Michael Flohr
Derek Harland
24. April 2009

QUANTIZATION OF THE FREE SCALAR BOSON

We come back to the prime example of quantum field theory, the free Boson. Up to now, we only dealt with classical theory. Now, we are going to quantize it. In this tutorial, you should repeat or learn the way of canonical quantization with a straight forward example. We consider a theory of N species of free massless scalar Bosons in a two-dimensional space time, governed by the action

$$S = \frac{1}{8\pi}\int \mathrm{d}^2x \sum_i \partial_\mu\Phi^i(x)\partial^\mu\Phi(x)^i = \frac{1}{8\pi}\int \mathrm{d}^2x \sum_i \left((\partial_0\Phi^i(x))^2 - (\partial_1\Phi(x)^i)^2\right)\,,$$

where the normalization has been chosen such that certain formulæ later will simplify nicely. We consider the Bosons on a cylinder, i.e. we chose the spatial coordinate to be compactified. This means that $\Phi^i(x^0, x^1 + 2\pi) = \Phi^i(x^0, x^1)$. This periodicity implies that the field Φ^i has a Fourier expansion of the form

$$\Phi^i(x^0, x^1) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{inx^1} f_n^i(x^0)\,,$$

where the functions $f_n^i(x^0)$ are, at this state, arbitrary.

[P1] *The mode expansion of the free field*
To further restrict the functions $f_n^i(x^0)$ proceed in the following way:

(a) Compute the equations of motion and the canonical conjugate momenta of the fields Φ^i.

(b) Use the equation of motion for Φ^i together with the Fourier expansion above to find differential equations for the functions $f_n^i(x^0)$. Give the general solutions for these differential equations (caveat: distinguish the cases $n \neq 0$ and $n = 0$).

(c) Put things together and write down the mode expansion for the free fields $\Phi^i(x^0, x^1)$. With appropriate normalizations, this should finally read

$$\Phi^i(x^0, x^1) = q^i + 2p^i x^0 + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^i \mathrm{e}^{-\mathrm{i}n(x^0+x^1)} + \tilde{\alpha}_n^i \mathrm{e}^{-\mathrm{i}n(x^0-x^1)}\right]\,.$$

(d) Verify that this mode expansion is manifestly in light cone coordinates $x^0 \pm x^1$. Thus, although only the spatial direction is compact and periodic, the free field propagates doubly periodic in both light cone coordinate directions.

[P2] *Canonical quantization*
In the preceding exercise, you should have obtained the canonical momenta $\Pi^i = \frac{1}{4\pi}\partial_0\Phi^i$. We now demand the canonical (equal time) commutation relations

$$\begin{aligned}
[\Phi^i(x^0, x^1), \Pi^j(x^0, y^1)] &= \mathrm{i}\delta^{ij}\delta(x^1 - y^1)\,,\\
[\Phi^i(x^0, x^1), \Phi^j(x^0, y^1)] &= 0\,,\\
[\Pi^i(x^0, x^1), \Pi^j(x^0, y^1)] &= 0\,.
\end{aligned}$$

Note that on the cylinder, the definition of the delta distribution is such that $\int_0^{2\pi}\mathrm{d}x^1\delta(x^1 - y^1) = 1$. Proceed now as follows:

(a) First, write down the mode expansions of the canonical momenta Π^i.

(b) Define the modes $X^i_m(x^0) = \int_0^{2\pi} \mathrm{d}x^1 \mathrm{e}^{\mathrm{i}mx^1} \Phi^i(x^0, x^1)$ and $P^j_n(x^0) = \int_0^{2\pi} \mathrm{d}x^1 \mathrm{e}^{\mathrm{i}nx^1} \Pi^j(x^0, x^1)$. Compute with the help of the canonical commutation relations the commutators $[X^i_m, X^j_n]$, $[P^i_m, P^j_n]$ and $[X^i_m, P^j_n]$.

(c) For later, we also need the explicit form of the modes $X^i_n(x^0)$ and $P^j_n(x^0)$ in terms of the coefficients of the full mode expansion you computed in [P1].(c). You should again distinguish the cases $n \neq 0$ and $n = 0$.

(d) Use your last result to find the commutation relations for the modes $\alpha, \tilde{\alpha}, p$ and q.

[P3] *From the cylinder to the plane*

By performing a Wick rotation $x^0 \mapsto -\mathrm{i}x^2$, let us introduce complex coordinates $z = \mathrm{e}^{\mathrm{i}(x^0+x^1)} = \mathrm{e}^{x^2+\mathrm{i}x^1}$ and $\bar{z} = \mathrm{e}^{\mathrm{i}(x^0-x^1)} = \mathrm{e}^{x^2-\mathrm{i}x^1}$. Write down the mode expansion in z and $\bar{z}$, i.e. for the field $\Phi^i(z, \bar{z})$.