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08. May 2009

OPERATOR PRODUCT EXPANSIONS AND THE FREE BOSON

We once more come back to the prime example of quantum field theory, the free Boson. Recall that classically the energy-momentum tensor for a theory of one massless scalar free Boson takes the form

$$\begin{aligned} T_{\mu\nu} &= -\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi \\ \Longrightarrow\ T_{zz} &= -\frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}\underbrace{g_{zz}}_{=0}g^{z\bar{z}}\bar{\partial}\phi\partial\phi \\ &= -\frac{1}{2}\partial\phi\partial\phi\,. \end{aligned}$$

In a quantized theory, the product of two operators at the same point in space-time is not defined. Instead, one has to use normal ordered products. Hence, in the conformally invariant quantum field theory of scalar Bosons, the quantum energy-momentum tensor reads

$$\begin{aligned} T(z) &= -\frac{1}{2}\sum_j :\partial\Phi^j(z)\partial\Phi^j(z): \\ &= -\frac{1}{2}\sum_j \lim_{w\to z}\left[\mathcal{R}\partial\Phi^j(w)\partial\Phi^j(z) + \frac{\delta^{jj}}{(z-w)^2}\right], \end{aligned}$$

where we now consider a theory of various species of free Bosons. To be specific, we call the number of Boson species c. To solve the following exercises, all you need is the operator product expansion (OPE)

$$\mathcal{R}\Phi^j(z,\bar{z})\Phi^k(w,\bar{w}) = -\delta^{jk}\left[\log(z-w) + \log(\bar{z}-\bar{w})\right] + :\Phi^j(z,\bar{z})\Phi^k(w,\bar{w}):$$

and Wick's theorem for normal ordered products of free fields.

[P1] *Derivatives of fields*
Verify that the OPE of $\partial\Phi^j(z)$ with $\partial\Phi^k(w)$ reads

$$\mathcal{R}\partial\Phi^j(z)\partial\Phi^k(w) \sim -\frac{\delta^{jk}}{(z-w)^2}\,.$$

[P2] *Primary fields*
Show that the fields $\partial\Phi^k(z)$ are primary fields of conformal weight $h = 1$ by computing the OPE $\mathcal{R}T(z)\partial\Phi^k(w)$. Recall that a field $\Psi(w)$ is primary of weight h if and only if its OPE with the energy-momentum temsor $T(z)$ has the form

$$\mathcal{R}T(z)\Psi(w) \sim \frac{h}{(z-w)^2}\Psi(w) + \frac{1}{(z-w)}\partial\Psi(w)\,.$$

[P3] *Energy-momentum tensor*
Show that the OPE of the energy-momentum tensor of a theory of c species of free scalar Bosons Φ^j, $j = 1, 2, \ldots, c$, with itself is given by

$$\mathcal{R}T(t)T(w) \sim \frac{c/2}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w)\,.$$

Thus, the energy-momentum tensor is not a primary field due to the presence of the $(z-w)^{-4}$ term. However, it is a quasiprimary field, i.e. a field which is a rank two tensor under global conformal transformations (the Möbius group), but not under all local conformal transformations.