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The free scalar Boson
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THE FREE SCALAR BOSON

So far, we have analysed conformal symmetry transformations and invariance of a (classical) field theory under them with the help of Weyl-scalings. In this tutorial, we discuss one very important example of a conformally invariant theory, the free massless scalar Boson. This theory is governed by the action

$$S = \int \mathrm{d}^d x \sqrt{\det g}\, g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x)\,,$$

where we omit a factor of proportionality for the sake of simplicity. We will also assume, for this tutorial, that we chose the signature of the metric $g_{\mu\nu}$ such that $\det g > 0$.

[P1] *Variation of the action*
Let $\delta g_{\mu\nu}$ be a given variation of the metric (note that the indices are lower one). In order to compute

$$\delta S = \frac{1}{2} \int \mathrm{d}^d x \sqrt{\det g}\, T^{\mu\nu} \delta g_{\mu\nu}\,,$$

proceed in the following way:

(a) Compute $\delta g^{\mu\nu}$ by using the triviality $\delta(\delta_\mu^{\ \nu}) = 0$.

(b) Compute $\delta\sqrt{\det g}$ by using the relation $\sqrt{\det g} = \exp(\frac{1}{2}\log(\det g))$. Use the chain rule and compute first $\delta \log(\deg g)$. For this, you will need the important relation $\det A = \exp(\mathrm{tr} \log A)$.

(c) Put things together and find δS. Read off the energy-momentum tensor.

[P2] *Energy-momentum tensor*
In the preceding exercise, you should have obtained the energy-momentum tensor for the theory of the free massless scalar Boson to read

$$T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi\,.$$

To find the symmetry invariances of this theory, proceed as follows:

(a) First, compute further the equation of motion for the free massless scalar Boson.

(b) Show that the energy-momentum tensor is conserved on-shell, i.e. that $\partial^\mu T_{\mu\nu} = 0$.

(c) Compute the trace of the energy-momentum tensor. For which dimension of space-time is the energy-momentum tensor traceless?

[P3] *Explicit check in flat space-time*
Check the validity of the results from the preceding exercise by computing the transformation of the action under a conformal coordinate transformation in case of the flat metric $g_{\mu\nu} = \eta_{\mu\nu}$. Use a convention such that $\sqrt{\det \eta} = 1$. Show that indeed the theory is conformally invariant if and only if $d = 2$.

[P4] *Massive case*
Contemplate, why in the case $d = 2$ the action $S_m = S + \int \mathrm{d}^2 x m^2 \phi^2$ is no longer conformally invariant.