
1/32 (1 + o5 G[Id, Id, Id, Id, e] + o4 G[Id, Id, Id, d, Id]
+ o4 o5 G[Id, Id, Id, d, e] + o3 G[Id, Id, c, Id, Id]
+ o3 o5 G[Id, Id, c, Id, e] + o3 o4 G[Id, Id, c, d, Id]
+ o3 o4 o5 G[Id, Id, c, d, e] + o2 G[Id, b, Id, Id, Id]
+ o2 o5 G[Id, b, Id, Id, e] + o2 o4 G[Id, b, Id, d, Id]
+ o4 o5  G[Id, b, Id, d, e] + o2 o3 G[Id, b, c, Id, Id]
+ o2 o3 o5 G[Id, b, c, Id, e] + o2 o3 o4 G[Id, b, c, d, Id]
+ o2 o3 o4 o5 G[Id, b, c, d, e] + o1 G[a, Id, Id, Id, Id]
+ o1 o5 G[a, Id, Id, Id, e] + o1 o4 G[a, Id, Id, d, Id]
+ o4 o5 G[a, Id, Id, d, e] + o1 o3 G[a, Id, c, Id, Id]
+ o1 o3 o5 G[a, Id, c, Id, e] + o1 o3 o4 G[a, Id, c, d, Id]
+ o1 o3 o4 o5  G[a, Id, c, d, e] + o1 o2 G[a, b, Id, Id, Id]
+ o1 o2 o5 G[a, b, Id, Id, e] + o1 o2 o4 G[a, b, Id, d, Id]
+ o1 o2 o4 o5 G[a, b, Id, d, e] + o1 o2 o3 G[a, b, c, Id, Id]
+ o1 o2 o3 o5 G[a, b, c, Id, e] + o1 o2 o3 o4 G[a, b, c, d, Id]
+ o1 o2 o3 o4 o5 G[a, b, c, d, e]
)]
```

Examples for coboundary maps for toric code

To calculate the coboundary maps we choose the example

$$\omega = \left(\begin{pmatrix} \underbrace{0101}_{A_{k,l+1}} & 1111 & 0001 \\ \underbrace{0000}_{A_{k,l}} & \underbrace{1000}_{A_{k+1,l}} & 1111 \end{pmatrix}, \begin{pmatrix} \underbrace{1001}_{P_{k+1,l}} & 0000 & 1010 \\ \underbrace{1100}_{P_{k,l}} & \underbrace{1111}_{P_{k+1,l}} & 0101 \end{pmatrix} \right)$$

and compute $\delta^0(\omega)(\sigma^{(1)}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)\prime}_{k,l})$, $\delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l})$ and $\delta^0(\omega)(\sigma^{(1)\prime\prime\prime}_{k,l})$.

$$\begin{aligned} j = 0\colon \ & \mathrm{res}^{|\partial_0 \sigma^{(1)}_{k,l}|}_{|\sigma^{(1)}_{k,l}|} \omega(\partial_0 \sigma^{(1)}_{k,l}) = \mathrm{res}^{A_{k+1,l}}_{\{X_{k+\frac{1}{2},l}\}} \omega(A_{k+1,l}) = 0 \\ j = 1\colon \ & \mathrm{res}^{|\partial_1 \sigma^{(1)}_{k,l}|}_{|\sigma^{(1)}_{k,l}|} \omega(\partial_1 \sigma^{(1)}_{k,l}) = \mathrm{res}^{A_{k,l}}_{\{X_{k+\frac{1}{2},l}\}} \omega(A_{k,l}) = 0 \\ \delta^0(\omega)(\sigma^{(1)}_{k,l}) =& 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} j = 0\colon \ & \mathrm{res}^{|\partial_0 \sigma^{(1)\prime}_{k,l}|}_{|\sigma^{(1)\prime}_{k,l}|} \omega(\partial_0 \sigma^{(1)\prime}_{k,l}) = \mathrm{res}^{A_{k,l+1}}_{\{X_{k,l+\frac{1}{2}}\}} \omega(A_{k,l+1}) = 1 \\ j = 1\colon \ & \mathrm{res}^{|\partial_1 \sigma^{(1)\prime}_{k,l}|}_{|\sigma^{(1)\prime}_{k,l}|} \omega(\partial_1 \sigma^{(1)\prime}_{k,l}) = \mathrm{res}^{A_{k,l}}_{\{X_{k,l+\frac{1}{2}}\}} \omega(A_{k,l}) = 0 \\ \delta^0(\omega)(\sigma^{(1)\prime}_{k,l}) =& 1 - 0 = 1 \end{aligned}$$

$$\begin{aligned} j = 0\colon \ & \mathrm{res}^{|\partial_0 \sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|} \omega(\partial_0 \sigma^{(1)\prime\prime}_{k,l}) = \mathrm{res}^{P_{k+1,l}}_{\{Z_{k+1,l-\frac{1}{2}}\}} \omega(P_{k+1,l}) = 1 \\ j = 1\colon \ & \mathrm{res}^{|\partial_1 \sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|} \omega(\partial_1 \sigma^{(1)\prime\prime}_{k,l}) = \mathrm{res}^{P_{k,l}}_{\{Z_{k+1,l-\frac{1}{2}}\}} \omega(P_{k,l}) = 0 \\ \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& 1 - 0 = 1 \end{aligned}$$

$$\begin{aligned} j = 0\colon \ & \mathrm{res}^{|\partial_0 \sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|} \omega(\partial_0 \sigma^{(1)\prime\prime}_{k,l}) = \mathrm{res}^{P_{k,l+1}}_{\{Z_{k+\frac{1}{2},l}\}} \omega(P_{k,l+1}) = 1 \\ j = 1\colon \ & \mathrm{res}^{|\partial_1 \sigma^{(1)\prime\prime}_{k,l}|}_{|\sigma^{(1)\prime\prime}_{k,l}|} \omega(\partial_1 \sigma^{(1)\prime\prime}_{k,l}) = \mathrm{res}^{P_{k,l}}_{\{Z_{k+\frac{1}{2},l}\}} \omega(P_{k,l}) = 1 \\ \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& 1 - 1 = 0 \end{aligned}$$

Generalizing these results for $\omega \in C^0(\mathcal{U}, \mathcal{E})$ we get

$$\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{k,l}) = \{0,1\}^{\times n \times n}$$

$$
\begin{aligned}
\ker \delta^0(\omega)(\sigma^{(1)}_{k,l}) =& \{M \in C^0(\mathcal{U},\mathcal{E}) \mid M_{1,k,l}[[3]] \oplus M_{1,k+1,l}[[2]] = 0\} \\
\text{Im } \delta^0(\omega)(\sigma^{(1)}_{k,l}|) =& \{0,1\}^{\times n \times n} \\
\ker \delta^0(\omega)(\sigma^{(1)\prime}_{k,l}) =& \{M \in C^0(\mathcal{U},\mathcal{E}) \mid M_{1,k,l}[[1]] \oplus M_{1,k,l+1}[[4]] = 0\} \\
\text{Im } \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& \{0,1\}^{\times n \times n} \\
\ker \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& \{M \in C^0(\mathcal{U},\mathcal{E}) \mid M_{2,k,l}[[4]] \oplus M_{2,k+1,l}[[2]] = 0\} \\
\text{Im } \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& \{0,1\}^{\times n \times n} \\
\ker \delta^0(\omega)(\sigma^{(1)\prime\prime}_{k,l}) =& \{M \in C^0(\mathcal{U},\mathcal{E}) \mid M_{2,k,l}[[1]] \oplus M_{2,k+1,l}[[3]] = 0\}
\end{aligned}
$$

On a disk

To understand the behavior of $\delta^0(\omega)(\sigma^{(1)}_{1,0,0})$, $\delta^0(\omega)(\sigma^{(1)}_{1,1,1})$, $\delta^0(\omega)(\sigma^{(1)}_{0,1,0})$ and $\delta^0(\omega)(\sigma^{(1)\prime}_{0,0,0})$ we choose

$$
\omega = (\{\underbrace{011}_{A_{110}}, \underbrace{110}_{A_{100}}, \underbrace{11}_{A_{111}}, \underbrace{00}_{A_{010}}\}, \{\underbrace{0101}_{P_{000}}, \underbrace{0011}_{P_{100}}, \underbrace{1111}_{P_{010}}\}) \in C^0(\mathcal{U},\mathcal{E})
$$

and compute examples.

$$
\begin{aligned}
j = 0:\ & \text{res}^{|\partial_0\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_0\sigma^{(1)}_{1,0,0}) = \text{res}^{A_{100}}_{A_{100}\cap A_{110}}\omega(A_{100}) = \text{res}^{A_{100}}_{\{X_{1,\frac{1}{2},0}\}}110 = 0 \\
j = 1:\ & \text{res}^{|\partial_1\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_1\sigma^{(1)}_{1,0,0}) = \text{res}^{A_{110}}_{A_{100}\cap A_{011}}\omega(A_{110}) = \text{res}^{A_{110}}_{\{X_{1,\frac{1}{2},0}\}}011 = 0
\end{aligned}
$$

$$
\delta^0(\omega)(\sigma^{(1)}_{1,0,0}) = \sum_{j=0}^{1}(-1)^j \text{res}^{|\partial_j\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_j\sigma^{(1)}_{1,0,0}) = 0 - 0 = 0
$$

$$
\begin{aligned}
j = 0:\ & \text{res}^{|\partial_0\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_0\sigma^{(1)}_{1,1,1}) = \text{res}^{A_{111}}_{A_{111}\cap A_{110}}\omega(A_{111}) = \text{res}^{A_{111}}_{\{X_{1,1,\frac{1}{2}}\}}11 = 1 \\
j = 1:\ & \text{res}^{|\partial_1\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_1\sigma^{(1)}_{1,1,1}) = \text{res}^{A_{110}}_{A_{111}\cap A_{110}}\omega(A_{110}) = \text{res}^{A_{110}}_{\{X_{1,1,\frac{1}{2}}\}}011 = 1
\end{aligned}
$$

$$
\delta^0(\omega)(\sigma^{(1)}_{1,1,1}) = \sum_{j=0}^{1}(-1)^j \text{res}^{|\partial_j\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_j\sigma^{(1)}_{1,1,1}) = 1 - 1 = 0
$$

$$
\begin{aligned}
j = 0:\ & \text{res}^{|\partial_0\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_0\sigma^{(1)}_{0,1,0}) = \text{res}^{A_{010}}_{A_{010}\cap A_{110}}\omega(A_{010}) = \text{res}^{A_{010}}_{\{X_{\frac{1}{2},1,0}\}}0 = 0 \\
j = 1:\ & \text{res}^{|\partial_1\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_1\sigma^{(1)}_{0,1,0}) = \text{res}^{A_{110}}_{A_{010}\cap A_{110}}\omega(A_{110}) = \text{res}^{A_{110}}_{\{X_{\frac{1}{2},1,0}\}}011 = 1
\end{aligned}
$$

$$
\delta^0(\omega)(\sigma^{(1)}_{0,1,0}) = \sum_{j=0}^{1}(-1)^j \text{res}^{|\partial_j\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_j\sigma^{(1)}_{0,1,0}) = 0 - 1 = 1
$$

$$j=0:\ \mathrm{res}^{|\partial_0\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_0\sigma^{(1)\prime}_{0,0,0})=\mathrm{res}^{P_{000}}_{P_{000}\cap P_{100}}\omega(P_{000})=\mathrm{res}^{P_{000}}_{\{Z_{1,\frac{1}{2},0}\}}0101=1$$

$$j=1:\ \mathrm{res}^{|\partial_1\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_1\sigma^{(1)\prime}_{0,0,0})=\mathrm{res}^{P_{100}}_{P_{000}\cap P_{100}}\omega(P_{100})=\mathrm{res}^{P_{100}}_{\{Z_{1,\frac{1}{2},0}\}}0011=1$$

$$\delta^0(\omega)(\sigma^{(1)\prime}_{0,0,0})=\sum_{j=0}^{1}(-1)^j\mathrm{res}^{|\partial_j\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_j\sigma^{(1)\prime}_{0,0,0})=1-1=0$$

Finally we notice the general structure of the images and kernels of these maps.

$$\begin{aligned}
\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{1,0,0}) &=\{0,1\}\\
\ker\delta^0(\omega)(\sigma^{(1)}_{k}) &=\{S\in C^0(\mathcal{U},\mathcal{E})\mid A_{100}[[3]]\oplus A_{110}[[1]]=0\}\\
\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{1,1,1}) &=\{0,1\}\\
\ker\delta^0(\omega)(\sigma^{(1)}_{k}) &=\{S\in C^0(\mathcal{U},\mathcal{E})\mid A_{111}[[1]]\oplus A_{110}[[2]]=0\}\\
\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{0,1,0}) &=\{0,1\}\\
\ker\delta^0(\omega)(\sigma^{(1)}_{k}) &=\{S\in C^0(\mathcal{U},\mathcal{E})\mid A_{010}[[2]]\oplus A_{110}[[3]]=0\}\\
\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)\prime}_{0,0,0}) &=\{0,1\}\\
\ker\delta^0(\omega)(\sigma^{(1)\prime}_{k}) &=\{S\in C^0(\mathcal{U},\mathcal{E})\mid P_{000}[[2]]\oplus P_{100}[[4]]=0\}
\end{aligned}$$

On an edge

As before we select one example out of $C^0(\mathcal{U},\mathcal{E})$ to compute $\delta^0(\omega)(\sigma^{(1)}_{1,0,0})$, $\delta^0(\omega)(\sigma^{(1)}_{1,1,1})$, $\delta^0(\omega)(\sigma^{(1)}_{0,1,0})$, $\delta^0(\omega)(\sigma^{(1)\prime}_{0,0,0})$, $\delta^0(\omega)(\sigma^{(1)\prime}_{1,0,0})$ and $\delta^0(\omega)(\sigma^{(1)\prime}_{0,1,0})$.

$$\omega=(\{\underbrace{011}_{A_{110}},\underbrace{110}_{A_{100}},\underbrace{000}_{A_{111}},\underbrace{101}_{A_{010}}\},\{\underbrace{0101}_{P_{000}},\underbrace{0011}_{P_{100}},\underbrace{1111}_{P_{010}}\})$$

$$j=0:\ \mathrm{res}^{|\partial_0\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_0\sigma^{(1)}_{1,0,0})=\mathrm{res}^{A_{100}}_{A_{100}\cap A_{110}}\omega(A_{100})=\mathrm{res}^{A_{100}}_{\{X_{1,\frac{1}{2},0}\}}110=0$$

$$j=1:\ \mathrm{res}^{|\partial_1\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_1\sigma^{(1)}_{1,0,0})=\mathrm{res}^{A_{110}}_{A_{100}\cap A_{011}}\omega(A_{110})=\mathrm{res}^{A_{110}}_{\{X_{1,\frac{1}{2},0}\}}011=0$$

$$\delta^0(\omega)(\sigma^{(1)}_{1,0,0})=\sum_{j=0}^{1}(-1)^j\mathrm{res}^{|\partial_j\sigma^{(1)}_{1,0,0}|}_{|\sigma^{(1)}_{1,0,0}|}\omega(\partial_j\sigma^{(1)}_{1,0,0})=0-0=0$$

$$j=0:\ \mathrm{res}^{|\partial_0\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_0\sigma^{(1)}_{1,1,1})=\mathrm{res}^{A_{111}}_{A_{111}\cap A_{110}}\omega(A_{111})=\mathrm{res}^{A_{111}}_{\{X_{1,1,\frac{1}{2}}\}}000=0$$

$$j=1:\ \mathrm{res}^{|\partial_1\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_1\sigma^{(1)}_{1,1,1})=\mathrm{res}^{A_{110}}_{A_{111}\cap A_{110}}\omega(A_{110})=\mathrm{res}^{A_{110}}_{\{X_{1,1,\frac{1}{2}}\}}011=1$$

$$\delta^0(\omega)(\sigma^{(1)}_{1,1,1}) = \sum_{j=0}^{1}(-1)^j \mathrm{res}^{|\partial_j\sigma^{(1)}_{1,1,1}|}_{|\sigma^{(1)}_{1,1,1}|}\omega(\partial_j\sigma^{(1)}_{1,1,1}) = 0 - 1 = 1$$

$$j=0\colon\ \mathrm{res}^{|\partial_0\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_0\sigma^{(1)}_{0,1,0}) = \mathrm{res}^{A_{010}}_{A_{010}\cap A_{110}}\omega(A_{010}) = \mathrm{res}^{A_{010}}_{\{X_{\frac{1}{2},1,0}\}}101 = 1$$

$$j=1\colon\ \mathrm{res}^{|\partial_1\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_1\sigma^{(1)}_{0,1,0}) = \mathrm{res}^{A_{110}}_{A_{010}\cap A_{110}}\omega(A_{110}) = \mathrm{res}^{A_{110}}_{\{X_{\frac{1}{2},1,0}\}}011 = 1$$

$$\delta^0(\omega)(\sigma^{(1)}_{0,1,0}) = \sum_{j=0}^{1}(-1)^j \mathrm{res}^{|\partial_j\sigma^{(1)}_{0,1,0}|}_{|\sigma^{(1)}_{0,1,0}|}\omega(\partial_j\sigma^{(1)}_{0,1,0}) = 1 - 1 = 0$$

$$j=0\colon\ \mathrm{res}^{|\partial_0\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_0\sigma^{(1)\prime}_{0,0,0}) = \mathrm{res}^{P_{000}}_{P_{000}\cap P_{100}}\omega(P_{000}) = \mathrm{res}^{P_{000}}_{\{Z_{1,\frac{1}{2},0}\}}0101 = 1$$

$$j=1\colon\ \mathrm{res}^{|\partial_1\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_1\sigma^{(1)\prime}_{0,0,0}) = \mathrm{res}^{P_{100}}_{P_{000}\cap P_{100}}\omega(P_{100}) = \mathrm{res}^{P_{100}}_{\{Z_{1,\frac{1}{2},0}\}}0011 = 1$$

$$\delta^0(\omega)(\sigma^{(1)\prime}_{0,0,0}) = \sum_{j=0}^{1}(-1)^j \mathrm{res}^{|\partial_j\sigma^{(1)\prime}_{0,0,0}|}_{|\sigma^{(1)\prime}_{0,0,0}|}\omega(\partial_j\sigma^{(1)\prime}_{0,0,0}) = 1 - 1 = 0$$

$$j=0\colon\ \mathrm{res}^{|\partial_0\sigma^{(1)\prime}_{1,0,0}|}_{|\sigma^{(1)\prime}_{1,0,0}|}\omega(\partial_0\sigma^{(1)\prime}_{1,0,0}) = \mathrm{res}^{P_{100}}_{P_{100}\cap P_{010}}\omega(P_{100}) = \mathrm{res}^{P_{100}}_{\{Z_{1,1,\frac{1}{2}}\}}0011 = 1$$

$$j=1\colon\ \mathrm{res}^{|\partial_1\sigma^{(1)\prime}_{1,0,0}|}_{|\sigma^{(1)\prime}_{1,0,0}|}\omega(\partial_1\sigma^{(1)\prime}_{1,0,0}) = \mathrm{res}^{P_{100}}_{P_{100}\cap P_{010}}\omega(P_{010}) = \mathrm{res}^{P_{100}}_{\{Z_{1,1,\frac{1}{2}}\}}1111 = 1$$

$$\delta^0(\omega)(\sigma^{(1)\prime}_{1,0,0}) = \sum_{j=0}^{1}(-1)^j \mathrm{res}^{|\partial_j\sigma^{(1)\prime}_{1,0,0}|}_{|\sigma^{(1)\prime}_{1,0,0}|}\omega(\partial_j\sigma^{(1)\prime}_{1,0,0}) = 1 - 1 = 0$$

$$j=0\colon\ \mathrm{res}^{|\partial_0\sigma^{(1)\prime}_{0,1,0}|}_{|\sigma^{(1)\prime}_{0,1,0}|}\omega(\partial_0\sigma^{(1)\prime}_{0,1,0}) = \mathrm{res}^{P_{010}}_{P_{010}\cap P_{000}}\omega(P_{010}) = \mathrm{res}^{P_{010}}_{\{Z_{\frac{1}{2},1,0}\}}1111 = 1$$

$$j=1\colon\ \mathrm{res}^{|\partial_1\sigma^{(1)\prime}_{0,1,0}|}_{|\sigma^{(1)\prime}_{0,1,0}|}\omega(\partial_1\sigma^{(1)\prime}_{0,1,0}) = \mathrm{res}^{P_{010}}_{P_{010}\cap P_{000}}\omega(P_{000}) = \mathrm{res}^{P_{010}}_{\{Z_{\frac{1}{2},1,0}\}}0101 = 0$$

$$\delta^0(\omega)(\sigma^{(1)\prime}_{0,1,0}) = \sum_{j=0}^{1}(-1)^j \mathrm{res}^{|\partial_j\sigma^{(1)\prime}_{0,1,0}|}_{|\sigma^{(1)\prime}_{0,1,0}|}\omega(\partial_j\sigma^{(1)\prime}_{0,1,0}) = 1 - 0 = 1$$

Calculating these examples we understood the behavior of the augmented complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0.$$

$$\mathrm{Im}\ \delta^0(\omega)(\sigma^{(1)}_{1,0,0}) = \{0,1\}$$

$$\begin{aligned}
\ker \delta^0(\omega)(\sigma_k^{(1)}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid A_{100}[[3]] \oplus A_{110}[[1]] = 0\}\\
\text{Im } \delta^0(\omega)(\sigma_{1,1,1}^{(1)}) =& \{0,1\}\\
\ker \delta^0(\omega)(\sigma_k^{(1)}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid A_{111}[[1]] \oplus A_{110}[[2]] = 0\}\\
\text{Im } \delta^0(\omega)(\sigma_{0,1,0}^{(1)}) =& \{0,1\}\\
\ker \delta^0(\omega)(\sigma_k^{(1)}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid A_{010}[[2]] \oplus A_{110}[[3]] = 0\}\\
\text{Im } \delta^0(\omega)(\sigma_{0,0,0}^{(1)\prime}) =& \{0,1\}\\
\ker \delta^0(\omega)(\sigma_k^{(1)\prime}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid P_{000}[[2]] \oplus P_{100}[[4]] = 0\}\\
\text{Im } \delta^0(\omega)(\sigma_{1,0,0}^{(1)\prime}) =& \{0,1\}\\
\ker \delta^0(\omega)(\sigma_k^{(1)\prime}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid P_{100}[[3]] \oplus P_{100}[[2]] = 0\}\\
\text{Im } \delta^0(\omega)(\sigma_{0,1,0}^{(1)\prime}) =& \{0,1\}\\
\ker \delta^0(\omega)(\sigma_k^{(1)\prime}) =& \{S \in C^0(\mathcal{U},\mathcal{E}) \mid P_{010}[[1]] \oplus P_{000}[[3]] = 0\}
\end{aligned}$$

Toric code on a cube

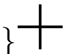
Contexts:

$$\begin{aligned}
A_{1,2,1} =& \quad \{X_{1,2,\frac{3}{2}}, X_{\frac{1}{2},2,1}, X_{\frac{3}{2},2,1}, X_{1,2,\frac{1}{2}}\}\\
A_{1,1,j} =& \quad \{X_{1,2,j}, X_{\frac{1}{2},1,j}, X_{\frac{3}{2},1,j}, X_{1,0,j}\},\ j \in \{0,2\}\\
A_{k,1,1} =& \quad \{X_{k,2,1}, X_{k,1,\frac{1}{2}}, X_{k,1,\frac{3}{2}}, X_{k,0,1}\},\ k \in \{0,2\}\\
A_{0,1,0} =& \quad \{X_{0,\frac{3}{2},0}, X_{0,1,\frac{1}{2}}, X_{\frac{3}{2},1,0}, X_{0,\frac{1}{2},0}\}\\
A_{2,1,0} =& \quad \{X_{2,\frac{3}{2},0}, X_{\frac{3}{2},1,0}, X_{0,1,\frac{1}{2}}, X_{2,\frac{1}{2},0}\}\\
A_{2,1,2} =& \quad \{X_{2,\frac{3}{2},2}, X_{2,1,\frac{3}{2}}, X_{\frac{3}{2},1,2}, X_{2,\frac{1}{2},2}\}\\
A_{0,1,2} =& \quad \{X_{0,\frac{3}{2},2}, X_{\frac{1}{2},1,2}, X_{0,1,\frac{3}{2}}, X_{0,\frac{1}{2},2}\}\\
A_{2,1,0} =& \quad \{X_{1,2,\frac{1}{2}}, X_{\frac{1}{2},2,0}, X_{\frac{3}{2},2,0}, X_{1,\frac{3}{2},0}\}\\
A_{2,2,1} =& \quad \{X_{\frac{3}{2},2,1}, X_{2,2,\frac{3}{2}}, X_{2,2,\frac{3}{2}}, X_{2,\frac{3}{2},1}\}\\
A_{1,2,2} =& \quad \{X_{1,2,\frac{3}{2}}, X_{\frac{3}{2},2,2}, X_{\frac{1}{2},2,2}, X_{1,1,2}\}\\
A_{0,2,1} =& \quad \{X_{\frac{1}{2},2,1}, X_{0,2,\frac{2}{2}}, X_{0,2,\frac{3}{2}}, X_{0,\frac{3}{2},1}\}\\
A_{0,2,0} =& \quad \{X_{0,2,\frac{1}{2}}, X_{\frac{3}{2},2,0}, X_{0,\frac{3}{2},0}\}\\
A_{2,2,0} =& \quad \{X_{\frac{3}{2},2,0}, X_{0,2,\frac{1}{2}}, X_{2,\frac{3}{2},0}\}\\
A_{2,2,2} =& \quad \{X_{2,2,\frac{3}{2}}, X_{\frac{3}{2},2,2}, X_{2,\frac{3}{2},2}\}\\
A_{0,2,2} =& \quad \{X_{\frac{1}{2},1,2}, X_{0,1,\frac{3}{2}}, X_{0,\frac{1}{2},2}\}\\
P_{k,2,j} =& \quad \{Z_{k+\frac{1}{2},2,j}, Z_{k,2,j-\frac{1}{2}}, Z_{k+\frac{1}{2},2,j-1}, X_{k+1,2,j-\frac{1}{2}}\},
\end{aligned}$$

$$
\begin{aligned}
& k \in \{0,1\},\ j \in \{1,2\} \\
P_{k,l,0} = \ & \{Z_{k+\frac{1}{2},l,0}, Z_{k+1,l+\frac{1}{2},0}, Z_{k+\frac{1}{2},l+1,0}, X_{k,l+\frac{1}{2},0}\}, \\
& k \in \{0,1\},\ l \in \{0,1\} \\
P_{k,l,2} = \ & \{Z_{k-\frac{1}{2},l,0}, Z_{k-1,l+\frac{1}{2},0}, Z_{k-\frac{1}{2},l+1,0}, X_{k,l+\frac{1}{2},0}\}, \\
& k \in \{1,2\},\ l \in \{0,1\} \\
P_{2,l,j} = \ & \{Z_{2,l,j+\frac{1}{2},j}, Z_{2,l+\frac{1}{2},j+1}, Z_{2,l+1,j+\frac{1}{2}}, X_{2,l+\frac{1}{2},j}\}, \\
& l \in \{0,1\},\ j \in \{0,1\} \\
P_{0,l,j} = \ & \{Z_{0,l,j-\frac{1}{2},j}, Z_{0,l+\frac{1}{2},j-1}, Z_{0,l+1,j-\frac{1}{2}}, X_{0,l+\frac{1}{2},j}\}, \\
& l \in \{0,1\},\ j \in \{1,2\}
\end{aligned}
$$

Simplices:

$\sigma^{(0)}_{k,l,j} = \langle A_{k,l,j} \rangle,\ k \in \{0,1,2\},\ l \in \{1,2\},\ j \in \{0,1,2\},$

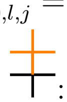
$klj \neq \{111\}$ 

$\sigma^{(0)\prime}_{k,l,j} = \langle P_{k,l,j} \rangle,\ k \in \{0,1,2\},\ l \in \{0,1,2\},\ j \in \{0,1,2\},$

$klj \notin \{111, 020, 120\}$ 

:

$$
\begin{aligned}
\sigma^{(1)}_{k,l,0} &= \langle A_{k,l,0}, A_{k+1,l,0} \rangle,\ k \in \{0,1\},\ l \in \{1,2\} \\
\sigma^{(1)}_{k,l,2} &= \langle A_{k,l,2}, A_{k-1,l,2} \rangle,\ k \in \{1,2\},\ l \in \{1,2\} \\
\sigma^{(1)}_{2,l,j} &= \langle A_{2,l,j}, A_{2,l,j+1} \rangle,\ l \in \{1,2\},\ j \in \{0,1\} \\
\sigma^{(1)}_{0,l,j} &= \langle A_{0,l,j}, A_{0,l,j-1} \rangle,\ l \in \{1,2\},\ j \in \{1,2\}
\end{aligned}
$$

:

$$
\begin{aligned}
\sigma^{(1)\prime}_{k,1,0} &= \langle A_{k,1,0}, A_{k,2,0} \rangle,\ k \in \{0,1\} \\
\sigma^{(1)\prime}_{k,1,2} &= \langle A_{k,1,2}, A_{k,2,0} \rangle,\ k \in \{1,2\} \\
\sigma^{(1)\prime}_{2,1,j} &= \langle A_{2,1,j}, A_{k,2,0} \rangle,\ j \in \{0,1\} \\
\sigma^{(1)\prime}_{0,1,j} &= \langle A_{0,1,j}, A_{k,2,0} \rangle,\ j \in \{1,2\}
\end{aligned}
$$

above:

$$
\begin{aligned}
\sigma^{(1)\prime\prime}_{1,2,0} &= \langle A_{1,2,0}, A_{1,2,1} \rangle \\
\sigma^{(1)\prime\prime}_{2,2,1} &= \langle A_{2,2,1}, A_{1,2,1} \rangle \\
\sigma^{(1)\prime\prime}_{1,2,2} &= \langle A_{1,2,2}, A_{1,2,1} \rangle \\
\sigma^{(1)\prime\prime}_{0,2,1} &= \langle A_{0,2,1}, A_{1,2,1} \rangle
\end{aligned}
$$

:

$$\sigma^{(1)'''}_{k,l,0} = \langle P_{k,l,0}, P_{k+1,l,0}\rangle\,,\ k \in \{0,1\},\ l \in \{0,1\}$$
$$\sigma^{(1)'''}_{k,l,2} = \langle P_{k,l,2}, P_{k-1,l,2}\rangle\,,\ k \in \{1,2\},\ l \in \{0,1\}$$
$$\sigma^{(1)'''}_{2,l,j} = \langle P_{2,l,j}, P_{2,l,j+1}\rangle\,,\ l \in \{0,1\},\ j \in \{0,1\}$$
$$\sigma^{(1)'''}_{0,l,j} = \langle P_{0,l,j}, P_{0,l,j-1}\rangle\,,\ l \in \{0,1\},\ j \in \{1,2\}$$

:

$$\sigma^{(1)''''}_{k,l,0} = \langle P_{k,l,0}, P_{k,l+1,0}\rangle\,,\ k \in \{0,1\},\ l \in \{0,1\}$$
$$\sigma^{(1)''''}_{k,l,2} = \langle P_{k,l,2}, P_{k,l+1,0}\rangle\,,\ k \in \{1,2\},\ l \in \{0,1\}$$
$$\sigma^{(1)''''}_{2,l,j} = \langle P_{2,l,j}, P_{k,l+1,0}\rangle\,,\ l \in \{0,1\},\ j \in \{0,1\}$$
$$\sigma^{(1)''''}_{0,l,j} = \langle P_{0,l,j}, P_{k,l+1,0}\rangle\,,\ l \in \{0,1\},\ j \in \{1,2\}$$

above:

$$\sigma^{(1)'''''}_{0,2,2} = \langle P_{0,2,2}, P_{1,2,2}\rangle$$
$$\sigma^{(1)'''''}_{1,2,2} = \langle P_{1,2,2}, P_{1,2,1}\rangle$$
$$\sigma^{(1)'''''}_{1,2,1} = \langle P_{1,2,1}, P_{0,2,1}\rangle$$
$$\sigma^{(1)'''''}_{0,2,1} = \langle P_{0,2,1}, P_{0,2,2}\rangle$$

Nerves:

$$\mathcal{N}(\mathcal{U})^0 = \Big\{\sigma^{(0)}_{k,l,j},\ k \in \{0,1,2\},\ l \in \{1,2\},\ j \in \{0,1,2\},\ klj \neq \{111\},\ \sigma^{(0)'}_{k,l,j},\ k \in \{0,1,2\},\ l \in \{0,1,2\},\ j \in \{0,1,2\},\ klj \notin \{111,020,120\}\Big\}$$

$$\mathcal{N}(\mathcal{U})^1 = \Big\{\sigma^{(1)}_{k,l,j},\ klj \in \{010,020,110,120,112,122,212,222,210,211,220,221,011,012,021,022\},$$
$$\sigma^{(1)'}_{k,l,j},\ klj \in \{010,110,112,212,210,211,011,012\},$$
$$\sigma^{(1)''}_{k,l,j},\ klj \in \{120,221,122,021\},$$
$$\sigma^{(1)'''}_{k,l,j},\ klj \in \{010,020,110,120,112,122,212,222,210,211,220,221,011,012,021,022\},$$
$$\sigma^{(1)''''}_{k,l,j},\ klj \in \{000,010,100,110,102,112,202,212,200,201,210,211,001,002,011,012\},$$
$$\sigma^{(1)'''''}_{k,l,j},\ klj \in \{022,122,121,021\},\Big\}$$

Sheaf:

$$\mathcal{E}(A_{k,l,j}) = \{0000,0011,0101,0110,1001,1010,1100,1111,$$

$$
\begin{aligned}
& k \in \{0,1,2\},\ l \in \{1,2\},\ j \in \{0,1,2\}, \\
& klj \notin \{111, 020, 220, 222, 022\}\} \\
\mathcal{E}(A_{k,l,j}) = \ & \{000, 011, 101, 110,\ klj \in \{020, 220, 222, 022\}\} \\
\mathcal{E}(P_{k,l,j}) = \ & \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111, \\
& k \in \{0,1,2\},\ l \in \{0,1,2\},\ j \in \{0,1,2\}, \\
& klj \notin \{111, 020, 120\}\} \\
\mathcal{E}(|\sigma^{(1)}_{k,l,0}|) = \ & \mathcal{E}(\{X_{k+\frac{1}{2},l,0}\}) = \{0,1\},\ k \in \{0,1\},\ l \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)}_{k,l,2}|) = \ & \mathcal{E}(\{X_{k-\frac{1}{2},l,2}\}) = \{0,1\},\ k \in \{1,2\},\ l \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)}_{2,l,j}|) = \ & \mathcal{E}(\{X_{2,l,j+\frac{1}{2}}\}) = \{0,1\},\ l \in \{1,2\},\ j \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)}_{0,l,j}|) = \ & \mathcal{E}(\{X_{0,l,j-\frac{1}{2}}\}) = \{0,1\},\ l \in \{1,2\},\ j \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{k,1,0}|) = \ & \mathcal{E}(\{X_{k,\frac{3}{2},0}\}) = \{0,1\},\ k \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{k,1,2}|) = \ & \mathcal{E}(\{X_{k,\frac{3}{2},2}\}) = \{0,1\},\ k \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{2,1,j}|) = \ & \mathcal{E}(\{X_{2,\frac{3}{2},j}\}) = \{0,1\},\ j \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime}_{0,1,j}|) = \ & \mathcal{E}(\{X_{0,\frac{3}{2},j}\}) = \{0,1\},\ j \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime}_{1,2,0}|) = \ & \mathcal{E}(\{X_{1,2,\frac{1}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime}_{2,2,1}|) = \ & \mathcal{E}(\{X_{\frac{3}{2},2,1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime}_{1,2,2}|) = \ & \mathcal{E}(\{X_{1,2,\frac{3}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime}_{0,2,1}|) = \ & \mathcal{E}(\{X_{\frac{1}{2},2,1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime}_{k,l,0}|) = \ & \mathcal{E}(\{Z_{k+1,l+\frac{1}{2},0}\}) = \{0,1\},\ k \in \{0,1\},\ l \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime}_{k,l,2}|) = \ & \mathcal{E}(\{Z_{k-1,l+\frac{1}{2},2}\}) = \{0,1\},\ k \in \{1,2\},\ l \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime}_{2,l,j}|) = \ & \mathcal{E}(\{Z_{2,l+\frac{1}{2},j+1}\}) = \{0,1\},\ l \in \{0,1\},\ j \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime}_{0,l,j}|) = \ & \mathcal{E}(\{Z_{0,l+\frac{1}{2},j-1}\}) = \{0,1\},\ l \in \{0,1\},\ j \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime}_{k,l,0}|) = \ & \mathcal{E}(\{Z_{k+\frac{1}{2},l+1,0}\}) = \{0,1\},\ k \in \{0,1\},\ l \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime}_{k,l,2}|) = \ & \mathcal{E}(\{Z_{k-\frac{1}{2},l+1,2}\}) = \{0,1\},\ k \in \{1,2\},\ l \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime}_{2,l,j}|) = \ & \mathcal{E}(\{Z_{2,l+1,j+\frac{1}{2}}\}) = \{0,1\},\ l \in \{0,1\},\ j \in \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime}_{0,l,j}|) = \ & \mathcal{E}(\{Z_{0,l+1,j-\frac{1}{2}}\}) = \{0,1\},\ l \in \{0,1\},\ j \in \{1,2\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime\prime}_{0,2,2}|) = \ & \mathcal{E}(\{Z_{1,2,\frac{1}{2}}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime\prime}_{1,2,2}|) = \ & \mathcal{E}(\{Z_{\frac{3}{2},2,1}\}) = \{0,1\} \\
\mathcal{E}(|\sigma^{(1)\prime\prime\prime\prime\prime}_{1,2,1}|) = \ & \mathcal{E}(\{Z_{1,2,\frac{3}{2}}\}) = \{0,1\}
\end{aligned}
$$

$$\mathcal{E}(|\sigma^{(1)'''''}_{0,2,1}|) = \mathcal{E}(\{Z_{\frac{1}{2},2,1}\}) = \{0,1\}$$

Cochains:
$$C^0(\mathcal{U},\mathcal{E}) = \prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^0} \mathcal{E}(|\sigma_{k,l,j}|) = (\mathcal{E}(A_{k,l,j})^{\times 2\times 2},\, \mathcal{E}(P_{k,l,j})^{\times 2\times 2})$$
$$C^1(\mathcal{U},\mathcal{E}) = \prod_{\sigma_{k,l,j}\in\mathcal{N}(\mathcal{U})^1} \mathcal{E}(|\sigma_{k,l,j}|) = (\{0,1\}^{\times 2\times 2\times 2},\, \{0,1\}^{\times 2\times 2\times 2},$$
$$\{0,1\}^{\times 2\times 2\times 2},\, \{0,1\}^{\times 2\times 2\times 2},\, \{0,1\}^{\times 2\times 2\times 2},\, \{0,1\}^{\times 2\times 2\times 2})$$