Exercise V for the lecture course CONFORMAL FIELD THEORY

Conformal Ward identity

Michael Flohr
Derek Harland
29. May 2009

CONFORMAL WARD IDENTITY

We learned in the lecture course that a *local* conformal transformation $z \mapsto z' = z(1-\varepsilon(z))$ with arbitrary meromorphic function $\varepsilon(z)$ is generated by the Noether charge

$$Q_\varepsilon = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,\varepsilon(z)T(z)\,.$$

Let us now consider a generic correlation function (a.k.a. vacuum expectation value)

$$G^{(N)}(\{w_i\}) = \langle 0|\phi_1(w_1)\phi_2(w_2)\ldots\phi_N(w_N)|0\rangle\,,$$

where ϕ_i are some, not necessarily primary, fields. The aim of this tutorial is to derive the conformal Ward identity and to use it in order to prove that all correlation functions involving descendant fields can entirely be computed in terms of correlation functions with solely primary fields.

[P1] *Conformal Ward identity*
To derive the conformal Ward identity, proceed in two steps:

(a) Show that the variation $\delta_\varepsilon G^{(N)}(\{w_i\})$ is given by

$$\delta_\varepsilon G^{(N)}(\{w_i\}) = \sum_k \langle 0|\phi_1(w_1)\ldots(\delta_\varepsilon\phi_k(w_k))\ldots\phi_N(w_N)|0\rangle$$

for all ε. This means that the variation of the correlation function works as a derivation, involving the variations of the individual fields. Hint: Deform the integration contour into a sum of small contours encircling the individual field insertions.

(b) Let us now assume that the fields ϕ_i are primary fields with conformal weights h_i. Since the above equation is valid for all ε, we can use it together with the OPE of the energy momentum tensor with primary fields to find the conformal Ward identity

$$\langle 0|T(z)\phi_1(w_1)\ldots\phi_N(w_N)|0\rangle = \sum_k\left[\frac{h_k}{(z-w_k)^2}+\frac{1}{(z-w_k)}\partial_{w_k}\right]\langle 0|\phi_1(w_1)\ldots\phi_N(w_N)|0\rangle\,.$$

Therefore, the action of the energy momentum tensor on a correlation function of primary fields is given in terms of a differential operator

$$\mathcal{L}_{-2}(z) = \sum_k\left[\frac{h_k}{(z-w_k)^2}+\frac{1}{(z-w_k)}\partial_{w_k}\right]\,,$$

[P2] *Correlation functions with descendant fields*
Let us now consider a correlation function with $N-1$ primary fields $\phi_i(w_i)$, and one descendant field

$$\phi_N^{(-k)}(w_N) = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,\frac{1}{(z-w_N)^{k-1}}T(z)\phi_N(w_n)\,.$$

We want to express this correlation function $\langle 0|\phi_1(w_1)\ldots\phi_{N-1}(w_{N-1})\phi_N^{(-k)}(w_n)|0\rangle$ in terms of the function $G^{(N)}(\{w_i\})$. To do so, we proceed as follows:

(a) Use contour deformation to replace the small contour encircling w_N by a large contour encircling all the w_i, $i=1,\ldots,N$ and $N-1$ small contours encircling the individual w_j, $j=1,\ldots,N-1$. Plug in the OPE of $T(z)$ with the primary fields ϕ_j in the latter $N-1$ contour integrals.

(b) The first term with the large contour can be deformed on the Riemann sphere into a small contour encircling the point infinity. Show that this term does not contribute for any $k \geq -1$ due to the highest weight property of the state $\langle 0|$.

(c) Put everything together to find the result

$$\langle 0|\phi_1(w_1)\dots\phi_{N-1}(w_{N-1})\phi_N^{(-k)}(w_N)|0\rangle = \mathcal{L}_{-k}(w_N)\langle 0|\phi_1(w_1)\dots\phi_N(w_N)|0\rangle$$

with the differential operator

$$\mathcal{L}_{-k}(w_N) = \sum_{j=1}^{N-1}\left[-\frac{(1-k)h_j}{(w_j-w_N)^k} + \frac{1}{(w_j-w_N)^{k-1}}\partial_{w_j}\right] .$$

(d) Contemplate in a hand waving way how this can be generalized to many multiple descendant fields $\phi_l^{(-k_1,-k_2,\dots,-k_l)}(w_l)$.

Thus, all correlation functions can be expressed in terms of correlation functions of solely primary fields. This means that conformal symmetry alone allows us to compute arbitrary correlation functions as long as we know how to compute the correlation functions of solely primary fields.